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Preface

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This book maps TED ESF Workshop, *CMP'04: Multiple Participants Decision Making*, held in Institute of Information Theory and Automation Academy of Sciences of the Czech Republic. It extends the series on *computer-intensive problems in data processing and control* or, generally, on dynamic decision making.

History: The joint UK-CS seminar (Warwick, et al. 1991), Prague 1990, is a predecessor of the series. The idea of the future workshop format and the emerging topic were formulated during it. In 1992, the IFAC sponsored workshop (Kárný & Warwick 1993) started the series. Then, the 1st and 2nd IEEE workshops *Computer-Intensive Methods in Control and Signal Processing* were held in 1994 (Kulhavá, et al. 1994, Kárný, et al. 1997) and 1996 (Warwick & Kárný 1997), respectively. The 3rd IEEE workshop held in 1998 (Rojíček, et al. 1998) extended the area covered to data processing. All referred events were successful but the last one indicated a loss of the characteristic series drive. A slight resignation on prolongation of the series made us to join in 2001 the conference ICANNGA (Kůrková, et al. 2001) and the workshop story seemed to be over.

Almost inevitably, the persistence of the addressed problems has brought a new impulse. The project of the European Science Foundation (ESF) *TED: Towards Electronic Democracy – Internet-based Approach* has revealed that the central series theme is even more important in societal than technical systems. This made us to prolong the series with the same aim but with a shift to decision problems with *multiple participants*.

Motivation: Due to the rapid increase in readily available computing power, a corresponding increase in the complexity of problems being tackled has occurred in many fields. A plethora of new methods has also arisen with a permanent desire deal with a more and more complex applications. Increasing the accuracy in models employed along with the use of the appropriate algorithms call for computations of a very high dimension. This bring with it a whole new breed of problem, which has come to be known as *The Curse of Dimensionality*. The need to solve this problem via distribution of modelling and decision making activities has brought qualitatively new dimension into the bunch of (sub)problems connected with the curse.

Book Aim: The Workshop has brought together researchers of the variety of fields affected by the curse to both learn from and swap experience. It has managed to cross usual boundary between researcher addressing technical and societal multiple-participant (distributed) decision making. The invited lectures of leading experts should help the reader to orient himself in the still messy workshop domain. Selected contributed papers could inspire him on important problems and on clever ideas to be transferred to other domains, to demonstrate advantages and deficiencies of existing solutions, to reveal similarities hidden behind field slang, etc. At general level, importance of *distributed dynamic* decision making as the only systematic methodology for overcoming the curse of dimensionality should become more obvious.

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Satisficing Games for Multiple-Participant Coordinated Decision Making

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Abstract. Nash equilibria requires each participant to optimize its own performance, regardless of the effect so doing has on the other participants. This feature limits the ability of von Neumann-Morgenstern game theory as a design paradigm for group behavior where coordination is required, since it cannot simultaneously accommodate both group and individual preferences. By replacing the demand for doing the best thing possible for the individual with a mathematically precise notion of being "good enough," satisficing game theory allows both group and individual interests to emerge from conditional local preferences. Altruistic behavior is an important aspect of cooperative behavior. In this paper we contrast *categorical altruism* (unconditionally modifying one's utilities to accommodate the interests of another) with *situational altruism* (conditional modification of one's utilities to accommodate the interests of another), where the conditioning is under the control of the one offering the accommodation.

Keywords: game theory, rationality, cooperation, altruism, emergent behavior

1 Introduction

Von Neumann-Morgenstern (von Neumann & Morgenstern 1944) game theory is perhaps the most well-known approach to multiple-participant decision making, and the most well-known game-theoretic solution concept is Nash equilibria (Nash 1951). A strategy is a Nash equilibrium if no participant can improve its level of satisfaction by making a unilateral change. In this sense, the Nash equilibrium is a constrained optimal solution to a multiple participant decision problem. This solution concept is appropriate when each participant is committed to individual rationality; that is, each desires to maximize its own expected utility (without regard for the welfare of others) under the assumption that all other participants will behave similarly. Conventional game theory loses much of its instrumentality, however, if the participants' spheres of concern extend beyond the individual. In this more general case, the participants are not narrowly self-interested, but are willing to entertain compromises that may require them to sacrifice some of their individual benefit in order to benefit a group. Unfortunately, it is generally impossible to reconcile both group and individual interests under conventional game theory, as Arrow's impossibility theorem attests (Arrow 1951), and most game theorists frown on attempts to couch notions of group welfare under the game-theoretic rubric (Shubik 1982). Hence, von Neumann-Morgenstern game theory is of limited value for characterizing decision problems where the participants entertain sociological concerns such as cooperation, fairness, altruism, etc. As adduced by Luce and Raiffa over forty years ago, "... general game theory seems to be in part a sociological theory which does not include any sociological assumptions ... it may be too much to ask that any sociology be derived from the single assumption of individual rationality" (Luce & Raiffa 1957, p. 196).

One way to address the limitations of individual rationality is to modify the game by redefining the individual utility functions so they represent an aggregate of the interests of all individuals in the group. The game is then played according to the standard rules. While this approach may have some value, it merely simulates concepts of cooperation, benevolence, and altruism with a regime that is fundamentally committed to competition, exploitation, and avarice. As Sen (1990, p. 19) observed: "It is possible to define a person's interests in such a way that no matter what he does he can be seen to be furthering his own interests in every isolated act of choice ... no matter whether you are a single-minded egoist or a raving altruist or a class-conscious militant, you will appear to be maximizing your own utility in this enchanted world of definitions." Such manipulations of utility functions *simply for the purpose of applying a particular solution methodology* are difficult to justify. It is much more reasonable for the methodology to fit the problem, rather than to attempt to force the problem to fit a particular methodology. A more natural approach to cooperative multiple participant decision making would be to employ a solution methodology that is designed, from its inception, expressly to accommodate the fact that the participants have interests that extend beyond the individual.

One reason why conventional game theory is attractive is that it is built on a firm mathematical principle of optimization. A useful feature of the theory is that it abstracts the problem scenario from its context

and expresses it in terms of mathematical functions (e.g., payoff arrays) that can be subjected to analytical investigations to arrive at Nash equilibria. This capability is of such enormous value that investigators may be motivated to impose the game theoretic structure on a problem primarily because doing so leads to a solution concept that is both systematic and defensible, even if it requires an unrealistic assumption of narrow self-interest on the participants.

This paper presents an alternative approach to multiple-participant decision making that does not rely upon the premise of individual rationality. It provides a mathematically rigorous and systematic solution concept that accommodates group as well as individual interests. The key feature of this approach is that *optimization* is no longer the theoretical foundation. Rather, this concept is replaced by a mathematically rigorous notion of good enough, or *satisficing*. Section 2 presents a new concept of satisficing and presents the mathematical foundation of *satisficing games*. Section 3 presents a new concept of altruism that is made possible by satisficing game theory, called *situational altruism*. Section 4 describes how group and individual preferences can emerge as conditional preferences propagate throughout a multiple participant system. Section 5 finishes with a discussion and conclusions.

2 Satisficing

Perhaps the most well-known principle of decision making is the classical economics hypothesis of Bergson and Samuelson, which asserts that individual interests are fundamental; i.e., that social welfare is a function of individual welfare (Bergson 1938, Samuelson 1948). This hypothesis leads to the doctrine of **rational choice,** which is that "each of the individual decision makers behaves as if he or she were solving a constrained maximization problem" (Hogarth & Reder 1986, p. 3). This paradigm is the basis of much of conventional decision theory that is used in economics, the social and behavioral sciences, and engineering. Optimization is the mathematical formalization of the concept "only the best will do." There is another concept of quality, however, that is more primitive than optimization, yet has a definite appeal, namely, the concept of "getting your money's worth," which we define as *satisficing*, or being good enough. The difference between these two criteria is that optimizing requires a global comparison of the attributes of all options against each other to select the one with the highest utility, while satisficing requires a local comparison of positive and negative attributes of each option individually. Thus, whereas optimization requires the formation of a single utility to aggregate all relevant attributes of an option into a single function, the satisficing approach presented in this paper employs a dual utilities approach, that is, it separately evaluates the attributes that are desirable to the decision maker (benefits) and the attributes that are undesirable (costs). We first formalize the notion of satisficing for a single decision maker, then generalize to the multiple participant case.

2.1 Single-Agent Satisficing

Definition 1. Let U denote a finite set of options. A *mass function*, denoted p, is a mapping of U onto the unit interval such that

$$p(u) \geq 0 \; \forall u \in U, \qquad \sum_{u \in U} p(u) = 1.$$

A common use of mass functions is to characterize the probability distribution of a random variable. In our application, however, we employ mass functions in a way analogous to probability theory, but with a different interpretation. In the interest of making local comparisons of attributes of options, we will define two mass functions.

Definition 2. Let U denote a finite set of options. A *selectability mass function*, denoted p_S, is a mapping of U onto the unit interval such that $p_S(u)$ characterizes the degree of support that is attributed to u in the interest of accomplishing whatever goal is relevant to the decision maker.

A *rejectability mass function*, denoted p_R, is a mapping of U onto the unit interval such that $p_R(u)$ characterizes the degree to which u consumes whatever resources are at the disposal of the decision maker as it performs its duties.

Definition 3. Let U denote a finite set of options. An option u is said to be *satisficing at caution level* q if the degree of support for its implementation is at least as great as q times the degree to which it consumes resources. The *satisficing set at caution level* q is

$$\Sigma_q = \{u \in U \colon p_S(u) \geq q p_R(u)\}. \tag{1}$$

The following example illustrates the difference between optimizing and satisficing.

Example 1. Consider the decision problem of choosing which automobile to purchase. To keep the problem simple, assume that the set of possibilities consists of five choices, which we denote as vehicles A through E. The option space is the set $U = \{A, B, C, D, E\}$. Only three criteria are important: performance, reliability, and affordability. Suppose we assign ordinal rankings to the vehicles in each of these attributes, as illustrated in Table 1. Vehicle B, for example, has the best performance, the median reliability, and the highest cost.

Vehicle	Performance	Reliability	Affordability
A	3	1	5
B	5	3	1
C	2	4	4
D	1	5	3
E	4	2	2

Key: 5 = best; 4 = next best; 3 = median; 2 = next worst; 1 = worst

Table 1. Ordinal rankings of vehicle attributes.

There are a number of questions that may be asked regarding the issue of how to choose from among the options presented. The optimizer's question is perhaps the most direct: "What is the best deal?" To address this question, we must define a preference function. Let us define this function as the equally-weighted sum of the ordinal rankings of the three attributes; that is,

$$J = \text{Performance} + \text{Reliability} + \text{Affordability},$$

where the affordability number is ordered such that the cheaper vehicle has the higher affordability. The values of this preference function are displayed in the second column of Table 2. Clearly, the uniquely optimal option is vehicle C, the choice that, although next worst in performance, is next best in both reliability and price. But asking for the best deal is not the only rational question one might compose. For example, suppose we frame the question as: "Am I really going to get what I pay for?" This question does not involve making inter-comparisons among options; rather, it involves intracomparisons of attributes for each individual vehicle. To make these intracomparisons, a natural procedure is to separate the attributes into two categories: one to involve the attributes that represent gains to the decision maker as a result of adopting the option, and the other to involve attributes that represent losses. A natural, but not unique, categorization of this problem is to identify performance and reliability as gains, and cost as loss. To compare gains and losses, they must be represented on the same scale. This may be done by creating selectability and rejectability functions and normalizing the problem so that the decision maker has a unit of gain utility and a unit of loss utility to apportion among the options. We may do this by normalizing ordinal rankings associated with each category, yielding the last two columns of Table 2, where we have reversed the ordering on the affordability attribute to convert it to cost. These values constitute the selectability and rejectability functions, respectively. We observe that selectability exceeds rejectability for options A and C, the gain equals the loss for D, and the loss exceeds the gain for B and E.

Vehicle	Global Preference (J)	Normalized Gain (p_S)	Normalized Loss (p_R)
A	9	0.133	0.067
B	9	0.267	0.333
C	10	0.200	0.133
D	9	0.200	0.200
E	8	0.200	0.267

Table 2. Global preference and normalized gain/loss functions.

Figure 1 provides a cross plot of selectability versus rejectability as u is varied over its domain, with p_R the abscissa and p_S the ordinate. The caution index q is taken as unity. Observe that although A has the lowest selectability, it also has the lowest rejectability, and a rational decision maker can legitimately come to

the conclusion that this is satisficing, since the benefits at least outweigh the costs. Option B is at the other extreme. It has the highest performance, but is also the most expensive. In the value system of the customer, however, the benefits do not outweigh the costs, and the option is not satisficing. Option E is also easily eliminated by the cost-benefit test. Now consider options C and D. Both are satisficing, and the customer would be justified in choosing either one. Choosing D, however, would cost more than C without offering increased benefit, hence should be preferred. Notice that a distinctive feature of the satisficing approach is that, in contrast to optimization, it does not seek a unique solution, but instead provides a set of solutions. The final step in the decision process, therefore, is to implement a tie-breaker to settle on a final choice.

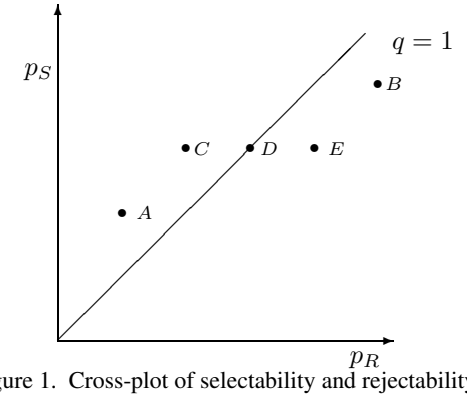

Figure 1. Cross-plot of selectability and rejectability.

2.2 Multiple Participant Satisficing

The advantage of using mass functions as utilities is that they may can be extended to the multiple-participant case in a way analogous to the way probability mass functions can be extended to the multivariate case to characterize random vectors.

Definition 4. Let $I = \{1, 2, \ldots, N\}$ denote a set of N decision makers, and let U_i denote the option set for each $i \in I$. The set of joint options is the product set $\mathbf{U} = U_1 \times U_2 \times \cdots \times U_N$, and the elements of $\mathbf{U}$ are vectors of the form $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, where $u_i \in U_i$. The *joint rejectability mass function*, denoted $p_{S_1 S_2 \cdots S_n}$ is a mapping from $\mathbf{U}$ to the unit interval such that

$$p_{S_1 \cdots S_N}(u_1, \ldots, u_N) \geq 0 \; \forall (u_1, \ldots, u_N) \in \mathbf{U}, \qquad \sum_{u_1 \in U_1} \cdots \sum_{u_N \in U_N} p_{S_1 \cdots S_N}(u_1, \ldots, u_N) = 1.$$

The *joint rejectability mass function* $p_{R_1 \cdots R_N}$ is defined similarly.

Analogous to the way univariate probability mass functions are obtained as marginals of joint probability mass functions, we may extract individual decision maker selectability and rejectability mass functions as marginals; namely,

$$p_{S_i}(u_i) = \sum_{u_1 \in U_1} \cdots \sum_{u_{i-1} \in U_{i-1}} \sum_{u_{i+1} \in U_{i+1}} \cdots \sum_{u_N \in U_n} p_{S_1 \cdots S_n}(u_1, \ldots u_{i-1} u_i, u_{i+1}, \ldots, u_n). \tag{2}$$

The rejectability marginal p_{R_i} for decision maker i is defined similarly.

Definition 5. A *satisficing game* (Stirling & Morrell 1991, Stirling 1994, Goodrich 1996, Goodrich, et al. 1996, Goodrich, et al. 1998, Stirling & Goodrich 1999, Goodrich, et al. 2000, Stirling, et al. 2002, Stirling 2002b, Stirling 2002a, Stirling 2003) is the triple $\{\mathbf{U}, p_{S_1 \cdots S_N}, p_{R_1 \cdots R_N}\}$. The jointly satisficing solution is the subset of all option vectors such that the joint selectability is at least as great as the index of caution times the joint rejectability; that is,

$$\mathbf{\Sigma}_q = \{(u_1, \ldots, u_N) \in \mathbf{U} \colon p_{S_1 \cdots S_N}(u_1, \ldots, u_N) \geq q p_{R_1 \cdots R_N}(u_1, \ldots, u_N). \tag{3}$$

The individually satisficing solutions for each participant is obtained from the marginal selectability and rejectability mass functions, yielding the *individually satisficing solutions*.

$$\Sigma_q^i = \{u_i \in U_i \colon p_{S_i}(u_i) \geq q p_{R_i}(u_i)\}. \tag{4}$$

The *satisficing rectangle* is the product set of the individually satisficing sets, namely,

$$\Re_q = \Sigma_q^1 \times \cdots \times \Sigma_q^N. \tag{5}$$

In general, the satisficing rectangle will not be the same as the jointly satisficing set; they may even be disjoint. However, the following theorem relates the two sets.

Theorem 1. *If u_i is individually satisficing for agent i, that is, $u_i \in \Sigma_q^i$, then it must be the ith element of some jointly satisficing vector $\mathbf{u} \in \mathbf{\Sigma}_q$.*

Proof: We will establish the contrapositive, namely, that if u_i is not the ith element of any $\mathbf{u} \in \mathbf{\Sigma}_q$, then $u_i \notin \Sigma_q^i$. Without loss of generality, let $i = 1$. By hypothesis, $p_{S_1 \cdots S_n}(u_1, \mathbf{v}) < q p_{R_1 \cdots R_N}(u_1, \mathbf{v})$ for all $\mathbf{v} \in U_2 \times \cdots \times U_N$, so $p_{S_1}(u_1) = \sum_{\mathbf{v}} p_{S_1 \cdots S_N}(u_1, \mathbf{v}) < q \sum_{\mathbf{v}} p_{R_1 \cdots R_N}(u_1, \mathbf{v}) = q p_{R_1}(u_1)$, hence $u_1 \notin \Sigma_q^1$. □

Thus, if an option vector is individually satisficing, it is part of a jointly satisficing vector, although it need not be part of all jointly satisficing vectors. The converse, however, is not true: if u_i is the ith element of a jointly satisficing vector, it is not necessarily individually satisficing for agent i. The content of this theorem is that *no one is ever completely frozen out of a deal—every decision maker has, from its own perspective, a seat at the negotiating table.* This is perhaps the weakest condition under which negotiations are possible.

2.3 Interdependence

The joint selectability and rejectability mass functions characterize all of the interconnections between the participants. They may be derived either from the perspective of individual rationality, or they may be derived from the perspective of coordination and collaboration. The jointly satisficing set $\mathbf{\Sigma}_q$ represents the subset of option vectors that are collectively satisficing for the group, in the sense that the benefits to the group dominate the costs to the group. However, it is important to appreciate that this concept does not presuppose that there is a cohesive notion of group preference. If the group is purely competitive, such as would be the case with a zero-sum game, then the group 'preference' may be to oppose each other, and the individual preferences as obtained as marginals will be consistent with narrow self-interest. On the other hand, if the group is committed to achieving a coherent collective goal, then a well-defined group preference may obtain, and the individual preferences marginals will be consistent with cooperative behavior, even at the expense of individual benefit.

Since the behavior of the group is dependent on the structure of the joint selectability and rejectability mass functions, it is imperative that we understand exactly how these functions are created. To understand this process, it is necessary to define a more fundamental concept, that of *interdependence*. An act by any individual member of a multiple-participant system has possible ramifications for the entire system. Some participants may be benefited by the act, some may be damaged, and some may be indifferent. Furthermore, although an individual may perform the act in its own interest or for the benefit of others or the entire system, the act is usually not implemented free of cost. Resources are expended, or risk is taken, or some other penalty or unpleasant consequence is incurred, perhaps by the individual whose act it is, perhaps by other participants, and perhaps by the entire system. Although these undesirable consequences may be defined independently from the benefits, the measures associated with benefits and costs cannot be specified independently of each other, due to the possibility of interaction. A critical aspect of modeling the behavior of such a system, therefore, is the means of representing the interdependence of both positive and negative consequences of all possible joint options that could be undertaken.

Definition 6. The *interdependence mass function*, denoted $p_{S_1 \cdots S_N R_1 \cdots R_N}$, is a mapping from $\mathbf{U} \times \mathbf{U}$ to the unit interval such that

$$p_{S_1 \cdots S_N R_1 \cdots R_N}(u_1, \ldots u_N; v_1, \ldots, v_N) \geq 0$$

and

$$\sum_{u_1 \in U_1} \cdots \sum_{u_N \in U_N} \sum_{v_1 \in U_1} \cdots \sum_{v_N \in U_N} p_{S_1 \cdots S_N R_1 \cdots R_N}(u_1, \ldots, u_N; v_1, \ldots, v_N) = 1.$$

The joint selectability and rejectability mass functions may then be obtained from the interdependence function as

$$p_{S_1 \cdots S_N}(u_1, \ldots u_N) = \sum_{v_1 \in U_1} \cdots \sum_{v_N \in U_N} p_{S_1 \cdots S_N R_1 \cdots R_N}(u_1, \ldots, u_N; v_1, \ldots, v_N) \tag{6}$$

$$p_{R_1 \cdots R_N}(v_1, \ldots v_N) = \sum_{u_1 \in U_1} \cdots \sum_{u_N \in U_N} p_{S_1 \cdots S_N R_1 \cdots R_N}(u_1, \ldots, u_N; v_1, \ldots, v_N). \tag{7}$$

A useful way to view the interdependence function is to consider each decision maker as possessing two personas, or roles. One persona considers only the positive, or selectable, attributes of the options under consideration, and the other persona considers only the negative, or rejectable, attributes. The interdependence function then describes the collective attitude of the group when considering both personas of every member of the group with respect to selecting option vector $(u_1, \ldots, u_N)$ and rejecting option vector $(v_1, \ldots, v_N)$. The special case when $v_i = u_i$ characterizes the conflict that arises between the two personas because of the desire to select an option on the basis of its expediency, but also desiring to reject it because of its expense. In the single-agent case, it is reasonable to assume that the criteria that define selectable attributes are distinct from the criteria the define rejectable attributes, so in that case interdependence function would factor into the product, that is $p_{SR}(u; v) = p_S(u)p_R(v)$ (this factorization, however, is not required by the theory). In the multiple-agent case, however, one participant's rejectability, say, may influence another player's selectability, so it is not generally true that $p_{S_1 \cdots S_N R_1 \cdots R_N}(u_1, \ldots, u_N; v_1, \ldots, v_N) = p_{S_1 \cdots S_N}(u_1, \ldots, u_N)p_{R_1 \cdots R_N}(v_1, \ldots, v_N)$.

The construction of the interdependence function is the most critical aspect of satisficing game theory. The reader may have already noticed, however, that the structure of a satisficing game is reminiscent of a Bayesian network. Recall that a Bayesian network is a directed acyclic graph (DAG)consisting of nodes and edges, where the nodes represent the variables of the system, and the edges represent a conditional probabilistic description of how the instantiation of a parent node at a particular value influences the likelihood of another child node assuming its values.

Similar to the way a Bayesian network is defined, we may also employ the tools of graph theory to express the multiple-participant system of decision makers as an directed acyclic graph with each participant having two nodes—one for its selectability persona and one for its rejectability persona. The variables associated with these nodes are the options available to the decision maker. The edges of the graph represent the influence that one agents persona has on another agent's persona. These linkages represent conditional selectability or conditional rejectability functions that characterize the way a parent persona influences a child persona. As an example, consider the graph displayed in Figure 2. This graph corresponds to a three-agent system where the rejectability of agent 1 influences the rejectability of agents 2 and 3, and the rejectability of agent 2 influences the selectability of agents 1 and, with the selectability of agent 1 influencing the selectability of agent 2. The corresponding interdependence function is

$$p_{S_1 S_2 S_3 R_1 R_2 R_3}(u_1, u_2, u_3; v_1, v_2, v_3) = p_{S_2|S_1}(u_2|u_1)p_{S_1 S_3|R_2}(u_1, u_3|v_2) \\ p_{R_2|R_1 R_3}(v_2|v_1, v_3)p_{R_3|R_1}(v_3|v_1)p_{R_1}(v_1). \quad (8)$$

For example, the quantity $p_{S_2|S_1}(u_2|u_1)$ is a conditional selectability mass function characterizing the degree of support that agent 2 places on selecting option u_2 given that agent 1 were to prefer to select option v_1. Also, $p_{R_3|R_1}(v_3|v_1)$ represents the conditional rejectability that agent 3 places on option v_3 given that agent 1 were to prefer to reject option v_1. The other terms in this expression may be interpreted similarly.

The key aspect of this representation of the interdependence function is that it is constructed from conditional relationships via the chain rule of probability theory (which also applies in this new context). It is a well known aspect of probability theory that it is often much easier to compose a joint distribution from conditional distributions by means of the chain rule (Pearl 1988). The conditional mass functions represent hypothetical situations that are often quite simple to evaluate. In the traditional application of Bayesian networks, they correspond to production rules of an expert system. The power of this approach is that, Once the network is constructed, it is not necessary to multiply out all of the terms to find the joint probability mass function and hence the marginals—all that is necessary is to invoke Pearl's belief propagation algorithm (Pearl 1988), which computes the marginals as efficiently as possible[1]

Obviously, Pearl's algorithm may also applied in our context to obtain the joint and marginal selectability and rejectability mass functions, and thereby solve the satisficing game. The conditional selectability and rejectability functions represent hypothetical situations of the form "If agent i were to select option u_i, then agent j should evaluate its rejectability of option v_j as $p_{R_j|S_i}(v_j|u_i)$." Such local hypothetical relationships are often much easier to evaluate than the global, or joint, relationship $p_{S_i R_j}(u_i; v_j)$.

3 Altruism

A critical difference between multiple-participant decision theory as developed by von Neumann and Morgenstern and the satisficing approach advocated by this paper is the structure of the preference functions. To

[1]It is well known that the general fully-linked problem is NP-hard, but, fortunately, most interesting systems are only sparsely linked, resulting in a greatly reduced computational burden that is often tractable.

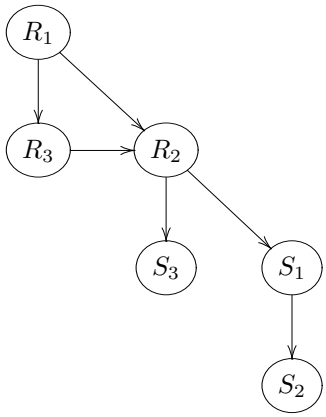

Figure 2. A DAG for a three-agent system.

appreciate this point, let us briefly review classical utility theory as it is employed by conventional game theory. Utility theory has developed as a mathematical way to encode individual preference orderings. It is built on a set of axioms that describe how a "rational man" would express his preference between two alternatives in a consistent way. An expected utility function is a mathematical expression that is consistent with the preferences and conforms to the axioms. Since, in a game-theoretic context, an individual's preferences are generally dependent upon the payoffs (expected utilities) that obtain as a result of the individual's strategies and of the strategies available to others, an individual's expected utility function must be a function of not only the individual's own strategies, but of the strategies of all other individuals. The important thing to note about this structure is that *it is not until the expected utilities are juxtaposed into an array so that the expected utility values for all players can be compared that the actual "game" aspects of the situation emerges.* It is the juxtaposition that reveals possibilities for conflict or coordination. These possibilities are not explicitly reflected in the individual expected utility functions by themselves. In other words, although the individual's expected utility is a function of other players' strategies, *it is not a function of other players' preferences.* This structure is completely consistent with narrow self-interest, where all a player cares about is its personal benefit as a function of its own and other players' strategies, without any regard for the benefit to the others. Under this paradigm, the only way the preferences of others factor into an individual's decision-making deliberations is to constrain behavior to limit the amount of damage they can do to oneself.

Seemingly, under the von Neumann-Morgenstern notion of preference formation, the decision maker can form its preferences in a social vacuum. It need not take into consideration the effect that its actions have on others; it only needs to take into consideration the effect that the actions of others has on itself. The agent is constrained to be selfish by its very nature. Many authors have argued that this selfish behavior can be overcome by redefining the utilities. As mentioned earlier, this is an unnatural approach that is motivated simply by the desire to apply a particular methodology, rather than to address the problem squarely. There is also another objection to such a procedure. To illustrate, let us consider the so-called Battle of the Sexes game. This is a game involving a man and a woman who plan to meet in town for a social function. She (S) prefers to go to the ballet (B), while he (H) prefers the dog races (D). Each also prefers to be with the other, however, regardless of venue. The classical way to formulate this game is via a payoff matrix, as given in Table 3 in ordinal form, with the payoff pairs representing the benefits to H and S, respectively.

	S	
H	D	B
D	(4, 3)	(2, 2)
B	(1, 1)	(3, 4)

Key: 4 = best; 3 = next best; 2 = next worst; 1 = worst

Table 3. Payoff matrix in ordinal form for the Battle of the Sexes game.

Rather than competing, these players wish to cooperate, but they must make their decisions without benefit of communication. Both players lose if they make different choices, but the choices are not all of equal value to the players. This game has two Nash equilibria, (D, D) and (B, B). Unfortunately, this does not provide an

unequivocal solution. One of the perplexing aspects of this game is that it does not pay to be altruistic (deferring to the venue preferred by the other), since, if both participants did, they would each receive the worst outcome. Nor does it pay for both to be selfish (demanding the venue preferred by oneself)—that guarantees the next worst outcome for each player.

The form of altruism invoked above does not distinguish between the state of *actually relinquishing* one's own self-interest and the state of *being willing to relinquish* one's own self-interest under the appropriate circumstances. To relinquish unconditionally one's own self-interest is a condition of *categorical* altruism—a decision maker unconditionally modifies its preferences to accommodate the preferences of others. A purely altruistic player would completely replace its preferences with the preferences of others. A state of being willing to modify one's preferences to accommodate others if the need arises is a state of *situational* altruism. Here, a decision maker is willing to accommodate, at least to some degree, the preferences of others in lieu of its own preferences if doing so would actually benefit the other, but otherwise retains its own preferences intact and avoids needless sacrifice.

Categorical altruism may be too much to expect from a decision maker who has its own goals to pursue. However, the same decision maker may be willing to engage, at least to a limited degree, in a form of situational altruism. Whereas it is one thing for an individual to modify its behavior if it is sure that doing so will benefit another individual (situational), it is quite another thing for an individual to modify its behavior regardless of its effect on the other (categorical). In the Battle of the Sexes, If H knew that S had a very strong aversion to D (even though S would be willing to put up with those extremely unpleasant surroundings simply to be with H and thus receive her second-best payoff), H might then prefer B to D. But if S did not have a strong aversion to D then H would stick to his preference for D over B.

This example seems to illustrate Arrow's claim that, when the assumption of perfect competition fails, "the very concept of [individual] rationality becomes threatened, because perceptions of others and, in particular, of their rationality become part of one's own rationality" (Arrow 1986). Arrow has put his finger on a critical weakness of individual rationality: it does not provide a way to incorporate another's rationality into one's own rationality without seriously compromising one's own rationality.

Example 2. We cast the Battle of the Sexes as a satisficing game. We must first establish each player's notions of selectability and rejectability;. Although there are many ways to frame this game, let us take selectability as the two players being with each other, regardless of where they go. Rejectability, on the other hand, deals with the costs of being at a particular function. According to the stereotypical roles of the players, H would prefer D if he did not take into consideration S's preferences; similarly, S would prefer B. Thus, we may express the myopic rejectabilities for H and S in terms of parameters h and s, respectively, as

$$\begin{aligned} p_{R_H}(D) &= h \\ p_{R_H}(B) &= 1-h \end{aligned} \tag{9}$$

and

$$\begin{aligned} p_{R_S}(D) &= 1-s \\ p_{R_S}(B) &= s, \end{aligned} \tag{10}$$

where h is H's rejectability of D and s is S's rejectability of B. The closer h is to zero, the more H is adverse to B with an analogous interpretation for s with respect to S attending D. To be consistent with the stereotypical roles, we may assume that $0 \leq h < \frac{1}{2}$ and $0 \leq s < \frac{1}{2}$. As will be subsequently seen, only the ordinal relationship need be specified, that is, either $s < h$ or $h < s$.

Since being together is a joint, rather than an individual objective, it is difficult to form unilateral assessments of selectability, but it is possible to characterize individually the conditional selectability. To do so requires the specification of the conditional mass functions $p_{S_H|R_S}$ and $p_{S_S|R_H}$; that is, H's selectability conditioned on S's rejectability and S's selectability conditioned on H's rejectability. If S were to place her entire unit mass of rejectability on D, H may account for this, if he cares at all about S's feelings, by placing some portion of his conditional selectability mass on B. S may construct her conditional selectability in a similar way, yielding

$$\begin{aligned} p_{S_H|R_S}(D|D) &= 1-\alpha \\ p_{S_H|R_S}(B|D) &= \alpha \\ p_{S_H|R_S}(D|B) &= 1 \\ p_{S_H|R_S}(B|B) &= 0 \end{aligned} \tag{11}$$

and

$$\begin{aligned} p_{S_S|R_H}(D|D) &= 0 \\ p_{S_S|R_H}(B|D) &= 1 \\ p_{S_S|R_H}(D|B) &= \beta \\ p_{S_S|R_H}(B|B) &= 1-\beta. \end{aligned} \tag{12}$$

The valuations $p_{S_H|R_S}(B|D) = \alpha$ and $p_{S_S|R_H}(D|B) = \beta$ can be considered conditions of situational altruism. If S were to place all of her rejectability mass on D, then H may defer to S's strong dislike of D by placing α of his selectability mass, *as conditioned by her preference,* on B. Similarly, S could show a symmetric conditional preference for D if H were to reject B strongly. The parameters α and β are H's and S's *indices of situational altruism,* respectively, and serve as a way for each to control the amount of deference he or she is willing to grant to the other. In the interest of simplicity, we shall assume that both players are maximally altruistic and set $\alpha = \beta = 1$. In principle, however, they may be set independently to any value in $[0, 1]$. Notice that, even in this most altruistic case, these conditional preferences do not commit one to categorical abdication of his or her own unilateral preferences. H still myopically (that is, without taking S into consideration) prefers D and S still myopically prefers B, and there is no intimation that either participant must "throw" the game in order to accommodate the other.

With these conditional and marginal functions, we may factor the interdependence function as follows:

$$\begin{aligned} p_{S_H S_S R_H R_S}(x, y; z, w) &= p_{S_H|S_S R_H R_S}(x|y; z, w) \cdot p_{S_S|R_H R_S}(y|z, w) \cdot p_{R_H R_S}(z, w) \\ &= p_{S_H|R_S}(x|w) \cdot p_{S_S|R_H}(y|z) \cdot p_{R_H}(z) \cdot p_{R_S}(w), \end{aligned}$$

where we have assumed that H's selectability conditioned on S's rejectability is dependent only on S's rejectability, that S's selectability conditioned on H's rejectability is dependent only on H's rejectability, and that the myopic rejectability values of H and S are independent.

The resulting joint selectability and rejectability functions are

$$\begin{aligned} p_{S_H S_S}(D, D) &= (1-h)s \\ p_{S_H S_S}(D, B) &= hs \\ p_{S_H S_S}(B, D) &= (1-h)(1-s) \\ p_{S_H S_S}(B, B) &= h(1-s) \end{aligned} \tag{13}$$

and

$$\begin{aligned} p_{R_H R_S}(D, D) &= h(1-s) \\ p_{R_H R_S}(D, B) &= hs \\ p_{R_H R_S}(B, D) &= (1-h)(1-s) \\ p_{R_H R_S}(B, B) &= (1-h)s. \end{aligned} \tag{14}$$

The marginal selectability and rejectability values for H and S are

$$p_{S_H}(D) = s \qquad p_{R_H}(D) = h \tag{15}$$

$$p_{S_H}(B) = 1-s \qquad p_{R_H}(B) = 1-h \tag{16}$$

and

$$p_{S_S}(D) = 1-h \qquad p_{R_S}(D) = 1-s \tag{17}$$

$$p_{S_S}(B) = h \qquad p_{R_S}(B) = s. \tag{18}$$

Setting the index of caution, q, equal to unity, we obtain the jointly satisficing set as

$$\boldsymbol{\Sigma}_q = \begin{cases} \{(D,B),(B,D),(B,B)\} & \text{for } s < h \\ \{(D,D),(D,B),(B,D)\} & \text{for } s > h \\ \{(D,D),(D,B),(B,D),(B,B)\} & \text{for } s = h \end{cases},$$

the individually satisficing sets are

$$\Sigma_q^H = \begin{cases} \{B\} & \text{for } s < h \\ \{D\} & \text{for } s > h \\ \{B, D\} & \text{for } s = h \end{cases}$$

$$\Sigma_q^S = \begin{cases} \{B\} & \text{for } s < h \\ \{D\} & \text{for } s > h \\ \{B, D\} & \text{for } s = h \end{cases},$$

and the satisficing rectangle is

$$\mathfrak{R}_q = \Sigma_q^H \times \Sigma_q^S = \begin{cases} \{B, B\} & \text{for } s < h \\ \{D, D\} & \text{for } s > h \\ \{\{B, B\}, \{D, D\}\} & \text{for } s = h \end{cases}.$$

Thus, if S's aversion to D is less than H's aversion to B, then both players will go to H's preference, namely, D, and conversely. This interpretation is an example of interpersonal comparisons of utility, a concept frowned upon by conventional game theorists. But such comparisons are essential to social choice theory, so long as the utilities are expressed in the same units and have the same zero-level.

Recall that, under a condition of categorical altruism, where each player defers to the other, the result is disastrous for both. By contrast, with the satisficing approach, even though both players are maximally situationally altruistic, the satisficing solution is far from disastrous—it results in a very natural cooperative strategy that is socially defensible.

4 Emergence

Another key difference between game theory *à la* von Neumann and Morgenstern and satisficing game theory is that, under the former, each player is obligated to define its preferences under every possible decision outcome of the group. Thus, all individual preference relationships must be determined *a priori* before the payoffs are juxtaposed into the payoff array. As a result, there is little facility available to account for group preferences. Shubik describes two concepts of group preference, neither of which is entirely satisfactory to game theorists: "Group preferences may be regarded either as derived from individual preferences by some process of aggregation or as a direct attribute of the group itself" (Shubik 1982, p. 109).

One way to aggregate a group preference from individual preferences is to adopt a bottom up approach, and to define a *social-welfare* function that provides a total ordering of the group's options as a function of individual preferences. The fundamental issue is whether or not, given arbitrary preference orderings for each individual in a group, there always exists a way of combining these individual preference orderings to generate a consistent preference ordering for the group. In an landmark result, Arrow (1951) showed that, in general, no social-welfare function exists that satisfies a set of reasonable and desirable properties, each of which is consistent with the notion of self-interested rationality and the retention of individual autonomy.

The Pareto principle provides a concept of social welfare as a direct attribute of the group, resulting in a top-down solution. A joint (group) option is a *Pareto equilibrium* if no single decision maker, by changing its decision, can increase its level of satisfaction without lowering the satisfaction level of at least one other decision maker. Adopting this view, in contrast to the Nash solution concept, would require the group to behave as a *superplayer*, or, as Raiffa puts it, the "organization incarnate" (Raiffa 1968), who functions as a higher-level decision maker. Shubik refers to the practice of ascribing preferences to a group as a subtle "anthropomorphic trap" of making a shaky analogy between individual and group psychology. He argues that, "It may be meaningful . . . to say that a group 'chooses' or 'decides' something. It is rather less likely to be meaningful to say that the group 'wants' or 'prefers' something" (Shubik 1982, p. 124).

Neither top-down nor bottom-up notions of emergence is a natural way to reconcile group and individual preferences. A bottom-up approach would to require that each participant specify his own preferences for each possible outcome and then work from there to a social choice that would be acceptable to the entire group. On the other hand, a top-down approach would require the specification of some group good, and each participant would have to make a choice that would be consistent with that good. If, however, the preferences are expressed by local conditional interdependencies; neither individual nor group orderings of preferences need be exhaustively provided. This situation seems to call for a new viewpoint, which we may characterize as an *inside-out* view, where intermediate-level conditional preferences propagate up to the group level and down

to the individual level. With this model, both individual and group behavior may emerge as consequences of local conditional interests that propagate throughout the group. If antagonism exists between members of the group, a harmonious group preference may not emerge. However, if the members of the group are willing to compromise, it may be possible to define a group preference which, although it does not represent ideal cooperation, does at least offer a weaker notion of acceptability as an alternative to the total failure of the group to function. The following example illustrates this principle.

Example 3. The Pot-Luck Dinner Larry, Curly, and Moe are going to have a pot-luck dinner. Larry will bring either soup or salad, Curly will provide the main course, either beef, chicken, or pork, and Moe will furnish the dessert, either lemon custard pie or banana cream pie. The choices are to be made simultaneously and individually following a discussion of their preferences, which discussion yields the following results:

1. In terms of meal enjoyment, if Larry were to prefer soup, then Curly would prefer beef to chicken by a factor of two, and would also prefer chicken to pork by the same ratio. However, if Larry were to prefer salad, then Curly would be indifferent regarding the main course.
2. If Curly were to reject pork as being too expensive, then Moe would strongly prefer (in terms of meal enjoyment) lemon custard pie and Larry would be indifferent regarding soup or salad. If, however, Curly were to to reject beef as too expensive, then Larry would strongly prefer soup and Moe would be indifferent regarding dessert. Finally, if Curly were to reject chicken as too expensive, then both Larry and Moe would be indifferent with respect to their enjoyment preferences.

Larry, Curly, and Moe all wish to conserve cost but consider both cost and enjoyment to be equally important. Table 4 indicates the total cost of each of the 12 possible meal combinations (using obvious abbreviations).

	lcst	bcrm
beef (soup/sald)	23/25	27/29
chkn (soup/sald)	22/24	26/28
pork (soup/sald)	20/22	24/26

Table 4. Meal cost structure for the Pot-Luck Dinner.

To examine this problem from the satisficing point of view, we first need to specify operational definitions for selectability and rejectability. Let us take rejectability as cost of the meal and take selectability as enjoyment of the meal. The interdependence function is a function of six independent variables and may be factored, according to the chain rule, as

$$p_{S_L S_C S_M R_L R_C R_M}(x, y, z; u, v, w) = p_{S_C|S_L S_M R_L R_C R_M}(y|x, z; u, v, w) \cdot p_{S_L S_M R_L R_C R_M}(x, z; u, v, w), \quad (19)$$

where the subscripts L, C, and M correspond to Larry, Curly, and Moe, respectively. The mass function $p_{S_C|S_L S_M R_L R_C R_M}(y|x, z; u, v, w)$ expresses the selectability that Curly places on y, given that Larry selects x and rejects u, that Curly rejects v, and that Moe selects z and rejects w. From the hypothesis of the problem, we realize that, conditioned on Larry's selectability, Curly's selectability is independent of all other considerations, thus we can simplify this conditional selectability to obtain

$$p_{S_C|S_L S_M R_L R_C R_M}(y|x, z; u, v, w) = p_{S_C|S_L}(y|x).$$

Next, we apply the chain rule to the second term on the right hand side of (19), which yields

$$p_{S_L S_M R_L R_C R_M}(x, z; u, v, w) = p_{S_L S_M|R_L R_C R_M}(x, z|u, v, w) \cdot p_{R_L R_C R_M}(u, v, w).$$

But, given Curly's rejectability, the joint selectability for Larry and Moe is independent of all other considerations, so

$$p_{S_L S_M|R_L R_C R_M}(x, z|u, v, w) = p_{S_L S_M|R_C}(x, z|v).$$

By making the appropriate substitutions, (19) becomes

$$p_{S_L S_C S_M R_L R_C R_M}(x, y; z, u, v, w) = p_{S_C|S_L}(y|x) \cdot p_{S_L S_M|R_C}(x, z|v) \cdot p_{R_L R_C R_M}(u, v, w). \quad (20)$$

Notice that, if the joint rejectability mass function is factored into $p_{R_L R_C R_M}(u, v, w) = p_{R_C|R_L R_M}(v|u, w) p_{R_m|R_L}(w|u) p_{R_L}(u)$, then this interdependence function corresponds to the DAG given in Figure 2.

We desire to obtain $\mathbf{\Sigma}_q$, the satisficing strategy profiles for the group and Σ_q^L, Σ_q^C, and Σ_q^M, the individually satisficing strategy sets for Larry, Curly, and Moe, respectively. To do so, we must specify each of the components of (20). To compute $p_{S_C|S_L}$, recall that Curly prefers beef to chicken to pork by respective factors of 2 conditioned on Larry preferring soup and that Curly is indifferent, conditioned on Larry preferring salad. We may express these relationships by the conditional selectability functions:

$$\begin{aligned} p_{S_C|S_L}(beef|soup) &= 4/7 \quad p_{S_C|S_L}(beef|sald) &= 1/3 \\ p_{S_C|S_L}(chkn|soup) &= 2/7 \quad p_{S_C|S_L}(chkn|sald) &= 1/3 \\ p_{S_C|S_L}(pork|soup) &= 1/7 \quad p_{S_C|S_L}(pork|sald) &= 1/3. \end{aligned}$$

To compute $p_{S_L S_M|R_C}$, we recall, given that Curly views pork as completely rejectable, Moe views lemon custard pie as highly selectable and Larry is indifferent. Given that Curly views beef as completely rejectable, Larry views soup as selectable, and Moe is indifferent; and given that Curly views chicken as completely rejectable, both Larry and Moe are indifferent. These relationships may be expressed as

$$\begin{aligned} p_{S_L S_M|R_C}(soup, lcst|pork) &= 0.5 \\ p_{S_L S_M|R_C}(soup, bcrm|pork) &= 0.0 \\ p_{S_L S_M|R_C}(sald, lcst|pork) &= 0.5 \\ p_{S_L S_M|R_C}(sald, bcrm|pork) &= 0.0, \end{aligned}$$

$$\begin{aligned} p_{S_L S_M|R_C}(soup, lcst|beef) &= 0.5 \\ p_{S_L S_M|R_C}(soup, bcrm|beef) &= 0.5 \\ p_{S_L S_M|R_C}(sald, lcst|beef) &= 0.0 \\ p_{S_L S_M|R_C}(sald, bcrm|beef) &= 0.0, \end{aligned}$$

and

$$\begin{aligned} p_{S_L S_M|R_C}(soup, lcst|chkn) &= 0.25 \\ p_{S_L S_M|R_C}(soup, bcrm|chkn) &= 0.25 \\ p_{S_L S_M|R_C}(sald, lcst|chkn) &= 0.25 \\ p_{S_L S_M|R_C}(sald, bcrm|chkn) &= 0.25. \end{aligned}$$

Lastly, we need to specify $p_{R_L R_C R_M}$, the joint rejectability function. This is done by normalizing the meal cost values in Table 4 by the total cost of all meals (e.g, $p_{R_L R_C R_M}(soup, beef, lcst) = 23/296$).

With the interdependence function so defined and letting $q = 1$, the jointly satisficing meals are as displayed in Table 5, each of which is good enough for the group, considered as a whole. Figure 3 provides a cross-plot of the joint selectability and joint rejectability. The individually satisficing items, as obtained by computing the selectability and rejectability marginals, are also provided in Table 5 to be *soup, beef,* and *lemon custard*. Fortunately, this set of choices is also jointly satisficing without lowering the index of caution. Thus, all of the preferences are respected at a reasonable cost and, if pies are thrown, it is only for recreation, not retribution.

With the Pot-Luck Dinner, we see that, although total orderings for neither individuals nor the group are specified, we can use the *a priori* partial preference orderings from the problem statement to generate emergent, or *a posteriori*, group and individual orderings. *A posteriori* individual orderings also emerge from this exercise: Larry prefers soup to salad, Moe prefers lemon custard pie to banana cream pie, and Curly prefers beef to either chicken or pork. Note, however, that Curly is not required to impose a total ordering on his preferences (chicken versus pork)—this approach does not force the generation of unwarranted preference relationships. We see that a group-wide preference of avoiding conflict emerges, since the individually satisficing strategies are also jointly satisficing. This group desideratum was *not* specified *a priori*.

5 Conclusions

Satisficing game theory is a significant departure from conventional methods of multiple participant decision making. The major differences include:

Jointly Satisficing

Meal	$p_{S_L S_C S_M}$	$p_{R_L R_C R_M}$
$\{soup,\ beef,\ lcst\}$	0.237	0.078
$\{soup,\ chkn,\ lcst\}$	0.119	0.074
$\{soup,\ beef,\ bcrm\}$	0.149	0.091
$\{sald,\ pork,\ lcst\}$	0.080	0.074

Individually Satisficing

Participant	Choice	p_S	p_R
Larry	$soup$	0.676	0.480
Curly	$beef$	0.494	0.351
Moe	$lcst$	0.655	0.459

Table 5. Jointly and individually satisficing choices for the Pot-Luck Dinner.

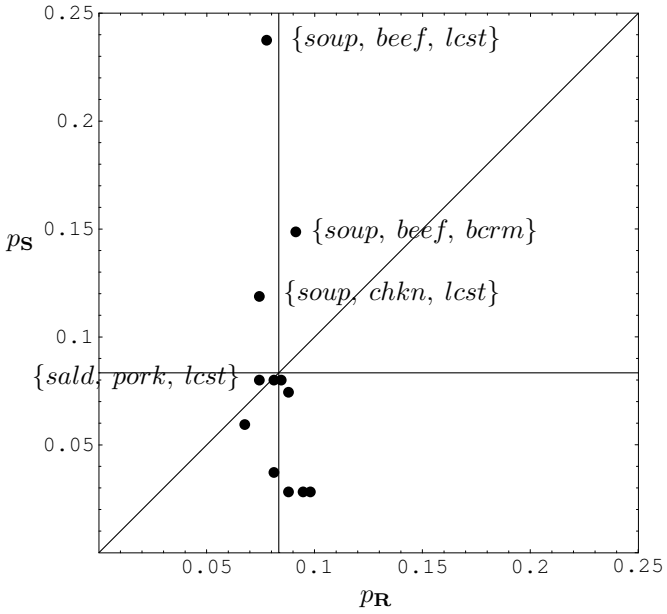

Figure 3. The cross-plot of joint rejectability and selectability for the Pot-Luck Dinner.

- replacement of single von Neumann-Morgenstern utility functions with dual utility functions that separate the desirable and undesirable attributes of the options;
- replacement of utility functions that express preferences of decision makers as functions of the options of other decision makers with utility functions that express preferences of decision makers as functions of preferences of other decision makers;
- replacement of unconditional utilities with conditional utilities which propagate through the system via the chain rule to create a joint interdependence function, thus allowing individual and group preference marginals to emerge as a consequence of social interaction;
- replacement of individual rationality (which does not accommodate compatible notions of group and individual interests) with satisficing rationality (which does accommodate notions of both group and individual interests).

When cooperation is essential in a multiple participant system, it is important to design the system according to principles that explicitly accommodate cooperation. However, under conventional individual rationality-based approaches, altruism (giving deference to others at one's own expense) is difficult to characterize. But under the notion of satisficing rationality, giving deference conditionally is easy to characterize and to specify via conditional preference relationships.

Optimization is a strongly entrenched procedure and dominates conventional decision-making methodologies. There is great comfort in following traditional paths, especially when those paths are founded on such a

rich and enduring tradition as rational choice affords. However, there is some danger in relying on only one principle of rational choice. Optimization is, by its very nature, an *individual* activity—either by a single entity or by a group acting in unison. It is a brittle and uncompromising standard of quality. Satisficing, on the other hand, is more elastic and conciliatory in its nature, since it does not demand a single "best" solution, but is willing to accommodate a set of solutions, each of which is "good enough." The approach described in this paper provides a way to account for both group and individual interests. Order can emerge through the local interactions that occur between agents who share common interests and who are willing to give deference to each other. Rather than depending upon the non-cooperative equilibria defined by individual rationality, this alternative may lead to the more socially realistic and valuable equilibrium of shared interests and acceptable compromises.

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On Dynamic Decision-Making Scenarios with Multiple Participants

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Abstract. Multiple-participant (MP) dynamic decision making (DDM) is encountered both in societal and technical systems. Unlike in single-participant (SP) DDM, no commonly accepted normative theory exists. The paper contributes to numerous attempts to overcome this state by classifying possible MP DM scenarios and connecting them with the normative SP DDM. This classification helps us to formulate open problems whose solutions will lead to feasible MP DDM.

Keywords: Bayesian dynamic decision making (DM), multiple participant DM, fully probabilistic design

1 Introduction

In SP scenario, Bayesian paradigm represents a normative theory of DDM that has – probably conceptually – no internally consistent competitors as it guarantees that its strategies are not dominated (DeGroot 1970, Berger 1985). This implies that a normative theory of MP DDM should be searched as an extension of the Bayesian DDM.

Participant's environment $\mathcal{E}$ is a limited sphere of its interests that the participant can influence. The environment is characterized by its possible behaviors defined on the considered data and action spaces as well as on the space of considered but directly unobservable quantities.

Fully probabilistic version of DDM (Kárný 1996, Kárný, et al. 2003), we focus on, models *environment* $\mathcal{E}$ of the *participant* $\mathcal{P}$ as well as its aims in probabilistic terms. Essentially, in SP DDM scenario, $\mathcal{P}$'s *strategy* is chosen so that its interconnection with *environment model* is as close as possible to the *ideal distribution* of environment behavior reflecting these aims. The paper inspects an extension of this methodology to MP DDM. The corresponding normative MP DDM requires specification of admissible strategies of participants, models of their environments $\mathcal{E}$ and relevant ideal distributions.

The paper starts with preliminaries containing a generalized version of the fully probabilistic design of decision strategies and a classification of its ingredients and cooperation stages. Then, a relevant part of Bayesian estimation is recalled. Conjugate prior distributions and the choice of the ideal *probability density function, (pdf)*, forming set-point in the fully probabilistic design, are also briefly discussed. Then, the following MP DDM scenarios are presented: selfish scenario, scenario for participants cooperating at design stage with and without a priori given hierarchy. Concluding remarks point to the important open problems.

2 Preliminaries

The section defines basic notions adopted; recalls the fully probabilistic SP DDM, introduces basic ingredients of DM and classifies possible cooperation stages.

2.1 Notation

The following agreements are used throughout the paper: $\equiv$ is equality by definition; x^* denotes a set of x-values; $\mathring{x}$ means cardinality of a finite set x^*; $f(\cdot|\cdot)$ is reserved for pdfs; E denotes expectation; t is discrete-time; $d = (\Delta, a)$ is data record consisting of (observable) innovation Δ and action a; d_t is data item at time t; $d(t)$ denotes sequence $(d_1, \ldots, d_t)$; $\propto$ is equality up to a normalizing factor; left-upper superscript distinguishes object categories, e.g. ${}^{\mathcal{P}}f$ means pdf related to the participant $\mathcal{P}$.

The pdfs are distinguished by the identifiers in their arguments. No formal distinction is made between random variable, its realization and an argument of a pdf.

The SP DDM exploits *Kullback-Leibler (KL) divergence* $\mathcal{D}\left(f||\tilde{f}\right)$ that measures of the proximity of a pair of pdfs $f, \tilde{f}$ acting on a common set x^*. It is defined

$$\mathcal{D}\left(f||\tilde{f}\right) \equiv \int f(x) \ln\left(\frac{f(x)}{\tilde{f}(x)}\right) dx. \tag{1}$$

The basic property of the KL divergence (Kullback & Leibler 1951)

$$\mathcal{D}(f||\tilde{f}) \geq 0,\ \mathcal{D}(f||\tilde{f}) = 0 \text{ iff } f = \tilde{f} \text{ almost everywhere}, \tag{2}$$

is used.

2.2 Fully probabilistic SP DDM

To make a decision, a participant $\mathcal{P}$ operates with *considered behaviors* $\mathcal{Q}$ of its environment $\mathcal{E}$. A behavior consists of time sequences of potential participant's actions $a(\mathring{t})$ and of observed innovations $\Delta(\mathring{t})$, informing about environment's state, as well as of considered but directly unobserved $x(\mathring{t})$ quantities. Thus,

$$\mathcal{Q} \equiv (a(\mathring{t}), \Delta(\mathring{t}), x(\mathring{t})) \equiv (d(\mathring{t}), x(\mathring{t})),$$

where $\mathring{t} < \infty$ is decision-making horizon and $d(\mathring{t})$ denotes *observable* data. The joint pdf

$${}^{\mathcal{P}}f(\mathcal{Q}) = {}^{\mathcal{P}}f(X|d(\mathring{t})) \underbrace{\prod_{t\in t^*} {}^{\mathcal{P}}f(\Delta_t|a_t, d(t-1))}_{\mathcal{M}\equiv \text{ model of environment}} \times \underbrace{\prod_{t\in t^*} {}^{\mathcal{P}}f(a_t|d(t-1))}_{\mathcal{R}\equiv \text{ decision strategy}} \tag{3}$$

describes possible behaviors of the closed loop formed by interconnection of $\mathcal{P}$ with its $\mathcal{E}$. It is product of the Bayesian estimate ${}^{\mathcal{P}}f(X|d(\mathring{t}))$ of the unobservable quantity $X \equiv x(\mathring{t})$, the model $\mathcal{M}$ of participant's environment $\mathcal{E}$ (more precisely *outer* model) and the decision strategy $\mathcal{R}$, selected by the participant. The Bayesian estimate of $X \equiv x(\mathring{t})$ and the model of $\mathcal{E}$ are gained via Bayesian filtering (Jazwinski 1970, Peterka 1981) and are fixed when strategy is designed. The decision strategy $\mathcal{R}$ is to be chosen so that aims of $\mathcal{P}$ are met. In the fully probabilistic setting, aims are expressed by the ideal counterpart of the joint pdf (3)

$${}^{I}f(\mathcal{Q}) = {}^{I}f(X|d(\mathring{t})) \prod_{t\in t^*} {}^{I}f(\Delta_t|a_t, d(t-1))\, {}^{I}f(a_t|d(t-1)). \tag{4}$$

Ideally, the optimized strategy $\mathcal{R}$ should move the influenced behavior $\mathcal{Q}$ to a desirable area. The ideal pdf ${}^{I}f(\mathcal{Q})$ expresses this aim. The fully probabilistic design simply searches for the strategy ${}^{o\mathcal{P}}\mathcal{R}$ that minimizes the KL divergence of the joint pdf ${}^{\mathcal{P}}f(\mathcal{Q})$ to its ideal counterpart ${}^{I}f(\mathcal{Q})$.

Proposition 1 (Fully probabilistic design) *The optimal decision strategy* ${}^{o\mathcal{P}}\mathcal{R}$ *minimizing* $\mathcal{D}\left({}^{\mathcal{P}}f||\,{}^{I}f\right)$ *has the form*

$${}^{o\mathcal{P}}f(a_t|d(t-1)) \propto {}^{I}f(a_t|d(t-1)) \exp[-\omega(a_t, d(t-1))]. \tag{5}$$

The functions $\omega(a_t, d(t-1))$ *are generated recursively in backward manner for* $t = \mathring{t}, \mathring{t}-1, \ldots, 1$

$$\begin{aligned}\omega(a_t, d(t-1)) &= \int {}^{\mathcal{P}}f(\Delta_t|a_t, d(t-1)) \ln\left[\frac{{}^{\mathcal{P}}f(\Delta_t|a_t, d(t-1))}{\gamma(d(t))\,{}^{I}f(\Delta_t|a_t, d(t-1))}\right] d\Delta_t \\ \gamma(d(t-1)) &= \int {}^{I}f(a_t|d(t-1)) \exp[\omega(a_t, d(t-1))]\, da_t.\end{aligned} \tag{6}$$

The recursion starts with

$$\ln(\gamma(d(\mathring{t}))) = -\int {}^{\mathcal{P}}f(X|d(\mathring{t})) \ln\left[\frac{{}^{\mathcal{P}}f(X|d(\mathring{t}))}{{}^{I}f(X|d(\mathring{t}))}\right] dX. \tag{7}$$

Proof: Definition of the KL divergence and Fubini theorem on multiple integration imply the following identity

$$\mathcal{D}\left({}^{\mathcal{P}}f||\,{}^{I}f\right) = \int {}^{\mathcal{P}}f(d(\mathring{t})) \left\{ \ln\left(\frac{{}^{\mathcal{P}}f(d(\mathring{t}))}{{}^{I}f(d(\mathring{t}))}\right) + \underbrace{\int {}^{\mathcal{P}}f(X|d(\mathring{t})) \ln\left(\frac{{}^{\mathcal{P}}f(X|d(\mathring{t}))}{{}^{I}f(X|d(\mathring{t}))}\right) dX}_{\equiv -\ln[\gamma(d(\mathring{t}))]} \right\} dd(\mathring{t}).$$

Using the chain rule for pdfs (Peterka 1981), we can write the part of the KL divergence that depends on ${}^{\mathcal{P}}f(a_{\mathring{t}}|d(\mathring{t}-1))$ in the form

$$E\left\{\int {}^{\mathcal{P}}f(a_{\mathring{t}}|d(\mathring{t}-1))\left[\ln\left(\frac{{}^{\mathcal{P}}f(a_{\mathring{t}}|d(\mathring{t}-1))}{{}^{I}f(a_{\mathring{t}}|d(\mathring{t}-1))}\right)+\omega(a_{\mathring{t}},d(\mathring{t}-1))\right]da_{\mathring{t}}\right\}$$

$$\text{with}\quad \omega(a_{\mathring{t}},d(\mathring{t}-1))\equiv\int {}^{\mathcal{P}}f(\Delta_{\mathring{t}}|a_{\mathring{t}},d(\mathring{t}-1))\ln\left(\frac{{}^{\mathcal{P}}f(\Delta_{\mathring{t}}|a_{\mathring{t}},d(\mathring{t}-1))}{\gamma(d(\mathring{t}))\,{}^{I}f(\Delta_{\mathring{t}}|a_{\mathring{t}},d(\mathring{t}-1))}\right)d\Delta_{\mathring{t}}.$$

The above expected value can be re-written as follows

$$E\left\{\int {}^{\mathcal{P}}f(a_{\mathring{t}}|d(\mathring{t}-1))\ln\left(\frac{{}^{\mathcal{P}}f(a_{\mathring{t}}|d(\mathring{t}-1))}{{}^{o\mathcal{P}}f(a_{\mathring{t}}|d(\mathring{t}-1))}\right)da_{\mathring{t}}-\ln(\gamma(d(\mathring{t}-1)))\right\},\quad\text{where}$$

$${}^{o\mathcal{P}}f(a_{\mathring{t}}|d(\mathring{t}-1))=\frac{{}^{I}f(a_{\mathring{t}}|d(\mathring{t}-1))\exp[-\omega(a_{\mathring{t}},d(\mathring{t}-1))]}{\gamma(d(\mathring{t}-1))}$$

$$\gamma(d(\mathring{t}-1))\equiv\int {}^{I}f(a_{\mathring{t}}|d(\mathring{t}-1))\exp[-\omega(a_{\mathring{t}},d(\mathring{t}-1))]\,da_{\mathring{t}}.$$

The last optimized pdf ${}^{\mathcal{P}}f(a_{\mathring{t}}|d(\mathring{t}-1))$ enters the first term having the form of (conditional) KL divergence. Due to (2), it is minimized by the claimed pdf (5). The normalizing factor $\gamma(d(\mathring{t}-1))$ represents remainder to be influenced by pdfs ${}^{\mathcal{P}}f(a_t|d(t-1))$ with $t<\mathring{t}$. Their optimization is, however, formally identical with the demonstrated last step. □

Remarks

1. The explicit solution depends on evaluation of respective integrals. Monte Carlo evaluation can be used in low dimensional case, whenever it is possible to store functions ω, γ.
2. The optimal strategy (5) respects constraints on actions expressed by the support of the ideal pdf ${}^{I}f(a_t|d(t-1))$ due to the inclusion of supports

$$\text{supp}\left[\,{}^{o\mathcal{P}}f(a_t|d(t-1))\right]\subset\text{supp}\left[\,{}^{I}f(a_t|d(t-1))\right].$$

3. The above version of the fully probabilistic design considers the unobservable quantity X, which is expressed in the initial condition (7). This extends the previous solution (Kárný et al. 2003). The extension reduces to it if the quantity X need not or cannot be influenced by the optimized strategy. In these cases, it sufficient to leave X to its fate and to choose ${}^{I}f(X|d(\mathring{t}))={}^{\mathcal{P}}f(X|d(\mathring{t}))$. This reduces the initial condition (7) to $\gamma(d(\mathring{t}))=1$. This start corresponds with the former design version in which $f(X|d(\mathring{t}))$ has no direct influence on the design. The estimate enters the (outer) model $\mathcal{M}$ only, cf. (Peterka 1981) and Section 2.3.2.
4. The design, similarly as general stochastic dynamic programming, relies on the assumption that admissible strategies work with non-decreasing information given by $d(t)=(d_t,d(t-1))$. This implies generally permanent increase of complexity of the resulting strategy with increasing horizon $\mathring{t}$.

2.3 Ingredients of Dynamic Decision Making

Specific instances of DDM are determined by ingredients occurring in Proposition 1. They are briefly discussed here.

2.3.1 Classification of ingredients and cooperation stages

Ingredients of the fully probabilistic design can be classified according to their origin. They can be *internal* with respect to $\mathcal{P}$ or *external*, i.e. received from $\mathcal{E}$ of this $\mathcal{P}$, or both. Variants are listed in the following table.

Ingredient	Source type	Comment
action a_t	internal	no alternative source
innovation Δ_t	external	no alternative source
inner quantity X	internal	$\mathcal{P}$ decides whether X will be included into its model of $\mathcal{E}$
model ${}^{\mathcal{P}}f(\mathcal{Q})$	both types	the model can be supplied externally in MP DDM
ideal ${}^{I}f(\mathcal{Q})$	both types	the ideal pdf can be modified externally in MP DDM; it reflects also constraints

Cooperation stages with other $\mathcal{P}$s can be classified according to the interaction of $\mathcal{P}$ in the design of the decision strategy (*design stage*) and during its application (*acting stage*).

External ingredient	Classification	Comment
behavior $\mathcal{Q}$	cooperation at the acting stage	data shared by $\mathcal{P}$s are specified
factors of ${}^{\mathcal{P}}f(\mathcal{Q})$	cooperation at the design stage	strategies of other $\mathcal{P}$s and/or their models help in building of the model of $\mathcal{E}$ for the considered $\mathcal{P}$
ideal pdf ${}^{I}f(\mathcal{Q})$	cooperation at the design and/or acting stages	knowledge of aims and restrictions of other $\mathcal{P}$s help in modifying $\mathcal{P}$ ideal pdf

Generally, design and acting stages can intertwine and the cooperation extent may vary. Numerous variants may arise in this way up to the powerful adaptive cooperation. These possibilities are too complex for this introductory inspection of MP DDM and their discussion is left aside. Thus, in the rest of the paper, *we take the definition of the external supply of data and models to the inspected $\mathcal{P}$ as given and fixed*. It leaves a few basic ways of cooperation in the design stage.

In the acting stage, the cooperation possible via (partial) sharing of data and (or) the ideal pdf. In the design stage, models, strategies and ideal pdfs can be (partially) shared.

Sharing of ingredients can be either compulsory or facultative. The former option introduces hierarchy into MP DDM. The latter one uses only hierarchy implied by differing "abilities" of individual $\mathcal{P}$s.

Before elaborating the corresponding scenarios, let us discuss freedom in *feasible constructions of ingredients*.

2.3.2 Bayesian Estimate and Model of Environment

For presentation simplicity, an unknown parameter is assumed to be the only unobserved quantity X. Then, the model ${}^{\mathcal{P}}f(\Delta_t|a_t, d(t-1))$ of $\mathcal{E}$ and the Bayesian parameter estimate ${}^{\mathcal{P}}f(X|d(t))$ are described by formulas (Peterka 1981)

$$
\begin{aligned}
{}^{\mathcal{P}}f(\Delta_t|a_t, d(t-1)) &= \int {}^{\mathcal{P}}f(\Delta_t|a_t, d(t-1), X)\, {}^{\mathcal{P}}f(X|d(t-1))\, dX \qquad (8)\\
{}^{\mathcal{P}}f(X|d(t)) &\propto {}^{\mathcal{P}}f(\Delta_t|a_t, d(t-1), X)\, {}^{\mathcal{P}}f(X|d(t-1)).
\end{aligned}
$$

$\mathcal{P}$ has to supply *observation model* ${}^{\mathcal{P}}f(\Delta_t|a_t, d(t-1), X)$ relating the current innovation to the current action a_t, observed data $d(t-1)$ and the unknown parameter X. Moreover, it has to provide prior pdf ${}^{\mathcal{P}}f(X)$ that expresses its prior knowledge about the unknown X. The formulas (8) are valid when X cannot be used for selecting a_t, i.e. when natural conditions of control (Peterka 1981) are met.

For DDM, pdfs as *functions* have to be available. Their dependence on data, whose amount is growing with time, can be feasibly represented only through a finite dimensional statistic. For existence of such a statistics, the observation model (8) has to depend at most on the *finite-dimensional regression vector* $\psi_t = [a_t, \ldots, a_{t-\partial_a}, \Delta_{t-1}, \ldots, \Delta_{t-\partial_\Delta}]$, ∂_a, $\partial_\Delta < \infty$, via a time-invariant function $M(\Psi, X)$ of the *data vector* $\Psi \equiv [\Delta, \psi]$ and the parameter X. Thus, feasible observation models have to have the form

$$
{}^{\mathcal{P}}f(\Delta_t|a_t, d(t-1), X) = M_t([\Delta_t, \psi_t], X) \equiv M(\Psi_t, X)
$$

with computationally feasible evaluation of M-values.

Existence of the finite-dimensional *sufficient* statistic V_t, that guarantees feasibility without information loss, is (practically) guaranteed only if $M(\Psi, X)$ belongs to the exponential family (Barndorff-Nielsen 1978)

$$
M(\Psi, X) = \exp\langle B(\Psi), C(X)\rangle \qquad (9)
$$

with $\langle\cdot,\cdot\rangle$ being a scalar product of compatible, finite-dimensional, practically manageable, functions $B(\cdot)$, $C(\cdot)$ of respective arguments.

A sort of approximation is needed whenever this family is not rich enough. Modelling by finite mixtures with components in the exponential family and development of specialized estimation procedures (Kárný et al. 2003, Titterington, et al. 1985) is one of a few generic approaches to this problem.

2.3.3 Conjugate Prior Pdf

Form of the prior pdf ${}^{\mathcal{P}}f(X)$ is determined by the need to describe Bayesian estimate ${}^{\mathcal{P}}f(X|d(t))$ and the model ${}^{\mathcal{P}}f(\Delta_t|a_t, d(t-1))$ of $\mathcal{E}$ by the finite-dimensional statistic V_t. For the exponential family (9), the prior pdf has to be conjugate prior pdf

$$ {}^{\mathcal{P}}f(X) \equiv {}^{\mathcal{P}}f(X|d(0)) \propto \exp\langle V_0, C(X)\rangle \, \Xi_{X^*}(X), \tag{10}$$

where $\Xi_{X^*}(X)$ is indicator of the set X^*. The optional prior statistic V_0 has to guarantee that right-hand side of (10) can be normalized to pdf. The posterior pdf ${}^{\mathcal{P}}f(X|d(t))$ preserves this form and the functional recursion (8) reduces to the simple algebraic updating

$$ V_t = V_{t-1} + B(\Psi_t), \; V_0 \text{ given}. \tag{11}$$

Specific prior knowledge is to be reflected in the statistic V_0. In (Kárný, et al. 2001a), an algorithmic translation of usual technical knowledge into V_0 is presented. Essentially, Bayesian estimation is performed on simulated or though-of data.

2.3.4 Ideal Pdf

The ideal pdf ${}^{I}f(d_t|d(t-1)) = {}^{I}f(\Delta_t|a_t, d(t-1)) \, {}^{I}f(a_t|d(t-1))$ (4) can be constructed through a specification of desirable ranges of respective arguments and their changes. It is done by defining locations and widths of pdfs taken from the exponential family. Restriction of the supports of these pdfs to desired ranges respects hard constraints on the solved DDM.

Often, it happens that the $\mathcal{P}$ is interested only in a few entries $d_{\mathcal{P};t}$ of $d_t \equiv (d_{\mathcal{P};t}, d_{\mathcal{E};t})$. Then, it leaves other variables, $d_{\mathcal{E};t}$, to their fate. It corresponds with the special choice of the ideal pdf

$$ {}^{I}f(d_t|d(t-1)) = {}^{\mathcal{P}}f(d_{\mathcal{E};t}|d_{\mathcal{P};t}, d(t-1)) \, {}^{I}f(d_{\mathcal{P};t}|d(t-1)), \tag{12}$$

where ${}^{\mathcal{P}}f(d_{\mathcal{E};t}|d_{\mathcal{P};t}, d(t-1))$ is conditional pdf derived from ${}^{\mathcal{P}}f(d_t|d(t-1))$, describing the estimated model of $\mathcal{E}$ connected with the designed decision strategy $\mathcal{R}$. It is straightforward to modify Proposition 1 to this special case.

It should be mentioned that *the use of finite mixtures as ideal pdfs allows us to express multiple aims*.

3 Decision-Making Scenarios with Multiple Participants

This section describes cooperations for different scenarios, outlines possible technical solutions and emphasizes open problems to be solved.

Participant's location (level of hierarchy) and type of interrelation with other participants (selfish, cooperative, hierarchic), determined by the communication structure, define its operation domain and type of information shared. As it has been mentioned before, the *acting stage* realizes the DM strategy determined at the design stage. From the cooperation viewpoint, the acting stages for different types of scenarios are distinguished by sharing the data about the common part of considered behaviors only. The highest cooperation level of cooperation at this stage can be based on sharing the statistics on that part of the behavior space, which is not "reachable" by another participant.

The more rich and efficacious is the cooperation in the *design stage* where the cooperation relies on sharing the models and ideal pdfs, possibly creating a common group model and (or) ideal, etc. The design stage of all types of scenarios consists of two phases: *communication phase*, when information exchange takes place and *updating phase*, when particular participant updates its models and (or) ideals. The strength of the mutual influence makes the main difference between individual variants.

In communication phase, the *selfish scenario* supposes that a communicating participant either send to or receive from another participant information. Then, the *facultative* updating phase for the information-receiving participant starts. The receiving participant may be selfish (egoistic), it may only partially respect the information provided. In extreme it does not make any change of its model or ideal pdf. The same sequence of phases is repeated for another participant of the communicating pair. Thus the whole design stage of selfish

scenario results in exchange of models and possible model updating. No common global model or ideal pdf for the both communicating participants is expected to be chosen. It means that each participant will use its own , possibly updated, model of $\mathcal{E}$ at the next acting stage.

Even selfish scenario requires at least minimum cooperation during communication phase: participant should provide its model or/and ideal at least partially. Providing correct models or real ideal is not however compulsory, i.e. possibility to lie consciously has been preserved for participant.

In contrast to the selfish scenario, the *cooperative scenario* assumes that the mutual exchange of models and ideals at communication phase which is followed by *obligatory* updating phase. The result of the negotiations at design stage must be *common model and ideal* of communicating participants concerning to their common part of data space.

The last considered *hierarchical scenario* assumes existence of a participant coordinating a whole group of participants. Exceptional role of the coordinator is given by its power to influence the members of the group either via their models or ideals or even via data supplied in acting stage.

The following subsections describe technical details of above mentioned scenarios.

3.1 Selfish Scenario and Its Estimation Aspects

In a non-trivial MP DDM, the environment of an individual $\mathcal{P}$ overlaps at least partially with that of other $\mathcal{P}$s.

The simplest MP DDM scenario discussed here takes individuals as selfish $\mathcal{P}$s. They simply act as if they were alone in their $\mathcal{E}$s, i.e. they act according to the theory recalled in Section 2.2. When modelling their environments, each of them relies on its observation model ${}^{\mathcal{P}}f(\Delta_t|,a_t,d(t-1),X)$ and prior pdf ${}^{\mathcal{P}}f(X)$. Presence of other acting participants makes the modelling and subsequent design are sensitive to modelling errors. Thus, inspection of achievable quality of estimation and consequently of resulting strategy becomes dominating aspect of this MP DDM scheme. The following proposition indicates how this problem can be addressed. It represents a version of large-deviations theorems (Sanov 1957, Kulhavý 1994).

Proposition 2 (Asymptotic properties of Bayesian estimation) *Let the observed data be generated by so called "objective" model (Berec & Kárný 1997) ${}^{o}f(\Delta_t|a_t,d(t-1))$, which is the best description of reality for the inspected DDM. Let also the used strategy meet natural conditions of control and bounds $\bar{B}(X)$, $\underline{B}(X)$ exist such that*

$$\infty > \bar{B}(X) \geq \frac{{}^{o}f(\Delta_t|a_t,d(t-1))}{{}^{\mathcal{P}}f(\Delta_t|\psi_t,X)} \geq \underline{B}(X) > 0 \tag{13}$$

for almost all $X \in \mathrm{supp}\left[{}^{\mathcal{P}}f(X)\right]$ and $d(\mathring{t}) \in d^*(\mathring{t})$.

Then, for $\mathring{t} \to \infty$,

$$\mathrm{supp}\left[{}^{\mathcal{P}}f(X|d(\mathring{t}))\right] \equiv \mathrm{Arg}\inf_{X\in\mathrm{supp}\left[{}^{\mathcal{P}}f(X)\right]} \frac{1}{\mathring{t}}\sum_{t\in t^*}\ln\left(\frac{{}^{o}f(\Delta_t|a_t,d(t-1))}{M(\Psi_t,X)}\right) \text{ almost surely.} \tag{14}$$

Proof: Using the Bayes rule and the definition of proportionality sign $\propto$, the posterior pdf can be given the form

$${}^{\mathcal{P}}f(X|d(\mathring{t})) \propto {}^{\mathcal{P}}f(X)\exp\left\{-\mathring{t}\left[\underbrace{\frac{1}{\mathring{t}}\sum_{t\in t^*}\ln\left(\frac{{}^{o}f(\Delta_t|a_t,d(t-1))}{M(\Psi_t,X)}\right)}_{\equiv\mathcal{H}(d(\mathring{t}),X)} - \inf_{X\in\mathrm{supp}\left[{}^{\mathcal{P}}f(X)\right]}\mathcal{H}(d(\mathring{t}),X)\right]\right\}$$

Due to the factor $-\mathring{t}$, for $\mathring{t}\to\infty$,

$$\mathrm{supp}\left[f(X|d(\mathring{t}))\right] = \mathrm{Arg}\inf_{X\in\mathrm{supp}\left[f(X)\right]}\mathcal{H}(d(\mathring{t}),X)$$

if the infimum is finite. To verify the finiteness, let us introduce, for a fixed X and the considered strategy fulfilling natural conditions of control, the deviations

$$e(d(t-1),X) \equiv \ln\left(\frac{{}^{o}f(\Delta_t|a_t,d(t-1))}{M(\Psi_t,X)}\right) - E\left[\ln\left(\frac{{}^{o}f(\Delta_t|a_t,d(t-1))}{M(\Psi_t,X)}\right)|d(t-1)\right].$$

With them,

$$\mathcal{H}(d(\mathring{t}),X) = \frac{1}{\mathring{t}}\sum_{t\in t^*}E\left[\ln\left(\frac{{}^{o}f(\Delta_t|a_t,d(t-1))}{M(\Psi_t,X)}\right)|d(t-1)\right] + \frac{1}{\mathring{t}}\sum_{t\in t^*}e(d(t-1),X).$$

The first term is average of (conditional) KL divergences and as such it is non-negative. The second term is average of uncorrelated zero-mean variables that are almost surely bounded due to (13). Consequently, (Loeve 1962), the second term converges almost surely to zero. Thus, $\mathcal{H}(d(\mathring{t}), X)$ is almost surely bounded from below that guarantees finiteness of the infimum. □

The fact that the posterior pdf concentrates on points in $\mathrm{supp}\left[\,{}^{\mathcal{P}}f(X)\right]$ minimizing asymptotically entropy rate $\mathcal{H}(d(\infty), X) \equiv$

$$\equiv \limsup_{t\to\infty} \frac{1}{\mathring{t}} \sum_{t\in t^*} \ln\left(\frac{{}^{o}f(\Delta_t|a_t, d(t-1))}{M(\Psi_t, X)}\right) = \limsup_{t\to\infty} \frac{1}{\mathring{t}} \sum_{t\in t^*} E\left[\ln\left(\frac{{}^{o}f(\Delta_t|a_t, d(t-1))}{M(\Psi_t, X)}\right)|d(t-1)\right]$$

is the known message of Proposition 2. For the inspected MP DDM, it is, however, important to notice that this divergence may be so large that the obtained model is useless. To see it more clearly, let us discuss influence of modelling error on DDM. The optimization has to deal with the estimated model and to minimize the KL divergence $\mathcal{D}\left({}^{\mathcal{P}}f||\,{}^{I}f\right)$.

Let us consider the ideal situation that a unique model parameterized ${}^{\mathcal{P}}X \in X^*$ is asymptotically obtained, i.e. ${}^{\mathcal{P}}f(\Delta_t|a_t, d(t-1)) =$

$$= \int {}^{\mathcal{P}}f(\Delta_t|a_t, d(t-1), X)\,{}^{\mathcal{P}}f(X|d(t-1))\,dX \to_{t\to\infty} {}^{\mathcal{P}}f(\Delta_t|a_t, d(t-1), {}^{\mathcal{P}}X) = M\left(\Psi_t, {}^{\mathcal{P}}X\right).$$

Let also assume that, with this model $M\left(\Psi_t, {}^{\mathcal{P}}X\right)$, the perfect matching of ${}^{\mathcal{P}}f(\mathcal{Q})$ with the ideal pdf ${}^{I}f(\mathcal{Q})$ is achieved, i.e. ${}^{\mathcal{P}}f(\cdot) = {}^{I}f(\cdot)$. The loss per time step, corresponding to the designed strategy ${}^{o\mathcal{P}}f(a_t|d(t-1))$ connected to the objective model ${}^{o}f(\Delta_t|a_t, d(t-1))$, becomes

$$\frac{1}{\mathring{t}}\mathcal{D}\left({}^{o}f||\,{}^{I}f\right) = \frac{1}{\mathring{t}}\mathcal{D}\left({}^{o}f||\,{}^{\mathcal{P}}f\right) = \frac{1}{\mathring{t}}E\left[\ln\left(\frac{\prod_{t\in t^*}{}^{o}f(\Delta_t|a_t, d(t-1))}{\prod_{t\in t^*}M(\Psi_t, {}^{\mathcal{P}}X)}\right)\right] \to_{\mathring{t}\to\infty} E\left[\mathcal{H}\left(d(\infty), {}^{\mathcal{P}}X\right)\right] \quad (15)$$

Thus, the loss is the higher the greater is asymptotically achieved minimum of the entropy rate. Continuity arguments extend this property to ${}^{\mathcal{P}}f$s in neighborhood of ${}^{I}f$. A finer robustness analysis is generally needed (Perez 1965, Martin, et al. 2003).

Remarks

1. The condition (13) given by bounds $\underline{B}(X)$, $\overline{B}(X)$ excludes significant differences in tails of the objective ${}^{o}f$ and ${}^{\mathcal{P}}f$ pdfs. It can be definitely modified and refined so that $\mathring{t}^{-1}\sum_{t\in t^*} e(d(t-1), X)$ still converges to zero.
2. The infimum is taken over the support of the prior pdf ${}^{\mathcal{P}}f(X)$ that is included in X^*, cf. (10). The proposition can be simply re-stated to other mutual relationships of $\mathrm{supp}\left[\,f(X)\right]$ and X^*.

3.2 Cooperation at Design Stage of Selfish Scenario

Even in selfish scenario, the participant is given a chance to cooperate. In the communication phase of the design stage, it gets information on knowledge, aims and restrictions of its neighbors and it *may* exploit them for updating of its model and ideal in accordance with its fidelity into the information obtained.

The exploitation of models of $\mathcal{E}$s as well as of prior, ideal and strategy-describing pdfs have a common formal structure as all these objects are pdfs. The common exploitation way is discussed here.

At abstract level, $\mathcal{P}$ deals with the pdf ${}^{\mathcal{P}}f\left({}^{\mathcal{P}}Q\right)$ and gets the pdf ${}^{\tilde{\mathcal{P}}}f\left({}^{\tilde{\mathcal{P}}}Q\right)$ from other participant $\tilde{\mathcal{P}}$. The cooperation makes sense if the intersection of sets of possible behaviors ${}^{\cap}Q^* \equiv {}^{\mathcal{P}}Q^* \cap {}^{\tilde{\mathcal{P}}}Q$ is non-empty. Let us decompose behavior ${}^{\mathcal{P}}Q \equiv (G, H)$ and ${}^{\tilde{\mathcal{P}}}Q \equiv (H, I)$ with $G \in {}^{\mathcal{P}}Q^* \setminus {}^{\cap}Q^*$, $H \in {}^{\cap}Q^*$, $I \in {}^{\tilde{\mathcal{P}}}Q^* \setminus {}^{\cap}Q^*$. Thus, we deal with a pair of pdfs ${}^{\mathcal{P}}f(G, H)$, ${}^{\tilde{\mathcal{P}}}f(H, I)$. Generally, there is a discrepancy between these pdfs, i.e. the marginal pdfs ${}^{\mathcal{P}}f(H)$, ${}^{\tilde{\mathcal{P}}}f(H)$ differ.

Let ${}^{\mathcal{P}}\alpha \in [0, 1]$ be belief of the participant into correctness (relevance) of its pdf ${}^{\mathcal{P}}f(G, H)$ and $1 - {}^{\mathcal{P}}\alpha$ to that provided by the participant $\tilde{\mathcal{P}}$. The addressed problem can be now formulated as a construction of estimate ${}^{\mathcal{P}}\hat{f}(G, H)$ of the unknown objective pdf (Berec & Kárný 1997) ${}^{o\mathcal{P}}f(G, H)$ using the available knowledge ${}^{\mathcal{P}}f(G, H)$, ${}^{\tilde{\mathcal{P}}}f(H, I)$.

The advocated solution is based on several assumptions formulated and justified below.

- The conditional expectation ${}^{\mathcal{P}}\hat{f}(G,H) \equiv E\left[{}^{o}f|{}^{\mathcal{P}}f,\,{}^{\tilde{\mathcal{P}}}f\right]$ is the best estimate of ${}^{o\mathcal{P}}f(G,H)$ minimizing the expected KL divergence $E\mathcal{D}\left({}^{o}f||{}^{\mathcal{P}}\hat{f}\right)$ that should serve as the adequate expected loss of the problem, as it follows from Proposition 2.
- The expectation is taken over the set of unknown objective pdfs ${}^{o}f^*$. Its conditional version will be constructed using the observations listed further on.
- The participant $\mathcal{P}$ is not aware of the space ${}^{\tilde{\mathcal{P}}}Q^* \setminus {}^{\cap}Q^*$ and thus it may exploit at most the marginal pdf ${}^{\tilde{\mathcal{P}}}f(H)$. Similarly, the participant $\tilde{\mathcal{P}}$ is not aware of the space ${}^{\mathcal{P}}Q^* \setminus {}^{\cap}Q^*$ thus it brings no information on ${}^{o\mathcal{P}}f(G,H)$.
- The only available information about relationship between G and H is in ${}^{\mathcal{P}}f(G|H)$ which should thus coincide with the corresponding conditional expectation. These observations motivate assumptions

$$E\left[{}^{o\mathcal{P}}f(G,H)|{}^{\mathcal{P}}f(G,H),\,{}^{\tilde{\mathcal{P}}}f(H,I)\right] \equiv E\left[{}^{o\mathcal{P}}f(G,H)|{}^{\mathcal{P}}f(G,H),\,{}^{\tilde{\mathcal{P}}}f(H)\right]$$
$$E\left[{}^{o\mathcal{P}}f(G|H)|{}^{\mathcal{P}}f(G,H),\,{}^{\tilde{\mathcal{P}}}f(H)\right] \equiv E\left[{}^{o\mathcal{P}}f(G|H)|{}^{\mathcal{P}}f(G|H),\,{}^{\mathcal{P}}f(H)\right] \equiv {}^{\mathcal{P}}f(G|H). \tag{16}$$

- No information on correlation of ${}^{o\mathcal{P}}f(G|H)$ and ${}^{o\mathcal{P}}f(H)$ is available. Thus, we can assume furthermore that

$$E\left[{}^{o\mathcal{P}}f(G|H)\,{}^{o\mathcal{P}}f(H)|{}^{\mathcal{P}}f(G|H),\,{}^{\mathcal{P}}f(H),\,{}^{\tilde{\mathcal{P}}}f(H)\right] \underbrace{=}_{(16)} {}^{\mathcal{P}}f(G|H)E\left[{}^{o\mathcal{P}}f(H)|{}^{\mathcal{P}}f(G|H),\,{}^{\mathcal{P}}f(H),\,{}^{\tilde{\mathcal{P}}}f(H)\right]. \tag{17}$$

- It remains to specify the expectation of ${}^{o\mathcal{P}}f(H)$ given (essentially) by a pair of alternatives with given degrees of belief. Thus, it makes sense to adopt the last assumption

$$E\left[{}^{o\mathcal{P}}f(H)|{}^{\mathcal{P}}f(G|H),\,{}^{\mathcal{P}}f(H),\,{}^{\tilde{\mathcal{P}}}f(H)\right] \equiv E\left[{}^{o\mathcal{P}}f(H)|{}^{\mathcal{P}}f(H),\,{}^{\tilde{\mathcal{P}}}f(H)\right] \equiv {}^{\mathcal{P}}\alpha\,{}^{\mathcal{P}}f(H)+\left(1-{}^{\mathcal{P}}\alpha\right){}^{\tilde{\mathcal{P}}}f(H). \tag{18}$$

This implies the final composition rule defining the new pdf ${}^{\mathcal{P}}\hat{f}$ of the participant $\mathcal{P}$ corrected by information provided by the participant $\tilde{\mathcal{P}}$. It is summarized in the following algorithmic form.

Algorithm 1 (Correction of pdf ${}^{\mathcal{P}}f$ by the pdf ${}^{\tilde{\mathcal{P}}}f$)

1. *Participant $\mathcal{P}$ selects belief ${}^{\mathcal{P}}\alpha \in [0,1]$ in correctness of its pdf*
2. *Participant $\mathcal{P}$ gets marginal pdf ${}^{\tilde{\mathcal{P}}}f(H)$ of its neighbor $\tilde{\mathcal{P}}$ describing the common part H of behaviors ${}^{\mathcal{P}}Q \equiv (G,H)$, ${}^{\tilde{\mathcal{P}}}Q \equiv (H,I)$ considered by the involved participants $\mathcal{P}$, $\tilde{\mathcal{P}}$*
3. *Participant $\mathcal{P}$ defines the corrected pdf ${}^{\mathcal{P}}\hat{f}(G,H)$ by the formula*

$${}^{\mathcal{P}}\hat{f}(G,H) = {}^{\mathcal{P}}f(G|H)\left[{}^{\mathcal{P}}\alpha\,{}^{\mathcal{P}}f(H) + \left(1-{}^{\mathcal{P}}\alpha\right){}^{\tilde{\mathcal{P}}}f(H)\right]. \tag{19}$$

Remarks

1. Application of Algorithm 1 to prior pdfs, models of environments as well as to ideal pdfs is generally non-trivial as the recognition of the common variable H and evaluation of the marginal pdf need not be straightforward. Examples of detailed evaluations are out of scope of this paper.
2. Section 3.4 that deals with hierarchical scenario combines individual aims indirectly via estimation. Even in this case, Algorithm 1 is useful as it shows that the combined model should be searched as the mixture model (19).
3. The improved model ${}^{\mathcal{P}}\hat{f}$ is supposed to serve to $\mathcal{P}$. Thus, $\mathcal{P}$ has a freedom to assign belief ${}^{\mathcal{P}}\alpha$ to itself. Obviously, it is wise to make this probability proportional to its predictive ability on the common observable part of H in comparison with this ability of information provider $\mathcal{P}$.
4. Extension of the proposed combination to a larger number of cooperating participants is nontrivial. Often, however, pair-wise comparison (possibly iterative) can be sufficient.

3.3 Cooperative Scenario

In the selfish scenario, communicating partners have right to decide on degree of exploitation of the information provided by their neighbors. In the cooperative scenario, they are obliged to come to a common model

and ideal concerning to the common part of their behavior. It means that have to negotiate on the belief values assigned to the partner. They have to reach the situation with ${}^{\mathcal{P}}\alpha = 1 - {}^{\tilde{\mathcal{P}}}\alpha \equiv \alpha$, they have to *negotiate*.

The fair negotiation requires to judge final loss on union of both considered behaviors, i.e. to define the common model ${}^{\mathcal{P}\tilde{\mathcal{P}}}f(G,H,I)$ and common ideal ${}^{I\mathcal{P}\tilde{\mathcal{P}}}f(G,H,I)$ as a function of the common belief(s) α and to minimize the smallest value of the KL divergence $\mathcal{D}\left({}^{\mathcal{P}\tilde{\mathcal{P}}}f||\,{}^{I\mathcal{P}\tilde{\mathcal{P}}}f\right)$ with respect to these beliefs.

Practically, the minimization can be performed by comparing values on a relatively sparse grid in belief-values space. Thus, the extension of the involved pdfs on the union of behaviors is the key problem to be addressed. There are indicators that the pdf

$$ {}^{\mathcal{P}\tilde{\mathcal{P}}}f(G,H,I) = {}^{\mathcal{P}}f(G|H)\,{}^{\tilde{\mathcal{P}}}f(I|H)\left[\alpha\,{}^{\mathcal{P}}f(H) + (1-\alpha)\,{}^{\tilde{\mathcal{P}}}f(H)\right] \tag{20}$$

is the adequate extension. No rigorous justification is, however, available.

3.4 Hierarchical Scenario

Hierarchical scenario supposes that actions of individual participants in the group $\mathcal{P}^*$ are coordinated by a single member of the group, referred to as *coordinator* $\mathcal{C} \in \mathcal{P}^*$. *The coordinator has the power to influence other participants in the group* by: i) enforcing ideal pdfs and models of their environments (at design stage); ii) data supplied (at acting stage). Although individual participants may behave as selfish or cooperative, they must accept some ingredients enforced to them by the coordinator $\mathcal{C}$. The degree of obedience is predefined by strength of hierarchy.

Type of coordination is determined by the way how $\mathcal{C}$ constructs its own ideal according to which it coordinates the group $\mathcal{P}^*$. A selfish coordinator respecting aims neither group nor particular participants behaves as a dictator. The coordinator that constructs its ideal pdf so that group aim as well as aims of individual members are respected as much as possible can be labelled as democrat. The construction of such a "democratic ideal" ${}^{I\mathcal{C}}f(d_t|d(t-1))$ is the major new task connected with the hierarchical scenario. The task solution is discussed below. For a complementary information see (Kárný & Kracík 2004).

In the task formalization, the coordinator $\mathcal{C}$ is assumed:

- To have at disposal data that inform it about the particular participants behavior and behavior of the group as a whole.
- To share a part of its data d_t with all participants of the group. These data may include actions of the coordinator as well as its innovations serving for coordination.
- To have information about externally supplied group aim (if any).
- To have at disposal information about aims of individual participants. It serves for harmonizing these individual aims both mutually and with the group aim.
- To obtain regularly data with respect to which members of the group have expressed their aims.

The construction of the coordinator's ideal ${}^{I\mathcal{C}}f$ is straightforward if the number of members $\mathring{\mathcal{P}}$ in the coordinated group is small. It has to describe the common data available to each participant and thus it suffices to create the ideal pdf from the corresponding marginal pdfs of respective ideals. Their convex combination solves this problem. The weights, selected by the coordinator, reflects the significance it assigns to participants. The constant weights $\mathring{\mathcal{P}}^{-1}$ can be taken as prototype of the fair weights. This simple weighting is possible due to the common expression of losses in terms of the KL divergences.

Real problem arises when the number of coordinated participants is large. It this case, however, it can be expected that participant aims can be clustered in relatively small amount $\mathring{\mathcal{P}} < \infty$ of clusters. A cluster contains an ${}^{\mathcal{P}}\alpha > 0$ part of the group participants $\left(\sum_{\mathcal{P}=1}^{\mathring{\mathcal{P}}} {}^{\mathcal{P}}\alpha = 1\right)$. Each cluster has data ${}^{\mathcal{P}}d_t$ on which cluster participants express their aims through a cluster ideal ${}^{I\mathcal{P}}f\left({}^{\mathcal{P}}d_t|\,{}^{\mathcal{P}}d(t-1)\right)$.

The coordinator processes the provided cluster ideals by employing data ${}^{\bar{\mathcal{P}}}d_t$ not available to the participants from this cluster. Actually, the extension (12) of the cluster ideal pdf with respect to the full data record $d_t = \left({}^{\mathcal{P}}d_t, {}^{\bar{\mathcal{P}}}d_t\right)$ available to the coordinator, is defined as

$$ {}^{I\mathcal{P}}f(d_t|d(t-1)) = {}^{I\mathcal{C}}f\left({}^{\bar{\mathcal{P}}}d_t|d_t, d(t-1)\right)\,{}^{I\mathcal{P}}f\left({}^{\mathcal{P}}d_t|\,{}^{\mathcal{P}}d(t-1)\right) \text{ for any } \mathcal{P} = 1,\ldots,\mathring{\mathcal{P}} \tag{21}$$

where the marginal pdf of the constructed coordinator's ideal ${}^{I\mathcal{C}}f(d_t|d(t-1))$ is used. With the extension (21), the coordinator's democratic ideal ${}^{I\mathcal{C}}f(d_t|d(t-1))$, respecting ideals of all clusters within a group,

is searched for. The pdf defined on the coordinator's data d_t and closest to the extended ideals of particular clusters is selected, i.e.:

$$ {}^{I\mathcal{C}}f(d_t|d(t-1)) = \mathrm{Arg}\min_{f(d_t|d(t-1))} \sum_{\mathcal{P}\in\mathcal{P}^*} {}^{\mathcal{P}}\alpha\mathcal{D}\left({}^{I\mathcal{P}}f(d_t|d(t-1)) \,||\, f(d_t|d(t-1))\right). \tag{22} $$

This formulation can be interpreted as minimization of the expected KL divergence of the objective but unknown coordinator's ideal ${}^{oI\mathcal{C}}f(d_t|d(t-1))$ and its approximation ${}^{I\mathcal{C}}f(d_t|d(t-1))$. For it, we have to assume that ${}^{oI\mathcal{C}}f(d_t|d(t-1)) \in {}^{oI\mathcal{C}}f^*(d_t|d(t-1)) \equiv \left\{{}^{I\mathcal{P}}f(d_t|d(t-1))\right\}_{\mathcal{P}\in\mathcal{P}^*}$ and

$$ \mathsf{Probability}\left[{}^{oI\mathcal{C}}f(d_t|d(t-1)) = {}^{I\mathcal{P}}f(d_t|d(t-1))|\,{}^{oI\mathcal{C}}f^*(d_t|d(t-1))\right] = {}^{\mathcal{P}}\alpha. $$

Unrestricted minimization gives implicit equation for the ideal search for

$$ {}^{I\mathcal{C}}f(d_t|d(t-1)) = \sum_{\mathcal{P}\in\mathcal{P}^*} {}^{\mathcal{P}}\alpha\,{}^{I\mathcal{P}}f(d_t|d(t-1)) \underbrace{=}_{(21)} \sum_{\mathcal{P}\in\mathcal{P}^*} {}^{\mathcal{P}}\alpha\,{}^{I\mathcal{C}}f\left({}^{\bar{\mathcal{P}}}d_t|d_t, d(t-1)\right) {}^{I\mathcal{P}}f\left({}^{\mathcal{P}}d_t|\,{}^{\mathcal{P}}d(t-1)\right). \tag{23} $$

Equation (23) has no analytic solution for generic cases with $\mathcal{P}$. This problem can be practically avoided by replacing the extension (21). Alternatively, the cluster ideal can be extended on the full data record $d = \left({}^{\mathcal{P}}d,\, {}^{\bar{\mathcal{P}}}d\right)$ by multiplying the marginal pdf ${}^{I\mathcal{P}}f\left({}^{\mathcal{P}}d_t|\,{}^{\mathcal{P}}d(t-1)\right)$ by a flat pdf ${}^{I\bar{\mathcal{P}}}\bar{f}\left({}^{\bar{\mathcal{P}}}d_t|d(t-1)\right)$ defined on the data ${}^{\bar{\mathcal{P}}}d_t$ out of the interests of the $\mathcal{P}$th cluster. Then, the *optimal joint ideal pdf of the whole group* equals

$$ {}^{I}f(d_t|d(t-1)) = \sum_{\mathcal{P}\in\mathcal{P}^*} {}^{\mathcal{P}}\alpha\,{}^{I\mathcal{P}}f\left({}^{\mathcal{P}}d_t|\,{}^{\mathcal{P}}d(t-1)\right) {}^{I\bar{\mathcal{P}}}\bar{f}\left({}^{\bar{\mathcal{P}}}d_t|d(t-1)\right). \tag{24} $$

The obtained explicit group ideal (24), representing compromise between ideals of all clusters entering the group, can be taken as the coordinator's ideal ${}^{I\mathcal{C}}f(d_t|d(t-1)) \equiv {}^{I}f(d_t|d(t-1))$. However even in this simple case, the direct use of the explicit group ideal, driving the fully probabilistic design of coordinator's strategy, is inhibited by the difficulty to cluster the participants properly. For it, we have to define which ideal pdfs are taken as practically identical and the results may be quite sensitive to this definition. Moreover, the well-established clustering techniques are not elaborated for such objects as pdfs are. Thus, it makes sense to search for an alternative way of constructing of the coordinator's ideal.

Constructing of the group ideal, addressed as *estimation problem*, seems to offer a remedy of all mentioned troubles. The above considerations indicate the ideal should be search in the form of finite mixture. The solution is described by the following algorithm.

Algorithm 2 (Data-based construction of the group ideal) *1. The coordinator's ideal is supposed to have a form of a finite mixture (Titterington et al. 1985) composed of $\mathring{\mathcal{P}}$ components ${}^{I}f\left(d_t|d(t-1),\,{}^{\mathcal{P}}\Theta\right)$ parameterized by finite collection of unknown parameters ${}^{\mathcal{P}}\Theta$ and weighted by unknown probabilistic weights ${}^{\mathcal{P}}\alpha$*

$$ {}^{I}f(d_t|d(t-1),\Theta) = \sum_{\mathcal{P}=1}^{\mathring{\mathcal{P}}} {}^{\mathcal{P}}\alpha\,{}^{I}f\left(d_t|d(t-1),\,{}^{\mathcal{P}}\Theta\right). $$

2. *Each participant is required to provide data records characterizing its ideal. The data format and number of data records is previously defined and fixed for all participants of the group. All entries of the data record available to the coordinator has to be filled in.*
3. *The data, provided by participants, are processed to estimate a group ideal in the form of mixture model suggested. The standard Bayesian methodology, see e.g. (Kárný et al. 2003) and references there, is used for estimation of unknown mixture parameters $\Theta = \{\alpha_c, \mathring{c}, \Theta_c\}$. Resulting mixture model approximates the whole group data with particular components modelling individual clusters of the group.*

Remarks

1. In all variants, the degree of compromise between contradictory ideals is defined by the cluster (component) weights ${}^{\mathcal{P}}\alpha$.
2. As a rule, $\mathcal{P}$ expresses its wishes about sub-selection of data entries only, i.e. $d = \left({}^{\mathcal{P}}d,\, {}^{\bar{\mathcal{P}}}d\right)$. This introduces the hard problem of missing data into the mixture estimation. One possibility is to ask the participant to provide lower and upper bound on entries of its interest and complement it by the lowest and the highest

bounds on the variables it is not interested in. Alternatively, the fact that often whole groups of entries are out of the participant interest can be exploited. Considering for presentation simplicity the static parameterized ideal ${}^{I}f(d_t|d(t-1),\Theta) = {}^{I}f(d_t|\Theta)$, we get

$$f(\Theta|d_{\mathcal{P}}) \propto f(\Theta)\, {}^{I\mathcal{P}}f\left({}^{\mathcal{P}}d|\Theta\right) = f(\Theta)\int {}^{I\mathcal{P}}f(d|\Theta)\, d\,{}^{\bar{\mathcal{P}}}d.$$

To get a practical algorithmic solution of this formally correct treatment is, however, difficult and generally un-elaborated problem.

3. The data-based presentation of ideals is "natural" in societal problems (Kárný & Kracík 2004). It may be advantageous even in technical problems. Let ${}^{I\mathcal{P}}f({}^{\mathcal{P}}d_t|\,{}^{\mathcal{P}}d(t-1))$ be the ideal expressing the decision aims and restrictions. As a rule, it is rather complex pdf and the fully probabilistic design with it or with mixtures having it as a component is quite hard. Then, it makes sense to take samples from this pdf and use it for estimation of a mixture with numerically tractable components. Then, the estimation-based fitting of the ideal pdf performs simultaneously projection of the original "wild" ideal onto a feasible set.
4. Standard mixture estimation procedures can be used (Titterington et al. 1985). An advanced initialization (Kárný, et al. 2001b) combined with structure estimation becomes inevitable in high-dimensional cases. It guarantees practically appropriate clustering of the group.
5. The participant may have at disposal data not available to the coordinator. Then, their existence does not influence $\mathcal{C}$ who can assume that such surplus data (taken from its perspective) do not exist. Availability of surplus data of $\mathcal{C}$ that are strongly related to aims of the coordinated $\mathcal{P}$s is the main source of its ability to construct the proper ideal.
6. If power of enforcing of the ideal and models is weaken for the coordinator, the *scenario of horizontally cooperating group* can be obtained. Practically it means that some participant (called *probabilistic advisor*) is assumed to have enough modelling and evaluation power allowing to support other participants in their environment (Kárný et al. 2003, Quinn, et al. 2003). Instead of enforcing, probabilistic advisor offers projections of the optimized models as advices to respective participants. Supported participants have freedom to take the ideal pdfs optimized by advisor as the ideal pdfs for the data shared with advisor. In extreme case, they can follow blindly up to copying the advised actions. They are, however, responsible for the achieved quality and thus they should harmonize these advices with the information and actions related to their surplus data that are not at disposal to advisor.

4 Concluding Remarks

MP DDM discussed in this paper has many facets and problems. Besides the problems unsolved for SP DDM and thus transferred to MP DDM, new specific problems arise. Let us list some of them, we are aware of.

- It is not clear whether classification introduced within this paper is general enough and that no significant aspect has been left aside.
- Communication ways are extremely important in MP DDM. The net of participants should be structured – both in time and space – in order to get a high overall performance or even just stable behavior.
- Efficient storing of function of many variables (both pdfs and functions appearing in the fully probabilistic design) becomes even more important in MP DDM than in SP DDM.
- Estimation with incomplete data becomes especially urgent in MP DDM.
- Sharing of probabilistic models leads directly to finite mixtures whose set is not closed with respect to learning and conditioning. These general problems still wait for their solutions.
- Extension of pdfs (a solution of general marginal problems) is to be elaborated as the outlined solution is neither unique nor sufficiently justified.

Each item of the above list represents difficult and fundamental research problems whose quick solution can hardly be expected. In spite of this, partial results, reflected in the paper, improve understanding of MP DDM and can be used in solution of particular problems like normative fair governing (Kárný & Kracík 2004), advising to other $\mathcal{P}$s or sensor fusion.

Concerning to relationships of discussed scenarios, it is clear that

- The hierarchical scenario has a significant potential to provide the highest performance if the cooperating group is small enough.
- The horizontal cooperation has potential to be better than selfish MP DDM. Comparing to groups manageable in hierarchical scenario, it is expected to be worse. It has, however, an extreme potential in *scalability*.

Horizontal cooperation of participants with a sufficiently dense subset of coordinators can be successfully applied to extremely large MP DDM where hierarchical scenario cannot be applied.

- Combination of small-scale hierarchies with horizontal cooperations, enhanced potentially by the adaptive shifting of boundaries between participant environments, seems to be the most powerful way of solving large scale MP DDM. It also seems that Nature came to the same conclusion.

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On combining expertise in Dynamic Linear Models

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Abstract. We consider different models combining opinions of two experts, given at specific time points, when forecasting with dynamic models. We describe a Bayesian approach for inference and prediction in such situations, using both historical data and current expert's opinions. We consider cooperation cases in which experts' opinions are merged into a class of priors, or one expert provides information to another, and compare these cases with the non-cooperative, independent one. The former case leads to robust Bayesian analyses, whereas in the latter, the expert's input is treated as information used to improve upon the basic model and learn about the forecasting ability of the expert. Our approach is motivated by the need of companies to forecast project costs in bidding processes.

Keywords: Dynamic model, Project costing, Expert's opinion, Bayesian forecasting, Bayesian robustness.

1 Introduction

We consider the case of two experts who are faced with the problem of formulating opinions on a series of similar events at different times (e.g. bidding in off-shore oil plant auctions). The cost structure is evolving over time and decisions are to be made combining both historical data and experts' opinions. Dynamic linear models (DLM's) are typical choices to address this problem and the Bayesian approach is the natural framework to combine past evidence and expertise. We suppose the two experts (the number could be larger) have different knowledge and we explore different ways to combine their expertise. We suppose the experts have to report decisions based on their opinions to the top management of a company. We explore three cases:

- the experts act independently and each of them reports his/her own decision;
- the experts combine their opinions into a class and consider consequences of such action;
- one expert expresses his/her own opinion and updates it using the other expert's opinion as data.

The first case is treated with the well-known methods in the Bayesian analysis of dynamic linear models; see (West and Harrison 1997) for a thorough review. The second case is treated with methods from Bayesian robustness; see (Rios Insua and Ruggeri 2000) for a review. The third case is an example of a general model developed in (Palomo et al. 2004).

Here we are also exploring issues such as the experts' quality and what happens when the information they provide is ignored. Some suggestions regarding these issues may be seen in (Spiegelhalter et al. 1994), such as when to ignore expert's opinions, and (Clemen and Winkler 1999), when more than one expert opinion is available.

The paper is structured as follows: in Section 2, we illustrate the project which has lead us to consider the problem of combining opinions in DLM's. Section 3 reviews the key issues in robust Bayesian analysis. In Section 4, we introduce the basic DLM and illustrate the case of independent experts. In Section 5, we apply methods from Bayesian robustness to the case of opinions joined into classes of priors. Section 6 illustrates a DLM in which an expert uses the other expert's opinion to update his/her own. Section 7 illustrates the proposed methods in an example. We finish with some discussion.

2 Project description

The paper stems from a consulting project in which a framework to cope with the problem of bid formulation in off-shore oil plant auctions was required. Project cost forecasting is an essential step in project management and is becoming increasingly complex because of financial uncertainties, sometimes caused by the use of innovative technologies. Several project management techniques have been proposed, see (Turner 1993) for a review. However, in the current economy where auctions have become a daily activity with which companies

must deal, these traditional techniques are no longer sufficient. In this competitive environment, project cost forecasting becomes especially crucial, since an accurate estimation of the project cost distribution allows bidders to prepare better bids, in terms of expected profits, while decreasing the probability of undercapitalization. In other words, auctions increase the project's level of uncertainty and make it so that a bad cost estimate may lead a bidder to either loose the auction, or win it with a bid lower than the actual cost.

Quite frequently, project costing techniques appeal to heuristics, with relatively few statistical methods being applied and, in general, do not account for all the uncertainties - nor can they handle expert's opinions, which may be essential when handling multi-component projects that span many professional areas in which the proposal manager (PM) will not be skilled. One example is the Activity-Based Costing (ABC) technique, see (Alan 1995) or (Raz and Elnathan 1998), which is used profusely in large scale industries to improve operational performance by providing product cost estimations. Although there are a large number of papers dealing with the design and implementation of the ABC technique, see e.g. (Berliner and Brimson 1988), (David and Robert 1995), or (Booth 1996), there is no procedure to account for the uncertainty in the cost estimations of each project's activity. Therefore, since several benefits and important insights could be obtained by applying modeling uncertainty to activity costs, see (Garvey 2000) for a review, (Palomo et al. 2004) addressed the problem of finding the project total cost distribution through aggregation of the cost distributions of all project activities. Combining ideas on expert modeling, as e.g. in (Campodonico and Singpurwalla 1995), with the dynamic model, as e.g. in (West and Harrison 1997), (Palomo et al. 2004) proposed a Bayesian approach which uses a formal framework to incorporate historical data and expert's opinions in a dynamic cost forecasting process. In the current paper we consider the cost of one project activity and we suppose two people are involved in forecasting its cost. The proposal manager (PM) is an expert on business matters, with scarce knowledge on the project activity but with a global view on the project and the economical/political/social situation (and its influence on costs) of the environment in which the plant is planned. The unit manager (UM) has a deeper technical knowledge on that particular project activity but little knowledge about the whole project and the external conditions under which the plant should be built. Therefore, the two sources of knowledge are different and we propose different methods of combining them, whereas (Palomo et al. 2004) considered only the case in which the the PM had beliefs about the uncertain cost component c_i at time i, and he asked for an expert's forecast t_i about it. Using this new information, the PM could update his beliefs to $c_i|t_i$ and assess the expert's forecasting quality along similar projects.

3 Bayesian robustness

As criticized by non-Bayesians and recognized by Bayesians, the choice of a prior distribution is a critical aspect of the Bayesian approach, since different choices could lead to very different results. Although the elicitation of a prior should be performed as carefully as possible, uncertainty in specifying prior beliefs cannot be avoided, in general. Robust Bayesian Analysis is concerned with the sensitivity of the results of a Bayesian analysis to the inputs for the analysis, mostly in replacing a single prior distribution by a class of priors, and developing methods of computing the range of the ensuing answers as the prior varies over the class. Interest naturally expands into study of robustness with respect to the likelihood and loss function as well, with the aim of having a general approach to sensitivity towards all the ingredients of the Bayesian paradigm (model/prior/loss).

The usual practical motivation underlying robust Bayesian analysis is the difficulty in assessing the prior distribution. A common elicitation technique about continuous random variables is to discretize their range Θ and to assess the probabilities of the resulting sets. One actually ends up only with some constraints on these probabilities. This is often phrased in terms of having bounds on the quantiles of the distribution. A related idea is having bounds on the distribution function corresponding to the random variable. Once again, qualitative constraints, such as unimodality, might also be added. Perhaps the most common procedure for a continuous random variable is to assess a parameterized prior, such as a conjugate prior, and place constraints on the parameters of the prior. While this can be valuable analysis, it does not serve to indicate robustness with respect to the specified form of the prior. Similar concerns apply to the other elements, likelihood and loss function. The main goal of Bayesian robustness is to quantify and interpret the uncertainty induced by partial knowledge of one (or more) of the three elements in the analysis. Also, ways to reduce the uncertainty are studied and applied until, hopefully, robustness is achieved.

As an example, suppose the quantity of interest is the posterior mean; the likelihood is considered to be known, but the prior distribution is known only to lie in a given class. The choice of the class is very important; it should have some desirable, yet competing properties, such as:

- computation of robustness measures should be as easy as possible;
- all reasonable priors should be in the class and unreasonable ones (e.g. discrete distributions in many problems) should not;
- class should be easily specified from elicited prior information.

The uncertainty might be quantified by specifying the range, i.e. the difference between upper and lower bounds on the quantity of interest, spanned by the posterior mean, as the prior varies over the class. One must appropriately interpret this measure; for instance, one might say that the range is "small" if it spans less than 1/4 of a posterior standard deviation (suitably chosen from the range of possible posterior standard deviations). If the range is small, robustness is asserted, and the analysis would be deemed to be satisfactory. If, however, the range is not small, then some way must be found to reduce the uncertainty; narrowing the class of priors or obtaining additional data would be ideal.

There are three main approaches to Bayesian robustness. The first is the *informal approach*, in which a few priors are considered and the corresponding posterior means are compared. The approach is valued for its simplicity, but it can easily "miss" priors that are compatible with the actually elicited prior knowledge and yet which would yield very different posterior means.

The second approach, called *global robustness*, considers the class of all priors compatible with the elicited prior information, and computes the range of the posterior mean as the prior varies over the class. This range is typically found by determining the "extremal" priors in the class that yield the maximum and minimum posterior means. Such computations can become cumbersome in multidimensional problems.

The third approach, called *local robustness*, is interested in the rate of change in inferences, with respect to changes in the prior, and uses differential techniques to evaluate the rate.

A general review of literature on Bayesian robustness is given in (Rios Insua and Ruggeri 2000). In this paper we consider both informal and global robustness. The former will be considered in Section 4 where effects of two priors will be considered and compared; the latter will be performed in Section 5 where the expert's opinions will be joined in a class of priors.

4 Independent DLM's

We suppose that the two experts are independent of each other. In this case a DLM describes their beliefs:

$$\begin{aligned} c_i|\theta_i &\sim N(F_i'\theta_i, V_i) \\ \theta_i|\theta_{i-1} &\sim N(G_i\theta_{i-1}, W_i) \\ \theta_0 &\sim N(m_0, \Delta_0) \end{aligned} \tag{1}$$

For each time i, the model is characterized by the tuple $\{F_i, G_i, V_i, W_i\}$, with F_i a known (n x 1) matrix, G_i a known (n x n) matrix, W_i a known (n x n) variance matrix, and V_i is a known scalar variance.

As illustrated in (West and Harrison 1997), the forecast on the cost c_i follows from the standard DLM equations, i.e. $\pi(c_i) \sim N(f_i, q_i)$, where

$$\begin{aligned} f_i &= F_i'G_im_{i-1} \\ q_i &= F_i'R_i^\theta F_i + V_i \\ R_i^\theta &= G_i\Delta_{i-1}G_i' + W_i \\ m_i &= G_im_{i-1} + A_i^\theta e_i^\theta \\ \Delta_i &= R_i^\theta - A_i^\theta q_i A_i^{\theta'} \\ A_i^\theta &= R_i^\theta F_i \tfrac{1}{q_i} \\ e_i^\theta &= \tilde{c}_i - f_i \end{aligned} \tag{2}$$

When the actual value $\tilde{c}_i$ arrives, the posterior on θ_i is obtained following standard DLM computations,

$$\pi(\theta_i|\tilde{c}_i) \sim N(m_i, \Delta_i)$$

with m_i, Δ_i as before, see (West and Harrison 1997).

The two experts can specify their own parameters and, once the values $(\tilde{c}_1, \ldots, \tilde{c}_n)$ have been observed, comparison of forecasting and posterior distributions is quite straightforward since the distributions involved are normal ones. We will show an example of comparison in Section 7.

5 Classes of priors in DLM

Experts may agree on a class of DLM's or priors as representative of their joint belief. The simple way is to consider classes of priors on θ_0 in the DLM (1). The simple class of priors is the one containing the two priors specified by the experts. Starting from them, the forecast distributions and the posterior estimators can be compared, performing an informal robustness check. A slightly wider class considers all the priors with parameters within a specified interval. Usually, the range of the posterior quantity of interest is achieved at endpoints of the interval. Ranges of other quantities in (1), e.g. F (when it is a scalar), can be easily considered.

Despite of the plethora of papers published, specially in the 90's, on Bayesian robustness, the application of these methods to DLM's is rather scarce. In this paper, we do not plan to pursue a systematic analysis of robustness in DLM but we just want raise the awareness of the importance, although neglected, of robust studies in DLM.

We will illustrate the robust Bayesian approach with an example in Section 7.

6 DLM with expert's opinion

Now we suppose that an expert (the PM in our consulting project) uses a DLM for his basic forecasts and the other expert (the UM) provides point forecasts. The PM does not know exactly how the UM produces these estimates, but he assumes that the UM aims at being accurate. He tries to learn the UM's forecasting quality for later periods and, hopefully, obtain a more informative forecasting distribution combining his model prediction with the UM's opinion.

6.1 Formulation

The case we consider is described by the influence diagram in Figure 1. Note that we are essentially combining

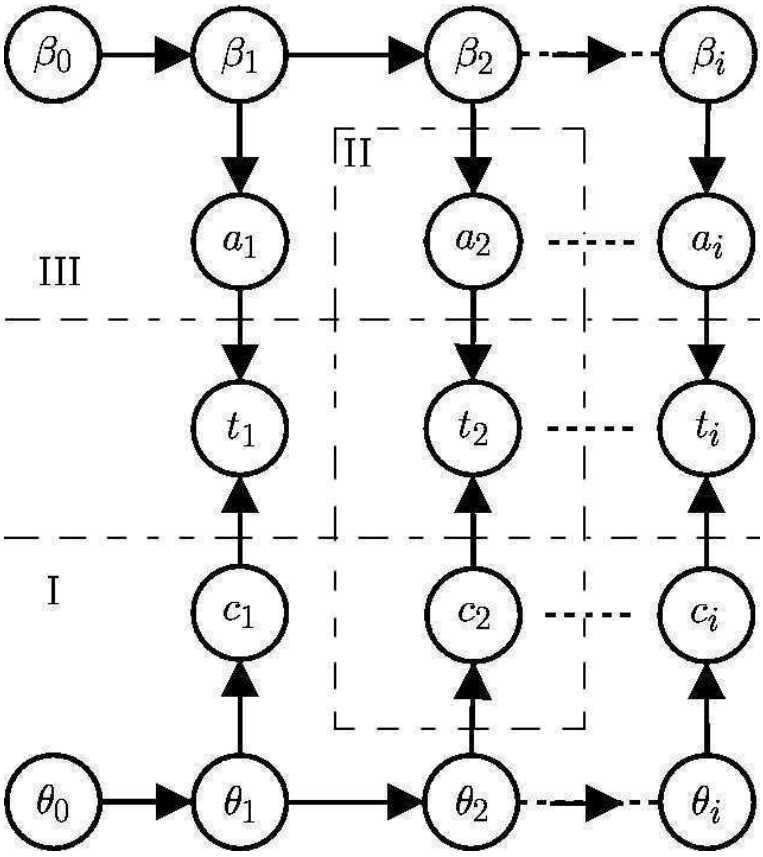

Figure 1. Influence diagram for the dynamic model with expert input

three standard models. Block **I** describes the linear state-space model we use; block **II** describes a standard

error model; block **III** describes a time-varying autoregressive model for the evolution of biases. Nonlinearity in Block **I** has been considered in the general model developed in (Palomo et al. 2004), and applied to a neural network example.

The PM is interested in forecasting the quantity c_i at time i, both before and after having heard the UM's opinion t_i. We assume that the evolution of the c_i's is appropriately described by a linear state-space model with state variable θ_i, which includes all unobservable factors affecting c_i, like the economy, actual demand,... An accuracy parameter a_i measures, somehow, the expert's forecasting ability. We shall assume that $a_i = t_i - c_i$ is a bias, which follows an autoregressive process.

6.2 Inference and Prediction

As previously mentioned, the key issue is the prediction of c_i, both before and after t_i is heard. At any instant i, before any forecasting task takes place, we assume the PM has distributions for the variables β_i and θ_i, which summarize all available information. For simplicity, we shall write them $\pi(\beta_i)$, $\pi(\theta_i)$, dropping any dependence from previously observed values $\tilde{c}^{i-1}$, $\tilde{t}^{i-1}$ (and $\tilde{a}^{i-1}$).

Specifically, the information processing is based on the following joint distribution

$$\pi(c_i, \theta_i, a_i, t_i, \beta_i) = \pi(\theta_i)\pi(c_i|\theta_i)\pi(t_i|c_i, a_i)\pi(a_i|\beta_i)\pi(\beta_i)$$

At each time i, we start with the distributions $\pi(\beta_i)$ and $\pi(\theta_i)$, and those given by the error model, the linear dynamic model and the autoregressive model. Before observing t_i, the PM may provide a forecast for c_i, based on $\pi(c_i)$. Once he hears $\tilde{t}_i$, he provides a forecast, based on $\pi(c_i|\tilde{t}_i)$. Then he observes $\tilde{c}_i$, and $\tilde{a}_i = \tilde{t}_i - \tilde{c}_i$ is found out. Finally, he obtains the priors $\pi(\beta_{i+1})$ and $\pi(\theta_{i+1})$ for the next period. The general model is described in (Palomo et al. 2004). Here we consider an important case in which computations may be done analytically. Let θ_i evolve as a DLM, with the following distributional assumptions

I Linear state-space model
$$\begin{cases} c_i|\theta_i & \sim N(F_i'\theta_i, V_i) \\ \theta_i|\theta_{i-1} & \sim N(G_i\theta_{i-1}, W_i) \\ \theta_0 & \sim N(m_0, \Delta_0) \end{cases}$$

II Standard error model
$$\{ t_i|c_i, a_i \sim N(c_i + a_i, \Gamma_i)$$

III Standard autoregressive model
$$\begin{cases} a_i|\beta_i & \sim N(\beta_i a_{i-1}, \Sigma_i) \\ \beta_i|\beta_{i-1} & \sim N(\beta_{i-1}, \Omega_i) \\ \beta_0 & \sim N(\mu_0, \Lambda_0) \end{cases}$$

For each time i, the model is characterized by the tuple $\{F_i, G_i, V_i, W_i, \Gamma_i, \Sigma_i, \Omega_i\}$, with F_i a known (n x 1) matrix, G_i a known (n x n) matrix, W_i a known (n x n) variance matrix, and V_i, Γ_i, Σ_i, Ω_i are known scalar variances.

The previous forecasting iterative process can be described analytically as follows

- Before hearing t_i, forecast with the standard DLM equations from $\pi(c_i) \sim N(f_i, q_i)$, as in (2) when independent experts were considered.
- After hearing $\tilde{t}_i$, forecast from $\pi(c_i|\tilde{t}_i) \sim N(f_i^*, q_i^*)$, where

$$\begin{aligned} f_i^* &= F_i'\delta_i G_i m_{i-1} + \tfrac{V_i(\tilde{t}_i - \tilde{a}_{i-1}\mu_{i-1})}{\Gamma_i + V_i} \\ q_i^* &= \tfrac{V_i^2\,\xi_i^2}{(\Gamma_i+V_i)^2(V_i\xi_i - F_i'\phi_i F_i \Gamma_i^2)} \end{aligned}$$

with

$$\begin{aligned} \delta_i &= \phi_i [R_i^\theta]^{-1} \tfrac{\Gamma_i(\Gamma_i+V_i)}{V_i\xi_i} q_i^* \\ \phi_i &= \left[[R_i^\theta]^{-1} + F_i \tfrac{V_i\xi_i}{\Gamma_i^2} F_i' \right]^{-1} \\ \xi_i &= V_i(R_i^\beta \tilde{a}_{i-1}^2 + \Sigma_i) + \Gamma_i(\Gamma_i + V_i) \\ R_i^\beta &= \Omega_i + \Lambda_{i-1} \end{aligned}$$

Observe the correction from f_i to f_i^* in the above results. Note that, when θ_i is unidimensional, we can

write the predictive mean as

$$f_i^* = \frac{\Gamma_i}{\Gamma_i + V_i}\left[\frac{V_i\,\xi_i}{R_i^\theta[V_i\xi_i\phi_i^{-1} - F_i'\Gamma_i^2 F_i]}\right] f_i + \frac{V_i}{\Gamma_i + V_i}(\tilde{t}_i - \tilde{a}_{i-1}\mu_{i-1})$$

where f_i is corrected by a factor which depends on the variances of the predictive distributions of c_i and a_i. The second term considers the expert's opinion corrected by a forecast of his own bias $(\tilde{t}_i - a_{i-1}\mu_{i-1})$ times the factor $\frac{V_i}{\Gamma_i+V_i}$.
It is also interesting to consider several extreme cases. When the expert is a clairvoyant ($a_i = 0, \Gamma_i = 0$, $\forall i = 1, \ldots, n$), we have $\delta_i = 0$, so $f_i^* = \tilde{t}_i$ and $q_i^* = \Sigma_i$: the predictive density is centered on the expert's opinion $\tilde{t}_i$, with variance equal to the expert's bias variance. When the expert is very unreliable ($\Gamma_i \to \infty$), we have $f_i^* \to \frac{V_i}{2V_i - q_i} f_i = \frac{q_i^*}{V_i} f_i$ and $q_i^* \to \frac{V_i^2}{2V_i - q_i}$. Notice that, in this case, we 'forget' the expert's opinion $\tilde{t}_i$ and his quality parameters (a_i,Σ_i,Γ_i).

- When the actual value $\tilde{c}_i$ arrives, the posteriors are obtained following standard DLM computations, see (West and Harrison 1997),

$$\begin{aligned}
\pi(\theta_i|\tilde{c}_i) &\sim N(m_i, \Delta_i), \text{ with } m_i,\, \Delta_i \text{ as before.}\\
\pi(\beta_i|\tilde{a}_i) &\sim N(\mu_i, \Lambda_i)\\
\mu_i &= \mu_{i-1} + A_i^\beta e_i^\beta\\
\Lambda_i &= \frac{R_i^\beta \Sigma_i}{R_i^\beta \tilde{a}_{i-1}^2 + \Sigma_i}\\
A_i^\beta &= \frac{R_i^\beta \tilde{a}_{i-1}}{R_i^\beta \tilde{a}_{i-1}^2 + \Sigma_i}\\
e_i^\beta &= \tilde{a}_i - \mu_{i-1}\tilde{a}_{i-1}
\end{aligned}$$

- The priors for the next period are obtained, using standard DLM computations, as follows

$$\begin{aligned}
\pi(\theta_{i+1}|\tilde{c}_i) &\sim N(G_{i+1}m_i, R_{i+1}^\theta)\\
\pi(a_{i+1}|\tilde{a}_i) &\sim N(\mu_i\tilde{a}_i, R_{i+1}^\beta\tilde{a}_i^2 + \Sigma_{i+1})
\end{aligned}$$

7 An example in project cost forecasting

(Palomo et al. 2004) addressed the estimation of a project total cost using data from a case study on ABC technique in (Gunasekaran and Singh 1999). The study is focused on estimating the total production cost of machines for the photo framing industry. Here we illustrate our methods just for the assembly activity which involves 82 units. The quantity of interest is the cost per unit.

We show now one iteration of the estimation process in the three cases we have illustrated in the previous Sections.

7.1 Independent DLM's

We suppose that the two experts act independently. We specify for each of them a DLM (1). Since all the distributions involved in the DLM are, at each stage, normal ones, then we consider, without loss of generality, the first cost c_1. We suppose that the first expert specifies

$$\begin{aligned}
c_1|\theta_1 &\sim N(\theta_1, 1)\\
\theta_1|\theta_0 &\sim N(\theta_0, 0.05)\\
\theta_0 &\sim N(11, 0.05).
\end{aligned}$$

With these assumptions, it follows that the forecasting distribution is

$$\pi(c_1) \sim N(11, 1.1),$$

whereas the posterior distribution becomes

$$\pi(\theta_i|\tilde{c}_1) \sim N(10.98, 0.091)$$

when the actual cost $\tilde{c}_1 = 10.81$ is observed.

The second expert specifies a model which starts with (possible) larger values due to a larger m_0 but it should shrink over time since $F = 0.9$. His DLM is

$$\begin{aligned} c_1|\theta_1 &\sim N(0.9\theta_1, 1) \\ \theta_1|\theta_0 &\sim N(\theta_0, 0.05) \\ \theta_0 &\sim N(12, 0.05), \end{aligned}$$

so that he gets

$$\begin{aligned} \pi(c_1) &\sim N(10.8, 1.85) \\ \pi(\theta_1|\tilde{c}_1) &\sim N(11.39, 2.04). \end{aligned}$$

It is worth mentioning that in both cases the posterior mean of $\theta_1|\tilde{c}_1$ lies between $\tilde{c}_1$ and m_0. The change in F affects the variances of the distribution of the second expert, more than the change in m_0 (which, alone, would have not changed the variances with respect to the first expert).

7.2 Classes of priors in DLM

Suppose the two experts agree in considering the class of DLM's given by

$$\begin{aligned} c_1|\theta_1 &\sim N(\theta_1, 1) \\ \theta_1|\theta_0 &\sim N(\theta_0, 0.05) \\ \theta_0 &\sim N(m_0, 0.05), \end{aligned}$$

with $m_0 \in [11, 12]$, i.e. within the two prior opinions of the experts. In this case, it can be shown that

$$\pi(\theta_1|\tilde{c}_1) \sim N(K, 0.091), 10.98 \leq K \leq 11.89.$$

7.3 DLM with expert's opinion

Following the model presented in Section 6, the first expert (PM) obtains

$$\begin{aligned} c_i|\theta_i &\sim N(\theta_i, 1) \\ \theta_i|\theta_{i-1} &\sim N(\theta_{i-1}, 0.05) \\ \theta_{i-1} &\sim N(11, 0.05) \\ t_1|c_i, a_i &\sim N(c_i + a_i, 1) \\ a_i|\beta_i &\sim N(-5\beta_i, 0.1) \\ \beta_i|\beta_{i-1} &\sim N(\beta_{i-1}, 0.1) \\ \beta_{i-1} &\sim N(1, 0.1) \end{aligned}$$

and the expert's error in the previous forecast for c_i was, say, $\tilde{a}_{i-1} = -1$. As for the independent model, we obtain the same forecast before asking any expert's opinion, i.e.

$$\pi(c_i) \sim N(11, 1.1).$$

Then, we can obtain a point estimate from the expert UM, say, $\tilde{t}_i = 12$, and, hopefully, a more accurate forecasting distribution is obtained

$$\pi(c_i|\tilde{t}_i) \sim N(11.13, 0.6).$$

Note that with our approach, the proposal manager obtains:

- posterior information about the costs for future products or updates of the current one, through a dynamic process. Following the detailed process for c_i, when the actual cost $\tilde{c}_i = 10.81$ arrives, the posteriors are obtained as follows:

$$\begin{aligned} \pi(\theta_i|\tilde{c}_i) &\sim N(10.98, 0.091) \\ \pi(\beta_i|\tilde{a}_i) &\sim N(-0.46, 0.067) \end{aligned}$$

where $\tilde{a}_i = \tilde{t}_i - \tilde{c}_i = 1.19$. For future products, we shall start with the following prior for assembly cost

$$\pi(\theta_{i+1}|\tilde{c}_i) \sim N(10.98, 0.141)$$

- posterior information about the expert's forecasting quality for future elicitations which, in the case of the assembly unit manager, has the following prior for the bias

$$\pi(a_{i+1}|\tilde{a}_i) \sim N(-0.55, 0.34)$$

8 Discussion

Although we have focused on project cost forecasting, the dynamic linear models combining different sources of expertise can be applied in many other fields. The same methodology we used for cost forecasting can also be applied to time forecasting. Indeed the company we were consulting for, is currently interested in extending the methodology to time forecasting. Currently, shorter delivery times are a key issue in assigning subcontracts, and controlling delivery time is crucial in order to meet the final deadline signed with the auctioneer. The company needs software capable of combining past data about subcontractors' timeliness and the evolving requests for quickier building of the plants.

These models can fit applications in electronic democracy as a decision support tool, since they relate opinions from different experts. Other scenarios could also be explored, especially ones where new evidence is available at some time points and the quantity of interest (which we have been referring to as cost throughout this paper) evolves dynamically. Different levels of cooperation between the experts were modeled in the previous sections, and they are suitable for a variety of practical situations. We can think of environmental activists and local government interested in building a power plant in an agricultural area. They can make forecast on future effects stemming from opposite positions (Section 4), combining their opinions (Section 5) or with environmental activists acting as consultant for the local government (Section 6).

As mentioned before, robustness has been rarely studied in DLM, probably because computations in more general DLM's are complex. Even more difficult, although worthwhile, robustness analysis could be performed in the more general model by (Palomo et al. 2004). A plethora of models have been proposed in robust Bayesian literature, see (Rios Insua and Ruggeri 2000), but no efficient simulation scheme has been proposed for robustness in complex systems. Algorithms for robustness in DLM's are challenging ones.

Our consulting project has lead to a prototype software which uses DLM's and expert's opinion to make forecasts. Sensitivity analysis, as suggested by the current paper, will be added in its future development.

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Approximate Recursive Bayesian Estimation of Dynamic Probabilistic Mixtures

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Abstract. Majority of complex non-linear systems can be successfully modelled by a finite probabilistic mixture of linear models. The mixture model can be handled analytically, which is important for control of the system as well as for decision making. Quality of the model is a crucial requirement of all tasks of this type. The exact Bayesian methodology can not be used for estimation of this model, because complexity of the posterior distribution grows exponentially with number of data. Therefore, approximation techniques such as the quasi-Bayes algorithm must be used. This paper introduces a new estimation algorithm, which is based on minimization of Kullback-Leibler distance between the proper Bayesian posterior density and an approximate posterior density. The approximate posterior distribution is chosen from the exponential family in order to achieve numerically efficient estimation.

Keywords: parameter estimation, recursive estimation, probabilistic mixture, control of the complex system, system model

1 Introduction

The choice of a suitable model is essential for both control and decision making when dealing with complex systems. One way to face complexity is the principle of adaptivity, i.e. using models which evolve during their use. The demand for adaptivity of the model leads to the recursive estimation of its parameters, i.e. permanent updating of its parameter estimates by the new data. In other words, statistics describing estimates are corrected by newly acquired data. The model should be chosen from a sufficiently rich family of models to capture all properties of the modelled system. Naturally, computational cost associated with estimation of parameters of the model grows with complexity of the model. If the modelled system is non-linear, its model should be non-linear too. In this paper, we study finite probabilistic mixture of linear models. The finite mixtures provide a universal approximation of almost any probabilistic density function (Titterington, et al. 1985) and thus can be successfully used in modelling of complex systems. Invoking the principle of adaptivity, we seek an efficient recursive estimation of the mixture model parameters.

The resulting model can be then used both for control and decision making tasks. Universal algorithms for mixture-based control (Kárný, et al. 2003) were derived, but quality of the resulting control strategy strongly depends on the quality of the estimated model. Practical experience indicates that this is a weak element of adaptive control and that an improvement of the estimation part improves the overall control quality. Hence, we try to develop better estimation algorithms for the mixture model. The control algorithms (Kárný et al. 2003) as well as efficient structure estimation algorithms were derived using the Bayesian theory (Peterka 1981). The unknown model parameters are treated as random variables and all subsequent task are defined in terms of posterior distributions of the parameters rather then thier point estimates.

The recursive Bayesian estimation evaluates the posterior distribution on parameters at time t as an update of the the posterior distribution at time $t-1$ using the Bayes' rule and the data acquired at time t. The recursion starts at $t = 1$ with update of the prior distribution which must be chosen before the estimation starts. The posterior distribution obtained by the Bayes' rule may not be, however, analytically tractable and thus unsuitable for the next update.

In practice, mostly such prior distribution is used so that the posterior distribution in each estimation step has the same functional form as the prior distribution. Hence, just the sufficient statistics determining the posterior density are updated. Such a prior distribution is then known as conjugate with the observation model. For example, conjugate prior distribution is available for all models from the exponential family. If the conjugate prior does not exists the exact recursive estimation can not be achieved. In such a case, we seek approximate recursive estimation. This is the case of the probabilistic mixture model. Using the exact Bayesian update, the complexity of posterior density grows exponentially with number of the data samples. The quasi-Bayes

algorithm (Kárný, et al. 1998) (Kárný et al. 2003) or a modification of the EM algorithm (Titterington et al. 1985) are examples of approximate algorithms facing this problem.

This paper introduces a new approximate estimation method, which can be viewed as a generalization of quasi-Bayes algorithm. The basis of both approaches is finding the approximate posterior density in particular (well manipulable) class of densities.

The new algorithm finds the optimal projection of the correct Bayesian density into the selected class of densities. The projection is optimal in the sense of Kulback Leibler distance (Kullback & Leibler 1951). It should be mentioned that the Kullback Leibler distance is not symmetric. Algorithm presented in this paper minimizes the Kulback Leibler distance with the argument order, which conforms with Bayesian principles (Berec & Kárný 1997). An algorithm minimizing the Kullback Leibler distance with arguments in different order can be found in (Roberts & Penny 2002).

2 Notions and notations

x^* *denotes the range of* x, $x \in x^*$.
$\mathring{x}$ *denotes the number of entries in the vector* x.
$\equiv$ *means the equality by definition.*
x_t *is a quantity (vector)* x *at the discrete time labelled by* $t \in t^* \equiv \{1, \ldots, \mathring{t}\}$.
$x_{i;t}$ *is an* i*-th entry of the vector* x_t. The semicolon in the subscript indicates that the symbol following it is the time index.
$x_{k\_l;t}$ *is a subvector of the vector* x_t. $x_{k\_l;t} = (x_{k;t}, \cdots, x_{l;t})$.
$x(k\_l) \equiv x_k, \ldots, x_l$.
$x(t) \equiv x(1\_t)$.
$x(t)$ *is an empty sequence and reflects just the prior information if* $t < 1$.
d *is data array,* d_t *is data record at time* t *(vector with entries* $(d_{1;t}, \cdots, d_{\mathring{d}_t;t})$ *).*
Θ *unknown parameter, finite-dimensional vector*
f, π *are the letters reserved for probability density functions(pdf).*
$f(d_t|d(t-1), \Theta)$ *means model of the system.*
$f_c(d_t|d(t-1), \Theta_c)$ *is component of the mixture.*
$\pi_0(\Theta)$ *denotes prior density of the unknown parameter* Θ.
$\pi_t(\Theta|d(t)) \equiv \pi_t(\Theta|\mathcal{G}_t)$ *means (approximate) posterior density of the parameter* Θ *determined by the sufficient statistic* $\mathcal{G}_t$.
$\propto$ *is the proportion sign,* $h \propto g$ *means that function* h *equals to the function* g *up to the normalization. I.e.* $\frac{h}{\int h} = \frac{g}{\int g}$.
∂ *is the model order.*
$\mathcal{D}(\,||\,)$ *means the Kullback-Leibler distance(Kullback & Leibler 1951). This "distance" is familiarly used in Bayesian analysis as the measure how good the second pdf approximates the first pdf. For conciseness, the Kullback-Leibler distance is referred to as the KL distance.* $\mathcal{D}\left(f \,\middle|\middle|\, g\right) = \int f \ln\left(\frac{f}{g}\right)$
$\Gamma(x)$ *means gamma function,* $\Gamma(x) = \int_0^{+\infty} t^{x-1} \exp(-t) dt$.
$\psi_0(x)$ *is digamma function,* $\psi_0(x) = \frac{\partial \ln \Gamma(x)}{\partial x}$.
δ *denotes identity matrix. I.e.* $\delta_{ij} = 1$ *iff* $i = j$*, otherwise* $\delta_{ij} = 0$.

Agreement 1 (Multimatrix, multivector) *Multimatrix of type* m, n

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & \ddots & \vdots \\ M_{m1} & \cdots & M_{mn} \end{pmatrix}$$

is a mathematical object, where M_{ij} *is either matrix or multimatrix. Hence matrix is a multimatrix. Multimatrix need not be a matrix. Definition of Multivector is analogical.*

Agreement 2 (Multimatrix indexing) *For* M *being a multimatrix of type* m, n *the following notation is used:*
M_{ij} *is* ij*-th entry of* M.

$M_{\bullet j}$ *is multimatrix* $\begin{pmatrix} M_{1j} \\ \vdots \\ M_{mj} \end{pmatrix}$.
$M_{i\bullet}$ *is multimatrix* $(M_{i1}, \cdots, M_{in})$.
$M_{\bullet\bullet}$ *means the same as* M.*We use this notation when we want to stress that* M *is a multimatrix (matrix).*

Agreement 3 (Other Matrix notations) *Let's* M *be a matrix of type* m, n *and* c *some scalar. Let's define the following operations:*
$M \pm c$ *is matrix of type* m, n, $(M \pm c)_{ij} = M_{ij} \pm c$.
$\exp(M)$ *is matrix of type* m, n, $(\exp(M))_{ij} = \exp(M_{ij})$.
$\max M$ *is scalar with maximal value of* M.

3 Basic elements and tools

3.1 Recursive parameter estimation

The task of recursive parameter estimation is to determine the posterior density $\pi_t(\Theta|d(t))$ based on the knowledge of

- last posterior density $\pi_{t-1}(\Theta|d(t-1))$
- new data record d_t
- model of the system $f\big(d_t|d(t-1),\Theta\big)$ parameterized by unknown parameter Θ .

The algorithm starts from prior pdf $\pi_0(\Theta) \equiv \pi_0(\Theta|d(0))$. We assume existence of the sufficient statistic $\mathcal{G}_t$ for posterior pdfs, i.e.

$$\pi_t(\Theta|d(t)) \equiv \pi_t(\Theta|\mathcal{G}_t).$$

Next, consider that the actual data record d_t doesn't depend on all historical data $d(t-1)$ but only on a subset $\phi_{t-1} = (d_{t-1}, d_{t-2}, \cdots, d_{t-\partial})$. Hence,

$$f\big(d_t|d(t-1),\Theta\big) \equiv f\big(d_t|\phi_{t-1},\Theta\big).$$

The standard Bayesian approach determines $\pi_t(\Theta|\mathcal{G}_t)$ as

$$\pi_t(\Theta|\mathcal{G}_t) \propto f\big(d_t|\phi_{t-1},\Theta\big)\pi_{t-1}(\Theta|\mathcal{G}_{t-1}). \tag{1}$$

3.1.1 Recursive parameter estimation with conjugate pdf

The considered approach (1) can be effectively used in the case when $\pi_0(\Theta)$ is conjugate pdf to the system model $f\big(d_t|\phi_{t-1},\Theta\big)$. In such a case, $\pi_t(\Theta|\mathcal{G}_t)$ has the same functional form as $\pi_0(\Theta)$. Hence, we can get

$$\pi_t(\Theta|\mathcal{G}_t) \equiv \pi(\Theta|\mathcal{G}_t), \quad \forall t.$$

When updating from $\pi(\Theta|\mathcal{G}_{t-1})$ to $\pi(\Theta|\mathcal{G}_t)$ it suffices to update the sufficient statistics: $(\mathcal{G}_{t-1}, d_t) \longrightarrow \mathcal{G}_t$.

3.1.2 Recursive parameter estimation without conjugate pdf

If the pdf conjugate to the system model doesn't exist, the dimension of sufficient statistic grows with number of data samples. Then, of course, complexity of π_t grows as well. In such a case we can proceed in the following way:

- we choose prior pdf in an arbitrary well manipulable functional form,
- we seek an approximate posterior pdf's of the same functional form,
- we set, in each step of estimation, the statistic determining the approximate posterior pdf in such a way that it is "closest" to the "correct Bayesian" pdf.

We need to specify what we mean by: "correct Bayesian" and "closest" . Let's have the approximate posterior pdf $\pi(\Theta|\mathcal{G}_{t-1})$, which depends on the statistic $\mathcal{G}_{t-1}$. If we handle the approximate posterior pdf $\pi(\Theta|\mathcal{G}_{t-1})$

as the correct posterior pdf, the "correct Bayesian" posterior pdf in the next step $\hat{\pi}(\Theta|\mathcal{G}_{t-1}, d_t, \phi_{t-1})$ is (according to (1)) obtained as

$$\hat{\pi}(\Theta|\mathcal{G}_{t-1}, d_t, \phi_{t-1}) = \frac{f\big(d_t|\phi_{t-1}, \Theta\big)\pi(\Theta|\mathcal{G}_{t-1})}{\int f\big(d_t|\phi_{t-1}, \Theta\big)\pi(\Theta|\mathcal{G}_{t-1})d\Theta}.$$

The term "closest" means closest in sense of the KL distance. It means that we want to find $\mathcal{G}_t$ so that

$$\mathcal{D}\left(\hat{\pi}(\Theta|\mathcal{G}_{t-1}, d_t, \phi_{t-1}) \,\middle\|\, \pi(\Theta|\mathcal{G}_t)\right) \tag{2}$$

is minimized.

Remarks 1

1. *Applicability of the presented algorithm strictly depends on the complexity of the KL distance. Except of trivial cases, it is usable only if the KL distance can be evaluated analytically.*
2. *The algorithm uses the approximate posterior pdf obtained in step $t-1$ as the true posterior pdf in step t. This leads to error accumulation.*

3.2 Dynamic probabilistic mixture

In this paper, we consider the parameterized model of the system in the form of a finite probabilistic mixture:

$$f(d_t|\phi_{t-1}, \Theta) \equiv \sum_{c\in c^*} \alpha_c f_c(d_t|\phi_{c;t-1}, \Theta_c),\ c^* = \{1, \ldots, \mathring{c}\},\ \mathring{c} < \infty,\ \text{ where} \tag{3}$$

$$f_c(d_t|\phi_{c;t-1}, \Theta_c) \equiv \text{c-th } \textit{component} \text{ given by component parameters } \Theta_c \text{ and the state}$$

$$\phi_{c;t-1} \equiv \text{subset of } \phi_{t-1}$$

$$\alpha_c \equiv \text{the probabilistic } \textit{component weight}$$

$$\Theta \equiv \text{mixture parameter formed by the component weights and parameters}$$

$$\Theta \in \Theta^* \equiv \left\{\{\Theta_c \in \Theta_c^*\}_{c\in c^*},\ \alpha \equiv [\alpha_1, \ldots, \alpha_{\mathring{c}}] \in \alpha^* \equiv \left\{\alpha_c \geq 0,\ \sum_{c\in c^*} \alpha_c = 1\right\}\right\}.$$

Before fixing and refining nomenclature related to the mixture, we split the individual components into so called *factors* that provide flexibility of the parametric description.

Using the chain rule, the pdfs $f_c(d_t|\phi_{c;t-1}, \Theta_c)$ can be written as a product of pdfs of individual entries of d_t. Before applying the chain rule, entries of d_t can be permuted and some permutations may lead to parameterizations with less parameters. This motivates inclusion of permutations into the model description

$$d \to d_c \ \text{ with } \ d_{ic} = d_{j_{ic}},\ \text{ where} \tag{4}$$

j_{ic} is i-th entry of the permuted indices $[1, \ldots, \mathring{d}]$. The assignment (4) is applied component-wise and together with the chain rule give

$$f_c(d_t|\phi_{c;t-1}, \Theta_c) = \prod_{i\in i^*} f_{ic}(d_{ic;t}|d_{(i+1)\_\mathring{d}c;t}, \phi_{c;t-1}, \Theta_{ic}) \equiv \prod_{i\in i^*} f_{ic}(d_{ic;t}|\psi_{ic;t}, \Theta_{ic}). \tag{5}$$

The additional subscript i of the parameter Θ_{ic} indicates that only some entries of Θ_c may occur in i-th pdf (*factor*) in (5). Similarly, the *regression vector* $\psi_{ic;t}$ is generally a sub-vector of the vector

$$[d_{(i+1)\_\mathring{d}c;t}, \phi'_{c;t-1}, 1]'. \tag{6}$$

Agreement 4 (Nomenclature related to mixtures)

Pdfs: *The pdf $f_c(d_t|\phi_{c;t-1}, \Theta_c)$ in (3) is called* parameterized component *of a mixture and* α_c *is the* weight of the c-th parameterized component.
The pdf $f_{ic}(d_{ic;t}|\psi_{ic;t}, \Theta_{ic})$ in (5) is called parameterized factor.

Data: *The vector d_t containing data measured at time t is called* data record.
The vector $\phi_{c;t-1}$ is the observable state of the parameterized component.
The parameterized factor is determined by regression vector $\psi_{ic;t}$ *defined as a sub-selection of the vector $[d_{i+1\_\mathring{d}c;t}, \phi'_{c;t-1}, 1]'$ (6).*
The coupling $\Psi_{ic;t} \equiv [d_{ic;t}, \psi'_{ic;t}]'$ is called data vector *of the factor.*

Remarks 2

1. *We added the number 1 to the definition of the regression vector, because it helps us to effectively express the constant shifts in mean values of factors.*
2. *The adopted dynamic mixture model is not sufficiently general. The component weights should also depend on the state vector. The choice is driven by our inability to estimate this "natural" and more realistic model. See discussion in (He & Kárný 2003)*

3.3 Form of the prior and the posterior pdf

According to the general hints in section 3.1.2 we need to choose the prior pdf in a form that is well manipulable, i.e. analytically tractable.

Agreement 5 (Considered forms of pdfs on Θ^*) *The prior $\pi(\Theta) \equiv \pi(\Theta|d(0))$ and the posterior $\pi(\Theta|d(t)) \equiv \pi(\Theta|\mathcal{G}_t)$ are considered to be of the common form:*

$$\pi(\Theta|\mathcal{G}_t) = Di_\alpha(\kappa_t) \prod_{i\in i^*, c\in c^*} \pi_{ic}(\Theta_{ic}|\mathcal{S}_{ic;t}),\ t \in \{0\} \cup t^* \quad \text{, where} \tag{7}$$

$$\mathcal{G}_t \equiv (\kappa_{\bullet;t}, \mathcal{S}_{\bullet\bullet;t}),$$

$$Di_\alpha(\kappa_t) \text{ is Dirichlet distribution, } Di_\alpha(\kappa_\bullet) \equiv \frac{\prod_{c\in c^*} \alpha_c^{\kappa_c - 1}}{\mathcal{B}(\kappa)},\ \mathcal{B}(\kappa) \equiv \frac{\prod_{c\in c^*} \Gamma(\kappa_c)}{\Gamma(\sum_{c\in c^*} \kappa_c)},$$

$$\text{each pdf } \pi_{ic}(\Theta_{ic}|\mathcal{S}_{ic;t}) \text{ is conjugate to the factor } f_{ic}(d_{ic;t}|\psi_{ic;t}, \Theta_{ic}).$$

Parameters $\Theta_{ic},\ i \in i^* \equiv \{1, \ldots, \mathring{d}\},\ c \in c^*$, of the individual parameterized factors are mutually conditionally independent, and also, independent of the component weights α. The component weights have Dirichlet distribution $Di_\alpha(\kappa)$ with support on the probabilistic simplex *α^*.*

Remarks 3

1. *The considered form of the posterior distribution restricts the class of mixture factors $f_{ic}(d_{ic;t}|\psi_{ic;t}, \Theta_{ic})$ to those having conjugate pdf.*
2. *Details about Dirichlet distribution $Di_\alpha(\kappa_t)$ can be found for example in (Andrýsek 2004).*
3. *The research report (Andrýsek 2004) contains all details and proofs, which were omitted in this paper.*

3.4 Notations related to mixtures

In the sequel, we use the following elements: $i \in i^* \equiv \{1, \ldots, \mathring{d}\},\ c \in c^*$

$$\text{Factor prediction} \quad \mathcal{I}_{ic;t} = \int f_{ic}(d_{ic;t}|\psi_{ic;t}, \Theta_{ic})\pi_{ic}(\Theta_{ic}|\mathcal{S}_{ic;t-1})d\Theta_{ic}$$

$$\text{Component prediction} \quad \beta_{c;t} = \prod_{i=1}^{\mathring{\Psi}} \mathcal{I}_{ic;t} \tag{8}$$

$$\text{Estimate of component weight} \quad \hat{\alpha}_{c;t} = \frac{\kappa_{c;t}}{\sum_{c\in c^*} \kappa_{c;t}} \tag{9}$$

$$\text{QB weight of data} \quad w_{c;t} = \frac{\hat{\alpha}_{c;t-1}\beta_{c;t}}{\sum_{c=1}^{\mathring{c}} \hat{\alpha}_{c;t-1}\beta_{c;t}} \tag{10}$$

$$\text{"Correct" estimate of factor parameters} \quad \pi_{ic}(\Theta_{ic}|\mathcal{S}^U_{ic;t}) = \frac{f_{ic}(d_{ic;t}|\psi_{ic;t}, \Theta_{ic})\pi_{ic}(\Theta_{ic}|\mathcal{S}_{ic;t-1})}{\mathcal{I}_{ic;t}} \tag{11}$$

Remarks 4

1. *The assumption of conjugacy of $\pi_{ic}(\Theta_{ic}|\mathcal{S}_{ic;t-1})$ to the factor $f_{ic}(d_{ic;t}|\psi_{ic;t}, \Theta_{ic})$ implies that $\frac{f_{ic}(d_{ic;t}|\psi_{ic;t}, \Theta_{ic})\pi_{ic}(\Theta_{ic}|\mathcal{S}_{ic;t-1})}{\mathcal{I}_{ic;t}}$ has the same functional form as $\pi_{ic}(\Theta_{ic}|\mathcal{S}_{ic;t-1})$, and thus we need to evaluate only the statistic $\mathcal{S}^U_{ic;t}$.*
2. *As values of $\mathcal{I}_{ic;t}$ can be very close to zero, it is numerically advantageous to evaluate the weights $w_{\bullet;t}$ using $\mathcal{L}_{ic;t} = \ln \mathcal{I}_{ic;t}$. $\mathcal{L}_{ic;t}$ can be computed directly without evaluating $\mathcal{I}_{ic;t}$.*

Algorithm 1 $w_{\bullet;t} = \text{EVAL\_WEIGHT}(\mathcal{L}_{\bullet\bullet;t}, \kappa_{\bullet;t-1})$

1. *For each component* c *evaluate* $\mathcal{H}_{c;t} = \ln \kappa_{c;t-1} + \sum_i \mathcal{L}_{ic;t}$
2. $\bar{\mathcal{H}}_{\bullet;t} = H_{\bullet;t} - \max H_{\bullet;t}$
3. $w_{\bullet;t} = \frac{\exp(\bar{\mathcal{H}}_{\bullet;t})}{\sum_c \exp(\bar{\mathcal{H}}_{\bullet;t})}$

Remarks 5 $w_{c;t}$ *evaluated in this algorithm is the same as defined in (10):*

$$w_{c;t} = \frac{\exp\left(\mathcal{H}_{c;t} - \max \mathcal{H}_{\bullet;t}\right)}{\sum\left(\exp\left(\mathcal{H}_{c;t} - \max \mathcal{H}_{\bullet;t}\right)\right)} = \frac{\exp\left(\mathcal{H}_{c;t}\right)\exp\left(\max \mathcal{H}_{\bullet;t}\right)}{\exp\left(\max \mathcal{H}_{\bullet;t}\right)\sum\exp\left(\mathcal{H}_{c;t}\right)} =$$
$$= \frac{\kappa_{c;t-1}\beta_{c;t}}{\sum \kappa_{c;t-1}\beta_{c;t}} = \frac{\frac{\kappa_{c;t-1}\beta_{c;t}}{\sum \kappa_{c;t-1}}}{\frac{\sum \kappa_{c;t-1}\beta_{c;t}}{\sum \kappa_{c;t-1}}} = \frac{\hat{\alpha}_{c;t-1}\beta_{c;t}}{\sum \hat{\alpha}_{c;t-1}\beta_{c;t}}.$$

4 Problem formulation and general solution

In this Section, we apply the approximation from section 3.1.2 to the introduced mixture model (3). We seek the statistic $\mathcal{G}_t$ that minimizes $\mathcal{D}\left(\hat{\pi}(\Theta|\mathcal{G}_{t-1}, \overbrace{d_t, \phi_{t-1}}^{\equiv \Psi_t}) \,\middle|\middle|\, \pi(\Theta|\mathcal{G}_t)\right)$, where

$$\hat{\pi}(\Theta|\mathcal{G}_{t-1}, \Psi_t) = \frac{f(d_t|\phi_{t-1}, \Theta)\pi(\Theta|\mathcal{G}_{t-1})}{\int f(d_t|\phi_{t-1}, \Theta)\pi(\Theta|\mathcal{G}_{t-1})d\Theta}$$

$$\pi(\Theta|\mathcal{G}_{t-1}) = Di_\alpha(\kappa_{t-1}) \prod_{i=1,c=1}^{\mathring{\Psi},\mathring{c}} \pi_{ic}(\Theta_{ic}|\mathcal{S}_{ic;t-1})$$

$$f(d_t|\phi_{t-1}, \Theta) = \sum_{c=1}^{\mathring{c}} \alpha_c \prod_{i=1}^{\mathring{\Psi}} f_{ic}(d_{ic;t}|\psi_{ic;t}, \Theta_{ic}).$$

In this case, the statistic $\mathcal{G}_t$ consist of vector κ_t (of $\mathring{c}$ elements) and multimatrix $\mathcal{S}_{\bullet\bullet;t}$ of type $(\mathring{\Psi}, \mathring{c})$.

The next proposition summarizes the form of $\hat{\pi}(\Theta|\mathcal{G}_{t-1}, \Psi_t)$.

Proposition 1

$$\hat{\pi}(\Theta|\mathcal{G}_{t-1}, \Psi_t) = \sum_{c=1}^{\mathring{c}} w_{c;t} Di_\alpha(\kappa_{t-1} + \delta_{\bullet c}) \prod_{\substack{j,r=1 \\ r\neq c}}^{\mathring{\Psi},\mathring{d}} \pi_{jr}(\Theta_{jr}|\mathcal{S}_{jr;t-1}) \prod_{j=1}^{\mathring{\Psi}} \pi_{jc}(\Theta_{jc}|\mathcal{S}^U_{jc;t}). \tag{12}$$

Proposition 2 (Minimization of KL distance) *For* $\mathcal{G}_t \equiv \{\mathcal{S}_{\bullet\bullet;t},\ \kappa_t\}$ *minimizing*

$$\mathcal{D}\left(\hat{\pi}(\Theta|\mathcal{G}_{t-1}, \Psi_t) \,\middle|\middle|\, \pi(\Theta|\mathcal{G}_t)\right),$$

it holds:

$$\kappa_t \in \operatorname{Arg\,min}_{\kappa_t} \left[\sum_{c=1}^{\mathring{c}} w_{c;t} \mathcal{D}\left(Di_\alpha(\kappa_{t-1} + \delta_{\bullet,c}) \,\middle|\middle|\, Di_\alpha(\kappa_t)\right)\right] \tag{13}$$
$$\mathcal{S}_{ic;t} \in \operatorname{Arg\,min}_{\mathcal{S}_{ic;t}} \left[(1 - w_{c;t})\mathcal{D}\left(\pi_{ic}(\Theta_{ic}|\mathcal{S}_{ic;t-1}) \,\middle|\middle|\, \pi_{ic}(\Theta_{ic}|\mathcal{S}_{ic;t})\right) + w_{c;t}\mathcal{D}\left(\pi_{ic}(\Theta_{ic}|\mathcal{S}^U_{ic;t}) \,\middle|\middle|\, \pi_{ic}(\Theta_{ic}|\mathcal{S}_{ic;t})\right)\right].$$

Remarks 6 *The previous proposition split the overall problem into two subproblems. The subproblem (13) can be solved in general, as presented in section 4.2. Solution of the second subproblem depends on the choice of the system model. Solution for the Normal models is presented in section 5.*

4.1 General algorithm

Following the proposition 2 we sketch the general algorithm of one mixture estimation step. We naturally suppose that $\Psi_{ic;t}$ can be obtained from $d(t)$.

Algorithm 2

Inputs - $\kappa_{\bullet;t-1},\ \mathcal{S}_{\bullet\bullet;t-1},\ \Psi_{\bullet\bullet;t}$
Outputs - $\kappa_{\bullet;t},\ \mathcal{S}_{\bullet\bullet;t}$

1. *For each factor ic evaluate* $\mathcal{L}_{ic;t} = \ln \mathcal{I}_{ic;t}$
2. $w_{\bullet;t} = \text{EVAL\_WEIGHT}(\mathcal{L}_{\bullet\bullet;t}, \kappa_{\bullet;t-1})$ *(Algorithm 1)*
3. $\kappa_t \in \text{Arg}\min_{\kappa_t>0}\left[\sum_{c=1}^{\mathring{c}} w_{c;t}\mathcal{D}\left(Di_\alpha(\kappa_{t-1}+\delta_{\bullet c})\ \middle|\middle|\ Di_\alpha(\kappa_t)\right)\right]$
4. *For each factor ic evaluate* $\mathcal{S}^U_{ic;t}$ *so that* $\pi_{ic}(\Theta_{ic}|\mathcal{S}^U_{ic;t}) = \frac{\pi_{ic}(\Theta_{ic}|\mathcal{S}_{ic;t-1})f_{ic}(d_{ic;t}|\psi_{ic;t},\Theta_{ic})}{\mathcal{I}_{ic}}$
5. *For each factor ic evaluate*
   $\mathcal{S}_{ic;t} \in \text{Arg}\min_{\mathcal{S}_{ic;t}}\left[(1-w_{c;t})\mathcal{D}\left(\pi_{ic}(\Theta_{ic}|\mathcal{S}_{ic;t-1})\ \middle|\middle|\ \pi_{ic}(\Theta_{ic}|\mathcal{S}_{ic;t})\right) + w_{c;t}\mathcal{D}\left(\pi_{ic}(\Theta_{ic}|\mathcal{S}^U_{ic;t})\ \middle|\middle|\ \pi_{ic}(\Theta_{ic}|\mathcal{S}_{ic;t})\right)\right]$

Steps 1,4,5 depends on the specific choice of the system model, step 2 is solved, and step 3 is discussed in the next section.

4.2 Minimization with respect to κ_t

The following proposition converts the problem of KL distance minimization of κ-part to minimization of an algebraic expression.

Proposition 3 (Minimization with respect to κ_t)
For κ_t minimizing

$$\sum_{c=1}^{\mathring{c}} w_{c;t}\mathcal{D}\left(Di_\alpha(\kappa_{t-1}+\delta_{\bullet c})\ \middle|\middle|\ Di_\alpha(\kappa_t)\right)$$

it holds

$$\kappa_{\bullet;t} \in \text{Arg}\min\left\{\sum_{c=1}^{\mathring{c}}\left[\ln\left(\Gamma\left(\kappa_{c;t}\right)\right) - \kappa_{c;t}\xi_{c;t}\right] - \ln\left(\Gamma\left(\sum_{c=1}^{\mathring{c}}\kappa_{c;t}\right)\right)\right\}$$

where

$$\xi_{c;t} = \left(\psi_0\left(\kappa_{c;t-1}\right) + \frac{w_{c,t}}{\kappa_{c;t-1}} - \psi_0\left(\sum_{c=1}^{\mathring{c}}\kappa_{c;t-1}+1\right)\right).$$

Proposition 3 yields the following algorithm.

Algorithm 3 $\kappa_{\bullet;t} = \text{NEW\_KAPPA}(w_{\bullet;t}, \kappa_{\bullet;t-1})$

1. *For each component c evaluate* $\xi_{c;t} = \psi_0\left(\kappa_{c;t-1}\right) + \frac{w_{c,t}}{\kappa_{c;t-1}} - \psi_0\left(\sum_{c=1}^{\mathring{c}}\kappa_{c;t-1}+1\right)$
2. $\kappa_{\bullet;t} \in \text{Arg}\min\left\{\sum_{j=1}^{\mathring{c}}\left[\ln\left(\Gamma\left(\kappa_{j;t}\right)\right) - \kappa_{j;t}\xi_{j;t}\right] - \ln\left(\Gamma\left(\sum_{c}^{\mathring{c}}\kappa_{c;t}\right)\right)\right\}$

Remarks 7

1. *Minimization of the term (13) can be simply approximated by changing* $\mathcal{D}\left(Di_\alpha(\kappa_1)\ \middle|\middle|\ Di_\alpha(\kappa_2)\right)$ *into square of the Euclidean norm* $||\kappa_1-\kappa_2||^2$*. The problem is then transformed into minimization of* $\min_x \sum_c w_c||x-x_c||^2$ *which has explicit solution:* $x = \sum_c w_c x_c$*. Applied to our case it yields* $\kappa_t = \kappa_{t-1} + w_t$*, which is identical to the solution obtained using the quasi-Bayes algorithm (Kárný et al. 1998).*

2. *The minimization problem in step 2 must be solved numerically or by suitable approximation. For detailed solution of this problem see (Nenutil 2004).*

We have completed all steps which can be done on this general level. In the next parts of the paper, we are dealing with the special case of the factors.

5 Application to normal factors

In this section, we assume the parameterized factor to be dynamic Gaussian pdf with parameters $\Theta_{ic} \equiv (\theta_{ic}, r_{ic})$, where θ_{ic} is so called vector of regression coefficients and r_{ic} is noise variance of the factor.

$$f_{ic}(d_{ic;t}|\psi_{ic;t}, \Theta_{ic}) = N_{d_{ic;t}}(\theta'_{ic}\psi_{ic;t}, r_{ic}) = \frac{1}{\sqrt{2\pi r_{ic}}} \exp\left(-\frac{(d_{ic;t} - \theta'_{ic}\psi_{ic;t})^2}{2r_{ic}}\right)$$

We don't need to introduce a shift in the mean value, because the regression vector can contain number 1. See Remarks 2. The shifting constant is then placed to the corresponding place of the vector of regression coefficients.

The prior conjugate to this model is the Gauss inverse Wishart pdf with parameters $\mathcal{S}_{ic;t} = (\nu_{ic;t}, V_{ic;t})$, where $\nu_{ic;t}$ is scalar count of degrees of freedom and $V_{ic;t}$ is so called extended information matrix (symmetric, positive definite, of type $(\mathring{\Psi}, \mathring{\Psi})$).

$$\pi_{ic}(\Theta_{ic}|\mathcal{S}_{ic;t}) = GiW_{\theta_{ic},r_{ic}}(V_{ic;t}, \nu_{ic;t}) \propto r_{ic}^{-0.5(\nu_{ic;t}+\mathring{\psi}_{ic;t}+2)} \exp\left\{-\frac{1}{2r_{ic}}\mathrm{tr}\left(V_{ic;t}[-1, \theta'_{ic}]'[-1, \theta'_{ic}]\right)\right\}$$

Note that the matrix V_{ic} can be equivalently manipulated through its $L'DL$ decomposition (i.e. with lower triangular matrix L_{ic} and diagonal matrix D_{ic} which fulfills the relation $V_{ic} = L'_{ic}D_{ic}L_{ic}$). Next, the matrices L_{ic} and D_{ic} can be equivalently expressed via matrix C_{ic}, vector $\hat{\theta}_{ic}$ and scalar ${}^{\lfloor d}D_{ic}$.

Because all three representations described above are equivalent, we will not formally distinguish between them. If V_{ic} is a statistic of GiW factor, under the terms $L_{ic}, D_{ic}, \hat{\theta}_{ic}, C_{ic}, {}^{\lfloor d}D_{ic}$ we automatically mean the parts of corresponding representation of the matrix V_{ic}.

Now, we specify the steps 1,4,5 in the general algorithm 2 for Normal factors.

5.1 Evaluating $\mathcal{I}_{ic;t}$

$\mathcal{I}_{ic}$ is defined as

$$\mathcal{I}_{ic;t} = \int f_{ic}(d_{ic;t}|\psi_{ic;t}, \Theta_{ic})\pi_{ic}(\Theta_{ic}|\mathcal{S}_{ic;t-1})d\Theta_{ic} = \int N_{d_{ic;t}}(\theta'_{ic}\psi_{ic;t}, r_{ic})GiW_{\theta_{ic},r_{ic}}(V_{ic;t-1}, \nu_{ic;t-1})d\theta_{ic}dr_{ic}.$$

$\mathcal{I}_{ic;t}$ for normal factors is evaluated as:

$$\mathcal{I}_{ic;t} = \frac{\Gamma(0.5(\nu_{ic;t-1}+1))\left[{}^{\lfloor d}D_{ic;t-1}(1+\zeta_{ic;t})\right]^{-0.5}}{\sqrt{\pi}\Gamma(0.5\nu_{ic;t-1})\left(1+\frac{\hat{e}^2_{ic;t}}{{}^{\lfloor d}D_{ic;t-1}(1+\zeta_{ic;t})}\right)^{0.5(\nu_{ic;t-1}+1)}} \tag{14}$$

where

$$\hat{e}_{ic;t} \equiv d_{ic;t} - \hat{\theta}'_{ic;t-1}\psi_{ic;t} \equiv \textit{prediction error}$$
$$\zeta_{ic;t} \equiv \psi'_{ic;t}C_{ic;t-1}\psi_{ic;t}$$

Remarks 8 *According to remarks 4, we need to evaluate $\mathcal{L}_{ic;t} = \ln\mathcal{I}_{ic;t}$. It can be done efficiently via the product form of (14). The following algorithm summarizes this task. Recall that $\Psi_{ic;t} = [d_{ic;t}, \psi_{ic;t}]$.*

Algorithm 4 (evaluation of $\mathcal{L}_{ic;t}$) $\mathcal{L}_{ic;t} = \mathrm{FACNORM}(C_{ic;t-1}, \hat{\theta}_{ic;t-1}, {}^{\lfloor d}D_{ic;t-1}, \nu_{ic;t-1}, \Psi_{ic;t})$

1. *Evaluate $\zeta_{ic;t} = \psi'_{ic;t}C_{ic;t-1}\psi_{ic;t}$*

2. *Evaluate* $\hat{e}_{ic;t} \equiv d_{ic;t} - \hat{\theta}'_{ic;t-1}\psi_{ic;t}$

3. *Evaluate*

$$\mathcal{L}_{ic;t} = \ln \mathcal{I}_{ic;t} = \ln \Gamma\left(0.5(\nu_{ic;t-1}+1)\right) - \ln \Gamma\left(0.5\nu_{ic;t-1}\right) - 0.5 \ln\left({}^{\lfloor d}D_{ic;t-1}\right) - 0.5 \ln\left(1+\zeta_{ic;t}\right) -$$
$$-0.5(\nu_{ic;t-1}+1) \ln\left(1 + \frac{\hat{e}^2_{ic;t}}{{}^{\lfloor d}D_{ic;t-1}(1+\zeta_{ic;t})}\right) - 0.5 \ln(\pi)$$

Remarks 9 *Function* $\ln\Gamma$ *can be evaluated without computing* Γ *first (Abramowitz & Stegun 1972).*

5.2 Evaluating $\mathcal{S}^U_{ic;t}$

$\mathcal{S}^U_{ic;t} \equiv [V^U_{ic}, \nu^U_{ic}]$ can be evaluated in the following way:

$$\begin{aligned} V^U_{ic;t} &= V_{ic;t-1} + \Psi_{ic;t}\Psi'_{ic;t} \\ \nu^U_{ic;t} &= \nu_{ic;t-1} + 1 \end{aligned} \tag{15}$$

The relation (15) can be rewritten in terms of $C, \hat{\theta}, {}^{\lfloor d}D$:

$$C^U_{ic;t} = C_{ic;t-1} - \frac{1}{1+\zeta_{ic;t}} z_{ic;t} z'_{ic;t} \quad , \; \hat{\theta}^U_{ic;t} = \hat{\theta}_{ic;t-1} + \frac{\hat{e}_{ic;t}}{1+\zeta_{ic;t}} z_{ic;t}$$
$${}^{\lfloor d}D^U_{ic;t} = {}^{\lfloor d}D_{ic;t-1} + \frac{\hat{e}^2_{ic;t}}{1+\zeta_{ic;t}} \qquad z_{ic;t} = C_{ic;t-1}\psi_{ic;t}$$

5.3 Minimizing the KL distance

According to the proposition 2, we need to minimize

$$(1-w_{c;t})\mathcal{D}\left(\pi_{ic}(\Theta_{ic}|\mathcal{S}_{ic;t-1}) \,\middle\|\, \pi_{ic}(\Theta_{ic}|\mathcal{S}_{ic;t})\right) + w_{c;t}\mathcal{D}\left(\pi_{ic}(\Theta_{ic}|\mathcal{S}^U_{ic;t}) \,\middle\|\, \pi_{ic}(\Theta_{ic}|\mathcal{S}_{ic;t})\right)$$

for each factor i within the component c. The minimization can be done factor-vise (see alg. 2), thus we can simplify the notation by considering one particular factor.

$$\begin{array}{lcll} \mathcal{S}_{ic;t-1} & \equiv & (V_{ic;t-1}, \nu_{ic;t-1}) & \rightarrow \; (V, \nu) \\ \mathcal{S}_{ic;t} & \equiv & (V_{ic;t}, \nu_{ic;t}) & \rightarrow \; (V^\spadesuit, \nu^\spadesuit) \\ \mathcal{S}^U_{ic;t} & \equiv & (V^U_{ic;t-1}, \nu^U_{ic;t-1}) & \rightarrow \; (V^U, \nu^U) \\ w_{c;t} & & & \rightarrow \; w \\ \psi_{ic;t}, \Psi_{ic;t}, d_t & & & \rightarrow \; \psi, \Psi, d \end{array}$$

Thus, we minimize

$$\min_{V^\spadesuit, \nu^\spadesuit} \left\{ (1-w)\mathcal{D}\left(GiW_{\theta,r}(V,\nu) \,\middle\|\, GiW_{\theta,r}(V^\spadesuit, \nu^\spadesuit)\right) + w\mathcal{D}\left(GiW_{\theta,r}(V^U,\nu^U) \,\middle\|\, GiW_{\theta,r}(V^\spadesuit, \nu^\spadesuit)\right) \right\}. \tag{16}$$

Remarks 10

1. *It can be proven that this minimization task can be divided into two independent algebraic subproblems. First of them is minimization on two dimensional space* $(\nu^\spadesuit, {}^{\lfloor d}D^\spadesuit)$*, the second is minimization on multi-dimensional space* $(\hat{\theta}^\spadesuit, C^\spadesuit)$*. Both subproblems are solved in the following sections.*
2. *If we approximate* $\mathcal{D}\left(GiW_{\theta,r}(V,\nu) \,\middle\|\, GiW_{\theta,r}(V^\spadesuit,\nu^\spadesuit)\right)$ *with* $||V - V^\spadesuit||^2 + ||\nu - \nu^\spadesuit||^2$*, we can quickly achieve the result* $V^\spadesuit = V + w\Psi\Psi'$*,* $\nu^\spadesuit = \nu + w$*, which is exactly the same as the quasi-Bayes update (Kárný et al. 1998).*

5.3.1 Searching for ${}^{\lfloor d}D^{\spadesuit}$ and $\nu^{\spadesuit}$

Proposition 4 *For $\nu^{\spadesuit}$, ${}^{\lfloor d}D^{\spadesuit}$ minimizing (16) it holds:*

$$\frac{\nu^{\spadesuit}}{{}^{\lfloor d}D^{\spadesuit}} = (1-w)\frac{\nu}{{}^{\lfloor D}D} + w\frac{\nu^U}{{}^{\lfloor D}D^U}$$

$$\ln\left(0.5\nu^{\spadesuit}\right) - \psi_0\left(0.5\nu^{\spadesuit}\right) = \Upsilon, \text{ where}$$

$$\Upsilon \equiv (1-w)\left(\psi_0\left(0.5\nu\right) - \ln\left({}^{\lfloor D}D\right)\right) + w\left(\psi_0\left(0.5\nu^U\right) - \ln\left({}^{\lfloor D}D^U\right)\right) -$$

$$- \ln\left(0.5(1-w)\frac{\nu}{{}^{\lfloor D}D} + 0.5w\frac{\nu^U}{{}^{\lfloor D}D^U}\right)$$

Straightforward application of Proposition 4 yields the following algorithm. Recall that $\Psi = [d, \psi]$.

Algorithm 5 (Updating ${}^{\lfloor d}D$ and ν) $\left({}^{\lfloor d}D^{\spadesuit}, \nu^{\spadesuit}\right) = \text{UPDATE\_DFM}(w, C, \nu, \hat{\theta}, {}^{\lfloor D}D, \Psi)$

1. $\hat{e} = d - \hat{\theta}'\psi, \quad \zeta = \psi' C \psi$
2. $\nu^U = \nu + 1, \quad {}^{\lfloor D}D^U = {}^{\lfloor D}D + \frac{\hat{e}^2}{1+\zeta}$
3. $X^S = (1-w)\frac{\nu}{{}^{\lfloor D}D} + w\frac{\nu^U}{{}^{\lfloor D}D^U}$
4. $\Upsilon = (1-w)\left[\psi_0\left(0.5\nu\right) - \ln\left({}^{\lfloor D}D\right)\right] + w\left[\psi_0\left(0.5\nu^U\right) - \ln\left({}^{\lfloor D}D^U\right)\right] - \ln\left(0.5X^S\right)$
5. *Solve the equation for $\nu^{\spadesuit}$:* $\ln\left(0.5\nu^{\spadesuit}\right) - \psi_0\left(0.5\nu^{\spadesuit}\right) = \Upsilon$
6. ${}^{\lfloor d}D^{\spadesuit} = \frac{\nu^{\spadesuit}}{X^S}$

Remarks 11

1. *Step 5 must be solved numerically or using some suitable approximation.*
   For detail description of the numerical solution and for proof of unicity of the solution see (Nenutil 2004).
2. *The proof of existence of solution fulfilling $\nu^{\spadesuit} > 0$, ${}^{\lfloor d}D^{\spadesuit} > 0$ can be found in (Andrýsek 2004).*

5.3.2 Searching for $\hat{\theta}^{\spadesuit}$ and $C^{\spadesuit}$

Proposition 5 *For $\hat{\theta}^{\spadesuit}$ and $C^{\spadesuit}$ minimizing (16) it holds:*

$$C^{\spadesuit} = C + w_c z z' \tag{17}$$

$$\hat{\theta}^{\spadesuit} = \hat{\theta} + w_\theta z \tag{18}$$

where

$$z = C\psi, \quad \hat{e} = d - \hat{\theta}'\psi, \quad \zeta = \psi' C \psi$$

$$w_c = \left[\frac{\hat{e}^2}{(1+\zeta)^2}\frac{XX^U}{X+X^U} - \frac{w}{1+\zeta}\right], \quad w_\theta = \left[\frac{\hat{e}}{1+\zeta}\frac{X^U}{X+X^U}\right]$$

$$X = (1-w)\frac{\nu}{{}^{\lfloor D}D}, \quad X^U = w\frac{\nu^U}{{}^{\lfloor D}D^U}$$

Algorithm 6 (Updating $\hat{\theta}$ and C) $(C^{\spadesuit}, \hat{\theta}^{\spadesuit}) = \text{UPDATE\_C}(w, C, \nu, \hat{\theta}, {}^{\lfloor D}D, \Psi)$

1. $\hat{e} = d - \hat{\theta}'\psi, \quad \zeta = \psi' C \psi$
2. $\nu^U = \nu + 1, \quad {}^{\lfloor D}D^U = {}^{\lfloor D}D + \frac{\hat{e}}{1+\zeta}$
3. $X = (1-w)\frac{\nu}{{}^{\lfloor D}D}, \quad X^U = w\frac{\nu^U}{{}^{\lfloor D}D^U}$
4. $z = C\psi$
5. $C^{\spadesuit} = C + \left[\frac{\hat{e}^2}{(1+\zeta)^2}\frac{XX^U}{X+X^U} - \frac{w}{1+\zeta}\right] zz'$

6. $\hat{\theta}^{\spadesuit} = \hat{\theta} + \left[\frac{\hat{e}}{1+\zeta}\frac{X^U}{X^S}\right] z$

Remarks 12

1. *The keystone of the previous algorithm is step 5. The formula is very simple, but its iterative use can cause numerical troubles. Therefore, in practice, we always work with matrix C in its $L'DL$ decomposition. Numerical stability of this operation is discussed in (Nenutil 2004).*
2. *It can be proven that $C^{\spadesuit}$ obtained by this algorithm is positive definite.*

6 Resulting PB algorithm

In this Section, we summarize all the elaborated parts into one consistent algorithm.

Algorithm 7 (PB)

Inputs - $\kappa_{\bullet;t-1},\ C_{\bullet\bullet;t-1},\ \hat{\theta}_{\bullet\bullet;t-1},\ {}^{\lfloor d}D_{\bullet\bullet;t-1},\ \nu_{\bullet\bullet;t-1},\ \Psi_{\bullet\bullet;t}$
Outputs - $\kappa_{\bullet;t},\ C_{\bullet\bullet;t},\ \hat{\theta}_{\bullet\bullet;t},\ {}^{\lfloor d}D_{\bullet\bullet;t},\ \nu_{\bullet\bullet;t}$

1. *For each factor ic:* $\mathcal{L}_{ic;t} = \text{FACNORM}(C_{ic;t-1}, \hat{\theta}_{ic;t-1},\ {}^{\lfloor d}D_{ic;t-1}, \nu_{ic;t-1}, \Psi_{ic;t})$. *(algorithm 4)*
2. *Evaluate* $w_{\bullet;t} = \text{EVAL\_WEIGHT}(\mathcal{L}_{\bullet\bullet;t}, \kappa_{\bullet;t-1})$. *(algorithm 1)*
3. *Evaluate* $\kappa_{\bullet;t} = \text{NEW\_KAPPA}(w_{\bullet;t}, \kappa_{\bullet;t-1})$. *(algorithm 3)*
4. *For each factor ic:* $\left({}^{\lfloor d}D_{ic;t}, \nu_{ic;t}\right) = \text{UPDATE\_DFM}(w_{c;t}, C_{ic;t-1}, \nu_{ic;t-1}, \hat{\theta}_{ic;t-1},\ {}^{\lfloor d}D_{ic;t-1}, \Psi_{ic;t})$. *(algorithm 5)*
5. *For each factor ic:* $\left(C_{ic;t}, \hat{\theta}_{ic;t}\right) = \text{UPDATE\_C}(w_{c;t}, C_{ic;t-1}, \nu_{ic;t-1}, \hat{\theta}_{ic;t-1},\ {}^{\lfloor d}D_{ic;t-1}, \Psi_{ic;t})$. *(algorithm 6)*

7 Comparison of PB and QB algorithms

In this Section, we compare the performance of the PB algorithm with the performance of the standard QB algorithm. The QB algorithm has been used extensively in real-life applications (Kárný, et al. 2003), and it is proven to be reliable and computationally efficient. Therefore, we study differences of the PB algorithm from the QB in terms of numerical properties and quality of estimation. The algorithms are based on different objective criteria for which they are optimal. Therefore, comparison of their behaviour is presented in a subjective way: arguing what seem to be more "rational".

In order to compare the analytical properties, we review the QB algorithm. Then, we investigate the differences between the two algorithms from analytical and computational point of view. Those finding are supported by experimental results.

7.1 The Quasi-Bayes algorithm

The general QB algorithm uses the following rule(see (Kárný et al. 1998)):

$$\kappa_t = \kappa_{t-1} + w_t$$
$$\pi_{ic}(\Theta_{ic}|\mathcal{S}_{ic;t}) \propto \left[f_{ic}(d_{ic;t}|\psi_{ic;t}, \Theta_{ic})\right]^{w_{c;t}} \pi_{ic}(\Theta_{ic}|\mathcal{S}_{ic;t-1})$$

Let's mark the statistics corresponding to the QB algorithm by the subscript Q. Application of the general algorithm to the case with Normal factors yields:

$$V_Q = V + w\Psi\Psi',\ \nu_Q = \nu + w,\ \kappa_{Q_{\bullet;t}} = \kappa_{\bullet;t-1} + w_{\bullet;t} \tag{19}$$

We would receive exactly this result, if we approximate the KL distances in the PB algorithm with squares of euclidian norms of the parameter difference (see remarks 7 and 10).

For better comparison of the QB algorithm with the PB algorithm, we rewrite the relations (19) in terms of $C, \hat{\theta},\ {}^{\lfloor d}D$:

$$C_Q = C + w_{QC} z z' \tag{20}$$

$$\hat{\theta}_Q = \hat{\theta} + w_{Q\theta} z\,, \quad {\lfloor^d D}_Q = {\lfloor^D D} + \frac{w\hat{e}^2}{1 + w\zeta} \tag{21}$$

where

$$z = C\psi, \quad \hat{e} = d - \hat{\theta}'\psi, \quad \zeta = \psi' C \psi$$
$$w_{QC} = \frac{-w}{1 + w\zeta}\,, \quad w_{Q\theta} = \frac{w\hat{e}}{1 + w\zeta}$$

7.2 Analytical comparison

Nature of both algorithms allows us to divide the analytical investigation into two parts. In the first part, we investigate the update of factors. This part is discussed next. In the second part, computing of the new component weights can be studied, see (Nenutil 2004).

7.2.1 Differences of the algorithms

Note that the expressions for the QB update (20), (21) are very similar to the expressions for the PB update (17),(18). Hence, it suffice to investigate differences between the pairs $(\nu^{\spadesuit}, \nu_Q), ({\lfloor^d D^{\spadesuit}}, {\lfloor^d D_Q}), (w_C, w_{QC}), (w_\theta, w_{Q\theta})$. This involves observation of 4 scalar variables, no matter what is the full dimension of the parameters.

We illustrates differences in behavior on the following examples. Consider the following situations:
a) $\nu = 165.39,\ {\lfloor^D D} = 9.77, \hat{e} = -0.0140, \zeta = 0.59$
b) $\nu = 102.82,\ {\lfloor^D D} = 1.14, \hat{e} = -0.7386, \zeta = 1.20$

The figures 1 and 2 shows the parameters $(\nu^{\spadesuit}, \nu_Q), ({\lfloor^d D^{\spadesuit}}, {\lfloor^d D_Q}), (w_C, w_{Q_c}), (w_\theta, w_{Q\theta})$ as functions of $w \in < 0, 1 >$. The parameters related to the PB algorithm are plotted with the thick line. It is clear, that values obtained using PB equals to those of QB for $w = 0, w = 1$.

7.2.2 Bahaviour of the PB algorithm

In this Section, we study two particular factors and evaluate marginal distributions of their updates provided by both algorithms. For better comparison, we will also show the marginal pdf of the correct Bayesian update (12) which is a mixture of two GiW factors.

Consider the GiW factor $\pi(\Theta|\mathcal{S}) = GiW_{\theta,r}(V, \nu)$ and denote the associated densities as follows:

trial update $\pi(\Theta\|\mathcal{S}^U) = GiW_{\theta,r}(V^U, \nu^U)$	$V^U = V + \Psi\Psi',\ \nu^U = \nu + 1$
QB update $\pi(\Theta\|\mathcal{S}_Q) = GiW_{\theta,r}(V_Q, \nu_Q)$	$V_Q = V + w\Psi\Psi',\ \nu_Q = \nu + w$
PB update $\pi(\Theta\|\mathcal{S}^{\spadesuit}) = GiW_{\theta,r}(V^{\spadesuit}, \nu^{\spadesuit})$	result of the algorithms 4 and 5
correct update $\hat{\pi}(\Theta) = (1-w)\pi(\Theta\|\mathcal{S}) + w\pi(\Theta\|\mathcal{S}^U)$	

Consider the statistics V, ν of the GiW factor, updating weights w and actual data vectors of the factor Ψ, to be:

$a)$	$b)$
$V = \begin{pmatrix} 1.16 & 0.12 \\ 0.12 & 0.83 \end{pmatrix}$	$V = \begin{pmatrix} 1.96 & -1.47 \\ -1.47 & 6.07 \end{pmatrix}$
$\nu = 102.82$	$\nu = 108.06$
$\Psi = (-0.59 \quad 1)'$	$\Psi = (-0.79 \quad 1)'$
$w = 0.43$	$w = 0.39$

The figures 3 and 4 shows marginal pdfs of all discussed densities for both cases. From visual inspection of these figures, we can conclude that the PB algorithm can provide results significantly different from those of the QB algorithm. We also consider behavior of the PB algorithms as reasonable.

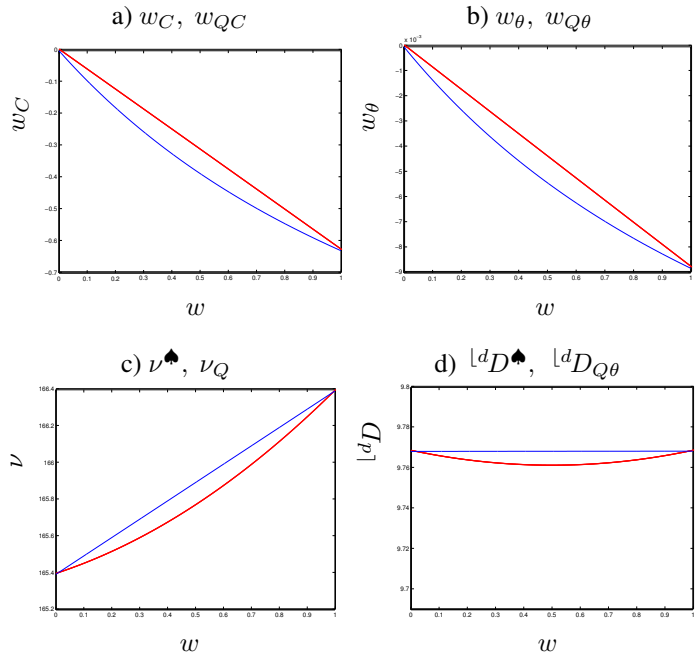

Figure 1. Similar behavior of the QB and PB algorithms for case a)

The figure shows the parameters $(\nu^\spadesuit, \nu_Q), (\lfloor^d D^\spadesuit, \lfloor^d D_Q), (w_C, w_{Qc}), (w_\theta, w_{Q\theta})$ *as the functions of* $w \in <0, 1>$ *for the case a)* $\nu = 165.39$, $\lfloor^D D = 9.77, \hat{e} = -0.0140, \zeta = 0.59$. *The parameters related to PB algorithm are plotted with the thick line. In this case the difference between the QB and PB algorithms is rather small.*

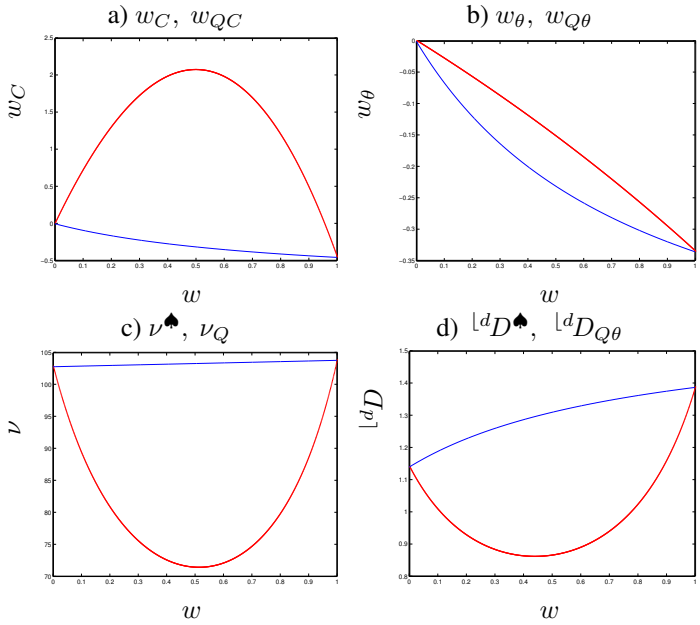

Figure 2. Different behavior of the QB and PB algorithms for case b)

The figure shows the parameters $(\nu^\spadesuit, \nu_Q), (\lfloor^d D^\spadesuit, \lfloor^d D_Q), (w_C, w_{Qc}), (w_\theta, w_{Q\theta})$ *as the functions of* $w \in <0, 1>$ *for the case b)* $\nu = 102.82$, $\lfloor^D D = 1.14, \hat{e} = -0.7386, \zeta = 1.20$. *The parameters related to the PB algorithm are plotted with the thick line. In this case, the difference between the QB and PB algorithms is significant.*

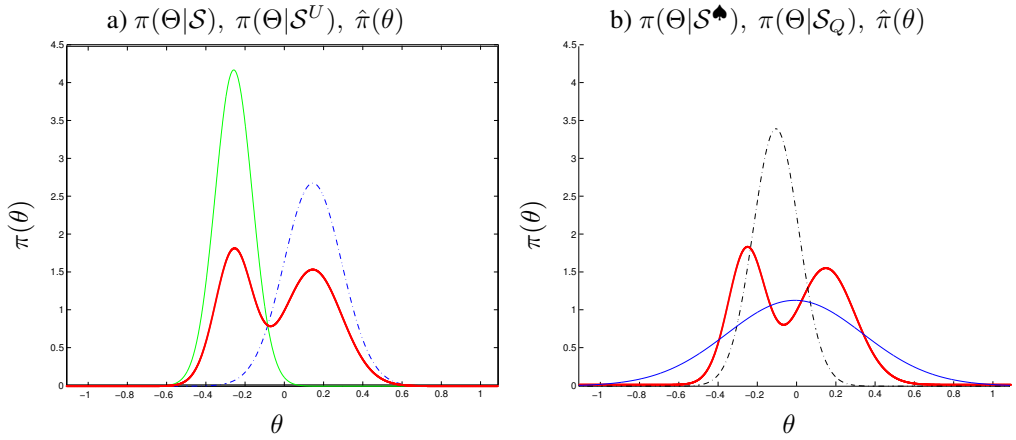

Figure 3. Marginal pdfs of the QB and PB updates for the case a)

The left part shows original factor (dashdot), its trial update (dotted) and the correct Bayesian update (thick), i.e. the mixture of the two mentioned factors. The right part shows how the QB update (dashdot) and the PB update (solid) approximates the correct Bayesian update (thick). It can be seen that the PB update is in this case flatter then the QB update which concentrates on smaller interval.

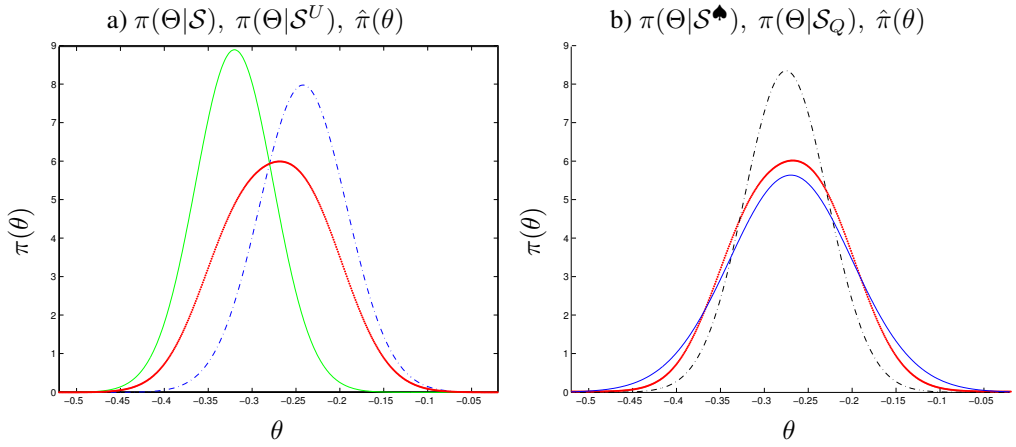

Figure 4. Marginal pdfs of the QB and PB updates for the case b)

The left part shows original factor (dashdot), its trial update (dotted) and the correct Bayesian update (thick), i.e. the mixture of the two mentioned factors. The right part shows how the QB update (dashdot) and the PB update (solid) approximates the correct Bayesian update (thick). It can be seen that the PB update in this case better approximates the correct pdf.

7.3 Experimental comparison

Intensive tests consisting of 1396 data sets were done. Data used for this test represent various types of systems (static, dynamic, multidimensional) and are part of standard testing procedure of new algorithms. As a quality measure, we used the likelihood (Kárný et al. 2003) of the estimated model. For each set, we evaluated a criterion h which is the difference between the likelihood obtained by the PB algorithm and the QB algorithm. (i.e $h > 0$ if the PB algorithm was better.) The table 1 shows the results. Mean value of h over all sets is 6.18.

7.4 Comparing of computational complexity

We compare all 5 steps of the PB algorithm (algorithm 7).

1. This step is needed in both algorithms.
2. This step is needed in both algorithms.

condition	number of sets	percentage
$h > 0$	1125	80.6%
$h < 0$	271	19.4%
abs$(h) < 2$	1126	80.6%
$h > 2$	251	18.0%
$h < -2$	19	1.4%

Table 1. Results of experimental comparison

The table shows some conditions for h and number of sets fulfilling each condition.

3. We have to find minimizer of a convex function with $\mathring{c}$ variables. There exist a good approximation of the starting point for iterative numerical algorithm, which warrants quick solution of this task (Nenutil 2004).
4. Solution of one-dimensional nonlinear equation must be found. However, a good approximation which always leads to solving the equation in a few steps was found (Nenutil 2004).
5. This step has the same complexity in both algorithms.

Addressing the previous considerations, we conclude that computational cost of numerical evaluation of the PB algorithm is comparable to the computational cost associated with the QB algorithms. Detailed case study of the computational costs of both algorithms can be found in (Nedoma & Andrýsek 2004).

8 Conclusions

This work describes a novel and efficient algorithm for recursive estimation of finite probabilistic mixture. The algorithm has the potential of providing more accurate results than the well-established quasi-Bayes estimator. This improvement is important as mixtures represent a universal approximating tool for modelling of non-linear stochastic systems. Therefore, mixture models can be used to address complex control and decision-making problems in changing environments, such as multiple-participants decision making. Each participant (or group of participants) can be modelled by a component of the overall mixture model. All subsequent decision-making task can be easily formalized within the consistent formal framework of probabilistic mixture models. We believe, that the algorithms presented in this paper will be an important part of this framework.

9 Acknowledgments

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DRAFT

A Multiple Objectives Optimization Approach to Distributed System Design and Control

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Abstract An optimization approach to distributed intelligent system design and control is presented. It is expected to enhance the autonomous decision-making capabilities of systems functioning as members of a team. It is applicable to autonomous distributed agents within homogeneous or heterogeneous clusters, when they must collaborate as a team to achieve a common goal. This approach inherently supports situation awareness, collision avoidance, and operations in a complex environment that may experience both degraded communications and sensor failures.

Keywords: Multi-Objective, Optimization, Control, Distributed, MOO, Team

1 Introduction

We propose a new, multiple objective optimization based approach to distributed intelligent system design and control, which will lead to an improved performance of many missions by enhancing the autonomous decision making capabilities of systems functioning as members of a team. It is applicable to autonomous multiple agents within homogeneous or heterogeneous clusters, when they must collaborate to achieve a common goal. This approach inherently supports situation awareness, collision avoidance, and operations within a complex environment that may experience both degraded communications and sensor failures. The complex system of autonomous systems is designed to operate in real-time with no single system indispensable to the team and with self monitoring of execution and recovery from faults. This hierarchical, multi-resolution approach provides coordination among the levels of control within a distributed architecture. The modularity of the architecture enables advanced debugging and a model development environment appropriate for concurrent engineering and design (due to its parallel nature).

An example is provided that makes use of this approach to solve a problem where a team of agents must traverse a two dimensional plane and are tasked with independently optimizing their control to satisfy both their local objectives and their joint team objective.

2 The Problem

At present, there is no consistent mathematical theory for the automated reasoning, control, and design of a large set of intelligent systems which must compete, collaborate, or even just share some space or resource. As a result, "suboptimal", often ineffective, heuristics based design 'rules' are used, leading to the well known unreliable, poorly scalable, and brittle behavior of these systems.

Fundamental questions exist in the design of such systems. For example, how does one design and execute the control, resource allocation, and coordination of a group of independent systems? What are the underlying principles for the "decentralization" of the design, communication, and control of these systems? Our proposed approach will attempt to offer a solution to these problems, which indeed are highly generic.

Many communication, networking, software development, system management, and engineering tasks fall under this class of problems.

3 Background

How does one distinguish between a group of agents and one "complex" agent or system? This question is central, since without such distinction, there is no "hope" for obtaining a scalable, distributed algorithm in order to improve upon the present state of the art offered by large scale system design. Present approaches do not distinguish between: i) a large scale dynamic system obtained as an aggregate of each subsystem, which results in a sub-optimal, often ineffective resource utilization via a heuristics based approach; ii) a set of dynamic systems that describe each agent that, though coordinated, are independently controlled, while potentially sharing a subset of the state space and constraints. This discrepancy results in confusion regarding the terminology used in the communication of research and engineering results within this field. Furthermore, what makes a group of systems a team is the fact that there is a task / objective to be shared by all the team members. In the case where various subsystems (of a system) have different, i.e., non-redundant, non-overlapping objectives / tasks / missions / constraints – we no longer are dealing with a single system but rather with a set of systems, which may or may not be coupled. It is also important to note that all systems have at least one (often implicit) self objective / task, examples include: maximize survivability, minimize metabolism / energy used, and minimize time.

4 MOO Approach

On the basis of several observations, which we make regarding teams of systems, a new constructive definition of a Cooperating Team Of Systems (CTOS) is offered. The tools of Multiple Objective Optimization (MOO) (Pareto 1964), dynamic systems control, mathematical programming, differential games, and other theories are employed to generate this new approach, which we name the MOO Approach. This approach is the framework that embodies well-defined architectural concepts for enabling the development of autonomous systems that meet high-level mission goals.

A CTOS is a set of systems for which, in addition to individual objectives for each system, the team jointly possesses a common objective. Thus a system becomes a member of a team of systems by "accepting" an additional shared objective. Such a team objective often conflicts with the individual objective(s) of each system.

The planning, design and control of a CTOS utilizing the MOO Approach is summarized below:

For each i-th member of the CTOS, let $\mathbf{x}_i$ be the member state vector, $\mathbf{u}_i$ be the member input vector; $\mathbf{y}_i$ be the member output vector; $\mathbf{f}_i(\mathbf{x}_i, \mathbf{u}_i, t)$ be the member dynamics, and o_i be the member performance index (objective function). Describe the team of systems dynamics by: $\mathbf{U}$=Union($\mathbf{u}_i$), the augmented input vector; $\mathbf{X}$=Union($\mathbf{x}_i$) the augmented state vector; $\mathbf{Y}$=Union($\mathbf{y}_i$) the augmented output vector, and $\mathbf{F}$=Union($\mathbf{f}_i$) the augmented system dynamic. Let $\mathbf{O}$ be the team of systems' performance index functional defined on the augmented state and input space $\{\mathbf{X}, \mathbf{U}\}$. Let $\mathbf{G}(\mathbf{X}, \mathbf{U}) <= 0$ be the set of input and state inequality constraints describing boundaries, obstacles (including the space occupied by other systems in the team), hardware limits, etc.

Note the only source of coupling is through the inequality constraints $\mathbf{G}$ and the common objective function $\mathbf{O}$. A control solution may be sought as follows:

Project $\mathbf{O}$ and $\mathbf{G}$ on the i-th state space and obtain the corresponding restricted constraints and objective functions for each i-th system. For each i-th system, independently and simultaneously solve the resulting multiple objective optimal control problem (Guez et. al. 1992, and Messac and Ismail-Yahaya 2001). Such a "solution" consists of identifying the set of non improvable decision variables, the so called Pareto set, which encompasses all the optimal tradeoffs that the i-th system may choose in compromising its self objective with that of the team and vice versa. The Pareto set may or may not be a connected set. The Pareto set is the key source of many of the expected benefits of the proposed approach. Since the total

design is reduced to N simultaneous and independent Pareto sets, it is vastly smaller than a single N dimensional Pareto set that would result from a large scale coupled system approach. The solution of our problem is then obtained as the aggregate control **U** consisting of all of the corresponding $\mathbf{u}_i$ solutions being selected from their corresponding Pareto sets. Note that each control can be calculated independently by each system once the projection is accomplished.

5 Example

Provided below is an example that demonstrates the benefits of the MOO Approach for a CTOS. The example is a MATLAB simulation, and the simulation results are depicted in the figures below. The example instantiates twenty systems, which could be robots, autonomous teams, software agents, etc…, and joins the systems together in a team. The team is placed in a two dimensional plane in which each system is initially located at random and can navigate based on its individual control. The premise is that each system is initially programmed with its objectives and relies on a local sensor to determine its nearest neighbor (a limited horizon problem). No additional sensors or communication is provided or required. The number of systems was chosen to be visually pleasing in the figures and also to convey that this approach is well suited for teams with a large number of members (e.g., tens, hundreds or even thousands) since the approach does not require a coupling of the member dynamics. The example problem provides a team objective for each system to follow a global path in two dimensions over a specified time interval. This team objective is accepted by each system by projecting the objective onto the set of local objectives. The local / self objectives are minimize energy and avoid collisions. The collision avoidance objective is formulated as a formation objective (stay a certain distance in the x and y dimension from your neighbor). The simulation follows the MOO Approach summarized in the previous section by formulating a control utilizing the techniques of multiple objective optimization (Guez et. al. 1992).

The dynamics of each individual system are defined as:

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), x_i(t_0) = x_0, i = 1,2,3,...N$$
$$y_i(t) = Cx_i(t)$$

where $\mathbf{x}_i$, $\mathbf{u}_i$, and $\mathbf{y}_i$ are previously defined.

The following cost functions are used for the competing objectives:

The team / global cost:

$$J_{gi} = \int_{t_o}^{t_f} [y_i(\tau) - y_g(\tau)]^T Q_{gi} [y_i(\tau) - y_g(\tau)] d\tau, i = 1,2,3..N.$$

where $\mathbf{y}_i$ is the position of each system, $\mathbf{Q}_{gi}$ is the cost weighting matrix, and $\mathbf{y}_g$ is the global path that should be followed.

The formation / collision avoidance cost:

$$J_{fi} = \int_{t_o}^{t_f} [y_i(\tau) - (y_j(\tau) - \vec{d}_{ij})]^T Q_{fi} [y_i(\tau) - (y_j(\tau) - \vec{d}_{ij})] d\tau, i = 1,2,3..N.$$

where $\mathbf{y}_j$ is the j-th system that is the minimum distance in front of the i-th system in the direction of the global path, $\mathbf{Q}_{fi}$ is the cost weighting matrix, $\mathbf{d}_{ij}$ is the desired distance between the two systems.

The energy expenditure cost (minimization of the effort):

$$J_{ei} = \int_{t_o}^{t_f} u_i(\tau)^T R_i u_i(\tau) d\tau, i = 1,2,3...N.$$

where u_i is the input vector and R_i is the cost weighting matrix.

Notice that the cost functions in this example are all quadratic. This allows us to project the global cost function onto the set of local costs by using the Weighted Sum of Objective Functions method (Guez et. al. 1992, Collete and Siarry 2003). The new cost function is:

$$J_i = \alpha_i J_{gi} + (1 - \alpha_i) J_{si} + J_{ei}$$

where J_i is the new cost function for each system, and α_i is the weight assigned to the global and local cost functions by the team leader / system engineer.

As described in the preceding section and detailed in (Guez et. al. 1992, Lewis 1986) a control is calculated utilizing the principles of optimal control for linear quadratic regulation and tracking problems. It is important to note that each system control is calculated independently after the global objective has been projected onto the system.

For this example, α_i would be specified by the system engineer or team leader to set the priority between self objectives and the global / team objective. The following figures show the results from various weightings to convey the benefits and tradeoffs that result from this approach.

Figure 1 shows the results when the control is weighted to follow the team objective only. As can be seen, each agent starts at a random x and y location and begins to move towards the global path as a result of the control. It is important to note that each agent is operating completely autonomously and is moving as a result of its local control. It is also important to note that for this and all subsequent simulations, the algorithms / control would scale linearly if the team grew in size (e.g., from 20 to 200).

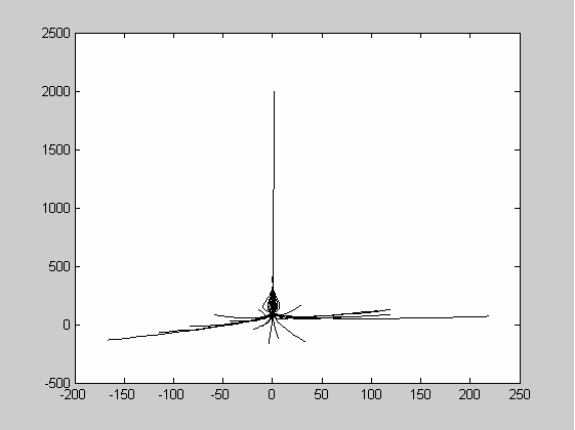

Figure 1: Team of Systems that have only a team objective of path following.

The objectives are now changed for the team members to ignore the team task and instead fully weight the local formation task / cost function. The results are shown in Figure 2. As can be seen, each agent that is not the leader follows the closest agent to the left and behind. The leader traverses a path that increases monotonically with random increments.

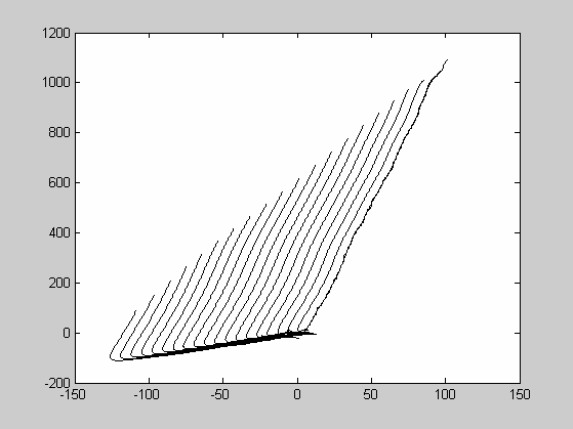

Figure 2: Systems that have only the formation objective.

In the next examples, the system engineer / team leader wishes each team member to perform its mission with a weighting of both the global and local objectives. Under the MOO Approach, this simply requires a changing of the weighting and a recalculation of the control, which is done locally by each team member. The results of these new team assignments are shown in Figures 3 through 5. Note that one of the 20 agents becomes the leader based on its initial random position. This leader uses the global path as its neighbor with a zero formation distance. The three simulations below show the results as the weighting is increased for the team objective of global path following over formation. In the case of Figure 5, the weights may be set to heavily in favor of the team objectives; therefore, it should be evident that some system level design is required to get good results over the entire tradeoff space that can be achieved by varying the weights.

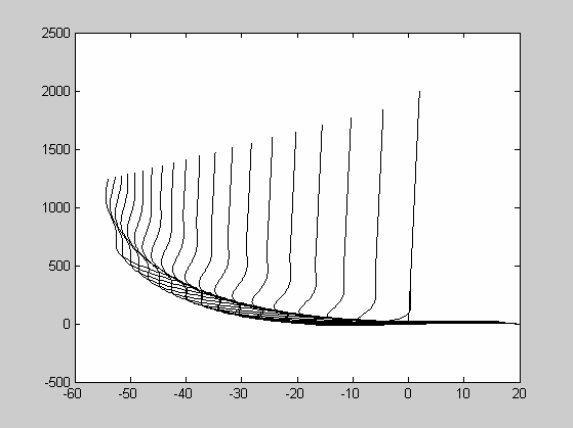

Figure 3: Mixture of local and team objectives.

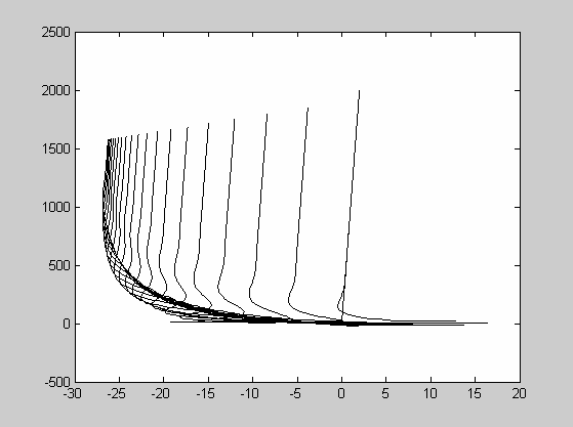

Figure 4: Mixture of local and team objectives; team weighting increased from Figure 3.

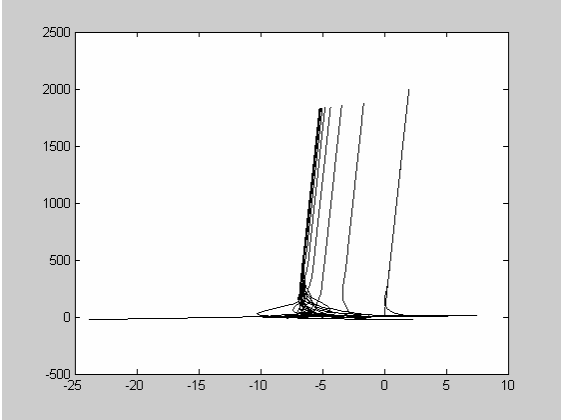

Figure 5: Mixture of objectives. Team perhaps too heavily weighted.

One of the key benefits of this MOO Approach for team design is the distribution of intelligence among the team members, which results in fault tolerance for the team if one or more of the team nodes fail. Figures 6 and 7 depict the results of one of the team members failing. Figure 6 shows the simulation results when each team member is programmed only with its self objectives, which include formation / collision avoidance but not the team objective of global path following. As can be seen in the figure, one of the systems fails and moves lower and to the right rather than following its closest neighbor. As a result, many of the other systems now follow the faulty system while the other systems follow the team leader. Figure 7 shows the results when the simulation is rerun with the same initial states / positions for the systems, but the team objective of global path following is now projected onto each team member. As a result, each system is able to maintain a heading towards the global path and in effect ignores the faulty team member.

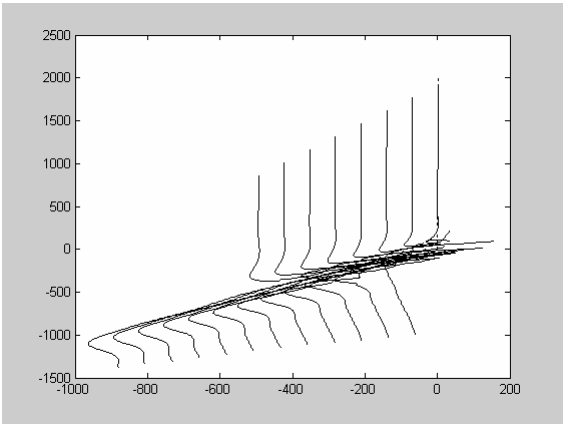

Figure 6: Results when one random system is faulty and weighting is fully formation.

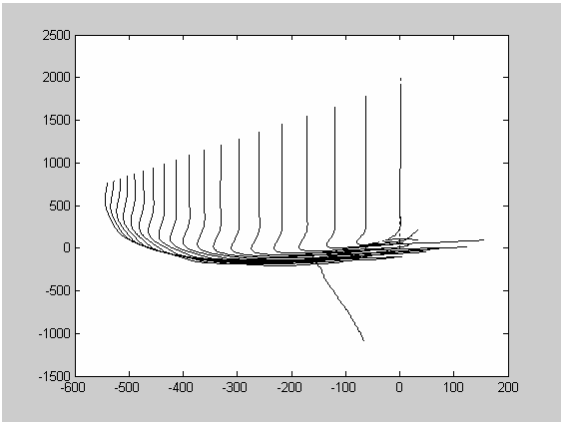

Figure 7: Same initial conditions as in Figure 6; however, the "global path following" team objective is now mixed with the formation objective.

6 Benefits

Due to the underlying mathematics involved in the MOO Approach as described above, the resulting distributed system is expected to possess the following features:

6.1 Decentralized / Parallel / Distributed Design Control and Communication

This is due to the fact that each subsystem is independently optimizing (planning) and executing its tasks. Notice that this benefit is both in the hardware as well as the software structure aspects of the design. Also, there are lower requirements on the communication channel data rate-capacity as the global task is communicated at low rate.

6.2 Linear Scalability

Notice that, unlike present centralized, large scale system design (Shi 1992) and/or current distributed (e.g. Neurocontrol) (Guez and Selinsky 1988) control / communication architectures, which scale geometrically in the aggregate system dimension (as measured by the number of subsystems), the MOO approach is expected to scale linearly!

6.3 Maximal Delegation / Autonomy

This is expected since each subsystem communicates with team members normally only through its onboard sensors (via the environment in which the entire team operates, and not through peer to peer channels. Also, since the only information downloaded from higher levels, is the teams task objective (i.e., higher level information and not detailed planned trajectories, the i-th system need not be informed about the tasks and plans of other systems) – there communication burden is reduced and maximal autonomy is delegated.

6.4 Robustness and Reliability

With reduced complexity we expect improved robustness. For example when a sudden cost increase is being observed at the i-th system Pareto set, it implies some failure or structural change, which will result automatically in some other system, say the j-th to select a different point in its own Pareto set since its payoff will increase. Note that such fault detection and recovery occurs without centralized re-planning.

6.5 Hierarchical and Multi Resolution Design

It is possible to apply the MOO approach to several resolution system levels and to provide coordination among the levels of control within a distributed architecture. This will enable:

6.6 Modularity

This enables advanced debugging and model-development environments for autonomy software. Easily and inexpensively programmed and modified due to their modular and invariant structure. Also, easily and inexpensively maintained and trained due to their modularity and invariant structure.

6.7 Optimal Performance

This benefit is the essence of the MOO approach.

These MOO Approach properties (6.1-6.7) are expected to provide: high performance, with dramatically lower complexity, reduced resource consumption, and intelligent and reflexive behavior. Also, 6.1 implies a naturally distributed architecture for autonomy with automatic coordination, which includes multiple types

of autonomous agents, or mixed human-artificial agents. We can also expect intelligent fault protection (6.4), providing model-based fault management capabilities into an autonomous executive control loop. Due to 6.7, planning and execution are done via real time optimization, which is viewed as automatic planning and execution, generating sequences of executable activities, as well as systems for robust execution. Indeed these are large-scale concurrent planning under uncertainty involving continuous quantities such as time and resources.

7 Concluding Remarks

We proposed a MOO based approach to distributed intelligent systems - analysis, design and control. Future publications will describe the results of applying this approach to the design, management and control of complex systems.

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Framework for Multisensory Convergence

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Abstract. In this paper we consider the multisensory convergence problem, that is when signals from several different sensors report on the same event, possible through the employment of different pieces and types of information. A unified approach based on non-linear filtering is proposed here. Firstly, Kalman filtering is applied to a simple linear state-space model. The filter takes input data from different sensors and produces estimate of the physical state describing, for example, environment of a robot. This shows structure of optimal sensor fusion. Then, a general version of such a filtering is presented with a careful account for computational complexity. This leads to modelling through finite probabilistic mixtures: both sensors and global state estimates are modelled by finite mixtures of uni-modal probability density functions (pdf). Bayesian non-linear estimation is then used to mix together the different sensor responses to a single information driven estimate of environment state. The proposed theory will be confronted with conditions offered by a prototype 5-sensory robot head that has been set up as a test-bed platform for the multisensory fusion algorithms.

1 Introduction

In the biological world the ability of a creature to integrate information obtained from a number of sensory mechanisms at its disposal in order to help it decide what action to take at any instant in time is paramount in its whole being. Because of such input a creature is able to orient itself in space, in relation to its perception of objects within a region of that space, respond to changes perceived in the environment, for such as nutrition, and perhaps, important of all, preserve itself as a dependable life form. In any search for artificially alive beings, autonomous devices that exist for long periods under their own adaptive control or even simply robots that interact with humans, efficient and functional multisensory fusion is an area of primary concern. Yet it is an area of research which has received relatively little attention to date, other than for a specific problem solution for a robot faced with two pieces of information and a quick decision to make e.g. (Engels & Schoner 1995, Tyrrell 1994, Foresti & Regazzoni 2002, Jassemi-Zourgani & Necsulescu 2002).

In particular, in any one biological system, operative sensors can be extremely different in their characteristics, they can operate for example with different time constants and even focus on different aspects of the environment. As has been pointed out (Howard 1998) sensors can operate with respect to each other, in a series, nested or parallel fashion, in particular with the possibility of being arranged in a ”multi-cue system” in order to provide information regarding the same event. Such information may either be redundant (competitive) or complementary. Combining such complementary information can then be employed as a necessary part of an overall control mechanism, as indeed appears to be the case with retina stabilization in humans (Robinson 1977) in which high frequency components of head rotation, sensed by the vestibular system, are mixed with low frequency optokinetic signals. Some attempts at multisensory integration (e.g. (Boss, et al. 2001)) have fused multisensory signals, in a theoretical environment by quite simply taking a weighted average of the two inputs involved. This has the advantage of dealing with redundancy when one input proves to be problematic or even becomes dysfunctional however it is far removed from biological reality (van Beers, et al. 99) and generally such occurrences can be dealt with by a straightforward hierarchical decision mechanism as that in the human vision system (Wolfe 1994). Indeed it is acknowledge in (Boss et al. 2001) that a straightforward averaging mechanism can only work if it is, as they put it, supported by a ”book keeping” stage, which has the ability to reconfigure the averaging process accordingly! Clearly what is needed is a more firmly based theoretical approach of sensor fusion that exhibits biological plausibility whilst proving to be practically feasible and operable in a robotic environment. It is, we believe, that which we are suggesting and striving for here.

Evidence suggests (Stein & Meredith 1993) that neurons in the superior colliculus of the brain form a sort of sensory map of the surroundings based on a combination of auditory and visual information. An overlap appears to exist between the neurons involved, i.e. a high proportion of them are truly multisensory. As a result it is often the case, when stimuli of different modality are complementary, that the overall neural response is larger than the sum of the uni-modal responses. Conversely where competitive instances occur overall neural activity can even decrease. One interesting feature is that the response enhancement is strongest when complementary signals are weak and is less apparent as such signals become uni-modally stronger.

It is with regard to the biological observations discussed that we wish to aim for a more plausible and practically operable overall sensor fusion approach. Whilst taking inspiration from the biological world to some extent, it is not felt to be a sensible approach, at this time, simply to throw further biologically inspired techniques at the problem and hope that it all "comes out in the wash", such would be the case with neural networks. What we need to gain, given the problem in hand, is a deeper understanding of what possibilities there are and to technically assess what is possible.

1.1 Framework, Layout and Presentation Way

In this paper we consider non-linear-filtering framework (Jazwinski 1970, Anderson & Moore 1979) with which to tackle the multisensory convergence problem. Applicability of this conceptually plausible approach depends strongly on availability of such observation and time-evolution models that can efficiently processed. Finite mixtures of uni-modal probability density functions (pdfs) (Titterington, et al. 1985, McLachlan & Peel 2001) seem to be adequate. It seems to be both biologically plausible (Berthouse & Kunniyoshi 1998) and has been successfully applied in audiovisual object tracking (Beal & Jojic 2003). Still the presented results depend too much on the specific problem and give a weak guide how to solve other types of sensor convergence.

The advocated approach is a relatively straightforward elaboration of stochastic filtering theory. Two seemingly different ways were originally inspected (i) joint state modelling of sensors and (ii) global modelling of the observed environment. At the end we found that the second option results in a specific version of the first one. It is demonstrated on intentionally simplified case in section 2. It shows that the state-space formulation has the added advantage of mathematically taking into account the correlations that exist between different sensors, directly in the formation of the state vector. Subsequently it is the separate state vector elements that we wish to fuse, rather than highly correlated, noise ridden, inaccurate measured sensor values.

The general discussion is in section 3. It is shown that computational obstacles caused by complexity of general stochastic filtering – that performs formally optimal sensor fusion – can be to significant extent suppressed. The key step is modelling of local sensors by probabilistic mixtures with local components and a flat background component. It leads naturally to the use of finite mixtures (Titterington et al. 1985, Quinn, et al. 2003) for description of global estimate, too. Feasible treatment of both data updating and time updating of estimates of high-dimensional field of physical states of the environment state are proposed.

The proposed theory will be tested on 5 sensory robot head that is described in section 4. The head can be goal directed, based on its multisensory input and can be given simple task, for instance, the head is to remain a fixed distance away from a (potentially moving) object which it can track through the variety of sensors at its disposal. The foreseen tests with it are briefly commented in section 5

1.2 Notation

In the exposition, the following notation is used: $\equiv$ definition by equality; a^* set of values of variable a; $a^{\mathcal{C}}$ complement of the set a^*, $f(a|b)$ probability density function (pdf) of a conditioned on b; $\propto$ proportionality; $t \in t^* \equiv \{1, 2, \ldots, \mathring{t}\}$ discrete time, always placed as the last subscript; q position in (possibly phase) space q^*; $s \in s^*$ sensor indices; $q^*_{s;t}$ observation range of the sth sensor at time t; $x_{q;t}$ physical state of the observed environment at position q and time t (a finite dimensional vector); $x_t(q^*)$ collection of values $x_{q;t},\ q \in q^*$, at time t; $x(t, q^*)$ collection of values $x_{q;\tau},\ q \in q^*$, at time moments $\tau \leq t$; $y_{s;t}$ (vector) output of the sensor $s \in s^*$, at time t; $u_{s;t}$ (vector) exogenous input to the sensor $s \in s^*$, at time t.

2 State-Space Description of Sensing

This section introduces gently into the problem and outlines the structure of the solution. Multiple sensors operating in high dimensions and their application within a robotics environment are discussed subsequently.

We consider two separate and potentially different sensors, labelled by $s \in s^* \equiv \{1, \mathring{s} \equiv 2\}$ reporting

on the same scalar physical quantity x_t at discrete time $t \in t^* \equiv \{1, \dots, \mathring{t}\}$ at the same spatial position $q \in q^*$. Outputs of sensors $y(\mathring{s}, t) \equiv [y_{1;1}, \dots, y_{1;t}, y_{2;1}, \dots, y_{2;t}]$ observed till time t should be converted into estimate $\hat{x}_{t|t}$ of the physical state x_t. Design of the estimator processing outputs of several sensors is referred to as the multisensory convergence problem (Meredith 2002) or sensor fusion.

Models of sensors relate their outputs $y_t(\mathring{s})$ to the physical state x_t. Here, linear static models are assumed:

$$y_{s;t} = h_s x_t + e_{ys;t}, \; s \in s^* \Leftrightarrow y_t(\mathring{s}) = [h_1, h_2] x_t + e_{y;t}(\mathring{s}), \tag{1}$$

where the known coefficient h_s characterize sensors and unobserved $e_{y;t}$ is white measurement noise.

Time evolution of the physical quantity is modelled by linear model:

$$x_t = a x_{t-1} + e_{x;t} \tag{2}$$

with a known coefficient a (degenerate version of state matrix) and white process noise $e_{y;t}$.

The estimate $\hat{x}_{t|t}$ is gained through a filter that fusses all available measurements into the posterior pdf $f(x_t|\mathcal{P}_t)$, $\mathcal{P}_t \equiv y(\mathring{s}, t)$. This pdf describes belief into possible values of the estimated x_t. Its mean or mode serves as the point estimate $\hat{x}_{t|t}$ usually searched for and variance determines confidence of this estimate.

Under appropriate conditions, the conditional pdf has a fixed functional form determined by a finite-dimensional information state V_t that evolves according to the state equation of the fussing filter, (Jazwinski 1970):

$$V_{t|t} = \mathcal{V}_1(V_{t|t-1}, y_t(\mathring{s})), \;\; V_{t+1|t} = \mathcal{V}_2(V_{t|t}) \tag{3}$$

and generates the desired estimate through output equation of the filter:

$$\hat{x}_{t|t} = \mathcal{X}(V_{t|t}). \tag{4}$$

The functions determining the state evolution $\mathcal{V}(\cdot) \equiv [\mathcal{V}_1(\cdot), \mathcal{V}_2(\cdot)]$ and output of the filter $\mathcal{X}(\cdot)$ are determined by models of sensors (1), by the model of the physical state evolution (2) and by the definition of the state estimate required. Considering whiteness and normality of the involved noises, the conditional pdfs preserves normal form. Its moments evolves according to the celebrated Kalman filter, (Jazwinski 1970):

Data updating

$$\begin{aligned} \mathcal{V}_1(\cdot): \quad \hat{x}_{t|t} &= \hat{x}_{t|t-1} + \big(y_t(\mathring{s}) - h\hat{x}_{t|t-1}\big) \underbrace{h' \left(h' h P_{t|t-1} + \mathrm{cov}[e_{y;t}(\mathring{s})]\right)^{-1}}_{\text{Kalman gain} \equiv g_{t|t}} \\ P_{t|t} &= (1 - g_{t|t} h') P_{t|t-1} \end{aligned} \tag{5}$$

Time updating

$$\begin{aligned} \mathcal{V}_2(\cdot): \quad \hat{x}_{t+1|t} &= a\hat{x}_{t|t} \\ P_{t+1|t} &= a^2 P_{t|t} + \mathrm{var}(e_{x;t}). \end{aligned}$$

The function $\mathcal{X}$ just selects $\hat{x}_{t|t}$ as the required estimate. The sufficient statistics are the conditional mean $\hat{x}_{t|\tau}$ and variance $P_{t|\tau}$ of $f(x_t|\mathcal{P}_\tau)$, $\mathcal{P}_\tau \equiv y(\mathring{s}, \tau))$, $\tau \in \{t-1, t\}$ and $'$ is transposition. The two-dimensional Kalman gain $g_{t|t}$ is deterministic function of noise characteristics and takes into account possible correlations between involved noises. The second term in the first equation of (5) shows how the sensors outputs are fused in this case. It is worth stressing that the evolved information state $V_{t|\tau} \equiv [\hat{x}_{t|\tau}, P_{t|\tau}]$ describes uni-modal pdf that can be interpreted as activity distribution of the fused sensors. Such a uni-modal pdf has biological interpretation as a target object being mapped by means of a retinotopical projection (Boss et al. 2001).

In our case however the pdf results from operational characteristics of sensors and considered evolution of the observed physical state. A similar view point can be found in biologically inspired neural representations of sensing stimuli. They exhibit extended receptive fields, effectively blurring the physical state on the sensor input. In essence it is assumed that an activity at a location q in a sensory input drives the state vector through a bell shaped distribution.

The above filtering structure can be also described in terms of observers (Warwick 1987) and extended to non-linear sensor and time-evolution models and multi-dimensional case (Jazwinski 1970). Computational feasibility of the resulting filter is decisive for applicability of any such extension. The subsequent section provide a relatively general solution while preserving the simple reasoning structure outlined above.

3 Probabilistic Mixing Sensory Data

References and results of section 2 indicate the expected structure of fusing sensory data. This section tries to embed the problem into a probabilistic framework that allows us to see the problem structure, to specify more precisely the involved elements and operations and propose a general solution.

At each $q \in q^* \equiv$ *position (possibly phase) space* and at each time instant t, the inspected environment is characterized by a finite-dimensional state vector $x_{q;t}$ of *physical quantities*. They are observed by several sensors. Their data serve for estimation of the state field $x_t(q^*)$ over whole space q^*. The estimation is formulated and solved as stochastic filtering recalled in section 3.1. Sensor models are discussed in subsection 3.2. Data updating of the physical-state-vector estimates by single sensor is discussed in subsection 3.3. Joint data updating that fuses data of several sensors into a common estimate of physical state is presented in subsection 3.4. Then, models of sensors belonging to tractable dynamic exponential family (DEF) (Barndorff-Nielsen 1978) are discussed with their use for data updating of conjugate prior pdf, subsection 3.5. Dimensionality problem present even in this nice family of models is addressed through mixture-based re-parameterization presented in subsection 3.6. Global estimate of the physical states $x(q^*)$ over whole considered environment is discussed in subsection 3.7 where its data updating is also discussed. Time evolution of these estimates is resolved in subsection 3.8 under simplifying assumption of slow temporal changes of physical state of the environment. This assumption is justified by a relatively high rate with which the sensors can inform us on the current state of the environment. The section is concluded by summarizing the overall fusion (filtering) algorithm, subsection 3.9. Questionable steps are discussed here, too.

3.1 Stochastic filtering

Let $X_t \equiv x_t(q^*)$ be a state to be estimated using measurement data $D_t \equiv d_t(\mathring{s}) \equiv [d_{1;t}, \ldots, d_{\mathring{s};t}]$, where data $d_{s;t}$ related to s-th sensor consist of the measured sensor output $y_{s;t}$ and possibly of external sensor inputs $u_{s;t}$.

Let us assume that *observation model* $f(D_t|\mathcal{P}_{t-1}, X_t)$ and *time evolution model* $f(X_{t+1}|\mathcal{P}_t, X_t)$ are at disposal. The symbols $\mathcal{P}_{t-1},\ \mathcal{P}_t$ used in these conditional pdfs, denote the information processed up to and including time moments $t-1,\ t$, respectively.

Stochastic filtering updates posterior pdf $f(X_t|\mathcal{P}_t)$ of the unknown state X_t. It is described by the coupled formulas, (Jazwinski 1970, Peterka 1981) (Kalman filter discussed in section 2 is its special version):

$$\text{Data updating} \quad f(X_t|\mathcal{P}_t) \propto f(D_t|\mathcal{P}_{t-1}, X_t) f(X_t|\mathcal{P}_{t-1}) \tag{6}$$

$$\text{Time updating} \quad f(X_{t+1}|\mathcal{P}_t) = \int f(X_{t+1}|\mathcal{P}_t, X_t) f(X_t|\mathcal{P}_t)\, dX_t. \tag{7}$$

The recursion starts with the externally supplied prior pdf $f(X_1|\mathcal{P}_0)$.

Remarks

1. Filtering processes optimally outputs of all sensors, i.e. fusses them optimally. Optimality means that all available information about the estimated state is exploited.
2. Specific models and approximations used in the functional recursion (6), (7) decide on the applicability.
3. Any reasonable characteristic of $f(X_t|\mathcal{P}_t)$, e.g. mean or mode, can be selected as a point estimate of X_t.
4. A pair of time indices occur in (6), (7). One refers to the estimated variable X_t, the other one to the information $\mathcal{P}_{t-1}$ or $\mathcal{P}_t$ included into the condition. In order to stress it, we shall use the subscript $t|t-1$ or $t|t$ at time positions whenever needed.
5. Recursive form of filtering implies that the posterior pdf $f(X_t|\mathcal{P}_{t-1})$ serves as a prior one for the time t.

3.2 Models of Sensors

Sensors are man-made imprecise devices of a known structure. Output $y_{s;t}$ of the sensor $s \in s^* \equiv \{1, \ldots, \mathring{s}\}$ at time t is corrupted by sensor dynamics, its non-linearity and measurement noise. General description of the sensor at a specific position is given by the conditional pdf $f(y_{s;t}|y_s(t-1), x(t, q^*))$.

Sensing is always to some extent *local*. It reflects some entries of physical state x of the environment distributed only on a subset $q^*_{s;t} \subset q^*$ of the space q^*. The subset varies with variations of sensor positions.

Sensing dynamics can be modelled by a dependence of the current sensor output $y_{s;t}$ on *regression vector* $\psi_{s;t}$ consisting of several delayed sensor outputs $y_{s;\tau}, \tau = t-1, t-2, \ldots, t-\partial_y$ and (possibly) on current and delayed *exogenous sensor input* $u_{s;\tau}, \tau = t, t-1, \ldots, t-\partial_u$. The exogenous sensor input is supposed to meet

natural conditions of control, NCC, (Peterka 1981), i.e. it uses at most the information about the environment state contained in the measured data. Thus, the sensor approximately exhibits the Markov property:

$$f(y_{s;t}|u_s(t), y_s(t-1), x(t,q^*)) = f(y_{s;t}|\psi_{s;t}, x_t(q^*_{s;t})). \tag{8}$$

The pdf (8) is either implied directly by the construction of the sensor or can be gained through the Bayesian estimation of its parameters (Peterka 1981).

3.3 Exploitation of Sensor Reading

Given a prior pdf on the environment state $f(x_t(q^*)|\mathcal{P}_{t-1})$, the measurement of s-th sensor refreshes it to:

$$f(x_t(q^*)|\mathcal{P}_{s;t}) \propto f(y_{s;t}|\psi_{s;t}, x_t(q^*_{s;t}))f(x_t(q^*)|\mathcal{P}_{t-1}) \tag{9}$$

where $\mathcal{P}_{s;t} \equiv (d_{s;t}, \mathcal{P}_{t-1}) \equiv (y_{s;t}, u_{s;t}, \mathcal{P}_{t-1})$ is the information $\mathcal{P}_{t-1}$, acquired up to time $t-1$ from all sensors, updated by s-th sensor data only. The relation (9), performing a version of data updating (6), is just a plain Bayes rule valid under NCC.

Decomposing $q^* = q^*_{s;t} \cup q^{\mathcal{C}}_{s;t}$, with the *complement of* $q^*_{s;t}$ defined $q^{\mathcal{C}}_{s;t} \equiv q^* \setminus q^*_{s;t}$, and using chain rule, we can re-write (9) into the form:

$$\begin{aligned} &f\left(x_t\left(q^{\mathcal{C}}_{s;t}\right)|x_t(q^*_{s;t}), \mathcal{P}_{s;t}\right) f(x_t(q^*_{s;t})|\mathcal{P}_{s;t}) \propto \\ &\propto f(y_{s;t}|\psi_{s;t}, x_t(q^*_{s;t}))f\left(x_t\left(q^{\mathcal{C}}_{s;t}\right)|x_t(q^*_{s;t}), \mathcal{P}_{t-1}\right) f(x_t(q^*_{s;t})|\mathcal{P}_{t-1}). \end{aligned} \tag{10}$$

Integrating (10) over $x_t\left(q^{\mathcal{C}}_{s;t}\right)$, we get:

$$f(x_t(q^*_{s;t})|\mathcal{P}_{s;t}) \propto f(y_{s;t}|\psi_{s;t}, x_t(q^*_{s;t}))f(x_t(q^*_{s;t})|\mathcal{P}_{t-1}), \tag{11}$$

i.e. the information about state in the neighborhood $q^*_{s;t}$ is updated by the Bayes rule irrespectively of the information on the state on its complement $q^{\mathcal{C}}_{s;t}$. By inserting this result into (10), we see that:

$$f\left(x_t\left(q^{\mathcal{C}}_{s;t}\right)|x_t(q^*_{s;t}), \mathcal{P}_{s;t}\right) = f\left(x_t\left(q^{\mathcal{C}}_{s;t}\right)|x_t(q^*_{s;t}), \mathcal{P}_{t-1}\right), \tag{12}$$

i.e. this conditional pdf of x_t on the complement $q^{\mathcal{C}}_{s;t}$ of the set $q^*_{s;t}$ is unchanged by the sensor output $y_{s;t}$.

This intuitively appealing result is practically important as the information change caused by the local sensor measurement causes a local change of $f(x(q^*)|\mathcal{P}_{t-1})$, which is extremely high-dimensional pdf defined over the environment states in all positions $q \in q^*$.

3.4 Fusion of Sensor Readings

Probabilistic fusion of several sensor reading is conceptually straightforward if their common model $f(y_t(\mathring{s})|y(\mathring{s}, t-1), u(\mathring{s}, t), x(t, q^*))$ is available. Sensors are, however, independently constructed devices whose outputs are correlated just due the overlapping observations of the environment physical state, see Figure 1. Thus, *without loss of generality*, their outputs can be assumed to be *conditionally independent*, i.e.:

$$\begin{aligned} f(y_t(\mathring{s})|y(\mathring{s}, t-1), u(\mathring{s}; t), x(t, q^*)) &= \prod_{s \in s^*} f(y_{s;t}|y_s(t-1), u_s(t), x(t, q^*)) \underbrace{\equiv}_{(8)} \\ &\equiv \prod_{s \in s^*} f(y_{s;t}|\psi_{s;t}, x_t(q^*_{s;t})). \end{aligned} \tag{13}$$

With this joint description of sensors, the fusion of their readings reduces simply to a straightforward application of the Bayes rule:

$$\begin{aligned} f(x_t(q^*)|\mathcal{P}_t) &\propto f(y_t(\mathring{s})|y(\mathring{s}, t-1), u(\mathring{s}, t), x_t(q^*))f(x_t(q^*)|\mathcal{P}_{t-1}) = \\ &= \prod_{s \in s^*} f(y_{s;t}|\psi_{s;t}, x_t(q^*_{s;t}))f(x_t(q^*)|\mathcal{P}_{t-1}). \end{aligned}$$

Arguments similar to that for (12) imply that the high-dimensional prior pdf $f(x_t(q^*)|\mathcal{P}_{t-1})$ is modified on:

$$x\left(q^*_t(\mathring{s})\right) \equiv x\left(\cup_{s \in s^*} q^*_{s;t}\right) \tag{14}$$

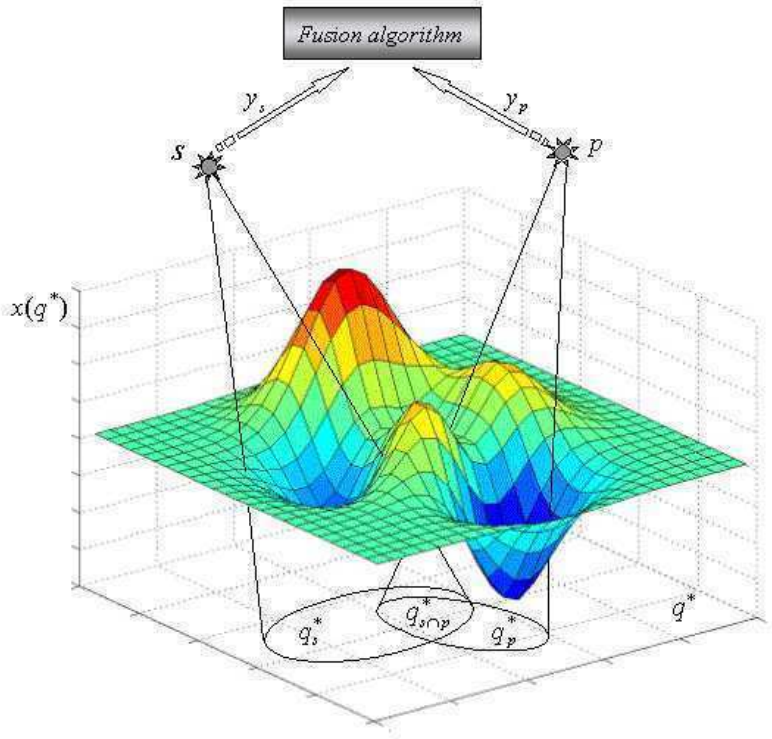

Figure 1. Multi-sensors with overlapping observed positions.

Remarks

1. The marginalization of the global pdf $f(x_t(q^*)|\mathcal{P}_{t-1})$ is the most time consuming operation. It need not be done unless sensor positions are changed. The remaining operations are computationally cheap as they work on – always very localized – position sets q^*_s defined by the used sensors.
2. The outlined data updating maps $f(x_t(q^*)|\mathcal{P}_{t-1})$ on $f(x_t(q^*)|\mathcal{P}_t)$. It is necessary to specify the time-updating mapping

   $$f(x_t(q^*)|\mathcal{P}_t) \to f(x_{t+1}(q^*)|\mathcal{P}_t)$$

   in order to complete the learning recursion. Generally, it needs the model of time evaluation $f(x_{t+1}(q^*)|x_t(q^*),\mathcal{P}_t)$, see (7). It is hard to get it even in much simpler situations and the non-linear stochastic filtering, subsection 3.1, is solvable in rare cases (Daum 1988). Thus, at the considered level of generality, we have to restrict ourselves to them or to the case slowly varying values of $x_t(q^*)$ and apply a version of forgetting (Kulhavý & Zarrop 1993). The latter version is adopted and discussed below.

3.5 Sensor Models in Dynamic Exponential Family

Inherent problem dimensionality forces us to approximate sensor models by pdfs in *dynamic exponential family (DEF)*. The parameterized model of a sensor (8) belongs to DEF iff it is described by the formula:

$$f(y_{s;t}|\psi_{s;t},x_t(q^*_{s;t})) = \exp\left\langle B_s(\Psi_{s;t}), H_sC(x_t(q^*_{s;t}))\right\rangle,\ s\in s^*,\ \text{where} \tag{15}$$

data vector $\Psi_{s;t}\equiv[y_{s;t},\psi_{s;t}]$ = [sensor output,regression vector];
the data vector $\Psi_{s;t}$ can be updated recursively, i.e. there is a function $\tilde{\Psi}_s$ such that $\Psi_{s;t}=\tilde{\Psi}_s(d_{s;t},\Psi_{s;t-1})$ with the *data item* $d_{s;t}\equiv[y_{s;t},u_{s;t}]$ =[sensor output, exogenous sensor input];
H_s is a fixed *linear selector* of x_q entries that are reflected in outputs of s-th sensor;
$\left\langle B_s(\Psi_{s;t}), H_sC(x_t(q^*_{s;t}))\right\rangle$ is a scalar product (a bilinear functional) of arrays $B_s(\cdot)$, $H_sC(\cdot)$, for which there is a *conjugated linear selector* $\bar{H}_s$ such that

$$\left\langle B_s(\Psi_{s;t})\bar{H}_s, C(x_t(q^*_{s;t})\right\rangle = \left\langle B_s(\Psi_{s;t}), H_sC(x(q^*_{s;t}))\right\rangle,$$

finite-dimensional functions $B_s(\cdot), H_sC(\cdot)$ of respective arguments are arranged into arrays of compatible dimensions,

the array $C(x(q^*_{s;t}))$ has always a constant (unit) entry: this allows us to include normalizing factor of the pdf as an entry of B.

Existence of *self-reproducing prior pdf*, conjugated with the model of the sensor in DEF, determined by a finite-dimensional sufficient statistics, explains its practical significance. DEF allows, moreover, to update the statistics recursively.

Under the assumption (15), the conjugated prior pdf has the form, see. (14):

$$f(x_t(q^*_t(\mathring{s}))|\mathcal{P}_{t-1}) \propto \prod_{s\in s^*} \exp\left\langle V_{s;t|t-1}, H_sC(x_t(q^*_{s;t}))\right\rangle f\left(x\left(q^{\mathcal{C}}_t(\mathring{s})\right)|x\left(q^*_t(\mathring{s})\right), \mathcal{P}_{t-1}\right), \tag{16}$$

where $V_{s;t|t-1}$ are individual sufficient statistics defining this prior pdf. The non-updated pdf on the complement $q^{\mathcal{C}}_t(\mathring{s}) \equiv q^* \setminus q^*_t(\mathring{s})$ can be formally arbitrary. Practically, it has to be carefully chosen to allow practical implementation of the full learning cycle consisting of data and time updating of the global estimate $f(x_t(q^*)|\mathcal{P}_{t-1})$. In any case, the updating by data from given sensors reduces to algebraic recursions:

$$V_{s;t|t} = V_{s;t|t-1} + B_s(\Psi_{s;t})\bar{H}_s,\ s \in s^* \tag{17}$$

and the posterior pdf $f(x_t(q^*)|\mathcal{P}_t)$ has the form (16) with $V_{s;t|t}$ replacing $V_{s;t|t-1}$.

3.6 Re-Parameterization of Sensor Models

The presented straightforward extension of the known data updating in DEF has significant drawback in the sensor convergence problem: the union of the spaces $q^*_t(\mathring{s})$ observed by sensors evolves with time. A correct determination of the conjugated prior pdf on this set requires computer intensive manipulations with the joint pdf $f(x_{t+1}(q^*)|\mathcal{P}_t)$. Its even approximate description has to be very detailed and consequently extremely high-dimensional. Otherwise, it is impossible to store fine "traces" of sensing made in past. The re-parameterization of the sensor models, proposed here, avoids this problem to a substantial degree. At the same time, the proposed solution seems to be both general and flexible enough.

Essentially, we re-parameterize sensor models so that they work formally on whole time-invariant space q^*. An individual sensor is modelled by a mixture of the sharp component $f_s(y_{s;t}|\psi_{s;t}, x_t(q^*))$ in DEF, that has practical support on $x(q^*_{s;t})$, and of a flat $\mathcal{U}(x(q^*))$ component (uniform when q^* is finite):

$$\begin{aligned} f(y_{s;t}|\psi_{s;t}, x(q^*)) &= \alpha_s f_s(y_{s;t}|\psi_{s;t}, x_t(q^*)) + (1-\alpha_s)\mathcal{U}(x(q^*)) \equiv \\ &\equiv \alpha_s \prod_{q\in q^*} \exp\left\langle B_s(\Psi_{s;t}), H_sC(x_{q;t})\right\rangle + (1-\alpha_s)\mathcal{U}(x(q^*)). \end{aligned} \tag{18}$$

The function $C(x_{q;t})$ of the local state $x_{q;t}$ is chosen as sensor independent. The sensor specificity is assumed to be concentrated into the selector H_s. The probabilistic weight $\alpha_s \in (0,1)$ is known (estimated) characteristics of the sensor chosen so that on the set $q^*_{s;t}$ the contribution of $\alpha_s f_s(y_{s;t}|\psi_{s;t}, x_t(q^*))$ is higher than that of $(1-\alpha_s)\mathcal{U}(x(q^*))$ and vice versa on the complement $q^{\mathcal{C}}_{s;t}$.

Remarks

1. The mixture models of sensors (18) are considered further on. Of course, specific physical knowledge may lead to other, often simpler, types of models.
2. Construction of the sensors often guarantees that the locality of sensing, reflected in data updating (11), (12), is approximately met when using the model (18) in Bayes rule.
3. Physical states $x_{q;t}$ considered on the whole space q^* are assumed to enter the model through $\sum_{q\in q^*} C(x_{q;t})$. This assumption can be generalized to $\sum_{q\in q^*} \beta_q C(x_{q;t})$, with fixed weights β_q. These weights are supposed to reflect dependence of physical properties on sensor position q. The states $x_{q;t}$ enter the model with weights respecting physical properties of the observed environment. Presentation simplicity makes us to consider the version without weighting.

3.7 Global Estimate and its Data Updating

The overall state space $x(q^*)$ is extremely large and cannot be efficiently described in an exact point-wise manner. At the same time, the dimensionality implies that the number of areas with interesting configurations

of physical states is very limited. Moreover, these areas represent as a rule "objects" with a physical meaning and occupy connected parts of the geometrical space. These parts can be further split into a finite amount of subsets on which physical states have very similar values at the inspected time moment.

Let us suppose, that each region with similar physical states is described by so called *component*. It implies that the joint prior pdf can be approximated by a finite mixture with (relatively sharp) components formed by pdfs conjugated to DEF and by a flat (uniform) pdf:

$$f(x_t(q^*)|\mathcal{P}_{t-1}) = \sum_{c=1}^{\mathring{c}_{t|t-1}} \alpha_{c;t|t-1} \prod_{q\in q^*} \exp\left\langle V_{c;t|t-1}, C(x_{q;t})\right\rangle + \alpha_{0;t|t-1}\mathcal{U}(x(q^*)). \tag{19}$$

The finite number of components $\mathring{c}_{t|t-1}+1$ may vary with time. The probabilistic weights $\alpha_{c;t|t-1}$ sum to unity. The values of statistics $V_{c;t|t-1}$ characterize individual components and specify their practical domain on which they are higher than the flat pdf describing unlearnt background of physical states.

Application of the Bayes rule to this prior pdf with the mixture models of sensors (18) gives, see. (6):

$$f(x(q^*)|\mathcal{P}_t) \propto \prod_{s\in s^*}\left[\alpha_s \prod_{q\in q^*} \exp\left\langle B_s(\Psi_{s;t}), H_s C(x_{q;t})\right\rangle + (1-\alpha_s)\mathcal{U}(x(q^*))\right] \times \tag{20}$$

$$\times \left[\sum_{c=1}^{\mathring{c}_{t|t-1}} \alpha_{c;t|t-1} \prod_{q\in q^*} \exp\left\langle V_{c;t|t-1}, C(x_{q;t})\right\rangle + \alpha_{0;t|t-1}\mathcal{U}(x(q^*))\right] \equiv$$

$$\equiv \sum_{c=1}^{\mathring{\tilde{c}}_{t|t}} \tilde{\alpha}_{c;t|t} \prod_{q\in q^*} \exp\left\langle \tilde{V}_{c;t|t}, C(x_{q;t})\right\rangle + \tilde{\alpha}_{0;t|t}\mathcal{U}(x(q^*)). \tag{21}$$

Thus, for the uniform background, the mixture form (19)) reproduces. The algebraic updating of statistics:

$$\left\{\alpha_{c;t|t-1}, V_{c;t|t-1}\right\}_{c=1}^{\mathring{c}_{t|t-1}} \underbrace{\rightarrow}_{\{B_s(\Psi_{s;t},H_s,\alpha_s\}_{s\in s^*}\}} \left\{\tilde{\alpha}_{c;t|t}, \tilde{V}_{c;t|t}\right\}_{c=1}^{\mathring{\tilde{c}}_{t|t}}$$

is just needed to perform the exact data updating of the global estimate. However, the number of terms $\mathring{\tilde{c}}_{t|t}$ is much higher than $\mathring{c}_{t|t-1}$ and the growth of the number of components must be limited by a suitable projection:

$$\tilde{\alpha}_{0;t|t},\ \left\{\tilde{\alpha}_{c;t|t}, \tilde{V}_{c;t|t}\right\}_{c=1}^{\mathring{\tilde{c}}_{t|t}} \rightarrow \alpha_{0;t|t},\ \left\{\alpha_{c;t|t}, V_{c;t|t}\right\}_{c=1}^{\mathring{c}_{t|t}} \quad \text{with } \mathring{c}_{t|t} = \mathring{c}_{t|t-1}.$$

There is a range of ways how to construct such a projection. The following one seems to be adequate.

Let us denote $\tilde{f}_{t|t}(x(q^*))$ the exact posterior pdf and $f_{t|t}(x(q^*))$ the constructed projection. We would like to select this projection as the minimizer of the Kullback-Leibler (KL) divergence $\mathcal{D}(\tilde{f}_{t|t}||f_{t|t}) \equiv \int \tilde{f}_{t|t}(\bullet)\ln\left(\tilde{f}_{t|t}(\bullet)/f_{t|t}(\bullet)\right) d\bullet$ (Kullback & Leibler 1951) that is known to be a good measure of proximity of pdfs. The choice is motivated by the specific role of this divergence in the considered Bayesian framework (Berec & Kárný 1997). The mixture forms of , $\tilde{f}_{t|t}, f_{t|t}$ imply that this projection cannot be practically found. Instead of it, we use the upper bound on the KL divergence implied by Jensen inequality. Its minimizer is, however, a model with a single component. In order to prevent this degeneracy, we also require proximity of the weights $\alpha_{t|t-1}$ and $\alpha_{t|t}$. Thus, we construct $\alpha_{0;t|t},\ \left\{\alpha_{c;t|t}, V_{c;t|t}\right\}_{c=1}^{\mathring{c}_{t|t}}$ as minimizing argument of:

$$-\sum_{c=1}^{\mathring{c}_{t|t}}\left[\alpha_{c;t|t}\left\langle V_{c;t|t}, \int \tilde{f}(x(q^*))\sum_{q\in q^*} C(x_q)\,dx(q^*)\right\rangle - \Lambda\alpha_{c;t|t-1}\ln(\alpha_{c;t|t})\right] + \alpha_{0;t|t} - \Lambda\alpha_{0;t|t-1}\ln(\alpha_{0;t|t}). \tag{22}$$

The optional weight $\Lambda > 0$ defines the degree of conservatism with respect to similarity of $\alpha_{t|t-1}$ and $\alpha_{t|t}$.

3.8 Time Updating of the Global Estimate

As recalled above, the time updating of the global pdf $f(x_t(q^*)|\mathcal{P}_t) \rightarrow f(x_{t+1}(q^*)|\mathcal{P}_t)$ requires unavailable time evolution model $f(x_{t+1}(q^*)|\mathcal{P}_t, x_t(q^*))$. This, in conjunction with the expected intensive informational flow from sensors, makes us to apply stabilized forgetting (Kulhavý & Zarrop 1993) on $f_{t|t}(\cdot)$. Again, mixture

form of the model prevent us to use it directly. Instead of it, we apply this forgetting component-wise. It gives the final correction of statistics

$$V_{c;t+1|t} = \lambda V_{c;t|t} + (1-\lambda)V_A.$$

The optional $\lambda \in (0,1)$ can be interpreted as the probability of the hypothesis that $x_t(q^*)$ remained constant between two consecutive measurements. The externally supplied statistic V_A describes our belief into moves of the $x_t(q^*)$ expected in the same time interval. Typically, V_A is recommended to coincide with the statistics describing the flat pdf. For bounded q^*, the flat pdf can be uniform and $V_A = 0$.

3.9 Overall Algorithm

Initial phase

- Specify the statistic V_A describing flat prior pdf $\mathcal{U}(x(q^*))$.
- Set the number of component $\mathring{c}_{1|0} = 1$ in mixture describing prior global estimate of physical state $f(x_1(q^*)|\mathcal{P}_0) \propto 1$ and initialize the corresponding statistic $V_{1;1|0} = V_A$.
- Specify structure of models of used sensors (18), i.e. their forms including selectors H_s, regression vectors $\psi_{s;t}$ and weights α_s defining mixture form of sensor models.
- Fill initial values into regression vectors of respective sensors.
- Choose the optional scalar $\Lambda > 0$ driving the mixture projection.
- Specify the upper bound $\mathring{c}$ on the number of components $\mathring{c}_{t|t}$ of the mixture describing the global estimate $f(x_t(q^*)|\mathcal{P}_{t-1})$.
- Choose the forgetting factor $\lambda \in (0,1)$ reflecting the expected rate of changes of physical states $x_t(q^*)$.

On line phase running for $t \in t^*$

- Apply external inputs $u_{s;t}$ to respective sensors (if present) demanding, for instance, to turn direction of sensing to the hottest point in space found up to now.
- Fix positions of sensors.
- Complete specification of those sensor-model characteristics that depend on their current position.
- Measure sensors outputs $y_{s;t}$ and complete data vectors $\Psi_{s;t} = [y_{s;t}, \psi_{s;t}]$.

Data updating % Fusion of Sensor Readings

- Update statistics $V_{c;t|t-1}$, $c = 1,\ldots,\mathring{c}_{t|t-1}$ to $\tilde{V}_{c;t|t}$, $c = 1,\ldots,\tilde{\mathring{c}}_{t|t}$ using values of functions $B_s(\Psi_{s;t})$, H_s.
- Update weights $\alpha_{c;t|t-1}$, $c = 0,1,\ldots,\mathring{c}_{t|t-1}$ to $\tilde{\alpha}_{c;t|t}$, $c = 0,1,\ldots,\tilde{\mathring{c}}_{t|t}$ using values of functions $B_s(\Psi_{s;t})$, H_s, α_s and normalization to their unit sum.

Projection

- Set $V_{c;t|t} = \tilde{V}_{c;t|t}$, $\alpha_{c;t|t} = \tilde{\alpha}_{c;t|t}$, $\mathring{c}_{t|t} = \tilde{\mathring{c}}_{t|t}$ if $\tilde{\mathring{c}}_{t|t} \leq \mathring{c}$ and go to **Time updating by forgetting**.
- Project $\tilde{V}_{c;t|t}$, $\tilde{\alpha}_{c;t|t}$ on $V_{c;t|t}$, $\alpha_{c;t|t}$ by minimization of (22) if $\tilde{\mathring{c}}_{t|t} > \mathring{c}$ and continue.

Time updating by forgetting

- Forget by defining $V_{c;t+1|t} = \lambda V_{c;t|t} + (1-\lambda)V_A$.
- Evaluate point estimates of state, if need be, and go to the beginning of On line phase.

Remarks

1. Extent of prior options determines to significant extent usefulness of algorithm. Let us discuss them
   (a) The choice of statistics V_A describing flat background and prior pdf is inevitable and relatively simple. Properties of the algorithm are expected to be reasonably insensitive to this choice.
   (b) Forgetting factor is relatively easy to choose and the algorithm will be robust to its choice.
   (c) The weighting factor $\Lambda > 0$ is introduced in heuristic way and there is no experience with its choice and influence. *Its choice can be almost surely avoided by approximating joint pdf of data and pointers to components*. Such a pdf is known as an extension that has the mixture model as its marginal. Similar "trick" can be applied when introducing forgetting.
   (d) The upper bound on the number of components $\mathring{c}$ should be chosen as high as computationally acceptable. Its value is determined by computation resources available and by desired sampling rate.
2. Computational demands of the algorithm depend on dimensionality of the problem (length of x_t, number of sensors and dimensions of their outputs and number of components in the global estimate). Preliminary

considerations and the particular case reported in (Beal & Jojic 2003) indicate that the computational load can be acceptable in a wide range of applications.

3. Updating of statistic $V_{c;t|t-1} \rightarrow \tilde{V}_{c;t|t}$ in the dominant case of normal components is equivalent to single step of recursive least squares (Peterka 1981). As such it is computationally relatively cheap.
4. The considered projection performs merging and cancelling components at one shot. Its computational complexity is expected to be around that needed for data updating. Simpler versions of projections are at disposal if this load will be found to high.

4 Robot Platform Considered for Tests

A special robot head has been constructed as a test bed for multi-sensor integration and convergence. The head has been positioned as a hand on the end of a 6 Degrees of Freedom Industrial Robot Manipulator Arm. As a result the head can take up any position and attitude in respect to an object in 3 dimensional space.

The robot head for experimentation (see Figure 2) has been named MORGUI, which is Mandarin Chinese for Magic (mor) Ghost (gui). It consists of a rapid action head containing 5-senses. Two of these, vision and audio, are human equivalents, whilst the remainder radar, infrared and ultrasonics are extrasensory as far as humans are concerned.

Morgui's camera vision can be either fixed stereo (i.e. the eyes cannot be separately rotated) or mono, with one camera being employed for sensory purposes within the robot, whilst the other is used for monitoring, observation and recording of (possibly human) object responses. Audio meanwhile consists of two receivers positioned on either side of Morgui's head, positioned a little forward from the possibly expected (human ear) location in order to reduce noise effects at central positions.

Ultrasonic sensors located in Morgui's forehead give a broad object appearance signal at a fairly accurate range, i.e. the sensor output give a reasonably accurate ($\pm$ 2%) indication of the distance to the nearest object of a reasonable size. In Morgui's nose is a Doppler radar system, which gives an accurate indication of the speed of movement of an object, to or from the robot head. Finally, positioned just above Morgui's top lip is a dual infrared sensory system, which can give a reasonable thermionic read out from a specified distance away or, more likely, can be used to give an alternative distance measure from the robot head to the hottest nearby object. In the sense of a distance measure the ultrasonic and infrared sensors can be deemed to be mutually corresponding in that they are indicating different views on the same measurement – distance from the robot head. Further views on the same measure can also be obtained through a depth calculation using the stereovision system, if the magnification of object audio output is known and if the object moves, through radar. For our own studies we will initially consider input on distance from the robot head through signals taken in via the ultrasonic and infrared sensors.

The Control problem with Morgui is one that with the robot being given a straightforward single goal, it is this goal only that it aims to achieve at all times. Decision making between multiple goals through competing actors for control of the robot (as in (Breazeal 2000)) is not the subject of study here. The critical factor in our case is multisensory correspondence, enabling the robot to achieve its single goal with the aid of diverse sensory information.

A general directive for Morgui could be to retain its head position in uniform 3 dimensional space with regard to a distant object, i.e. to always retain a specified distant object in the center of the robot's sensory system, with the object remaining a set, pre-chosen, distance away. The aim being that Morgui focuses on, and tracks in real-time, a distant, specific moving objects. To restrict the goal, in the first instance to something a little simpler – we can merely give Morgui the task of keeping the distance, in the z-direction, between itself and an object, constant with respect to time. The aim is therefore for Morgui's head position to remain a set distance from an object as it moves in z-space only.

It is worth noting that the control algorithm employed with Morgui, for target tracking, is not of major concern here, in fact a PI control will invariably suffice. The only requirement is for the algorithm to ensure that the robot responds in a reasonable time frame (with regard to the object's speed) and with reasonable accuracy. Our concern in foreseen tests is in essence not about the type of control to be employed, but rather that whichever controller it is, the accuracy of the information it has at its disposal is improved through the use of multisensory convergence.

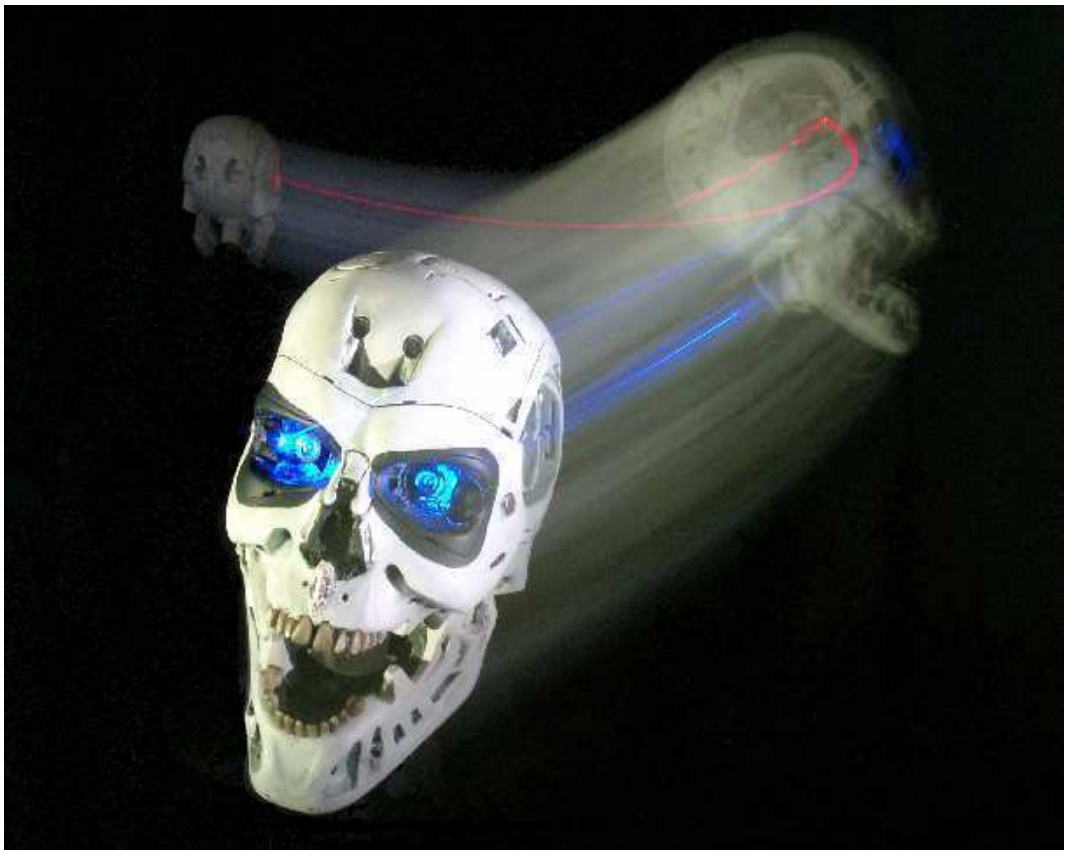

Figure 2. Morgui – Multisensory Robot Head in Action

5 Concluding Remarks

This preliminary inspection of the general but applicable framework of sensor fusion has been started from two distinctly different schools of research, namely state-space analysis of sensors as dynamic system and stochastic filtering reduced to Bayesian estimation of mixtures combined with advanced forgetting techniques. The paper presents a promising but still preliminary synthesis of this attempt to tackle the general sensor fusion problem. The current state of the research outlines further steps:

- Modelling of Morgui's sensors in the form described in section 3.5.
- Elaboration of algorithmic details of global fusion algorithm summarized in section 3.9.
- Implementation of the algorithm and extensive tests on the robot head test bed that should help to refine and if need be to modify the described approach.
- Modification and possible adaptation of the theory behind to the discussed area.

This is an ambitious plan but the framework presented in the paper promises that it is feasible. The generic nature of the solution indicates that it will be applicable not only to Murgui's head.

Acknowledgments

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Computationally Intensive Methods in Bayesian Model-Structure Identification

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Abstract. An important sub-problem of multiple participant decision-making is selecting of these participants who influence given process by their measured outputs. We propose to use the Bayesian model-structure identification based on maximizing the likelihood function. Already known model-structure identification method is based on repeated random restarts in the space of structures which result in local maxima of the likelihood function. The size of structure space grows exponentially with dimension of the model. Therefore it is not possible to search through the structure space completely and global maxima is searched for only with the given probability. The paper also propose improvement of the way that structure space is searched through by avoiding repeated searches.

Keywords: structure identification, system identification, structure estimation

1 Introduction

We are assuming that each participant has its set of outputs (which are measured) and we are studying the way how these outputs influence one common measured variable. Based on that, only selected participants are used in future decision-making process. This may cut-off necessity to communicate with other participants.

Assumed model for the way how participant's outputs influence the common output is linear regression model. Outputs of participants are called regressors, and the common variable is called regressand. Our aim is to find subset of regressors, that influence regressand, based on data measured. This task is equivalent to searching for the most optimal structure of linear regression model. considered model is discrete-time, which means that all measured variables are measured in discrete time instants $t = 1, 2, \ldots, T$.

The choice of the most optimal structure for modeling of given data is the vital problem in System Identification as described in (Peterka 1981). Various aspects of this problem were covered in (Kárný 1983), (Kárný & Kulhavý 1988), and (Kárný, et al. 1995). The method proposed in these papers is based on these ideas (structure identification is described in Section 2.1):

1. Data are used to calculate likelihood for given structure.
2. Likelihoods are used to judge between structures. The goal is to find the structure with maximal likelihood.
3. A very clever algorithm is used to calculate likelihoods for different structures without touching data again and again.
4. The term "surrounding" is introduced in the discrete structure space in order to have topology in this space. Moreover, likelihood function is easily enumerated in such surrounding.
5. Having topology in structure space, we are able to speak about local and global maxima of the given function, support of which is the whole structure space.
6. The structure space is searched through, using the method of random restarts, continuing towards the local maxima, each time selecting maximal value of likelihood in surrounding of current point in the structure space.
7. After sufficient number of random restarts, the structure space is considered covered and the biggest local maxima is announced to be global maxima pointing to the most optimal structure for given data.

The last idea of the method as originally proposed does not specify the stopping rule for the whole algorithm. Another paper (Tesař, et al. 2003) specifies the stopping by quantification of the proportion of structure space which was already searched through. This adds another point to above mentioned:

8. The stopping rule for random restarts is based on the desired proportion of structure space that needs to be searched through.

The stopping rule is described in Section 2.2.

This paper proposes another the new idea in order to improve the search through the structure space:

9. Blind repeated random restarts often lead to the structure that was already searched and time is lost in seeking the local maxima, from the point where the process of search was already started. The proposal is to make them less blind by trying to remember points that were already in search and to avoid such points in later searches. The problem of this approach is in time consumption of the operation of searching for the structure in the memory. Proposal is to use hash functions or trees in order to reduce time needed for structure space search. The proposed algorithm is described in Section 3. The hash function typically randomly maps a big space into a much smaller space. Hash functions were first suggested in (Luhn 1953) and published in (Dumey 1956). The basic book on practical usage of hash functions is (Knuth 1973).

2 Notation

Notation used in this paper follows.

Symbol	Meaning
$\sim$	random variable distributed as pdf on right-hand side
	equality by definition
x^*	a set of all possible values of x
x	the number of elements in a finite set x^*

The probability density functions (pdfs) are distinguished by the identifiers in their arguments. No formal distinction is made between random variable, its realization and an argument of a pdf. The correct meaning follows from the context. The elementary properties of pdfs are only used, see e.g. (Peterka 1981).

2.1 Structure identification algorithm

Modeled variable (also called regressand) in time t is denoted by y_t. This variable have to be modeled by data (column) vector d_t, which is vector of input and output channels and delayed inputs and outputs in time t. This data vector (also called regressand) have to include all possible channels which could be used for modeling of regressand variable. Let dimension of d_t is n.

Model is given by formula:

$$y_t = \quad S(s)d_t + e_t \tag{1}$$

where $e_t \sim \mathcal{N}(0, \sigma)$. Variable denote parameter vector (line vector) belonging to structure s. Vector $s = s_1, s_2, \ldots, s_n$ denotes yet unknown structure of the model. If $s_i = 1$, than ith element of regressor d_t belongs to the structure, if $s_i = 0$, then it does not. Matrix $S(s)$ (with element $(S_{i,j})$, have to be created from s, so that $S_{i,j} = 1$, if s_j is ith nonzero element of vector s, and $S_{i,j} = 0$ otherwise.

We are trying to find the most optimal structure s, for given data d_1 to d_t and y_1 to y_t. Likelihood function $L(s)$ is the function of data and structure. The way of calculating likelihood function is explained in (Kárný 1983), (Kárný & Kulhavý 1988), and (Kárný et al. 1995) and is not important for this paper.

For every element $s \in s^*$ we define the set called, surrounding of the element s like this:

$$U(s) = \{x \in s^*; \exists \text{ at most one } i \in \{1, 2, \ldots, n\} : x_i \neq s_i\} \tag{2}$$

As was already mentioned above, if $L(s)$ is already calculated, for every $x \in U(s)$ it is fast (algorithm of n^2 type) to calculate $L(x)$.

The element $m \in s^*$ is called local maximum, if $\forall x \in s^* : L(x) \quad L(m)$.

The problem of structure identification in space s^* is reduced now to searching for global maximum of function L in s^*.

The basic algorithm for structure identification described in (Kárný 1983) and (Kárný & Kulhavý 1988) is as follows:

1. Select random element $s \in s^*$.
2. Select $x \in U(s)$ with maximal $L(x)$.
3. If $L(x) > L(s)$, $s := x$ (i.e. put value of x into s), and go to Step 2. Otherwise continue by the next step.
4. s is local maximum and we are continuing by Step 1 until most of the set s^* is covered.
5. Global maxima is found by maximization from all elements collected in previous step.

2.2 Stopping rule of structure identification

An important improvement of structure identification algorithm described in Section 2.1 is introduction of the stopping rule applied in Step 4 of the algorithm.

The stopping rule described in paper (Tesař et al. 2003) uses these variables:

Number of unique local maxima found in Step 4 of structure identification algorithm. We denote this variable μ_t
Number of restarts in Step 1, denoted by t
k is the parameter of the model for prior distribution of number of local maxima.
is desired probability that all local maxima were found.

The stopping rule:

$$\frac{t \quad k \quad 1}{\mu_t + t \quad 1} \tag{3}$$

gives the estimation of the moment to finish algorithm described in Section 2.1.

3 Improving optimal structure search using structure memory

Due to the blind local maxima search in Step 3, what is very often happening is the duplicated search for the same local maxima again and again. The situation is shown in Figure 1. This happens whenever we start

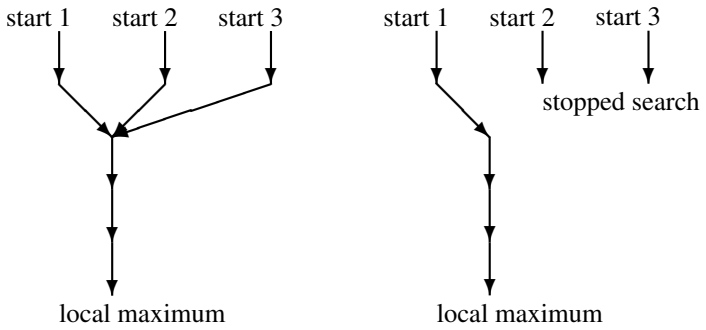

Figure 1. Figure of the undesirable convergence of the two branches and consequently repeated calculations. On the right-hand side, the search is stopped in order not to repeat the search

hitting the same local maxima repeatedly. This happens quite often at the end of the global maxima search. It is no longer necessary to continue the search after hitting the first element of the set s^*, which was already visited. In order to achieve this, the memory can be used to remember already visited elements (structures). Unfortunately, searching through such memory would be very computationally intensive and the time needed to consider whether to continue would be growing. Consequently, the time would be rather wasted than saved. It is why, two possible ways to store visited members were considered:

Using trees would reduce time needed to search through the memory proportional to $\log q$, where q is number of structures already visited. Also, number of computations needed to make one record to the memory would be proportional to $\log q$.

Using hash functions is very successful, as it reduces time for searching and storing almost to zero.

The principle of hash function method is that structures are randomly distributed in memory array based on the deterministic function (called hash function). Collisions may occur, which are dealt with by a few different methods. This results in almost zero search times.

Let's denote hash function's bits n_1 to n_2 by $H(s, n_1, n_2)$. Because it is not known in advance how much of memory will be needed for hash functions, the memory array it will be allocated in leveled structure with a fixed size of block. We can use for example size of block 512, then we need 9 bits to address it. Every time, we need to add a new structure s, we need to calculate Hash function $H(s, 1, 9)$. According to this we are going to select the place in top-most level of the memory array. If the place is already occupied, we are going to allocate new block of the size 512 and using $H(s, 10, 18)$, we find a place to the second level.

If such array needs to be searched through, parts of hash function, like $H(s, 1, 9)$, $H(s, 10, 18)$, are calculated until appropriate place is found and decided, whether the structure was already visited.

4 Illustrative examples

The first example is to illustrate possibilities of structure estimation method for multiple participant decision making and to evaluate the sensitivity of the method by Monte-Carlo simulation. The second example compares speed of the original algorithm with proposed improvement.

4.1 Sensitivity test

The model selected for this example has 21 regressors, denoted x_1 to x_{21}. They are all independent random variables with Gaussian distribution $\mathcal{N}(0, 1)$. Regressand y is random variable with the model:

$$y = \sum_{i=1}^{21} p_i x_i + \varepsilon \tag{4}$$

Random variable ε has Gaussian distribution $\mathcal{N}(0, 1)$. Sensitivity of correct estimate of structure on the magnitude of nonzero parameters is tested. Eight of parameters p_i were selected to be equal to the value k rest was set to zero. Figure 2 shows dependence of percentage of correct structure estimations on the value of k. We can see, that for all values of k bigger than 0.07 the chance of finding wrong structure is almost zero.

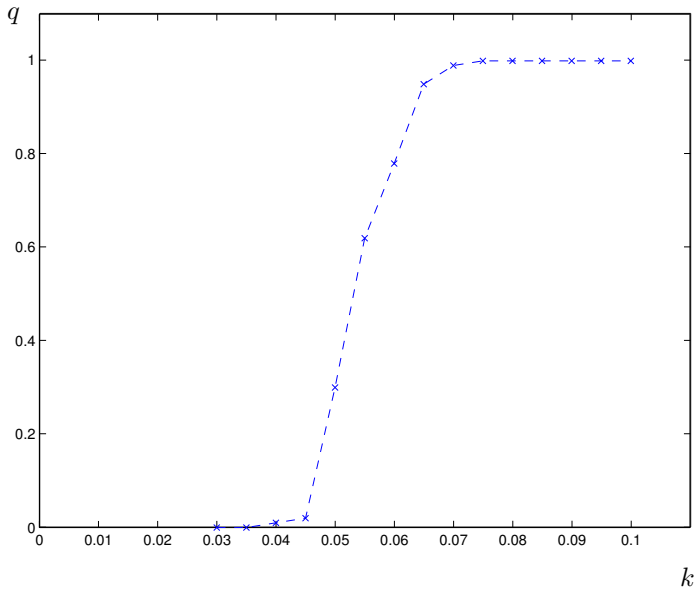

Figure 2. Result of Monte-Carlo simulation. q is the percentage(proportion) of correct structure estimates, k is value of independent variable

It has to be said that each point of graph in Figure 2 consist of 100 simulations. Each of these simulations consist of generating of 10000 individual data vectors $(x_1, x_2, \ldots, x_{21})$.

4.2 Speed Comparison

Average improvement in computation time between pure structure estimation algorithm and the algorithm with improvement from section 3. the same example as in section 4.1 was used.

The value of k used was 0.08.

The time needed for finding the correct structure (on the same machine) was 0.88 for original algorithm versus 0.79 for improved algorithm. It is not very big increase in speed, but every such increase is improvement.

5 Conclusions

The proposed method reliably selects only those participants who best describe given variable. It complements other methods in multiple-participant decision making in a way that only few participants may be selected which are used in future decision-making process, which eliminates expensive communication and coordination with other participants.

Also, an improvement of structure identification algorithm was proposed. It proposes to use structure memory in order to avoid continuous repeating of the search of the same local maxima, which is not necessary to repeat. It slightly speeds-up the algorithm for structure-estimation.

Acknowledgments

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Agent-based Computational Economics in Power Markets – Multi-agent Based Simulation as a Tool for Decision Support*

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Abstract. The aim of this research in progress is to examine the effect of CO_2-emission trading and the intensified application of renewable energy sources to the liberalized power market. The research approach applies methods from a combination of economics, natural sciences and computer science in order to obtain decision support for the planning of future power plant structures, investment decisions, and emission scales.

Besides the liberalization, future power markets are increasingly influenced by regulatory acts which are mainly responding to the necessity of a structured reduction of CO_2-emissions. For CO_2-eminssions, which are generated by the production of electric power, an emission trading system with obligatory participation is being launched from 2005 on. Since in the same period, the amount of power which has to be produced from renewable energy sources is increased dramatically, an additional incoming aspect is the fluctuating character of many of those sources.

These changed general conditions in the industrial power sector cause novel structures at international power markets and additionally a new emissions situation. The aim of this research is the conception of a methodologically innovative concept to simulate power markets based on multi-agent systems (MAS) and agent-based computational economics (ACE) which fulfils the described desiderata.

Since heterogeneous utility functions for individual market participants can be modelled – in comparison to existing centrally steered simulation methods – a multi-agent based simulation approach promises more realistic simulation results. Specific behaviour and the dynamic adoption of the strategies of individual participants based on the experiences derived from market behaviour can explicitly be simulated. The aim of the research approach is to obtain a strengthened understanding of the mutual effects among heterogeneous participants and the dynamics of the market of CO_2-emissions as well as renewable energy resources. To disseminate these results, acting recommendations for participants at international power markets as well as political and regulatory authorities are derived.

Keywords: Power Markets, Renewable Energy, CO_2-Emission Trading, Agent-based Computational Economics (ACE), Multi-Agent Systems (MAS), Simulation.

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1 Introduction and Aims of Research

1.1 Motivation

Motivated by the possible consequences of a global temperature increase which is partly based on anthropogenic generated climate relevant greenhouse gases climate protection is currently in the center of the discussion in environmental policy. At international level, United Nations have committed to a reduction of six mayor greenhouse gases[1] of 8 % until the timeline of 2008 – 2012 compared to emission level of 1990. These targets were formulated at the United Nations Framework Convention on Climate Change (UNFCCC) in Rio de Janeiro (1992) as well as the protocol accepted at the third conference of the parties in Kyoto (COP3). Within the so called EU burden sharing these 8 % (about 340 Bill. t of CO_2 equivalents) were allocated to the individual countries of the EU. An actual status report (Gugele and Ritter 2001) shows that most EU countries as well as the over-all EU are far away from reaching this target.

Recently, based on actual emission developments in many European countries, the European Commission released a green book concerning an Europe wide harmonized trade of emission certificates. Based on this green book, a suggestion for an EU directive and its implementation has been worked out. Although strongly discussed, EU Environment Ministers Council released the implementation plan for emission certificate trading in early December 2002 (w.a. 2002) which is planned to take effect in the year 2005. By the induced reduction of a good which was free until now, the possibility to emit greenhouse gases into atmosphere reaches – from economic point of view – the status of a production factor. From this reduction of a formerly free product a pricing process is induced. The aim of this pricing is the adoption of planning processes within energy intense domains. These do not only provide implications on CO_2-emissions but also on other air pollutant[2]. However, strengthened by certificate trading, local emissions may – compared to a CO_2-emission reduction without certificate trading – drastically rise.

In case future climate politically motivated changes at the electricity market are regarded, fluctuating and renewable energy sources like wind and potovoltarics will play a key role. Under the assumption that these energies are especially supported in future years, different prognoses find that until 2025 in Germany wind energy plants with a capacity of 30 to 50 GW could be installed. This is equivalent to about 50% of the entirely installed capacity of electricity production in Germany. The effects of this development towards the electricity market regarding the power plant structures (amongst others backup plants, regular energy amounts, transport and distribution power lines, power prices) and the effects on CO_2-emission certificate trading are very strong (Ehrhart, Hoppe et al. 2003). Furthermore, the question rises, how the subsidization of renewable energies can be combined with a system of emission certificate trading.

Methods from Operations Research (OR) and System Analytics are often applied in order to analyze the impact of climate political instruments on the planning and investment decisions of the concerned market participants such as energy supply companies, energy intense industrial branches or investors of wind farms. However, in this analysis the behaviours of the market participants on a competitive market are modelled only partial since alternative objectives of the participants are neglected.

Hence, from this discussion the necessity rises to develop methodologically novel approaches for market simulation of the European power market. This is in order to model the decision processes of all groups participating in the market more detailed and in this way to quantify the effects of the implementation of a Europe wide emission trading as well as the large-scale input of fluctuating energy resources in a better way.

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1 Carbon dioxide (CO_2), methane (CH_4), nitrous oxide (N_2O), partly halogenated chlorofluorocarbons (H-FKW), perfluorinated hydrocarbons (FKW) and sulfur hexafluoride (SF_6).

2 Based on the increased usage of gas, CO_2-reduction mostly entails a significant lowering of SO_2 emissions as well as a moderate reduction of NOx emissions.

1.2 Objectives

In order to achieve CO_2 reduction obligations e.g. according to the EU burden sharing targets substantial changes in the energy systems towards innovative and efficient production technologies are necessary. However, such a conversion process demands strategic decisions concerning innovations, which are connected with entrepreneurial risks. Besides market liberalization the necessities of a reduction of CO_2-emissions and the expansion of the use of renewable energies cause two more restrictions, which have to be considered in investment planning in the energy sector. Different participants will react with different strategies in taking risks and developing technical innovations under the modified framework conditions. However, the resulting participant-specific behaviour will be vital for the conversion of the energy systems. The different participants will influence each other and will modify their strategies continuously.

Due to the fact that so-called multi-agent systems are suited to model markets with the characteristics mentioned above it is the objective of this research project to simulate the development of the European electricity market based on this new modelling concept. With the help of a multi-agent system the effects of the liberalised energy market, the European emission trading scheme and the integration of fluctuating energy sources will be analysed. The multi-agent system shall be able to evaluate the European emission trading scheme with regard to its ecological efficiency and ecological effectiveness. Furthermore, conclusions about adjustments of the European emission trading scheme and the promotion of renewable energies will be drawn.

These objectives will be achieved with the help of a multi-agent system to simulate the liberalised electricity market in Germany considering the relevant neighbour-states for a time horizon between 20 and 30 years The different market participants will be modelled as autonomous units (autonomous software agents), which operate and communicate autonomously within the electricity market. Whereas the modelling of conventional technologies is state-of-the-art in optimising energy models, the problem of modelling plants with fluctuating energy production (like wind energy converters) is not yet solved. Therefore another objective of this research project is to integrate these plants into the simulation model. Thereby the exact characterisation as well as the modelling and prediction of fluctuations of renewable energies is of crucial importance.

2 Methodologies

2.1 Decision Support by Using Energy Models

For the elaboration of decision support in the energy sector, optimizing energy models have been subsequently designed and applied since the 1970s. These models can be differentiated into bottom-up models and top-down models (w.a. 1999). The main advantage of bottom-up models is the possibility to depict individual technologies. On the other hand, bottom-up models have the disadvantage that they neglect the economic cycle of production and usage of goods and services. Thus, effects on economic growth and employment cannot be modelled. Inverse advantages and disadvantages can be found in top-down models.

Bottom-up models are based on a detailed representation of energy conversion technologies and the interconnecting flows of energy (i. e. electricity and heat) and material (i. e. primary energy carriers, emissions of pollutants and greenhouse gases). The complete energy sector – starting from the resources via several energy-conversion steps up to the supply of final energy – is modelled in a consistent approach. Technologies and flows are characterised by technical (e. g. efficiency, lifetime), economic (e. g. investment, fixed and variable costs) and environmental (emission factors) parameters. Emissions resulting from electricity and heat generation as well as from distribution of energy carriers (e. g. natural gas) are calculated. Restrictions on the resulting emissions can be imposed on individual or cumulated emission levels. By comparing model results in scenarios with and without emission ceilings emission reduction strategies can be elaborated. The models usually employ a mixed-integer linear programming approach (see e.g. the PERSEUS model or the EUDIS model (Kreuzberg 1999, Wietschel, Fichtner et al. 1999)) or a system dynamics approach (see e.g. (Grobbel 1999)).

In bottom-up models, possible technical reduction options are represented, making it possible to consider the interdependencies between individual measures. For example, the cost-efficiency of the rational use of

electricity depends on the structure of the electricity supply system. If structural changes of the system (e. g. due to a switch from coal to gas or renewable energy carriers) lead to half the CO_2-emission factor for end use electricity, the emission reduction caused by the rational electricity use project will be halved.

In the context of this research project the advantages of optimizing energy models - with their all-encompassing target function negating alternative goals for different market players - will be combined with the advantages of multi-agent systems – with their agents acting autonomously.

2.2 Multi-Agent Systems and Simulation

A software agent or autonomous agent in the sense of the research community of Distributed Artificial Intelligence (DAI) is a piece of software that represents a participant and acts in its place. Software agents provide the following capabilities in full or in part. See also (Ferber 1999, Weiss 1999).

- Relation towards a function: i.e. they act within a specific domain.
- Locality (knowledge, data): i.e. they consider a specific situation
- Social behaviour: i.e. they communicate with other users and other agents.
- Autonomy: i.e. they autonomously search for negotiation counterparts and other agents.
- Reactivity on their environment: i.e. they monitor their environment and derive certain actions.
- Rationality: i.e. they maximize an individual utility function or preference rule.
- Adaptivity: i.e. they are able to adapt to new circumstances and learn from their results.
- (Mobility: i.e. they are able to move to other servers and continue working there.)

A set of two or more software agents are called multi-agent system. Based on above listed properties, the agents within a MAS provide the following properties: Interaction with their environment, autonomous acting, reactivity towards other agents, proactiveness, social behaviour in understanding and communication, rationality as well as adaptivity. These properties allow a modelling of distributed problem solving processes in a realistic way. The paradigm of multi-agent systems, originally stemming from computer science and DAI is applied now increasingly in economic research (Jennings and Wooldridge 1998, Kirn 2002, Veit 2003).

Based in its scalability, the anatomy of the individual software agents as well as the flexibility in the generation of role profiles, multi-agent systems are predestined to model and simulate market based coordination problems. Here, role profiles are defined based on a classification of market participants into groups, where each group obtains a specific role. Since several years, multi-agent systems are applied to model real markets as well as to simulate markets and future market developments. Two prominent platforms have been established on which this topic is addressed: The Workshop und Agent-based Simulation (Urban 2000, Urban 2001) and the Workshop on Multi-agent Based Simulation (Sichman, Conte et al. 1998, Moss and Davidsson 2001), bi-yearly held at the conference on Autonomous Agents and Multi-agent Systems (AAMAS). First approaches and models for the simulation of the German and English power markets exist (Bower, Bunn et al. 2000, Bunn and Oliveira 2001). These approaches solely focus the short-term horizon – long-term simulations have not been launched so far. In these approaches, the market situation under changed general conditions since the deregulation is addressed. In case of the German market, a simulation model has been developed. In the case of the English market a simulation model as well as an agent-based simulation environment. In the papers cited above, a simple reinforcement-learning approach is applied, using that the agents learn how to trade under consideration of auction rules. The insights from these approaches are included into the conception of the simulation model which will be implemented during this research initiative.

The application of multi-agent systems in economic research has lead to the novel and promising discipline of Agent-based Computational Economics (ACE). Within this discipline, a bottom-up approach to simulation economic markets is chosen. In contrast to the models of conventional simulation (e.g. system dynamics), in which participants are modeled in an aggregated way (top-down), participants are treated individually. This procedure enables a more realistic simulation.

By designing such a model a self evolving dynamic system is defined, which allows better insight into the processes and the results of strategic decisions in the power sector (Edmonds 2001, Tesfatsion 2003).

As shown in the mentioned work as well as within the development in different research areas in economics and computer science, the paradigm of multi-agent systems is very well suited for the design and the

implementation of simulation systems. Especially deregulated power markets provide properties, which can be modeled using MAS in a neat way. Of course a set of assumption about the behavior of market participants has to be made. However, such a simulation system hast to be calibrated and validated using past data from reality. After this realistic setup of the system, future decisions can be simulated and their market outcome can be compared. These results can be used as acting recommendations for strategic decisions of certain participants as well as political decision makers.

2.3 Modeling of Fluctuating Energy Sources

We will model fluctuating renewable energy sources using suitable stochastic processes. Methods applied stem from the domains of statistics, signal processing and time series analysis.

Using recorded weather data (wind velocity, radiation data), time series of the electrical power generation are simulated by means of known plant characteristics and subsequently aggregated to load curves in typical production regions based on geographical plant distribution. The multivariate data set thus generated can be studied using conventional statistical methods. These include among others the analysis of average values, variance, probability distributions, linear and cross correlations as well as of power spectra.

Analyzing the predictability of these data using data-driven models has particular significance since this determines how the rest of the power generation system is able to cope with fluctuations. In order to determine the predictability, different models such as multivariate autoregressive models, neural networks, phase space methods are adapted to the data and their predictive accuracy is determined. From this examination, the fluctuations observed can be divided into deterministic and stochastic ones. Furthermore, the proportion of non-linear fluctuations in the data can be ascertained.

3 The Principle Design of the Market Simulation Model

Alongside energy-supply side participants such as public utilities, operators of plants using renewable energies and energy traders, the aim is also to model households and small commercial users as well as industrial companies via demand-side agents. Furthermore, regulatory authorities can be incorporated into the model as neutral agents. In the following more details are given based on the example of the different software agents who represent the various planning departments of an electric power company.

The investment planner in the electric power company, who is modelled as an autonomous software agent, analyses the expansion decisions of his company for a longer-term period (e.g. 20 years). To do so he refers, on the one hand, to exogenously given factors such as the current legal requirements and information about the already existing capacities of competitors. On top of this, market parameters are also entered into the decision process, such as the current electricity price, available to the investment planner as endogenous factors and the result of previous activity rounds.

Based on this information, forecasts are drawn up by the investment planner for the primary energy prices, the energy demand and finally the electricity prices which are necessary for planning expansion decisions. The (de)investments in power stations suggested by the investment planner are passed on to management and are evaluated by them based on the strategy they have selected. The information presented is weighted in such a way as to take the risk behaviour portrayed into account.

The short-term planner’s job is offer amounts of electricity on the electricity market. To do so he makes decisions using the management's strategy about how much electricity he should offer at which price at any one time on the market as well as under which conditions bilateral electricity contracts should be offered. Considering the hourly demand the market clearing will be achieved by the proper dispatching of the available power plants. The software agent can learn from his previous behaviour, i.e. he analyses which price/quantity strategies achieved the best results in past periods and then develops new trading strategies from these.

After these short-term market activities have been simulated for several months or after the frame conditions have changed, the investment planner gets actively involved again. The short-term planner and market intermediary provide the investment planner with information who then conducts his investment planning using this information and finally transmits the results of his plans to the management.

4 Expected Results

This new research approach of energy system modelling should result in fundamental insights into understanding market dynamics in the electricity sector after the introduction of CO_2-certificate trading and the increased use of renewable energies. In particular, the results can make an essential contribution to understanding the future development of air pollutants and greenhouse gases. Where today there is still a wall between energy models and policy-oriented analyses and ex-ante evaluations, this methodological approach ultimately offers the possibility of consistently and transparently simulating group-specific behavioural patterns and obstacles with energy policy measures.

Previous modelling concepts used to analyse electricity markets usually stem from the domain of operations research (OR). However, these modelling concepts have natural limits and the following shortcomings:

- Participant specific market strategies which go beyond simply maximising profits are not realisable or only to a limited extent.
- The interaction between electricity and certificate market is not or only insufficiently taken into account.
- Adaptations of the participants' strategies due to experience gained from success or failure on the market are not integrated.
- The interaction between strategic planning and the more short-term quantity-price strategies on the electricity market is not considered.
- The influence of expanding renewable energy sources on participant specific investment decisions is not sufficiently considered.
- Price formation based on marginal costs is frequently used which negates that profits are achievable by forming prices above marginal costs in certain market constellations or in certain market segments (such as households).
- The dynamics of the market due to the multitude of existing and new participants with their different functions is not depicted.
- The limited access of the participants to information is not considered.

Fundamental insights with reference to the current state of research are expected in the following areas:

- A more realistic picture of the structure of the electricity system in Germany for a planning horizon of 20 – 30 years depending on the prices for CO_2-certificate.
- Evaluation of CO_2 abatement possibilities among the various participants of the European Electricity Market.
- Development of regional pollutant emissions.
- The balancing power required for the inclusion of fluctuating energy forms.
- Forecasting the development of the energy demand.
- Composition of electricity production (share of nuclear, lignite etc.).

To summarize, it is expected that, based on flexible and agent-based simulation systems, a new method of simulation for the electricity market can be developed. Under application of this novel method it is expected that it will be possible to study the impacts of emission trading and increased usage of renewable energies on the market.

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Probabilistic mixture control with multimodal target

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Abstract. Fully probabilistic control combined with probabilistic mixture modelling has created a base for efficient solution of the multiobjective control problem. Paper formulates the problem in appropriate terms and proposes general solution of the multiobjective control design with both system model and target represented by finite probabilistic mixtures.

A complete feasible algorithmic solution for mixtures with components formed by normal auto-regression models with external variable is provided .

Keywords: Fully probabilistic control, Probabilistic mixture modelling, Multiobjective optimal control.

1 Introduction

Searching of reasonable compromise between different, often contradictory, demands and wishes attracts the attention of the ”decision-making” society of human beings starting with Adam and Eve. And from time to time the actuality of this very general problem, applied to particular areas, returns, becoming the topic of the day. A *multiobjective control problem*, being particular case of *multiobjective decision making*, is not an exception (Zadeh 1963), (Keeny & Raiffa 1978), (Yu 1985), (Bendoly & Bachrach 2003).

Recent developments in science technology has caused the growing complexity of control problems, accompanied by increasing multiple performance criteria that should be reached simultaneously. Besides, the most of the criteria are conflicting or competing. Despite a lot of approaches to multiobjective control developed (Salukvadze 1982), (Stadler 1988), (Toivonen 1984), (Toivonen 1989), (Liao & Li 2002), there is still a lack of systematic methodology guaranteeing satisfactory solution of the problem. Creating efficient numerical algorithms of general structure is much complicated by the high dimensionality and uncertainty of the modern processes to be controlled.

The approach advocated here belongs to multi-model framework (Murray-Smith & Johansen 1997) with the controlled system described by dynamic probabilistic mixture model (Kárný, et al. 2003). Historical process data, fully describing the closed loop system behaviour are processed by quasi-Bayes algorithm (Kárný, et al. 1998) to build the mixture model. The model obtained reflects all significant operational modes of the system, each associated with so-called *mixture component*, while their weights indicate the probability of occurrence of a particular mode component. The mixture description of a system is especially useful in cases of complex, large-scale systems, when the system behaviour exhibits several different modes, i.e when the system behaviour cannot be described by a linear model with fixed parameters. The underlining probabilistic extraction of information from process data does not require a detailed knowledge of system dynamics and thus supports the generality of the approach.

To design a control strategy, fully probabilistic design methodology (Kárný 1996) is adopted. This methodology gives interesting insight into the *probabilistic decision making problem*. The approach designs a strategy that minimises the distance of the joint probability density function (pdf) describing closed-loop system behaviour to a joint pdf describing desired closed-loop behaviour. Kullback-Liebler divergence (Kullback & Leibler 1951) serves as a measure of the closeness and thus as universal quality criterion.

Algorithmic solution proposed has been obtained for the *system model* and the *user ideal* in the form of *finite mixtures* of uni-modal pdfs. This corresponds to the situations frequently met in practice, when there exist: i) several requirements imposed on single data item; ii) different ideals given by different users (for instance, in technology).

Paper is organised as follows. Preparatory Section 1 introduces the probabilistic mixture modelling and basic elements of the fully probabilistic control design. The probabilistic mixture control with multimodal target is described in Section 3. Section 4 provides algorithmic solution of the control design applied to the normal ARX model. Sections 5 and 6 contain an illustrative example and concluding remarks, respectively.

2 Preliminaries

This Section introduces the notions used, recalls basic elements of mixture modelling and fully probabilistic mixture control.

2.1 Notions

Throughout the text, $x(t)$ stands for $(x_1, \ldots, x_t)$, x^* denotes set of possible values of a variable x and $\mathring{x}$ is the number of entries in x. The symbol $f(x_{i;t}),\ t \in t^*$, denotes probability (density) function (p(d)f) of the ith entry of quantity x at discrete time $t \in t^* \equiv \{1, \ldots, \mathring{t}\}$, where $\mathring{t} \leq \infty$. Time index follows the semicolon in the subscript when multiple indices are used. The symbol x' means transposition of x.

In the paper, Kullback-Leibler (KL) divergence (Kullback & Leibler 1951) is used as a measure of proximity of a pair of pdfs f, g acting on a common set x^*. Their Kullback-Leibler divergence $\mathcal{D}(f||g)$, with the basic property $\mathcal{D}(f||g) \geq 0$, $\mathcal{D}(f||g) = 0$ iff $f = g$ almost everywhere, is defined by the formula

$$\mathcal{D}(f||g) \equiv \int f(x) \ln\left(\frac{f(x)}{g(x)}\right) dx. \tag{1}$$

2.2 Probabilistic mixture model

The approach advocated employs probabilistic modelling for the system description. The modelling operates with joint pdf on uncertain system quantities considered. They are supposed to consist of: the observed controlled system output y_t; the directly manipulated system input u_t; unobserved unknown time-invariant parameter $\Theta \in \Theta^*$ and a random pointer to the active mixture component $c_t \in c^* \equiv \{1, \ldots, \mathring{c}\},\ \mathring{c} < \infty$.

The relation between the quantities involved is assumed to be (approximately) described by the mixture in the *component form*

$$f(d(\mathring{t}), c(\mathring{t}), \Theta) \equiv Di_\alpha(\kappa_0) \prod_{t \in t^*} f(d_t|\phi_{c_t;t-1}, \Theta_{c_t}, c_t)\alpha_{c_t} \prod_{c \in c^*} f(\Theta_c), \quad \text{where} \tag{2}$$

$f(d_t|\phi_{c_t;t-1}, \Theta_{c_t}, c_t)$ is parameterised component, usually represented by uni-modal pdf of *data item* $d_t = [y_t, u_t]'$; the number of components $\mathring{c}$ is assumed to be finite and fixed;

$\phi_{c_t;t-1}$ is observable *state of c_t-th component*; the phase form $\phi_{c;t-1} \equiv [d'_{t-1}, \ldots, d'_{t-\partial_c}, 1]'$ of the state is considered; finite fixed orders ∂_c are assumed;

α_{c_t} is constant probability of the pointer c_t, called *component weight*; their collection forms probabilistic vector $\alpha \in \alpha^* \equiv \left\{\alpha_c \geq 0,\ \sum_{c \in c^*} \alpha_c = 1\right\}$;

Θ is *mixture parameter* parameterising the model; it is formed of the component parameters and weights, i.e. $\Theta = \{\Theta_c, \alpha_c\}_{c \in c^*}$;

$f(\Theta_c)$ are prior pdfs of unknown $\Theta_c \in \Theta_c^*$ parameterising individual components $c \in c^*$;

$Di_\alpha(\kappa_0) \propto \prod_{c \in c^*} \alpha_c^{\kappa_{c;0}-1}$ is prior Dirichlet pdf (Fergusson 1973) of the component weights, determined by the vector statistic κ_0 with non-negative entries $\kappa_{c;0}$; α^* is its support; the symbol $\propto$ means that right hand side has to be normalised to the unit integral to get equality.

In the closed loop, the data items $d_t \equiv (d_{1;t}, \ldots, d_{\mathring{d};t})$ are multivariate, i.e. $\mathring{d} > 1$. Using the chain rule, the individual components can be decomposed into parameterised *factors*

$$f(d_t|\phi_{c_t;t-1}, \Theta_{c_t}, c_t) = \prod_{i=1}^{\mathring{d}} f(d_{i;t}|d_{i+1;t}, \ldots, d_{\mathring{d};t}, \phi_{c_t;t-1}, \Theta_{c_t}, c_t) \equiv \prod_{i=1}^{\mathring{d}} f(d_{i;t}|\psi_{ic_t;t}, \Theta_{ic_t}, c_t), \tag{3}$$

with *regression vectors* $\psi_{ic_t;t} = [d_{i+1;t}, \ldots, d_{\mathring{d};t}, \phi'_{c_t;t-1}]'$. The decomposition allows to model jointly scalar entries $d_{i;t}$, called *factor outputs*, and to parameterise factors individually. Moreover, only some parameters Θ_{ic} from Θ_c may be needed to describe the i-th factor in (3). The factorised description allows to treat factor outputs of a different nature, for instance, this enables joint modelling of mixed continuous and discrete data. To simplify the presentation, the model description at component level (2) is used below.

The model introduced is the *mixture model*. This becomes obvious when the unobserved pointers to components are excluded by marginalisation

$$f(d_t|d(t-1), \Theta) = f(d_t|\phi_{1;t-1}, \ldots, \phi_{\mathring{c};t-1}, \Theta) = \sum_{c_t \in c^*} \alpha_{c_t} f(d_t|\phi_{c_t;t-1}, \Theta_{c_t}, c_t). \tag{4}$$

The expression (4) extends usual finite mixtures as it consists of dynamic components. It is, however, still restricted to have constant component weights, i.e. $f(c_t = c|d(t-1),\Theta) = \alpha_c$. The assumption is used to get a feasible recursive estimation and can be weakened so that slow changes are admitted by employing stabilised forgetting (Kulhavý & Zarrop 1993). This extension is often sufficient as the pf $f(c_{t+1} = c|d(t),\Theta)$ forms bounded martingale with respect the -algebra generated by observations $d(t)$. Thus, it converges *almost surely (a.s.)* for $t \to \infty$, (Loeve 1962). The approximate recursive quasi-Bayes mixture estimation (Kárný et al. 1998) is used. The classic definition of probabilistic mixture and the variety of techniques of its estimation can be found in (Titterington, et al. 1985). Subsequent control design adopts certainty-equivalence strategy and is based on recursively estimated model. Thus, parameter Θ can be dropped in pdfs from here onwards.

2.3 Mixture probabilistic control

The joint pdf $f(d(\mathring{t}), c(\mathring{t})) \equiv f(y(\mathring{t}), u(\mathring{t}), c(\mathring{t}))$ characterising possible *closed loop behaviours* of the system and an input generator can be factorised in the following way:

$$f(d(\mathring{t}), c(\mathring{t})) = \prod_{t\in t^*} f(y_t|u_t, d(t-1), c_t) \prod_{t\in t^*} f(u_t|d(t-1), c_t) \prod_{t\in t^*} f(c_t|d(t-1)). \tag{5}$$

The factors $\{f(y_t|u_t, d(t-1), c_t)\}_{t\in t^*}$ are learned during the estimation and describe observable system reactions on the control actions u_t under the available experience reflected in the data $d(t-1)$ and for the fixed c_t. Pdfs $\{f(u_t|d(t-1), c_t)\}_{t\in t^*}$ describe models of the considered randomised control strategy. The last factors in (5) $\{f(c_t|d_{t-1})\}_{t\in t^*}$ represent probabilities of pointers c_t to the particular components that determine component weights of the mixture model (2).

Control strategy designed is intended to make the closed loop behaviour as close as possible to a desired behaviour, when respecting given restrictions. Under the adopted probabilistic modelling, a control strategy can be searched as minimiser of the KL divergence of the joint pdf (5) to a its pre-specified ideal counterpart ${}^I f(d(\mathring{t}), d(\mathring{t}))$. The last pdf is called *user's ideal* or *ideal* and describes the desired closed loop behaviour and restrictions given.

The methodology providing the solution of the problem is known as *fully probabilistic design* (Kárný 1996) and is recalled in the following agreement.

Agreement 1 (Fully probabilistic design) *The fully probabilistic design specifies its target through an ideal pdf*

$${}^I f(d(\mathring{t}), c(\mathring{t})) = \prod_{t\in t^*} {}^I f(d_t|c_t, d(t-1))\, {}^I f(c_t|d(t-1)) = \prod_{t\in t^*} {}^I f(y_t|u_t, c_t, d(t-1))\, {}^I f(u_t, c_t|d(t-1)).$$

The optimal strategy is selected among causal, possibly randomised, decision strategies $\{f(u_t, c_t|d(t-1)\}_{t\in t^*}$. *It is defined as a minimiser of the Kullback-Leibler divergence (1) of* $f(d(\mathring{t}), c(\mathring{t}))$ *to* ${}^I f(d(\mathring{t}), c(\mathring{t}))$

$$\mathcal{D}\left(f||\,{}^I f\right) \equiv \int f(d(\mathring{t}), c(\mathring{t})) \ln\left(\frac{f(d(\mathring{t}), c(\mathring{t}))}{{}^I f(d(\mathring{t}), c(\mathring{t}))}\right) d(d(\mathring{t}), c(\mathring{t})). \tag{6}$$

The general formulation of fully probabilistic design guarantees its applicability to a wide class of problems.

While components, $f(y_t|u_t, d(t-1), c_t)$ (5), are obtained from the estimation and should be considered as given, the remaining two factors in (5) can be optimised and, by this, influence the closed loop behaviour of the system considered. Dependently on which factor is chosen for optimisation, there exist three types of the design: *academic*, *industrial* and *simultaneous*.

Academic design optimises probabilities $f(c_t|d(t-1))$ of pointers to particular components c_t.

Industrial design considers optimisation of randomised control strategy $f(u_t|d(t-1), c_t)$ without changing the probabilities of components $f(c_t|d(t-1))$ and assuming $f(u_t|d(t-1), c_t) = f(u_t|d(t-1))$. This type of design has to be used whenever the component weights have an objective meaning that cannot be influenced by the optional controllers used. An example of this design can be found in (Böhm & Kárný 2001).

Simultaneous design (Kárný et al. 2003) combines features both academic and industrial designs. It optimises probabilities of the components $f(c_t|d(t-1))$ *and* randomised control strategies of particular controllers $f(u_t|d(t-1), c_t)$. In fact this type of the design optimises the joint pdf $f(u_t, c_t|d(t-1)) \equiv f(u_t|d(t-1), c_t) f(c_t|d(t-1))$. It takes the model (4) as an approximation of a non-linear dynamic model and searches for the proper system inputs while respecting possible changes of operation mode.

3 Control design with multimodal target

Probabilistic control for the described types of design was elaborated only for the case when the *user's ideal* is given by *uni-modal pdf* (Kárný et al. 2003), (Böhm & Kárný 2001). The paper provides the extension of the technique to the case of when *desired closed loop behaviour is described by a mixture*. The mixture form of user's ideal allows to respect different user's demands expressed via particular components of the ideal mixture. Thus the resulting optimal strategy will provide the compromise between these demands. The theoretical and algorithmic details are elaborated here for the academic type of design. The other cases can be approached in a similar way.

The considered *academic* design selects the optimal pdf within the set of causal randomised strategies

$$\{f(c_t|d(t-1))\}_{t\in t^*}\,. \tag{7}$$

The optimal pdf defines such probabilities of particular components that make the resulting mixture closest to the user's ideal. Unlike uni-modal ideal case, where the KL divergence of the type "mixture-to-pdf" optimised, the suggested *mixture form* of user's ideal needs evaluation the KL divergence between two mixtures. As this evaluation is difficult, the divergence between two *joint* pdfs describing *optimised* and *desired closed-loop behaviour* is used in minimisation.

The behaviour of the optimised closed loop is described by the joint pdf

$$f(d(\mathring{t}),c(\mathring{t}))=\prod_{t\in t^*}f(d_t,c_t|d(t-1))=\prod_{t\in t^*}f(d_t|d(t-1),c_t)f(c_t|d(t-1)), \tag{8}$$

where $f(d_t|d(t-1),c_t)$ are learned components of the system model and $f(c_t|d(t-1))$ is a strategy from (7).

The *ideal pdf*, describing the desired behaviour of the closed loop, is expressed through the *finite mixture* of $\mathring{\tilde{c}}$ components:

$$^I f(d(\mathring{t}),\tilde{c}(\mathring{t}))=\prod_{t\in t^*}{}^I f(d_t|d(t-1),\tilde{c}_t)\,{}^I f(\tilde{c}_t|d(t-1)) \tag{9}$$

with their *supports* nested in the following way:

$$\text{supp}\,[f(d_t|d(t-1),c_t)]\subseteq\text{supp}\,\left[{}^I f(d_t|d(t-1),\tilde{c}_t)\right]. \tag{10}$$

The first factor in (9) describes user's wishes and restrictions on particular components $\tilde{c}\in\tilde{c}$ with respect to the system data, for instance, desired mean values and covariances of system outputs and restrictions on controller inputs. The second factor ${}^I f(\tilde{c}_t|d(t-1))$ expresses preferences on the choice of these components. The number of components $\mathring{\tilde{c}}$ in the ideal mixture can generally be different from the number of components $\mathring{c}$ learned (2). The minimisation of KL divergence of two joint pdfs $\mathcal{D}(f(d(\mathring{t}),c(\mathring{t}))||\,{}^I f(d(\mathring{t}),\tilde{c}(\mathring{t})))$ is simplified when the numbers of components equal, i.e. $\mathring{c}=\mathring{\tilde{c}}$. To get this, the following agreement is used.

Agreement 2 (Extension of components number) *Let mixtures f_1 and f_2 have different number of the components $\mathring{c}_2>\mathring{c}_1$ and $m=\mathring{c}_2-\mathring{c}_1$ is missing number of components of the mixture f_1.*

Then, the amount of components $\mathring{c}_1$ of the mixture f_1 can be extended on missing number m by:

1. *"virtual" duplication of any m components of f_1, such that the probability of each original component is equally distributed among two "virtually" created components;*
2. *"artificial" creation of a m new components of mixture f_1, such that each of new components is described by a flat prior pdf with a small weight.*

A possible variant of the first way can be m-times duplication only one of the original components. The second choice guarantees that the inclusion (10) holds also for the joint pdfs on (d_t,c_t) . Note, that the first way cannot be applied to the extension of ideal mixtures as it causes decreasing the weights of those m components chosen for duplication.

Besides, the components of both ideal and learned mixtures can be ordered arbitrarily, so the design (Agreement 1) should take into account all possible permutations of the components. It means an optimal strategy from (7) for a permutation of components ${}^{opt}\pi_{\mathring{t}}(c(\mathring{t}))$ minimising the KL divergence between optimised and desired closed-loop behaviour, will be searched for:

$$^{opt}f(\,{}^{opt}\pi_{\mathring{t}}(c(\mathring{t})))=\min_{\{\pi_{\mathring{t}}(c(\mathring{t}))\}}\ \arg\min_{\{f(\pi_t(c_t|d(t-1)))\}}\mathcal{D}(f(d(\mathring{t}),\pi_{\mathring{t}}(c(\mathring{t}))||\,{}^I f(d(\mathring{t}),c(\mathring{t}))). \tag{11}$$

Assuming a fixed, time-invariant permutation of the components $\pi_{t+1}(c_t)=\pi_t(c_t)=\pi(c_t)$, the solution of the addressed academic fully probabilistic design (Agreement 1) is described by the following proposition.

Proposition 1 (Solution fully probabilistic design with multimodal target and fixed $\pi(c_t)$) *The optimal strategy minimising the KL divergence (1) has the form*

$$
\begin{aligned}
f(\pi(c_t)|d(t-1)) &= {}^{I}f(c_t|d(t-1))\frac{\exp[-\omega_\gamma(\pi(c_t),d(t-1))]}{\gamma(d(t-1))}, \quad \textit{where} \qquad (12)\\
\gamma(d(t-1)) &\equiv \sum_{c_t\in c^*} {}^{I}f(c_t|d(t-1))\exp[-\omega(\pi(c_t),d(t-1))]\\
\omega_\gamma(\pi(c_t),d(t-1)) &\equiv \int f(d_t|\pi(c_t),d(t-1))\ln\left(\frac{f(d_t|\pi(c_t),d(t-1))}{\gamma(d(t))\,{}^{I}f(d_t|c_t,d(t-1))}\right)dd_t\\
\gamma(d(\mathring{t})) &= 1.
\end{aligned}
$$

The solution is performed against the time course, starting at $t=\mathring{t}$.

Proof: Using the chain rule, the KL divergence can be written in the form

$$
\mathcal{D}\left(f||\,{}^{I}f\right) = \mathcal{E}\left\{\sum_{t\in t^*}\sum_{c_t\in c^*} f(\pi(c_t)|d(t-1))\left[\ln\left(\frac{f(\pi(c_t)|d(t-1))}{{}^{I}f(c_t)|d(t-1))}\right)+\omega(\pi(c_t),c_t,d(t-1))\right]\right\}
$$

$$
\omega(\pi(c_t),d(t-1)) \equiv \int f(d_t|\pi(c_t),d(t-1))\ln\left(\frac{f(d_t|\pi(c_t),d(t-1))}{{}^{I}f(d_t|c_t,d(t-1))}\right)dd_t.
$$

Let us denote

$$
\begin{aligned}
-\ln(\gamma(d(t))) &\equiv \min_{\{f(\pi(c_{\tau+1})|d(\tau))\}_{\tau=t}^{\mathring{t}}}\\
&\mathcal{E}\left\{\sum_{\tau=t+1}^{\mathring{t}}\sum_{c_\tau\in c^*} f(\pi(c_\tau)|d(\tau-1))\left[\ln\left(\frac{f(\pi(c_\tau)|d(\tau-1))}{{}^{I}f(c_\tau|d(\tau-1))}\right)+\omega(\pi(c_\tau),d(\tau-1))\right]|d(t)\right\}.
\end{aligned}
$$

Then, this definition implies that $\gamma(d(\mathring{t}))=1$ and

$$
\begin{aligned}
-\ln(\gamma(d(t))) &\equiv \min_{f(\pi(c_{t+1})|d(t))}\sum_{c_{t+1}\in c^*} f(\pi(c_{t+1})|d(t))\left[\ln\left(\frac{f(\pi(c_{t+1})|d(t))}{{}^{I}f(c_{t+1}|d(t))}\right)+\omega_\gamma(\pi(c_{t+1}),d(t))\right]\\
\omega_\gamma(\pi(c_{t+1}),c_{t+1},d(t)) &\equiv \int f(d_{t+1}|\pi(c_{t+1}),d(t))\ln\left(\frac{f(d_{t+1}|\pi(c_{t+1}),d(t))}{\gamma(d(t+1))\,{}^{I}f(d_{t+1}|c_{t+1},d(t))}\right)dd_{t+1}.
\end{aligned}
$$

It implies the identity $-\ln(\gamma(d(t)))\equiv$

$$
\begin{aligned}
&\equiv \min_{f(\pi(c_{t+1})|d(t))}\sum_{c_{t+1}\in c^*} f(\pi(c_{t+1})|d(t))\times\left[\ln\left(\frac{f(\pi(c_{t+1})|d(t))}{\frac{{}^{I}f(c_{t+1}|d(t))\exp[-\omega_\gamma(\pi(c_{t+1}),d(t))]}{\sum_{c_{t+1}\in c^*}{}^{I}f(c_{t+1}|d(t))\exp[-\omega_\gamma(\pi(c_{t+1}),d(t))]}}\right)-\right.\\
&\left.-\ln\left(\sum_{c_{t+1}\in c^*}{}^{I}f(c_{t+1}|d(t))\exp\left[-\omega_\gamma(\pi(c_{t+1}),d(t))\right]\right)\right].
\end{aligned}
$$

The first term in the above identity is the KL divergence, that reaches its smallest zero value for the claimed pdf. At the same time, it defines the form of the reached minima. ♢

The case, when the permutation of the components is time invariant, corresponds with a frequently met real situation. It happens when particular components describe known operation modes or physical states of the system considered. Then the permutation is defined by physical properties of the system and should be taken as given. If the permutation of components varies with time, an additional minimisation over possible component permutations is to be performed, see (11).

Let us stress that the exact evaluation of the KL divergence and thus, direct practical application of Proposition 1, is problematic as the reached minimum has the mixture form. The feasible variant of the design is then obtained when Jensen upper bound of it is used and upper bound KL divergence is minimised (Rao 1987).

4 Control design for mixtures of normal ARX models

The solution developed is applied to the mixtures of normal auto-regression models with exogenous variables (ARX). Models from this class can be easily identified (Peterka 1981), (Kárný et al. 1998), besides they represent one of a few classes suitable for a numerical solution of large dimensional problems.

The system is modelled by a mixture with normal components

$$f(d_t|d(t-1),\Theta_c,c)=\mathcal{N}_{d_t}(\theta_c^{'}\phi_{c;t-1},r_c), \tag{13}$$

where $\mathcal{N}_x(\bar{x},r) \equiv |2\pi r|^{-0.5}\exp\left[-0.5(x-\bar{x})'r^{-1}(x-\bar{x})\right]$; parameters $\Theta_c=[\theta_c,r_c]'$ consist of matrix regression coefficients θ_c and covariance matrix r_c. The state vector $\phi_{c;t-1}$ contains the delayed data items $\phi_{c;t-1}=[d'_{(t-1)\_(t-\partial_c)},1]'$, $\partial_c\geq 0$. The regression coefficients are complemented by zeros so that all factors within a single component have a common state vector. In the academic design, the regression vectors of individual factors are nested in the following way:

$$\begin{aligned}&\psi_{i;t}\equiv[d'_{i+1\_\mathring{d};t},\phi'_{t-1}]'\equiv[d_{i+1;t};\psi'_{i+1;t}]',\ i=1,\ldots,\mathring{d}-1,\\ &\psi_{\mathring{d};t}\equiv\phi_{t-1}\equiv[d'_{(t-1)\_(t-\partial)},1]',\ \partial\geq 0,\ \ \psi_{0;t}\equiv\Psi_t\equiv[d'_{t\_(t-\partial)},1]'.\end{aligned} \tag{14}$$

The relation of the regression, state and data vectors is schematically depicted as follows:

$$\left.\begin{bmatrix} d_{1;t}\\ \cdots\\ d_{i;t}\\ d_{i+1;t}\\ \cdots\\ d_{\mathring{d};t}\\ ---\\ \cdots\\ ---\\ d_{1;t-\partial}\\ d_{2;t-\partial}\\ \cdots\\ d_{\mathring{d};t-\partial}\\ ---\\ 1\end{bmatrix}\right\}\equiv\psi_{i;t}\equiv\begin{bmatrix} d_{i+1;t}\\ ---\\ \psi_{i+1;t}\end{bmatrix}\equiv\begin{bmatrix} d_{i+1;t}\\ \cdots\\ d_{\mathring{d};t}\\ ---\\ \phi_{t-1}\end{bmatrix}. \tag{15}$$

The pdf modelling the closed loop, written in the factor form, reads

$$f(d(\mathring{t}),c(\mathring{t}))=\prod_{t\in t^*}f(d_t|d_{t-1},c_t)f(c_t|d(t-1))=\prod_{t\in t^*}\prod_{1}^{\mathring{d}}\mathcal{N}_{d_{ic;t}}(\theta'_{ic}\psi_{ic;t},r_{ic})f(c_t|d(t-1)). \tag{16}$$

The parameterised factors determining the ideal mixture are also supposed to be normal with parameters ${}^I\Theta_c=[{}^I\theta_c,\ {}^Ir_c]'$ consisting of matrix regression coefficients ${}^I\theta_c$ and covariance matrix ${}^Ir_c$:

$${}^If(d_t|d(t-1),\ {}^I\Theta_c,c)=\mathcal{N}_{d_t}({}^I\theta_c^{'}\phi_{c;t-1},\ {}^Ir_c), \tag{17}$$

and the desired closed loop behaviour can be written as

$${}^If(d(\mathring{t}),c(\mathring{t}))=\prod_{t\in t^*}{}^If(d_t|d_{t-1},c_t)\,{}^If(c_t|d(t-1))=\prod_{t\in t^*}\prod_{1}^{\mathring{d}}\mathcal{N}_{d_{ic;t}}({}^I\theta'_{ic}\psi_{ic;t},\ {}^Ir_{ic})\,{}^If(c_t|d(t-1)). \tag{18}$$

The algorithmic realisation of the solution developed is based on factorised form of the components. Thus, to complete the solution the following proposition is needed. Its proof exploits just basic operations with quadratic forms and as such is omitted.

Proposition 2 (Expected quadratic form for a factorised normal component) *Let us consider a normal parameterised component described by*

$$\begin{aligned}&f(d_t|\phi_{t-1},\Theta)\equiv\prod_{i\in i^*}\mathcal{N}_{d_{i;t}}(\theta'_i\psi_{i;t},r_i)\ \ \textit{with}\ \Theta\equiv(\Theta_1,\ldots,\Theta_{\mathring{d}}),\ \ \Theta_i\equiv[\theta_i,r_i]\\ &\psi'_{i;t}\equiv[d'_{(i+1)\_\mathring{d};t},\phi'_{t-1}]\equiv[d_{i+1;t},\psi'_{i+1;t}],\ i=0,1,\ldots,\mathring{d}-1,\ \psi_{0;t}\equiv\Psi_t,\ \psi'_{\mathring{d};t}\equiv[\phi'_{t-1}].\end{aligned} \tag{19}$$

Let us consider a quadratic form in $\psi_{0;t} \equiv \Psi_t$ with the decomposed kernel $L_0 D_0 L_0'$, where L_0 is a lower triangular matrix with unit diagonal and D_0 is a diagonal matrix with non-negative diagonal entries.

Then, the expected quadratic form, lifted by a constant k_0, reads

$$\begin{aligned}
&\mathcal{E}[k_0 + \Psi_t' L_0 D_0 L_0' \Psi_t | \phi_{t-1}] \equiv \mathcal{E}[k_0 + \psi_{0;t}' L_0 D_0 L_0' \psi_{0;t} | \psi_{\mathring{d};t}] = k_{\mathring{d}} + \psi_{\mathring{d};t}' L_{\mathring{d}} D_{\mathring{d}} L_{\mathring{d}}' \psi_{\mathring{d};t}, \ \textit{where} \\
&L_{i+1} D_{i+1} L_{i+1}' = {}^{\psi}L_i \, {}^{\psi}D_i \, {}^{\psi}L_i' + \left(\theta_{i+1} + {}^{d\psi}L_i\right) {}^{d}D_i \left(\theta_{i+1} + {}^{d\psi}L_i\right)' \\
&k_{i+1} = k_i + {}^{d}D_i r_{i+1}, \quad \textit{for } i = 0, \dots, \mathring{d} - 1, \ \textit{with} \\
&L_i \equiv \begin{bmatrix} 1 & 0 \\ {}^{d\psi}L_i & {}^{\psi}L_i \end{bmatrix}, \ D_i \equiv \operatorname{diag}\left[{}^{d}D_i, {}^{\psi}D_i\right], \ \textit{where } {}^{d}D_i \textit{ is scalar and } \mathring{D}_{i+1} = \mathring{D}_i - 1.
\end{aligned} \tag{20}$$

The updating the LDL' decomposition is described in (Bierman 1977).

This proposition and definition of the KL divergence for a fixed normal factorised component imply:

Proposition 3 (The weighted conditional KL divergence) *Let*

$$f(d_t|d(t-1)) = \prod_{i=1}^{\mathring{d}} \mathcal{N}_{d_{i;t}}(\theta_i' \psi_{i;t}, r_i), \quad {}^{I}f(d_t|d(t-1)) = \prod_{i=1}^{\mathring{d}} \mathcal{N}_{d_{i;t}}\left({}^{I}\theta_i' \psi_{i;t}, {}^{I}r_i\right).$$

The regression coefficients ${}^{I}\theta_i$ of the user ideal pdf ${}^{I}f(\cdot)$ are complemented by zeros so that regression vectors $\psi_{i;t}$ of the corresponding factors coincide. Recall (15),

$$\psi_{i;t} = [d'_{(i+1)\_\mathring{d};t}, \phi_{t-1}']' \equiv [d'_{(i+1)\_\mathring{d};t}, \psi'_{\mathring{d};t}]' \equiv [d_{(i+1);t}, \psi'_{i+1;t}]', i \in \{0, \dots, \mathring{d} - 1\}, \ \psi_{\mathring{d};t} = \phi_{t-1}.$$

Let, moreover, $\gamma(\phi_t) \equiv \exp\left[-0.5(k_\gamma + \phi_t' L_\gamma D_\gamma L_\gamma' \phi_t)\right]$, $\phi_t \equiv [d'_{t\_(t-\partial+1)}, 1]' \Rightarrow \psi_{0;t} \equiv \Psi_t \equiv [d_t', \phi_{t-1}']'$,

$L_\gamma \equiv$ *a lower triangular matrix with unit diagonal,* $D_\gamma \equiv$ *a diagonal matrix with non-negative diagonal.*

Then,
$$\omega_\gamma(\phi_{t-1}) \equiv 2 \int f(d_t|\phi_{t-1}) \ln\left(\frac{f(d_t|\phi_{t-1})}{\gamma(\phi_t) \, {}^{I}f(d_t|\phi_{t-1})}\right) dd_t = k_{\mathring{d}} + \psi_{\mathring{d};t}' L_{\mathring{d}} D_{\mathring{d}} L_{\mathring{d}}' \psi_{\mathring{d};t}, \tag{21}$$

and the lift and kernel of the KL divergence are found recursively as follows:

For $i = 1, \dots, \mathring{D}$

$$\begin{aligned}
L_i D_i L_i' &= {}^{\psi}L_{i-1} \, {}^{\psi}D_{i-1} \, {}^{\psi}L_{i-1}' + \left(\theta_i + {}^{d\psi}L_{i-1}\right) {}^{d}D_{i-1} \left(\theta_i + {}^{d\psi}L_{i-1}\right)' + \frac{\left(\theta_i - {}^{I}\theta_i\right)\left(\theta_i - {}^{I}\theta_i\right)'}{{}^{I}r_i} \\
k_i &= k_{i-1} + {}^{d}D_{i-1} r_i + \left[\ln\left(\frac{{}^{I}r_i}{r_i}\right) + \frac{r_i}{{}^{I}r_i}\right]
\end{aligned}$$

end of the cycle over i

$$\begin{aligned}
&k_0 = -\mathring{d}_o + k_\gamma, \ L_0 D_0 L_0' = \mathcal{K} L_\gamma D_\gamma L_\gamma' \mathcal{K}' \\
&\mathcal{K}' \equiv \begin{bmatrix} I_{\mathring{d}(\partial-1)} & 0 & 0 \\ 0 & 0_{1,\mathring{d}} & 1 \end{bmatrix}, \ \textit{where } I_{\mathring{d}(\partial-1)} \textit{ is unit matrix of dimension } (n \times n) \\
&L_i \equiv \begin{bmatrix} 1 & 0 \\ {}^{d\psi}L_i & {}^{\psi}L_i \end{bmatrix}, \ D_i \equiv \operatorname{diag}\left[{}^{d}D_i, {}^{\psi}D_i\right], \ \textit{where } {}^{d}D_i \textit{ is scalar, } \mathring{D}_{i+1} = \mathring{D}_i - 1.
\end{aligned} \tag{22}$$

The proposition above provides the kernel of the algorithmic solution of the academic design for the normal mixtures. Its version for a fixed permutation is summarised in the following algorithm.

Proposition 4 (Academic fully probabilistic design with multimodal target applied to ARX mixtures) *The optimal strategy minimising upper bound of KL divergence (Kárný et al. 2003) with models (13) and (17) is described by the pfs*

$$\begin{aligned}
{}^{opt}f(c_t|d(t-1)) &\propto {}^{I}f(c) \exp\left[-0.5\omega(c_t, \phi_{c_t;t-1})\right] \ \textit{where} \\
\omega(c_t, \phi_{c_t;t-1}) &\equiv k_{c_t;t-1} + \phi'_{c_t;t-1} L_{c_t;t-1} D_{c_t;t-1} L'_{c_t;t-1} \phi_{c_t;t-1}
\end{aligned}$$

with recursively updated lifts $k_{c_t;t-1}$ and kernels $L_{c_t;t-1} D_{c_t;t-1} L'_{c_t;t-1}$. The solution is performed against

the time course, starting at $L_{\gamma;\mathring{t}} = I_{\mathring{\phi}}$, $D_{\gamma;\mathring{t}} = 0$, $k_{\gamma;\mathring{t}} = 0$.

For $t = \mathring{t}, \dots, 1$

$$L_{\mathring{d}} = I_{\mathring{d}},\ D_{\mathring{d}} = 0$$

For $c = 1, \dots, \mathring{c}$

$$k_{0c} = -\mathring{d}_c + k_{\gamma;t},\ L_{0c} = \begin{bmatrix} {}^{\psi_0}L_{\gamma;t} & 0 & 0 \\ 0 & L_{\mathring{d};t} & 0 \\ {}^{\psi_k}L_{\gamma;t} & 0 & 1 \end{bmatrix}$$

$$D_{0c} = \mathrm{diag}\left[{}^{\psi_0}D_{\gamma;t}, D_{\mathring{d};t}, {}^{\psi_k}D_{\gamma;t}\right]$$

$$nor = 0$$

For $i = 1, \dots, \mathring{d}$

$$L_{ic}D_{ic}L'_{ic} = {}^{\psi_{i-1}}L_{(i-1)c}\,{}^{\psi_{i-1}}D_{(i-1)c}\,{}^{\psi_{i-1}}L'_{(i-1)c} +$$

$$+ \left(\theta_{ic} + {}^{d\psi}L_{(i-1)c}\right){}^{d}D_{(i-1)c}\left(\theta_{ic} + {}^{d\psi}L_{(i-1)c}\right)' + \frac{\left(\theta_{ic} - {}^{I}\theta_{ic}\right)\left(\theta_{ic} - {}^{I}\theta_{ic}\right)'}{{}^{I}r_{ic}}$$

$$k_{ic} = k_{(i-1)c} + {}^{d}D_{(i-1)c}r_{ic} + \left[\ln\left(\frac{{}^{I}r_{ic}}{r_{ic}}\right) + \frac{r_{ic}}{{}^{I}r_{ic}}\right]$$

end of the cycle over i

$$k_{c;t-1} = k_{\mathring{d}c}$$

$$L_{c;t-1}D_{c;t-1}L'_{c;t-1} = L_{\mathring{d}c}D_{\mathring{d}c}L'_{\mathring{d}c}$$

$$\beta_c = {}^{I}f(c)\exp[-0.5k_{c;t-1}],\ nor = nor + \beta_c$$

end of the cycle over c

$$L_{\gamma;t-1} = I_{\mathring{\phi}}, D_{\gamma;t-1} = 0, k_{\gamma;t-1} = 0$$

For $c = 1, \dots, \mathring{c}$

$$L_{\gamma;t-1}D_{\gamma;t-1}L'_{\gamma;t-1} = L_{\gamma;t-1}D_{\gamma;t-1}L'_{\gamma;t-1} + \frac{\beta_c}{nor}L_{c;t-1}D_{c;t-1}L'_{c;t-1}$$

$$k_{\gamma;t-1} = k_{\gamma;t-1} + \frac{\beta_c}{nor}k_{c;t-1}$$

end of the cycle over c

end of the cycle over t

Remarks 1

- *For a known state vector* ϕ_{t-1}*, it is possible to evaluate the achieved minimum as* $\mathcal{D}(f(c)||{}^{I}f(c)) + \exp\left[-0.5\left(k_{\gamma;t} + \phi'_{t-1}L_{\gamma;t-1}D_{\gamma;t-1}L'_{\gamma;t-1}\phi_{t-1}\right)\right]$ *and compare quality of various permutations. It is always possible for static systems with* $\phi_{t-1} = 1$*. Otherwise approximation of the future unknown states by the current known one is possible.*
- *The total number of such permutations is* $\mathring{c}!$ *and there are many ready-to-use programs for their systematic generation.*
- *The algorithm is presented in quadratic forms for simplicity of presentation. Real algorithm works with square roots of kernels of quadratic form that guarantee numerical robustness (Bierman 1977).*

5 Illustrative example

To illustrate the basic features of the academic control design with a multimodal target, the data describing static system with two data items $d_t = (d_{1;t}, d_{2;t})$ have been simulated. The system model used for simulation is supposed to be mixture with three normal components (13). The factors $f_i,\ i = \{1, \dots, 6\}$ corresponding to the particular data items are also static normal $\mathcal{N}_{d_i} \propto (\bar{x}, r)$ with known mean values $\bar{x}$ and variances r,

that enter the particular components as follows:

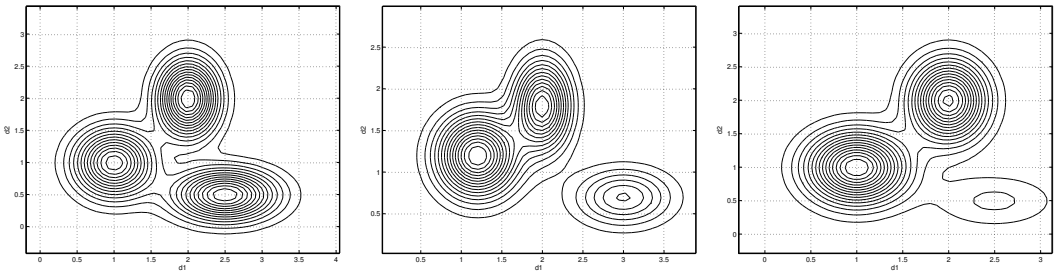
$$\begin{aligned} &\textit{1st component is composed of factor } f_1 \propto \mathcal{N}(1.0, 0.12) \quad \textit{and factor } f_4 \propto \mathcal{N}(1.00, 0.12) \\ &\textit{2nd component is composed of factor } f_2 \propto \mathcal{N}(2.00, 0.07) \quad \textit{and factor } f_5 \propto \mathcal{N}(2.00, 0.15) \\ &\textit{3rd component is composed of factor } f_3 \propto \mathcal{N}(2.5, 0.20) \quad \textit{and factor } f_6 \propto \mathcal{N}(0.50, 0.07) \end{aligned} \tag{23}$$

Probabilities of pointers to the particular components are set to be equal $f(c) = [1/3, 1/3, 1/3]$, for the components ordering $c = \{1\ 2\ 3\}$. The equiprobability curves describing pdf of the data simulated by the mixture (23) are shown on Figure 1A.
The user ideal has been chosen in a similar mixture form but with different ordering of components. Particular factors parameters are:

$$\begin{aligned} &\textit{1st component is composed of factor } {}^I f_3 \propto \mathcal{N}(3.0, 0.15) \quad \textit{and factor } {}^I f_6 \propto \mathcal{N}(0.7, 0.05) \\ &\textit{2nd component is composed of factor } {}^I f_1 \propto \mathcal{N}(1.2, 0.10) \quad \textit{and factor } {}^I f_4 \propto \mathcal{N}(1.2, 0.10) \\ &\textit{3rd component is composed of factor } {}^I f_2 \propto \mathcal{N}(2.0, 0.05) \quad \textit{and factor } {}^I f_5 \propto \mathcal{N}(1.8, 0.12) \end{aligned} \tag{24}$$

Probabilities of pointers to the particular components of the target mixture (24) are set to ${}^I f(\tilde{c}) = [1/6, 1/2, 1/3]$, with components ordering $\tilde{c} = \{1\ 2\ 3\}$. The equiprobability curves describing pdf of data simulated by the ideal mixture (24) are shown in the Figure 1B. Comparing the factors contributing to the particular components, it is easy to see that identically labelled components in simulated and user ideal mixtures do not correspond. Taking the ordering of components in ideal mixture as *fixed*, $\tilde{c} = \{\tilde{1}\ \tilde{2}\ \tilde{3}\}$, the optimisation over 3! possible permutations $\{\pi_1(c), \ldots, \pi_6(c)\}$ of the system mixture components was performed.

The optimised probabilities ${}^{opt} f(c) = [0.07, 0.42, 0.51]$, corresponding to the fixed component ordering accepted $\tilde{c} = \{\tilde{1}\ \tilde{2}\ \tilde{3}\}$, are found for the permutation $\pi(c) = \{2\ 3\ 1\}$ of components in simulated mixture (23). Logarithm of the upper bound of the corresponding KL divergence for the optimal solution equals 1.75, while the remaining permutations possess values $27, 40, 28, 40$ and 25. The strategy designed gives the optimised system model density as shown in the Figure 1C.



A. Simulated mixture B. Desired mixture C. Optimised mixture

Figure 1. Projections of mixture models

The numerical results obtained are well justified by comparison of mixture projections, Figure 1. Respecting the demands on the system behaviour, expressed by user's ideal (24), the strategy chosen strengthens desirable components and suppresses undesirable one.

6 Concluding remarks

The paper proposes an efficient solution of multiobjective control design problem. The probabilistic mixture modelling complemented by the fully probabilistic control adopting certainty equivalence strategy is used to that. To express the multiple control objectives, the mixture modelling is employed, where particular objectives are represented by uni-modal mixture components.

The solution of the simplest design, called academic, where only probabilities of components are optimised is presented. The obtained solution has been transformed into computationally feasible algorithm for mixtures with components formed by normal auto-regression models with external variable. Comparing to the case when user ideal is described by uni-modal pdf, this solution became more complicated both theoretically and computationally, however, it substantially broads the applicability of the whole approach.

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Decision making in an uncertain environment: the scenario-based optimization approach

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Abstract. A central issue arising in financial, engineering and, more generally, in many applicative endeavors is to make a decision in spite of an uncertain environment. Along a *robust* approach, the decision should be guaranteed to work well in *all* possible realizations of the uncertainty. A less restrictive approach consists instead of requiring that the risk of failure associated to the decision should be *small* in some – possibly probabilistic – sense. From a mathematical viewpoint, the latter formulation leads to a *chance-constrained* optimization program, i.e. to an optimization program subject to constraints in probability. Unfortunately, however, both the robust approach as well as the chance-constrained approach are computationally intractable in general.

In this paper, we present a computationally efficient methodology for dealing with uncertainty in optimization based on sampling a finite number of instances (or *scenarios*) of the uncertainty. In particular, we consider uncertain programs with convexity structure, and show that the scenario-based solution is, with high confidence, a feasible solution for the chance-constrained problem. The proposed approach represents a viable way to address general convex decision making problems in a risk-adjusted sense.

Keywords: Convex optimization, scenario approach, confidence levels, decision making.

1 Introduction

In this paper we consider decision making problems that can be formalized as robust convex optimization programs, where the constraints are parameterized by an uncertain parameter δ which describes the situation or 'states of nature' that can possibly occur. Precisely, letting $\delta \in \mathbf{\Delta} \subseteq R^{\ell}$ be the vector of uncertain variables, the robust convex program writes:

$$\text{RCP}: \min_{x \in \mathcal{X} \subseteq R^n} c^T x \quad \text{subject to } f(x,\delta) \leq 0, \quad \forall \delta \in \Delta, \tag{1}$$

where $\mathcal{X}$ is a convex and closed set that represents the feasible set for the solutions and the function $f(x,\delta) : \mathcal{X} \times \mathbf{\Delta} \rightarrow R$ is convex in x for any fixed $\delta \in \mathbf{\Delta}$. In typical situations, (1) is a semi-infinite optimization problem since it contains an infinite number of constraints (i.e. $\mathbf{\Delta}$ has infinite cardinality), while x represents a finite number of optimization variables.

The function $f(x,\delta)$ is here assumed to be scalar-valued without loss of generality, since multiple scalar-valued convex constraints $f_i(x,\delta) \leq 0$, $i = 1,\ldots,n_f$, can always be converted into a single scalar-valued convex constraint by the position $f(x,\delta) = \max_{i=1,\ldots,n_f} f_i(x,\delta) \leq 0$. We also note that the cost function $c^T x$ is certain (i.e. it does not depend on δ) and is linear in x without loss of generality, since any convex uncertain program can be reformulated so that it exhibits a linear certain cost. To see this, simply note that a problem $\min_{\xi \in \Xi} g(\xi,\delta)$ subject to $h(\xi,\delta) \leq 0, \forall \delta \in \Delta$, can be rewritten as $\min_{\xi \in \Xi, \gamma} \gamma$ subject to $g(\xi,\delta) - \gamma \leq 0$ and $h(\xi,\delta) \leq 0, \forall \delta \in \Delta$. Finally, we remark that no assumption is made on the functional dependence of f on δ.

In general situations, finding a solution to (1) is a formidable task and no general solution methodology for (1) is to date available. In special cases, (1) can be solved by reconducting it to a problem with a finite number of constraints, possibly by conservative relaxation techniques. We refer the reader to (A. Nemirovski 1993) and (G. Calafiore and M.C. Campi 2003) and the references therein for a discussion on the complexity

of solving (1) and on possible relaxation techniques within the specific context of robust control problems.

Program (1) is 'robust' since the optimal solution is guaranteed against all possible occurrences of the uncertain parameter δ, i.e. the constraint $f(x,\delta) \leq 0$ must be satisfied for all possible values of $\delta \in \Delta$. In certain problems it may be convenient to relax such a strong requirement by allowing that a small fraction of constraints can possibly be violated, so leaving a small chance that $f(x,\delta) \leq 0$ is not satisfied in correspondence of the found solution. The resulting optimization problem is named a chance-constrained convex optimization problem and is formalized as follows.

Let Prob denote a probability measure over Δ (i.e. Δ is endowed with a -algebra $\mathcal{D}$ and Prob is assigned for every set of $\mathcal{D}$). Depending on the situation at hand, Prob can have different interpretations. Sometimes, it is the actual probability with which the uncertainty parameter δ takes on value in a certain set, while other times Prob simply describes the relative importance we attribute to different uncertainty instances. The chance-constrained (or probabilistic) convex optimization problem then writes:

$$\text{PCP}(\epsilon): \min_{x \in \mathcal{X} \subseteq R^n} c^T x \quad \text{subject to Prob}\{f(x,\delta) > 0\} \leq \epsilon, \tag{2}$$

where the parameter $\epsilon \in (0,1)$ represents the admissible *probabilistic risk* of violating the constraint $f(x,\delta) \leq 0$. It is clear that allowing for a probability ϵ of violation results in an optimal solution that outperforms the optimal robust solution for the uncertainty instances that are indeed feasible at the optimum. Moreover, ϵ represents a 'tuning knob' that can be freely selected and the lower ϵ, the closer the chance-constrained optimization problem to the robust optimization problem.

Chance-constrained optimization has a long history, dating back to the work of Charnes and Cooper for linear programs in 1959 , (A. Charnes and W.W. Cooper 1959). Still to date, most of the available computational results on chance-constraints refer to the important but very specific case when $f(x,\delta)$ is linear in x, i.e. to probability-constrained *linear* programs. One reason for this is that chance-constrained problems are extremely difficult to solve exactly. This can be easily realized by noticing that the mere evaluation of the probability with which a constraint is satisfied amounts to solving a multi-dimensional integral, which is a formidable task in general. Moreover, even when $f(x,\delta)$ is convex (or even linear) in x for any fixed $\delta \in \mathbf{\Delta}$, the feasible set $\{x : \text{Prob}\{f(x,\delta) > 0\} \leq \epsilon\}$ may be nonconvex, and hence PCP is *not* a convex program in general. We direct the reader to the monograph by Prékopa (A. Prékopa 1995) and to (S. Vajda 1972) for an extensive presentation of many available results in this area.

Problems of the form (2) appear frequently in manufacturing and financial problems. For instance, in the Value-at-Risk (VaR) framework (see e.g. (T. J. Linsmeier and N. D. Pearson 1996, M. Pritsker 1997)) one considers the possible loss $-r(\xi,\delta)$ of a portfolio of risky assets, where ξ describes the allocations of assets in the portfolio, and δ represent the (uncertain) returns of the assets. The VaR is then defined as the minimal level γ such that the probability that the portfolio loss exceeds γ is below a fixed (small) level $\epsilon \in (0,1)$. Hence, the problem of minimizing the VaR over admissible portfolios takes the form

$$\min_{\xi,\gamma} \gamma \quad \text{subject to Prob}\{\gamma < -r(\xi,\delta)\} \leq \epsilon,$$

which is a PCP, with $x \doteq (\xi,\gamma)$ and $f(x,\delta) \doteq -r(\xi,\delta) - \gamma$.

The previous example indicates also that the class of PCP problems embodies probabilistic counterparts of classical min-max games. Indeed, a min-max game

$$\min_{\xi \in \Xi} \max_{\delta \in \mathbf{\Delta}} g(\xi,\delta)$$

may be expressed as the robust program

$$\min_{\xi \in \Xi, \gamma} \gamma \quad \text{subject to } g(\xi,\delta) \leq \gamma, \ \forall \delta \in \mathbf{\Delta}. \tag{3}$$

The probabilistic counterpart of this semi-infinite program is a PCP of the form

$$\min_{\xi \in \Xi, \gamma} \gamma \quad \text{subject to Prob}\{g(\xi,\delta) > \gamma\} \leq \epsilon, \quad \epsilon \in (0,1). \tag{4}$$

Notice that, for fixed ξ, the optimal γ in problem (3) is the max of $g(\xi,\delta)$ over δ, while in (4) it is the *probable approximate maximum* of $g(\xi,\delta)$ over δ, i.e. a value which is exceeded only over a set of δ's having small probability measure.

1.1 A computationally feasible paradigm: Scenario-based convex programs

Motivated by the computational complexity of RCP and PCP, in this paper we pursue a solution methodology which is based on randomization of the parameter δ.

To this end, we collect N samples $\delta^{(1)}, \ldots, \delta^{(N)}$ of the uncertain parameter randomly chosen in an independent fashion according to probability Prob (these instances shall be referred to as the *scenarios*), and construct the randomized convex program

$$\text{RCP}_N: \quad \min_{x \in \mathcal{X} \subseteq R^n} c^T x \ \text{ subject to } \ f(x, \delta^{(i)}) \leq 0, \ i = 1, \ldots, N.$$

This randomized program, called the *scenario program* in the sequel, has a distinctive advantage over RCP and PCP: it is a standard convex program with N constraints, and hence it is typically efficiently solvable. However, a fundamental question need be addressed: what can we say about the chance-constraint satisfaction for an optimal solution of RCP_N? More explicitly, the crucial question to which this paper is devoted is the following:

> How many samples (scenarios) need to be drawn in order to guarantee that the solution of the scenario problem violates at most a portion ϵ of the constraints (i.e. it is a feasible solution in a chance-constrained sense)?

Using statistical learning techniques, we provide an explicit bound on the measure (probability) of the set of original constraints that are possibly violated by the scenario solution. This measure rapidly decreases to zero as N is increased, and therefore the obtained randomized solution is feasible for the chance-constrained problem (2).

2 Scenario-Based Convex Programs

We first formalize the concept of violation probability.

Definition 1 (Violation probability). The *probability of violation* of $x \in \mathcal{X}$ is defined as

$$V(x) \doteq \text{Prob}\{\delta \in \mathbf{\Delta} : \ f(x, \delta) > 0\}$$

(here, it is assumed that $\{\delta \in \mathbf{\Delta} : \ f(x, \delta) > 0\}$ is an element of the -algebra $\mathcal{D}$). $\star$

For example, if a uniform (with respect to Lebesgue measure) probability density is assumed, then $V(x)$ measures the volume of the uncertainty instances δ such that the constraint $f(x, \delta) \leq 0$ is violated. We next have the following definition.

Definition 2 (ϵ-level solution). Let $\epsilon \in (0, 1)$. We say that $x \in \mathcal{X}$ is an ϵ-level robustly feasible solution if $V(x) \leq \epsilon$. $\star$

Notice that, by the above definition, the ϵ-level solutions are the feasible solutions for the chance-constrained optimization problem (2). Our goal is to devise an algorithm that returns a ϵ-level solution, where ϵ is any fixed small level. To this purpose, we now formally introduce the scenario convex program as follows.

Definition 3 (Scenario convex program). Let $\delta^{(1)}, \ldots, \delta^{(N)}$ be N independent identically distributed samples extracted according to probability Prob. The randomized convex program (or scenario convex program) derived from (2) is

$$\text{RCP}_N: \quad \min_{x \in \mathcal{X} \subseteq R^n} c^T x \quad \text{subject to } f(x, \delta^{(i)}) \leq 0, \ i = 1, \ldots, N. \tag{5}$$

$\star$

For the time being we assume that RCP_N admits a unique solution. The uniqueness assumption is temporarily made for clarity in the presentation and proof of the main result, and it is later removed in Appendix A.

Assumption 1 *For all possible extractions $\delta^{(1)}, \ldots, \delta^{(N)}$, the optimization problem (5) attains a unique optimal solution.* $\star$

Let then $\hat{x}_N$ be the unique solution of problem RCP$_N$. Since the constraints $f(x, \delta^{(i)}) \leq 0$ are randomly selected, the optimal solution $\hat{x}_N$ is a random variable that depends on the extraction of the multi-sample $\delta^{(1)}, \ldots, \delta^{(N)}$. The following key theorem pinpoints the properties of $\hat{x}_N$.

Theorem 1. *Fix two real numbers $\epsilon \in (0,1)$ (level parameter) and $\beta \in (0,1)$ (confidence parameter). If*

$$N \geq N(\epsilon, \beta) \doteq \frac{2}{\epsilon} \ln \frac{1}{\beta} + 2n + \frac{2n}{\epsilon} \ln \frac{2}{\epsilon}, \tag{6}$$

then, with probability no smaller than $1-\beta$, the optimal solution $\hat{x}_N$ of the scenario problem RCP$_N$ *is ϵ-level robustly feasible.* $\star$

In this theorem, probability $1-\beta$ refers to the probability Prob^N (= Prob $\times \cdots \times$ Prob, N times) of extracting a 'bad' multi-sample, i.e. a multi-sample $\delta^{(1)}, \ldots, \delta^{(N)}$ such that $\hat{x}_N$ does not meet the ϵ-level feasibility property. β cannot be sent to zero, since otherwise N goes to infinity. This corresponds to the natural fact that, no matter how large N is, a 'bad' multi-sample returning a $\hat{x}_N$ with poor violation properties can always be extracted. For any practical purpose, however, β plays a very marginal role. The reason is that β appears under the sign of logarithm in (6) so that it can be taken to be a really tiny number (10^{-10} or even 10^{-20}) without making N too large. If β is neglected, in simple words Theorem 1 states that if N (specified by (6)) random scenarios are drawn, the optimal solution of RCP$_N$ is a feasible solution of the chance-constrained problem (2). Thus, while problem (1) admits no solution methodologies in general, (5) represents a viable way to solve (1) in a risk-adjusted sense, namely the found solution is feasible for the corresponding chance-constrained problem.

The proof of Theorem 1 is postponed to Section 3 to avoid breaking the continuity of discourse.

2.1 Discussion on main result

We next comment more closely on the proposed randomized approach. Theorem 1 says that if we extract a *finite* number N of constraints, then the solution of the randomized problem satisfies most of the other unseen constraints. This is a *generalization* property: the explicit satisfaction of some scenarios generalizes automatically to the satisfaction of other scenarios. It is interesting to note that generalization calls for some kind of structure, and the only structure used here is convexity. So, convexity in the scenario approach is fundamental in two different respects: on the computational side, it allows for an efficient solution of the ensuing optimization problem, while on the theoretical side it allows for generalization.

Remark 1 (Sample complexity and VC-dimension). The 'sample complexity' of the scenario problem (i.e. the number N of random scenarios that need to be drawn in order to achieve the desired probabilistic level in the solution) exhibits a substantially linear scaling with $\frac{1}{\epsilon}$ and a logarithmic scaling with $\frac{1}{\beta}$. If e.g. $\epsilon = 0.01$, $\beta = 0.0001$ and $n = 10$, we have $N \geq 12459$. In general, for reasonable probabilistic levels, the required number of these constraints is manageable by current convex optimization numerical solvers. We also mention that the reader can find a tighter bound than (6) in Section 3.2: in Theorem 1 we have been well advised to provide bound (6) – derived from the bound in Section 3.2 – to improve readability.

Bound (6) depends on the problem structure through n, the number of optimization variables, only. It is not difficult to conceive situations where the class of sets $\{\delta \in \mathbf{\Delta} : f(x, \delta) > 0\} \subseteq \mathbf{\Delta}$, parameterized in x, has infinite VC-dimension, even for small n; see for instance (V. Vapnik 1996) for definition of VC dimension and an exposition of learning theory. In these situations, estimating Prob$\{\delta \in \mathbf{\Delta} : f(x, \delta) > 0\} = V(x)$ uniformly with respect to x is impossible and the VC-theory is of no use. Theorem 1 says that, if attention is restricted to $\hat{x}_N$, then estimating $V(\hat{x}_N)$ becomes possible at a low computational cost. $\star$

*Remark 2 (*Prob*-independent bound).* In some applications, probability Prob is not explicitly known, and the scenarios are directly made available as 'observations'. This could for example be the case when the instances of δ are actually related to various measurements or identification experiments made on a plant at different times and/or different operating conditions, see e.g. (G. Calafiore and M.C. Campi 2002, G. Calafiore and M.C. Campi 2003). In this connection, notice that the bound (6) is probability independent, i.e. it holds irrespective of the underlying probability Prob, and can therefore be applied even when Prob is unknown. $\star$

Remark 3 (A measurability issue). Theorem 1 states that Prob$^N\{V(\hat{x}_N) \leq \epsilon\} \geq 1-\beta$. However, without any further assumption, there is no guarantee that $V(\hat{x}_N)$ is measurable, so that Prob$^N\{V(\hat{x}_N) \leq \epsilon\}$ may not be well-defined. Similar subtle measurability issues are often encountered in learning theory, see e.g. (A. Blumer, A. Ehrenfeucht, D. Haussler, and M. Warmuth 1989). Here and elsewhere, the measurability of $V(\hat{x}_N)$, as well as that of other variables defined over $\mathbf{\Delta}^N$, is taken as an assumption. $\star$

Remark 4 (Comparison between RCP_N *and* RCP*).* Since the solution of RCP_N is obtained by looking at N constraints only, it is certainly superoptimal for the robust convex program RCP. Thus, RCP_N alleviates the conservatism inherent in the robust approach to uncertain optimization. On the other hand, we also remark that in general the optimal solution of PCP outperforms the optimal solution of RCP_N when such a solution is feasible for PCP. We do not insist here on this point and refer the reader to (G. Calafiore and M.C. Campi 2003) for more discussion. ⋆

Remark 5 (Computational complexity). Through the scenario approach, the initial semi-infinite optimization problem is reduced to an optimization problem that can be solved efficiently. On the other hand, a side problem arising along the scenario approach is that one has to extract constraints out of the uncertainty set. This is not always a simple task to accomplish, see (G. Calafiore, F. Dabbene and R. Tempo 2000) for a discussion of this topic and polynomial-time algorithms for the sample generation. ⋆

2.2 A-priori and a-posteriori assessments

It is worth noticing that a distinction should be made between the a-priori and a-posteriori assessments that one can make regarding the probability of constraint violation. Indeed, *before* running the optimization, it is guaranteed by Theorem 1 that if $N \geq N(\epsilon, \beta)$ samples are drawn, the solution of the scenario program will be ϵ-level robustly feasible, with probability no smaller than $1 - \beta$. However, the a-priori parameters ϵ, β are generally chosen to be not too small, due to technological limitations on the number of constraints that one specific optimization software can deal with.

On the other hand, once a solution has been computed (and hence $x = \hat{x}_N$ has been fixed), one can make an a-posteriori assessment of the level of feasibility using standard Monte-Carlo techniques. In this case, a new batch of $\tilde{N}$ independent random samples of $\delta \in \mathbf{\Delta}$ is generated, and the *empirical probability* of constraint violation, say $\hat{V}_{\tilde{N}}(\hat{x}_N)$, is computed according to the formula $\hat{V}_{\tilde{N}}(\hat{x}_N) = \frac{1}{\tilde{N}} \sum_{i=1}^{\tilde{N}} 1(f(\hat{x}_N, \delta^{(i)}) > 0)$, where $1(\cdot)$ is the indicator function. Then, the classical Hoeffding's inequality, (W. Hoeffding 1963), guarantees that

$$|\hat{V}_{\tilde{N}}(\hat{x}_N) - V(\hat{x}_N)| \leq \tilde{\epsilon}$$

holds with confidence greater than $1 - \tilde{\beta}$, provided that

$$\tilde{N} \geq \frac{\ln 2/\tilde{\beta}}{2\tilde{\epsilon}^2} \tag{7}$$

test samples are drawn. This latter a-posteriori verification can be easily performed using a very large sample size $\tilde{N}$, because no numerical optimization is involved in such an evaluation.

2.3 Multi-participant decision making

A decision making has sometimes to be made in presence of collective opinions coming from different participants in the optimization problem. In general situations, moving from a single decision maker to a multiple decision maker situation introduces a great deal of complexity into the analysis. Here, we merely refer to a simple situation where the different participants share the same objective to be minimized, while having different opinions on the uncertain environment in which the decision has to be made.

Let us enumerate the participants with $j = 1, \ldots, M$. We assume that all participants aim at minimizing the same objective ($\min_{x \in \mathcal{X} \subseteq R^n} c^T x$) while having different perception of the uncertain environment. In mathematical terms, this is expressed by saying that the constraints are $f(x, \delta_1) \leq 0$, $\delta_1 \in \Delta_1$, for the first participant; $f(x, \delta_2) \leq 0$, $\delta_2 \in \Delta_2$, for the second participant; and so on for all other participants. In a probabilistic chance-constrained framework, letting Prob_j be the probability associated to Δ_j, the problem is as follows:

$$\begin{aligned} \min_{x \in \mathcal{X} \subseteq R^n} c^T x \quad \text{subject to} \quad & \text{Prob}_1\{f(x, \delta_1) > 0\} \leq \epsilon, \\ & \text{Prob}_2\{f(x, \delta_2) > 0\} \leq \epsilon, \\ & \vdots \\ & \text{Prob}_M\{f(x, \delta_M) > 0\} \leq \epsilon. \end{aligned} \tag{8}$$

As an example of this setting, consider the VaR framework of Section 1. Suppose that more financial partners (the participants) want to decide how to allocate the assets ξ so that the loss be minimized and each participant is willing to run a risk at most ϵ (according to his/her viewpoint on the uncertain reality) that the loss is bigger than the found minimum value γ. Then, the problem can be mathematically formulated as:

$$\begin{aligned} \min_{\xi,\gamma} \gamma \quad \text{subject to} \quad & \text{Prob}_1\{\gamma < -r(\xi,\delta_1)\} \leq \epsilon, \\ & \text{Prob}_2\{\gamma < -r(\xi,\delta_2)\} \leq \epsilon, \\ & \vdots \\ & \text{Prob}_M\{\gamma < -r(\xi,\delta_M)\} \leq \epsilon. \end{aligned}$$

A feasible solution for problem (8) can be found by treating the problem as a set of M separate problems and by applying Theorem 1 to each single problem. This corresponds to selecting a β and then drawing $N(\epsilon,\beta)$ constraints $\delta_1 \in \Delta_1$, an equal number of constraints $\delta_2 \in \Delta_2$ and so on for all participants. An application of Theorem 1 leads then to the conclusion that each participant has a confidence β that his constraints are satisfied with a violation probability at most ϵ. However, a different standpoint can be taken by asking a deeper question: to what extent the constraints associated to one participant are going to help the constraint satisfaction of other participants?

For the sake of simplicity, consider the case of two participants ($M = 2$). A very simple situation arises when the two participants share the same set of constraints ($\Delta_1 = \Delta_2 = \Delta$) with , say, Prob_2 absolutely continuous with respect to Prob_1 with a uniform coefficient α: $\text{Prob}_2(A) \leq \alpha \text{Prob}_1(A)$, $\forall A$ in the -algebra on Δ. A simple reasoning then reveals that if the first participant extracts $N(\epsilon_1,\beta_1)$ constraints according to Prob_1, then, with no extractions whatsoever, the second participants has, with confidence $\beta_2 = \alpha^N \beta_1$, a violation probability $\epsilon_2 = \alpha\epsilon_1$.

The above result is rather obvious since the extractions of the first participant correspond to extractions for the second participant, even though according to a different probability. A more intriguing situation occurs when Δ_1 and Δ_2 are different. Is then still true that the two participants are going to help each other? The following result holds: *fix ϵ and β and extract $N(\epsilon,\beta)$ constraints δ_1 according to* Prob_1 *and $N(\epsilon,\beta)$ constraints δ_2 according to* Prob_2. *Solve the optimization problem with all the constraints in place. Then, with probability no smaller than $1-\beta$, the optimal solution is robustly feasible for the first and the second participant with a violation probability ϵ_1 and ϵ_2 respectively that have to satisfy the relation: $\epsilon_1 + \epsilon_2 - \epsilon_1\epsilon_2 \leq \epsilon$.* Thus, if $\epsilon_1 = \epsilon$, then ϵ_2 has to be zero. This correspond to say that if the extractions of the second participant do not help the first participant, then the constraints of the second participant are 'dominated' by those of the first participant, and are therefore automatically satisfied. Intermediate situations can be studied as well. So, in all cases, the two participants are helping each other somehow, even though it may a-priory be unknown who is helping whom.

A sketch of proof of the above result is as follows. The $N(\epsilon,\beta)$ extractions of δ_1's and δ_2's can be seen as $N(\epsilon,\beta)$ extractions of couples (δ_1,δ_2) from $\Delta_1 \times \Delta_2$ with probability $\text{Prob}_1 \times \text{Prob}_2$. Theorem 1 can then be applied to this space, so concluding that, with confidence β, the violation probability in $\Delta_1 \times \Delta_2$ is at most ϵ. Consider a multi-sample such that the violation probability is $\leq \epsilon$. If a given $\bar{\delta}_1$ is violated, then $(\bar{\delta}_1,\delta_2)$ is violated in the product space, $\forall \delta_2$. So, if ϵ_1 is the probability of violation in Δ_1, this leads to a probability of violation ϵ_1 in the product space too. Similarly, if ϵ_2 is the probability of violation in Δ_2, this leads to a probability of violation ϵ_2 in the product space. This two sets overlap with a probability $\epsilon_1\epsilon_2$ and there union has a probability bounded by ϵ, so leading to the result above.

3 Technical preliminaries and proof of Theorem 1

This section is technical and contains the machinery needed for the proof of Theorem 1. The reader not interested in these mathematical aspects may skip to Section 4 without any loss of continuity.

3.1 Preliminaries

We first recall a classical result due to Helly, see (R.T. Rockafellar 1970).

Lemma 1 (Helly). *Let $\{\mathcal{X}_i\}_{i=1,\ldots,p}$ be a finite collection of convex sets in R^n. If every sub-collection consisting of $n+1$ sets has a non-empty intersection, then the entire collection has a non-empty intersection.*

⋆

Next, we prove a key instrumental result. Consider the convex optimization program

$$\mathcal{P}: \min_{x \in R^n} c^T x \quad \text{subject to } x \in \bigcap_{i \in \{1,\ldots,m\}} \mathcal{X}_i, \tag{9}$$

where $\mathcal{X}_i$, $i = 1, \ldots, m$, are closed convex sets. Define the convex programs $\mathcal{P}_k$, $k = 1, \ldots, m$, obtained from $\mathcal{P}$ by removing the k-th constraint:

$$\mathcal{P}_k: \min_{x \in R^n} c^T x \quad \text{subject to } x \in \bigcap_{i \in \{1,\ldots,m\} \setminus k} \mathcal{X}_i. \tag{10}$$

Let x^* be any optimal solution of $\mathcal{P}$ (assuming it exists), and let x_k^* be any optimal solution of $\mathcal{P}_k$ (again, assuming it exists). We have the following definition.

Definition 4 (support constraint). The k-th constraint $\mathcal{X}_k$ is a *support constraint* for $\mathcal{P}$ if problem $\mathcal{P}_k$ has an optimal solution x_k^* such that $c^T x_k^* < c^T x^*$. ⋆

The following theorem holds.

Theorem 2. *The number of support constraints for problem $\mathcal{P}$ is at most n.* ⋆

A proof of this result was originally given by the authors of the present paper in (G. Calafiore and M.C. Campi 2003). We here report an alternative and more compact proof whose outline was suggested us by A. Nemirovski in a personal communication.

Proof. Let problem $\mathcal{P}$ have q support constraints $\mathcal{X}_{s_1}, \ldots, \mathcal{X}_{s_q}$, where $\mathcal{S} \doteq \{s_1, \ldots, s_q\}$ is a subset of q indices from $\{1, \ldots, m\}$. We next prove (by contradiction) that $q \leq n$.

Let x^* be any optimal solution of $\mathcal{P}$, with corresponding optimal objective value $J^* = c^T x^*$, and let x_k^* be any optimal solution of $\mathcal{P}_k$, $k = 1, \ldots, m$, with corresponding optimal objective value $J_k^* = c^T x_k^*$. Consider the smallest objective improvement obtained by removing a support constraint

$$\eta_{\min} \doteq \min_{k \in \mathcal{S}} (J^* - J_k^*)$$

and, for some η with $0 < \eta < \eta_{\min}$, define the hyperplane

$$\mathcal{H} \doteq \{x : c^T x = J^* - \eta\}.$$

By construction, the q points x_k^*, $k \in \mathcal{S}$, lie in the half-space $\{x : c^T x < J^* - \eta\}$, while x^* lies in the half-space $\{x : c^T x > J^* - \eta\}$, and therefore $\mathcal{H}$ separates x_k^*, $k \in \mathcal{S}$, from x^*. Next, for all indices $k \in \mathcal{S}$, we denote with $\overline{x}_k$ the point of intersection between the line segment $\overline{x_k^* x^*}$ and $\mathcal{H}$.

Since $x_k^* \in \bigcap_{i \in \{1,\ldots,m\} \setminus k} \mathcal{X}_i$, $k \in \mathcal{S}$, and $x^* \in \bigcap_{i \in \{1,\ldots,m\}} \mathcal{X}_i$, then by convexity we have that $\overline{x}_k \in \bigcap_{i \in \{1,\ldots,m\} \setminus k} \mathcal{X}_i$, $k \in \mathcal{S}$, and therefore (since, by construction, $\overline{x}_k \in \mathcal{H}$)

$$\overline{x}_k \in \left(\bigcap_{i \in \{1,\ldots,m\} \setminus k} \mathcal{X}_i \right) \bigcap \mathcal{H}, \quad k \in \mathcal{S}.$$

For $i = 1, \ldots, m$, define the convex sets $\Omega_i \doteq \mathcal{X}_i \bigcap \mathcal{H}$, and consider any collection $\{\Omega_{i_1}, \ldots, \Omega_{i_n}\}$ of n of these sets.

Suppose now (for the purpose of contradiction) that $q > n$. Then, there must exist an index $j \notin \{i_1, \ldots, i_n\}$ such that $\mathcal{X}_j$ is a support constraint, and by the previous reasoning, this means that there exists a point $\overline{x}_j$ such that $\overline{x}_j \in \left(\bigcap_{i \in \{1,\ldots,m\} \setminus j} \mathcal{X}_i \right) \bigcap \mathcal{H}$. Thus, $\overline{x}_j \in \Omega_{i_1} \cap \cdots \cap \Omega_{i_n}$, that is the collection of convex sets $\{\Omega_{i_1}, \ldots, \Omega_{i_n}\}$ has at least a point in common. Now, since the sets Ω_i, $i = 1, \ldots, m$, belong to the hyperplane $\mathcal{H}$ — i.e. to R^{n-1}, modulo a fixed translation — and all collections composed of n of these sets have a point in common, by Helly's lemma (Lemma 1) there exists a point $\tilde{x}$ such that $\tilde{x} \in \bigcap_{i \in \{1,\ldots,m\}} \Omega_i$. Such a $\tilde{x}$ would therefore be feasible for problem $\mathcal{P}$; moreover, it would yield an objective value $\tilde{J} = c^T \tilde{x} < c^T x^* = J^*$ (since $\tilde{x} \in \mathcal{H}$). This is a contradiction, because x^* would no longer be an optimal solution for $\mathcal{P}$, and hence we conclude that $q \leq n$. ⋆

We are now ready to present a proof of Theorem 1.

3.2 Proof of Theorem 1

We prove that the conclusion in Theorem 1 holds with

$$N \geq \frac{1}{1-\gamma}\left(\frac{1}{\epsilon}\ln\frac{1}{\beta} + n + \frac{n}{\epsilon}\ln\frac{1}{\gamma\epsilon} + \frac{1}{\epsilon}\ln\left(\left(\frac{n}{e}\right)^n \frac{1}{n!}\right)\right), \tag{11}$$

where γ is a parameter that can be freely selected in $(0,1)$. To prove that bound (6) follows from (11), proceed as follows. Since $n! \geq (n/e)^n$, the last term in (11) is not positive and can be dropped, leading to the bound

$$N \geq \frac{1}{1-\gamma}\left(\frac{1}{\epsilon}\ln\frac{1}{\beta} + n + \frac{n}{\epsilon}\ln\frac{1}{\gamma\epsilon}\right), \tag{12}$$

Bound (6) is then obtained from (12), by taking $\gamma = 1/2$. We also note that further optimizing (12) with respect to γ always leads to $\gamma \leq 1/2$ with a corresponding improvement by at most of a factor 2.

Proof of Theorem 1 with (11) in place of (6)

We shall prove that, with probability $1-\beta$, the solution of RCP$_N$ is ϵ-level robustly feasible. This part of the proof in inspired by a similar proof given in a different context in (S. Floyd and M. Warmuth 1995).

Given N scenarios $\delta^{(1)}, \ldots, \delta^{(N)}$, select a subset $I = \{i_1, \ldots, i_n\}$ of n indexes from $\{1, \ldots, N\}$ and let $\hat{x}_I$ be the optimal solution of the program

$$\min_{x \in \mathcal{X} \subseteq R^n} c^T x \quad \text{subject to } f(x, \delta^{(i_j)}) \leq 0,\ j = 1, \ldots, n. \tag{13}$$

Based on $\hat{x}_I$ we next introduce a subset $\mathbf{\Delta}_I^N$ of the set $\mathbf{\Delta}^N$ defined as

$$\mathbf{\Delta}_I^N \doteq \{(\delta^{(1)}, \ldots, \delta^{(N)}) : \ \hat{x}_I = \hat{x}_N\} \tag{14}$$

($\hat{x}_N$ is the optimal solution with all N constraints $\delta^{(1)}, \ldots, \delta^{(N)}$ in place).

Let now I range over the collection $\mathcal{I}$ of all possible choices of n indexes from $\{1, \ldots, N\}$ (notice that $\mathcal{I}$ contains $\binom{N}{n}$ sets). We want to prove that

$$\mathbf{\Delta}^N = \bigcup_{I \in \mathcal{I}} \mathbf{\Delta}_I^N. \tag{15}$$

To show (15), take any $(\delta^{(1)}, \ldots, \delta^{(N)}) \in \mathbf{\Delta}^N$. From the set of constraint $\delta^{(1)}, \ldots, \delta^{(N)}$ eliminate a constraint which is not a support constraint (this is possible in view of Theorem 2 since $N > n$). The resulting optimization problem with $N-1$ constraints admits the same optimal solution $\hat{x}_N$ as the original problem with N constraints. Consider now the set of the remaining $N-1$ constraints and, among these, remove a constraint which is not a support constraint for the problem with $N-1$ constraints. Again, the optimal solution does not change. If we keep going this way until we are left with n constraints, in the end we still have $\hat{x}_N$ as optimal solution, showing that $(\delta^{(1)}, \ldots, \delta^{(N)}) \in \mathbf{\Delta}_I^N$, where I is the set containing the n constraints remaining at the end of the process. Since this is true for any choice of $(\delta^{(1)}, \ldots, \delta^{(N)}) \in \mathbf{\Delta}^N$, (15) is proven.

Next, let

$$B \doteq \{(\delta^{(1)}, \ldots, \delta^{(N)}) : \ V(\hat{x}_N) > \epsilon\}$$

and

$$B_I \doteq \{(\delta^{(1)}, \ldots, \delta^{(N)}) : \ V(\hat{x}_I) > \epsilon\}$$

We now have:

$$\begin{aligned} B &= B \cap \mathbf{\Delta}^N \\ &= B \cap (\cup_{I \in \mathcal{I}} \mathbf{\Delta}_I^N) \quad \text{(apply (15))} \\ &= \cup_{I \in \mathcal{I}} (B \cap \mathbf{\Delta}_I^N) \\ &= \cup_{I \in \mathcal{I}} (B_I \cap \mathbf{\Delta}_I^N). \quad \text{(because of (14))} \end{aligned} \tag{16}$$

A bound for Prob$^N(B)$ is now obtained by bounding Prob$(B_I \cap \mathbf{\Delta}_I^N)$ and then summing over $I \in \mathcal{I}$.

Fix any I, e.g. $I = \{1, \ldots, n\}$ to be more explicit. The set $B_I = B_{\{1,\ldots,n\}}$ is in fact a cylinder with base in the cartesian product of the first n constraint domains (this follows from the fact that condition $V(\hat{x}_{\{1,\ldots,n\}}) > \epsilon$ only involves the first n constraints). Fix $(\bar{\delta}^{(1)}, \ldots, \bar{\delta}^{(n)}) \in$ base of the cylinder. For a point $(\bar{\delta}^{(1)}, \ldots, \bar{\delta}^{(n)}, \delta^{(n+1)}, \ldots, \delta^{(N)})$ to be in $B_{\{1,\ldots,n\}} \cap \mathbf{\Delta}^N_{\{1,\ldots,n\}}$, constraints $\delta^{(n+1)}, \ldots, \delta^{(N)}$ must be satisfied by $\hat{x}_{\{1,\ldots,n\}}$, for, otherwise, we would not have $\hat{x}_{\{1,\ldots,n\}} = \hat{x}_N$, as it is required in $\mathbf{\Delta}^N_{\{1,\ldots,n\}}$. But, $V(\hat{x}_{\{1,\ldots,n\}}) > \epsilon$ in $B_{\{1,\ldots,n\}}$. Thus, by the fact that the extractions are independent, we conclude that

$$\begin{aligned}\text{Prob}^{N-n}\{(\delta^{(n+1)}, \ldots, \delta^{(N)}) : \ & (\bar{\delta}^{(1)}, \ldots, \bar{\delta}^{(n)}, \delta^{(n+1)}, \ldots, \delta^{(N)}) \\ & \in B_{\{1,\ldots,n\}} \cap \mathbf{\Delta}^N_{\{1,\ldots,n\}}\} < (1-\epsilon)^{N-n}.\end{aligned}$$

The probability on the left hand side is nothing but the conditional probability that $(\delta^{(1)}, \ldots, \delta^{(N)}) \in B_{\{1,\ldots,n\}} \cap \mathbf{\Delta}^N_{\{1,\ldots,n\}}$ given $\delta^{(1)} = \bar{\delta}^{(1)}, \ldots, \delta^{(n)} = \bar{\delta}^{(n)}$. Integrating over the base of the cylinder $B_{\{1,\ldots,n\}}$, we then obtain

$$\text{Prob}^N(B_{\{1,\ldots,n\}} \cap \mathbf{\Delta}^N_{\{1,\ldots,n\}}) < (1-\epsilon)^{N-n} \cdot \text{Prob}^n(\text{base of } B_{\{1,\ldots,n\}}) \leq (1-\epsilon)^{N-n}. \tag{17}$$

From (16), we finally arrive to the desired bound for $\text{Prob}^N(B)$:

$$\text{Prob}^N(B) \leq \sum_{I \in \mathcal{I}} \text{Prob}^N(B_I \cap \mathbf{\Delta}_I) < \binom{N}{n} (1-\epsilon)^{N-n}. \tag{18}$$

The last part of the proof is nothing but algebraic manipulations on bound (18) to show that, if N is chosen according to (11), then

$$\binom{N}{n} (1-\epsilon)^{N-n} \leq \beta, \tag{19}$$

so concluding the proof.

Any of the following inequality implies the next in a top-down fashion, where the first one is (11):

$$\begin{aligned}
N &\geq \frac{1}{1-\gamma}\left(\frac{1}{\epsilon}\ln\frac{1}{\beta} + n + \frac{n}{\epsilon}\ln\frac{1}{\gamma\epsilon} + \frac{1}{\epsilon}\ln\left(\left(\frac{n}{e}\right)^n \frac{1}{n!}\right)\right) \\
(1-\gamma)N &\geq \frac{1}{\epsilon}\ln\frac{1}{\beta} + n + \frac{n}{\epsilon}\ln\frac{1}{\gamma\epsilon} + \frac{1}{\epsilon}\ln\left(\left(\frac{n}{e}\right)^n \frac{1}{n!}\right) \\
(1-\gamma)N &\geq \frac{1}{\epsilon}\ln\frac{1}{\beta} + n + \frac{n}{\epsilon}\left(\ln\frac{n}{\gamma\epsilon} - 1\right) - \frac{1}{\epsilon}\ln(n!) \\
N &\geq \frac{1}{\epsilon}\ln\frac{1}{\beta} + n + \frac{n}{\epsilon}\left(\ln\frac{n}{\gamma\epsilon} - 1 + \frac{\gamma N\epsilon}{n}\right) - \frac{1}{\epsilon}\ln(n!) \\
N &\geq \frac{1}{\epsilon}\ln\frac{1}{\beta} + n + \frac{n}{\epsilon}\ln N - \frac{1}{\epsilon}\ln(n!),
\end{aligned} \tag{20}$$

where the last implication can be justified by observing that $\ln x \geq 1 - \frac{1}{x}$, for $x > 0$, and applying this inequality with $x = \frac{n}{\gamma N\epsilon}$. Proceeding from (20), the next inequalities in the chain are

$$\begin{aligned}
\ln\beta &\geq -\epsilon N + \epsilon n + n\ln N - \ln(n!) \\
\beta &\geq \frac{N^n}{n!} e^{-\epsilon(N-n)} \\
\beta &\geq \frac{N(N-1)\cdot\cdots\cdot(N-n+1)}{n!}(1-\epsilon)^{N-n},
\end{aligned}$$

where, in the last implication, we have used the fact that $e^{-\epsilon(N-n)} \geq (1-\epsilon)^{N-n}$, as it follows by taking logarithm of the two sides and further noting that $-\epsilon \geq \ln(1-\epsilon)$. Proceeding,

$$\beta \geq \binom{N}{n} (1-\epsilon)^{N-n}, \tag{21}$$

which is (19).

4 A numerical example

As a simple numerical example of application of the scenario methodology, we consider the following linear program with uncertain constraint matrix

$$\min_{x \in R^5} c^T x \quad \text{subject to } (A + A_\delta)x \leq b, \tag{22}$$

where

$$A \begin{bmatrix} 13 & -3 & -24 & 7 & -4 \\ 19 & 2 & -11 & 7 & 14 \\ 7 & 6 & -4 & 6 & -6 \\ 8 & -6 & -21 & -1 & 2 \\ -2 & 2 & 15 & -12 & 7 \\ -1 & 3 & 2 & 21 & -10 \\ -9 & 5 & 6 & -14 & 6 \\ 4 & -7 & -12 & 4 & 17 \\ 12 & 13 & 1 & 3 & 0 \\ 12 & 9 & 16 & 20 & 25 \end{bmatrix}$$

$$c^T = \begin{bmatrix} 0 & -1 & -1 & 0 & 0 \end{bmatrix}$$

$$b^T = \begin{bmatrix} -23 & 39 & -5 & -18 & 51 & 61 & 23 & 17 & -22 & 1 \end{bmatrix},$$

and where the entries of the uncertainty matrix $A_\delta \in R^{10,5}$ are independent Normal random variables with zero mean and variance equal to 0.5. If the uncertain constraints are imposed up to a given level of probability ϵ, we obtain the chance-constrained problem

$$\min_{x \in R^5} c^T x \quad \text{subject to Prob}\{(A + A_\delta)x \not\leq b\} \leq \epsilon. \tag{23}$$

Notice that this problem is readily put in the form (2), by rewriting the element-wise constraints in scalar form, i.e. setting $f(x, \delta) = \max_j [A + A_\delta]_j x - [b]_j$, where $[\cdot]_j$ here denotes the j-th row of its argument, and vector δ contains the uncertain entries of matrix A_δ.

Now, fixing probability levels $\epsilon = 0.01$, $\beta = 0.001$, and using bound (6), we obtain

$$N(\epsilon, \beta) \simeq 6689.9.$$

Fixing then $N = 6690$, we solve the scenario problem

$$\min_{x \in R^5} c^T x \quad \text{subject to } (A + A_{\delta^{(i)}})x \leq b, \quad i = 1, \ldots, N = 6690, \tag{24}$$

where $A_{\delta^{(i)}}$ are the sampled uncertainty matrices, extracted in accordance with the assumed probability distribution (i.e. each entry is Normal with zero mean and variance equal to 0.5). One instance of the scenario problem then yielded an optimal objective $J_{RCP_N} = -3.2750$, which was attained for

$$\hat{x}_N = \begin{bmatrix} -2.0973 \\ -1.1436 \\ 4.4186 \\ 0.8752 \\ -2.4418 \end{bmatrix}.$$

Notice that the nominal problem (i.e. the problem obtained by fixing $A_\delta = 0$) yields optimal objective $J_{\text{nom}} = -5.3901$, which is attained for

$$x_{\text{nom}} \begin{bmatrix} -2.0958 \\ -0.0719 \\ 5.4620 \\ -0.4594 \\ -5.7843 \end{bmatrix}.$$

We next proceeded with an a-posteriori Monte-Carlo test on the obtained solutions. Fixing $\tilde{\epsilon} = 0.001$ and $\tilde{\beta} = 0.00001$, we have from (7) that the a-posteriori test should be run using at least 6.1030×10^6 samples. Setting $\tilde{N} = 6.1030 \times 10^6$ we obtained

$$\hat{V}_{\tilde{N}}(\hat{x}_N) = 0.0009,$$

leading to the estimate $V_{\tilde{N}}(\hat{x}_N) \leq \hat{V}_{\tilde{N}}(\hat{x}_N) + \tilde{\epsilon} = 0.0019$. Interestingly, running the same test on the nominal solution yielded

$$\hat{V}_{\tilde{N}}(x_{\text{nom}}) = 0.969$$

showing, as expected, that while the nominal problem provides a better optimal objective with respect to the scenario problem, the nominal solution is actually infeasible for most of the scenarios that may happen in reality. Contrary, the scenario solution provides a remarkable level of robustness, being in practice feasible in all but 0.2% of the possible situations.

A Relaxing the assumption that RCP$_N$ has a unique solution

In Section 2, the theory has been developed under Assumption 1 requiring that RCP$_N$ admits a unique optimal solution. Here, we drop Assumption 1 and consider the general case allowing for non-uniqueness of the solution or non-existence of the solution (i.e. the solution escapes to infinity), or even infeasibility of RCP$_N$.

A.1 Non-uniqueness of the solution

Suppose that when problem RCP$_N$ admits more than one optimal solution we break the tie by a tie-break rule as follows:

Tie-break rule: *Let $t_i(x)$, $i = 1, \ldots, p$, be given convex and continuous functions. Among the optimal solutions for RCP$_N$, select the one that minimizes $t_1(x)$. If indetermination still occurs, select among the x that minimize $t_1(x)$ the solution that minimizes $t_2(x)$, and so on with $t_3(x), t_4(x), \ldots$. We assume that functions $t_i(x)$, $i = 1, \ldots, p$, are selected so that the tie is broken within p steps at most. As a simple example of a tie-break rule, one can consider $t_1(x) = x_1, t_2(x) = x_2, \ldots$.* $\star$

From now on, by 'optimal solution' we mean either the unique optimal solution, or the solution selected according to the Tie-break rule, if multiple optimal solutions occur.

Theorem 1 holds unchanged if we drop the uniqueness requirement in Assumption 1, provided that 'optimal solution' is intended in the indicated sense.

To see this, generalize Definition 4 of support constraints to: *The k-th constraint $\mathcal{X}_k$ is a support constraint for $\mathcal{P}$ if problem $\mathcal{P}_k$ has an optimal solution x_k such that $x_k \neq x$* . Indeed this definition generalizes Definition 4 since, in case of a single optimal solution, $x_k \neq x$ is equivalent to $c^T x_k < c^T x$. In (G. Calafiore and M.C. Campi 2003), Section 4.1, it is proven that Theorem 2 holds true with this extended definition of support constraint (i.e. the number of support constraints is at most n), and then an inspection of the proof of Theorem 1 reveals that this proof goes through unaltered in the present setting, so concluding that Theorem 1 still holds.

A.2 Infeasibility of RCP$_N$

It may happen that RCP$_N$ is infeasible (i.e. the intersection of the domains where $f(x, \delta^{(i)}) \leq 0, i = 1, \ldots, N$ is empty), in which case the initial RCP is clearly infeasible too. In this case, going through the proof of Theorem 1 reveals that this theorem remains valid with small amendments: the first part remains unchanged, while the final part reads: "*..., with probability no smaller than $1 - \beta$, either the scenario problem* RCP$_N$ *is infeasible; or it is feasible, and then its optimal solution $\hat{x}_N$ is ϵ-level robustly feasible.*"

A.3 Non-existence of the solution

Even when RCP$_N$ is feasible, it may happen that no optimal solution exists since the set for x allowed by the extracted constraints is unbounded in such a way that the optimal solution escapes to infinity. In this section, we further generalize Theorem 1 so as to cope with this situation too and then provide a reformulation of Theorem 1 (Theorem 3 below) that covers all possible situations as indicated in Sections A.1, A.2 and the first part of this section.

Suppose that a random extraction of a multi-sample $\delta^{(1)}, \ldots, \delta^{(N)}$ is rejected when the problem is feasible but no optimal solution exists, and another extraction is performed in this case. Then, the result of Theorem 1 holds if attention is restricted to the accepted multi-samples. This idea is now formalized.

Let $D \subseteq \mathbf{\Delta}^N$ be the set where RCP_N is feasible but an optimal solution does not exist (it escapes to infinity) and assume that its complement $A = D^c$ has positive probability: $\mathrm{Prob}^N(A) > 0$. Moreover, let Prob^N_A be the probability Prob^N restricted to A: $\mathrm{Prob}^N_A(\cdot) \doteq \mathrm{Prob}(\cdot \cap A)/\mathrm{Prob}^N(A)$. Prob^N_A is therefore a conditional probability. In addition, assume that if a problem with, say, m constraints is feasible and admits optimal solution, then, after adding an extra $(m+1)$-th constraint, if the problem remains feasible, than an optimal solution continues to exist (this rules out the possibility of pathological situations where adding a constraint forces the solution to drift away to infinity).

The following theorem holds:

Theorem 3. *Fix two real numbers $\epsilon \in (0,1)$ (level parameter) and $\beta \in (0,1)$ (confidence parameter). If*

$$N \geq N(\epsilon, \beta) \doteq \frac{2}{\epsilon} \ln \frac{1}{\beta} + 2n + \frac{2n}{\epsilon} \ln \frac{2}{\epsilon},$$

then, with probability Prob^N_A no smaller than $1-\beta$, either the scenario problem RCP_N is infeasible; or it is feasible, and then its optimal solution $\hat{x}_N$ (unique after the Tie-break rule has been applied) is ϵ-level robustly feasible. $\star$

The proof of this theorem is here omitted and can be found in (G. Calafiore and M.C. Campi 2003).

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On Composition of Probability Density Functions

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Abstract. This paper deals with composition of probability density functions (pdfs) in a sense of composition of partial information pieces. There are many approaches dealing with the problem of pdfs composition, however, they are not suitable for the purpose of multiple participant decision making. Therefore, the problem of pdf composition is formulated in a more suitable way – as a minimization of weighted sum of Kullback-Leibler divergences. Both variants of this formulation differing in the order of arguments of Kullback-Leibler divergence are discussed. Finally, a link between the mentioned approach and Bayesian estimation with incomplete data is outlined.

Keywords: Multi-participant decision making, sensor fusion

1 Introduction

A task of composition of information pieces to a global information arises in various fields. For example, every animal forms its image of the environment on the basis of information provided by its senses. This information pieces are of different nature, they may be partially contradictory to each other, or some of them can be missing. In spite of that, animals are able to process such information. In a similar way, a group of experts reaches their common attitude as a compromise according to knowledge of particular members – sometimes also even though the information available to the experts partially differ or they are incomplete.

The information is often represented as a probability density function (pdf) of a considered quantity (Jensen 2001). In this sense, the composition of information pieces corresponds with composition of pdfs. The aim of the paper is to formulate the task of pdf composition so it is suitable for decision making problems with multiple participants (Guy & Kárný 2004), and point out to the approaches, which could be suitable for the solution. Examples of such multiple participant decision making are right the above mentioned.

The paper is structured as follows: In the second section two simple illustrative examples are presented. They, among others, form basis for the formulation of the task in the next section. State of the art is briefly discussed in Section 4, and in Section 5 several, hopefully, novel approaches are outlined and discussed. Section 6 includes concluding remarks.

2 Examples

This section consists of two examples on which the problem of pdf composition is illustrated. At the same time, the examples are used as a basis for a problem formulation below.

2.1 E-Democracy

The example described below belongs to a field of multiple-participant decision making. For details see (Kárný & Kracík 2003). A system consisting of a group of local decision-makers and one global decision-maker is considered. The local decision makers may have different interests, different goals, and different decision strategies. The aim of the global decision-maker is to find a decision strategy (influencing the whole system) so that a certain compromise among goals of local decision-makers is achieved.

Our approach to this problem is to formulate it as a control task and to find the optimal strategy using, so called, *fully probabilistic design (FPD)* (Kárný 1996) . FPD has proved itself many times, nevertheless it has been never used with the control objective consisting of few partial and generally opposing objectives.

Before we approach to the example itself we briefly explain the basic idea of the FPD.

2.1.1 Ideal Pdf and Fully Probabilistic Design

The fully probabilistic design is formulated in the following manner (Kárný 1996).

Let us consider a stochastic dynamic system described by random quantities $\__t$, a_t, where $t \in t^* \equiv \{1, \ldots, t\}$. a_t are called *actions* – values of these quantities are directly chosen by a decision maker. Quantities $\__t$, called *innovations*, represent observable consequences of the previous action a_t and all previous data $d(t-1) \equiv (d_{t-1}, d_{t-2}, \ldots, d_1)$, where $d_t \equiv (\__t, a_t)$.

The joint pdf $f(d(t))$, describing the entire system, can be factorized by a repetitive use of the chain rule (Peterka 1981)

$$f(d(t)) \equiv f((\_(t)), a(t)) = \prod_{t \in t^*} f(\__t | a_t, d(t-1)) f(a_t | d(t-1)).$$

The pdfs $\{f(\__t | a_t, d(t-1))\}_{t \in t^*}$ form, so called, outer model of the system, and describe the response of the system to the actions a_t for given past data $d(t-1)$. The pdfs $\{f(a_t | d(t-1))\}_{t \in t^*}$ represent the selected randomized decision strategy.

A target of the fully probabilistic design is specified by an *ideal pdf*

$$f^I(d(t)) = \prod_{t \in t^*} f^I(\__t | a_t, d(t-1)) f^I(a_t | d(t-1)).$$

The ideal pdf is assumed to be given. It characterizes the objectives of control.

An optimal decision strategy $\{f^o(a_t | d(t-1))\}_{t \in t^*}$ is then selected among randomized decision strategies $\{f(a_t | d(t-1))\}_{t \in t^*}$ as a minimizer of the Kullback-Leibler divergence (Kullback & Leibler 1951) of pdfs f and f^I

$$\mathcal{D}(f || f^I) = \int f(\_(t), a(t)) \ln \frac{f(\_(t), a(t))}{f^I(\_(t), a(t))} \mathrm{d}(\_(t), a(t)).$$

Note that a decision strategy $\{f(a_t | d(t-1))\}_{t \in t^*}$ forms a part of $f(\_(t), a(t))$.

2.1.2 FPD of Government Strategy

The example is as follows. Let us consider a group of local decision makers – e.g. *citizens*, and one global decision maker – e.g. *government*. Each citizen deals with his data sequence $d_c(t) \equiv (\__{c;t}, a_{c;t})_{t \in t^*}$, $c \in c^* \equiv \{1, \ldots, c\}$, consisting of his actions and his innovations. Individual citizens may deal with different data (in this sense, the term *data* coincides with *quantity*) – it means that these citizens have different fields of interests. Citizens are assumed to formulate their objectives in a form of the ideal pdfs, i.e., they specify $f_c^I(d_c(t)), \forall c \in c^*$.

It is natural to expect that there are groups of citizens with the same interests, i.e., the same data $d_c(t)$, the same decision strategies, and the same ideal pdfs $d_c^I(t)$, i.e., with the same objectives and restrictions. From now on, under the term *citizen* we will understand the *group of citizens* with the same data, decision strategy, and ideal pdf.

The government generates its actions $a_g(t)$, which influence the citizens. Furthermore, we can suppose that data available to the government $d_g(t)$ include data of all individual citizens, i.e., $d_c(t) \subset d_g(t), \forall c \in c^*$.

To be able to select a decision strategy using the FPD, the government needs to have an outer model of the "world" $\{f(\__{g;t} | a_{g;t}, d_g(t-1))\}_{t \in t^*}$ and its own ideal pdf $f_g^I(d_g(t))$. The government is supposed to have the outer model at disposal, or, at least, to be able to construct the model (e.g. using combination of expert knowledge and information contained in past observed data). Note that the outer model, among others, describes the citizens' decision strategies – we do not suppose that government knows these strategies, but only that it has its own model of them. The question is *how to construct the government ideal pdf* $f_g^I(d_g(t))$.

Let us suppose that the government is not dictatorial, and it wants to govern in a fair way, i.e., in accordance with citizens wishes. In words of the FPD, it means that the government ideal pdf is, in a certain sense, "close" to the ideal pdfs of individual (groups of) citizens. When more citizens have a nonempty intersection of data spaces and different ideal pdfs on this intersection then the government ideal pdf have to reach a compromise among them. This compromise should respect sizes of these groups – the bigger group the closer is the relevant marginal pdf to the ideal pdf of this group.

Note that having a government ideal pdf (in a suitable form) the design of government decision strategy is straightforward because the exact solution of the FPD is known. *The key problem is a construction of government ideal pdf "close" to the citizens ideal pdfs while considering the sizes of groups of citizens with the same ideal pdfs.*

2.2 Sensor Fusion

This example belongs to a field referred to as a *sensor fusion*. In general, problems of sensor fusion can be characterized as follows.

Let us consider a group of several sensors – e.g. sensors of a mobile robot. The sensors give information on the quantities measured in their neighbourhoods. Naturally, the quantities measured by different sensors may overlap, and the information on the same quantity, given by different sensors, may differ. It is caused by different sensor construction, different way of data processing, measurement noise, etc. The problem in hand is how to put the information about the individual sensors together so it gives a global information of the whole neighbourhood of the robot.

For simplicity, the example is described as a static task. In this example we use the following notation.

Sensors: Let us consider a group of n sensors denoted $s_1, \ldots, s_n$.
Data: Data measured by the sensor s_i (to a certain time) we denote $\mathcal{P}_i$.
Positions: q denotes a position in the robot's neighbourhood. The set of all positions is denoted q^*.
State: Physical state of the environment in the position q is x_q (a finite dimensional vector).
Observed areas: q_i^* q^* is the area (i.e. set of positions) observed by the sensor s_i.
States in an area: $x(q_i^*)$ $(x_q)_{q \in q_i^*}$ is a vector of states in the area q_i^*. Vector of states in the whole observable area is x_U $x(\bigcup_i q_i^*)$.

Our approach (Kárný, et al. 2004) to the problem of sensor fusion is based on the Bayesian data processing, therefore the information obtained from the sensor (say i-th) is represented by the posterior pdf $f_i(x(q_i^*)|\mathcal{P}_i)$ of states $x(q_i^*)$ in the area observed by the sensor. In this sense, an information fusion corresponds to finding a joint pdf $\tilde{f}(x_U)$ of all observable quantities. Of course, the joint pdf (global information) $\tilde{f}(x_U)$ should correspond, in a certain sense, to the individual partial pdfs (local information) $f_i(x(q_i^*))$ $(\forall i \in \{1, \ldots, n\})$.

In an ideal case, the distribution $\tilde{f}(x_U)$ should fulfil

$$\tilde{f}(x(q_i^*)) = f_i(x(q_i^*)|\mathcal{P}_i),\ \forall i \in \{1, \ldots, n\}. \tag{1}$$

It is obvious that a necessary (but not sufficient) condition for an existence $\tilde{f}(x_U)$ with the property 1 is

$$f_i(x(q_i^* \cap q_j^*)|\mathcal{P}_i) = f_j(x(q_i^* \cap q_j^*)|\mathcal{P}_j),\ \forall i, j \in \{1, \ldots, n\}. \tag{2}$$

In real applications, of course, we can not expect that the conditions 2 are fulfilled because the posterior pdfs $f_i(x(q_i^*)|\mathcal{P}_i)$ are provided by different sensors working with different imperfection and modelled by different imprecise models, and, furthermore, the pdfs are based on different finite data samples. For these reasons it is necessary to formulate the relation between the searched pdf $\tilde{f}(x_U)$ and given posterior pdfs $f_i(x(q_i^*)|\mathcal{P}_i)$ in another – less strict – way.

If the conditions 2 are fulfilled they, in general, do not determine the pdf $\tilde{f}(x_U)$ unambiguously. It is illustrated by the following example.

Example: Let us consider a simple situation when $q^* = \{1, 2\}$ and sensors provide two pdfs, for a simplicity, denoted $f_1(x_1)$ and $f_2(x_2)$. Let us suppose the pdfs $f_1(x_1)$, $f_2(x_2)$ to be normal, i.e., $f_i(x_i) = \mathcal{N}_{x_i}(\mu_i, \sigma_i),\ \mu_i, \sigma_i \in \mathbb{R},\ \sigma_i > 0,\ i \in \{1, 2\}$. Then, obviously, a two-dimensional normal pdf

$$\tilde{f}(x_1, x_2) = \mathcal{N}_{(x_1, x_2)}(\tilde{\mu}, \tilde{\sigma}), \text{where } \tilde{\mu} = (\mu_1, \mu_2),\ \tilde{\sigma} = \begin{matrix} \sigma_{11} & \sigma \\ \sigma & \sigma_{22} \end{matrix}$$

satisfies the condition 1 for an arbitrary $\sigma_{11}\sigma_{22}$ σ^2.

The presented ambiguity is caused by the fact that we know the marginal distributions of $\tilde{f}(x_1, x_2)$ but we know nothing about a correlation of this two quantities. There is a question whether some of possible values of σ, and possibly which one, express this ignorance better than others. Also in the case that the conditions 2 and 1 are not satisfied, the problem of ambiguity of the pdf $\tilde{f}(x_U)$ may arise, and thus it is necessary to formulate additional conditions, preventing the possible ambiguity.

Another thing we have to take into account is that different sensors may produce information of different "quality". The joint pdf $\tilde{f}(x_U)$ searched should look at this differences. The "quality" of information provided by sensors can be specified, for example, by their likelihood ratio. Note that in this case the dimensions of individual quantities have to be taken into account because of their impact on the likelihood value.

Summarized, we are looking for the pdf $\tilde{f}(x_U)$ which *is close to given posterior pdfs $f_i(x(q_i^*)|\mathcal{P}_i)$, respects the different "quality" of information provided by particular sensors, and respects the presented ignorance.*

3 Problem formulation

In this paragraph we formulate a problem outlined in the two previous examples, and stand required features of its solution. In the same time, we introduce a notation used in the rest of the paper.

3.1 Assumptions

Now, let us consider we have given the following items:

Multidimensional random quantity $\mathbf{X} \quad (X_1, X_2, \ldots, X_N),\ N \in \mathbb{N}$; X_i has values in X_i^*. $\mathbf{x}$ denotes a vector $(x_1, x_2, \ldots, x_N)$ of values of the quantity $\mathbf{X}$.
Note: This multidimensional quantity comports with all government data in the first example, or with states in all observable positions in the second example.
Nonempty sets $q_1^*, q_2^*, \ldots, q_n^*,\ n \in \mathbb{N}, q_r^* \quad \{1, 2, \ldots, N\}$, for $r \in \{1, 2, \ldots, n\}$. $\mathbf{X}_r$, denotes multidimensional random quantities $(X_i)_{i \in q_r^*}$; $\mathbf{x}_r$ denotes their values, i.e., $\mathbf{x}_r \quad (x_i)_{i \in q_r^*}$.
Note: In general, *the sets q_r^* need not to be disjoint*; In the examples, random quantities $\mathbf{X}_r$ comports with data of individual citizens , and with states in positions observed by individual sensors respectively.
Joint pdfs $f_r(\mathbf{x}_r)$ of quantities $\mathbf{X}_r,\ r \in \{1, 2, \ldots, n\}$.
Note: This pdfs represent ideal pdfs of individual citizens, or information provided by individual sensors. In general, it need not hold $f_r((x_i)_{i \in q_r^* \cap q_s^*}) = f_s((x_i)_{i \in q_r^* \cap q_s^*})$, for $r \neq s$, i.e., different citizens may have different preferences of the same quantity, and different sensors may provide different information about states in the same position.
Weights $\quad_1, \quad_2, \ldots, \quad_n. \quad_r \quad 0, \sum_{r=1}^{n} \quad_r = 1$. This weights represents a "significance" of individual pdfs f_r.
Note:The weights may correspond to a ratio of citizens in the particular groups, or to our belief in the quality of the particular sensors (expressed, e.g., by the ratio of likelihood).

3.2 The Addressed Task

The task, in general, can be formulated as follows.

Find a joint pdf $\tilde{f}(\mathbf{x})$ of the quantity $\mathbf{X}$ so it is, as much as possible, harmonized with given pdfs $f_r(\mathbf{x}_r)$ of quantities $\mathbf{X}_r$. The information represented by the joint pdf $\tilde{f}(\mathbf{x})$ should have a clear relationship with information represented by pdfs $f_r(\mathbf{x}_r)$ with respect to weights $\quad_1, \quad_2, \ldots, \quad_n$. At the same time, we require the pdf $\tilde{f}(\mathbf{x})$ to contain no information which is not contained in particular pdfs $f_r(\mathbf{x}_r)$.

Of course, this formulation is absolutely imprecise, nevertheless, it expresses our objectives quite well. That is to say, the assignment in hand is not only to find a solution but to find a suitable (exact) problem formulation, which is in accord with the "fuzzy" one mentioned above and which leads to a practically solvable task. Therefore, the individual (exact) problem formulations are discussed together with the possible solutions in the following paragraphs. Naturally, different task formulations give different interpretation of their results, nevertheless every such problem formulation corresponding to the "fuzzy" one, and practically solvable, (together with its solution) would be valuable result.

Because the task in hand is closely connected with practical problems, the solution must be applicable, i.e., it must be possible to implement it, and its time and memory complexity must be bearable. From this point of view, problems may be caused by time-demanding optimization algorithms, memory- demanding representation of pdfs (especially in discrete case), or a problematic representation of continuous pdfs in computer.

4 State of the Art

Nowadays a fully suitable solution of the problem presented in the preceding section probably does not exist. Of course, there are many publications dealing with similar problems, however, all of them differs from what we require in important details.

In general, we can say that available solutions do not meet our requirements from the following reasons:

The connection between the information represented by the original pdfs and the resulting one is not clear.
The resulting pdf depends on the order in which the partial pdfs are processed.

The resulting pdf is not unambiguous.
Majority of available solutions are oriented towards discrete pdfs only.

The composition of pdfs arises, e.g., in (Jensen 2001) in connection with disagreement of information (in form of pdfs) provided by a few experts. The compromise is taken as a mean of these pdfs or weighted mean in case that the confidence in the experts varies. Such an operation seems to be reasonable but a relationship between the original expert information and the "new" one is unclear. In (Jiroušek 2003) a composition of pdfs is used as an approach to description and processing pdfs of huge dimensions. The pdf resulting from this composition is dependent on the order of particular processed pdfs. This is not a flaw because the goal of the author varies from our one, nevertheless, this type of composition is not suitable for our purposes.

5 Other Approaches

In this paragraph several more approaches are presented and discussed. None of them represents an applicable solution. A goal is rather to demonstrate Other possibilities, their properties, and problems related to them.

5.1 Minimization of Weighted sum of KL Divergences f_r and $\tilde{f}$

Let us search for $\tilde{f}(\mathbf{x})$ among minimizers of weighted sum of Kullback-Leibler divergences of pdfs f_r and corresponding marginal pdfs $\tilde{f}$, i.e.,

$$\tilde{f}(\mathbf{x}) \in \arg\min_{\tilde{f}(\mathbf{x})} \sum_{r=1}^{n} {}_r\mathcal{D}(f_r(\mathbf{x}_r)||f(\mathbf{x}_r)). \tag{3}$$

If more than one such pdfs exist then we want the pdf which preserves the presented uncertainty best.

This definition corresponds to the idea that $f_r(\mathbf{x}_r)$ are the "true" pdfs and $\tilde{f}(\mathbf{x})$ is their common approximation. However, this idea is not compatible with the reality because $f_r(\mathbf{x}_r)$ do not have to, and probably they will not, satisfy the conditions 2, and thus they cannot be marginal pdfs of one "true" joint pdf. On the other hand, the pdfs $f_r(\mathbf{x}_r)$ are the only connection to the "reality". From this reason, this approach seems being more suitable for a type of problems similar to that described in the example "E-Democracy" 2.1.

Following examples demonstrate some properties of this approach.

Example 1: Let us consider $\mathbf{X} = (X_1)$, two pdfs $f_1(x_1),\ f_2(x_1)$ of X_1, and weights ${}_1,\ {}_2$. Then for $\tilde{f}(x_1)$ defined according with 3 it holds

$$\tilde{f}(x_1) \in \arg\min_{\tilde{f}(x_1)} \left(\sum_{r=1}^{2} {}_r \int f_r(x_1) \ln f_r(x_1) \mathrm{d}x_1 + \int \left(\sum_{r=1}^{2} {}_r f_r(x_1) \right) \ln \frac{1}{\tilde{f}(x_1)} \mathrm{d}x_1 \right).$$

The first sum in the expression is independent of $\tilde{f}(x_1)$, and the integral on the right hand side is Kerridge inaccuracy (Kulhavý 1996) is known to be minimal for $\tilde{f}(x_1)$ equal to the pdf standing before the logarithm, therefore $\tilde{f}(x_1) = \sum_{r=1}^{2} {}_r f_r(x_1)$. Thus, the resulting $\tilde{f}(x_1)$ is a probabilistic mixture of given pdfs $f_1,\ f_2$ with weights equal to the given coefficients ${}_r$. Of course, the similar result we obtain also for a higher number of given pdfs f_r with identical supports . Note that in this case the result is unambiguous, at least for discrete or absolutely continuous pdfs $f_r(\mathbf{x}_r)$.

Example 2: Now, let us consider $\mathbf{X} = (X_1, X_2, X_3)$, two pdfs $f_1(x_1, x_2),\ f_2(x_2, x_3)$, and weights ${}_1,\ {}_2$. Then, for $\tilde{f}(x_1, x_2, x_3)$ it holds

$$\begin{aligned}\tilde{f}(x_1, x_2, x_3) \in \operatorname*{argmin}_{f(x_1,x_2,x_3)} \Bigg(& {}_1 \int f_1(x_2) \int f_1(x_1|x_2) \ln \frac{f_1(x_1|x_2)}{f(x_1|x_2)} \mathrm{d}x_1 \mathrm{d}x_2 + \\ & + {}_2 \int f_2(x_2) \int f_2(x_3|x_2) \ln \frac{f_2(x_3|x_2)}{f(x_3|x_2)} \mathrm{d}x_3 \mathrm{d}x_2 + \int \sum_{r=1}^{2} {}_r f_r(x_2) \ln \frac{f_r(x_2)}{f(x_2)} \mathrm{d}x_2 \Bigg).\end{aligned}$$

The first integral is minimized by $f(x_1|x_2) = f_1(x_1|x_2)$ independently of $f(x_2)$; Similarly, the second integral is minimized by $f(x_3|x_2) = f_2(x_3|x_2)$ independently of $f(x_2)$ too. The last integral is minimized by $f(x_2) = \sum_{r=1}^{2} {}_r f_r(x_2)$ for the same reasons as in the previous example. Thus, $\tilde{f}(x_1, x_2, x_3)$ is an arbitrary

pdf satisfying conditions

$$\tilde{f}(x_1|x_2) = f_1(x_1|x_2), \quad \tilde{f}_2(x_3|x_2) = f(x_3|x_2), \quad \tilde{f}(x_2) = \sum_{r=1}^{2} {}_r f_r(x_2). \tag{4}$$

A question is whether all of these derivates of $\tilde{f}(x_1, x_2, x_3)$ are equivalent from the point of view of a knowledge composition, and, if not, which of them is the most suitable one. The answer results from the following consideration.

The pdfs $f_1(x_1, x_2)$ and $f_2(x_2, x_3)$ gives no information about a dependence between x_1 and x_3 "directly" but only via their dependencies on x_2. Therefore we should select such $\tilde{f}(x_1, x_2, x_3)$ (satisfying 4) for which the conditional mutual information between x_1 and x_3 given x_2 is minimal.

This is satisfied for

$$\tilde{f}(x_1, x_2, x_3) = f_1(x_1|x_2) f_2(x_3|x_2) \left(\sum_{r=1}^{2} {}_r f_r(x_2) \right). \tag{5}$$

Obviously, it corresponds to conditions 4 and, besides, under this joint pdf, the quantities x_1, x_3 are conditionally independent given x_2, therefore the conditional mutual information is 0. In addition to that, the conditions 4 and the requirement of the minimal conditional mutual information determine the pdf $\tilde{f}(x_1, x_2, x_3)$ unambiguously. Note that 5 does not say that in reality the quantities x_1 and x_3 are conditionally independent given x_2. It only expresses our partial ignorance – everything we know about the relation between x_1 and x_3 is given by their dependence on x_2.

The procedure from Example 2 can be directly used for problems where $f_1(x_1), \ldots, f_r(x_r)$ are given, and there exists $p^* \in \{1, \ldots, N\}$ such that corresponding $q_1^*, \ldots q_r^*$ fulfill $q_r^* \cap q_s^* = p^*, \forall r, s \in \{1, \ldots, n\}$. Nevertheless, this is not enough general and rarely applicable in practical problems. More general, but also much more difficult, is a problem represented in the following example.

Example 3: Let us consider $\mathbf{X} = (X_1, X_2)$, three pdfs $f_1(x_1),\ f_2(x_2),\ f_3(x_1, x_2)$, and weights ${}_1,\ {}_2,\ {}_3$. Then, for $\tilde{f}(x_1, x_2)$ it holds

$$\begin{aligned} \tilde{f}(x_1, x_2) \in \underset{f(x_1,x_2)}{\arg\min} \quad & {}_1 \int f_1(x_1) \ln \frac{f_1(x_1)}{f(x_1)} \mathrm{d}x_1 + \ {}_2 \int f_2(x_2) \ln \frac{f_2(x_2)}{f(x_2)} \mathrm{d}x_2 + \\ & + \ {}_3 \int f_3(x_1, x_2) \ln \frac{f_3(x_1, x_2)}{f(x_1, x_2)} \mathrm{d}x_1 \mathrm{d}x_2 \quad . \end{aligned} \tag{6}$$

Unlike the previous example, we cannot minimize any term in 6 independently on others. This is the simplest example of the task 3 in which the analytical solution cannot be found easily by the mere rearrangement of the Kullback-Leibler divergence. It can be shown (Kracík 2004) that $\tilde{f}(x_1, x_2)$ minimizes 6 iff it holds

$${}_1\tilde{f}(y|x) f_1(x) + \ {}_2\tilde{f}(x|y) f_2(y) + \ {}_3 f_3(x, y) = \tilde{f}(x, y),$$

at least for $f_1,\ f_2,\ f_3$ from "reasonable" classes like, e.g., discrete or absolutely continuous pdfs.

The general analytical solution of these type problems is not known. Nevertheless, there are promising attempts (Kracík 2004) to find the solution, or its approximation, by an iterative algorithm. The algorithm uses the following operator A

$$A(f)(\mathbf{x}) = \sum_{r=1}^{n} {}_r f((x_i)_{i \in q^* \setminus q_r^*} | \mathbf{x}_r) f_r(\mathbf{x}_r) \tag{7}$$

defined for an arbitrary pdf $f(\mathbf{x})$. The key property of this operator is that for an arbitrary pdf $f(\mathbf{x})$ it holds

$$\sum_{r=1}^{n} {}_r \mathcal{D}(f_r(\mathbf{x}_r) || f(\mathbf{x}_r)) \quad \sum_{r=1}^{n} {}_r \mathcal{D}(f_r(\mathbf{x}_r) || A f(\mathbf{x}_r)) \quad \mathcal{D}(A(f(\mathbf{x})) || f(\mathbf{x})), \tag{8}$$

i.e., for every approximation $f(\mathbf{x})$, which is not the optimal one, $A(f)(\mathbf{x})$ gives an approximation which is strictly better then $f(\mathbf{x})$ (in the sense of weighted sum of KL-divergences). A convergence to the optimal pdf is just a hypothesis nowadays, nevertheless partial results and experiments indicate that this hypothesis could be correct.

The most serious problem of the algorithm based on this iterative approach is that for continuous pdfs the particular approximations have a form which is difficult to handle. Therefore, the approximations must be restricted to a certain class of pdfs which is easy to handle and the algorithm must be adjusted to find the best approximation within this class. This problem does not appear in a case of discrete pdfs.

5.2 Minimization of Weighted Sum of KL Divergences $\tilde{f}$ and f_r

In the preceding paragraph the optimal pdf $\tilde{f}(\mathbf{x})$ was defined as a minimizer of weighted sum of the KL-divergences $f(\mathbf{x}_r)$ and $f(\mathbf{x}_r)$. As mentioned, this approach corresponds to the idea that $\tilde{f}(\mathbf{x})$ is a common approximation of "true" pdfs $f_r(\mathbf{x}_r)$, nevertheless, as $f_r(\mathbf{x}_r)$ are not marginal pdfs of one joint pdf, they cannot be "true" (i.e., marginal pdfs of the "true" joint pdf). For these reasons, it could be reasonable to formulate the problem like this: We are search for the "true" joint pdf $\tilde{f}(\mathbf{x})$ so that the weighted sum of KL-divergences (i.e., errors measured by the KL-divergence) $\tilde{f}(\mathbf{x})$ and given pdfs f_r is minimal. The idea that the pdf $\tilde{f}(\mathbf{x})$ is the "true" one determines the inverse order of arguments of KL-divergence in comparison with the preceding paragraph, i.e.,

$$\tilde{f}(\mathbf{x}) \in \arg\min_{f(\mathbf{x})} \sum_{r=1}^{n} {}_r\mathcal{D}(f(\mathbf{x}_r)||f_r(\mathbf{x}_r)). \tag{9}$$

For a brief comparison with the approach presented in the preceding paragraph we use the same examples.

Example 4: Let us consider $\mathbf{X} = (X_1)$, two pdfs $f_1(x_1),\ f_2(x_1)$ of X_1, and weights ${}_1,\ {}_2$. Then for $\tilde{f}(x_1)$ defined according with 9 it holds

$$\tilde{f}(x_1) \in \arg\min_{f(x_1)} \int f(x_1) \ln \frac{f(x_1)}{f_1{}^{1}(x_1) f_2{}^{2}(x_1)} \mathrm{d}x_1.$$

From basic properties of KL-divergence it follows $\tilde{f}(x_1) = \frac{f_1^{\alpha_1}(x_1) f_2^{\alpha_2}(x_1)}{\int f_1^{\alpha_1}(x_1) f_2^{\alpha_2}(x_1) \mathrm{d}x_1}$.

This example demonstrates the main difference between 3 and 9 – for particular pdfs $f_r(\mathbf{x}_r)$ given for the "whole" quantity $\mathbf{x}$, i.e. $\mathbf{x}_r = \mathbf{x}$, $\tilde{f}(\mathbf{x})$ defined by 9 is a normalized weighted geometric mean (while $\tilde{f}(\mathbf{x})$ defined by 3 is a probabilistic mixture). It has, among others, the following important consequence: If for an arbitrary $\mathbf{x} \in X^*$ (i.e., value of $\mathbf{X}$) there exists r such that $f_r(\mathbf{x}) = 0$ then $\tilde{f}(\mathbf{x}) = 0$. Thus, a single pdf $f_r(\mathbf{x}_r)$, even with an arbitrary small weight, may exclude points from the support of the resulting $\tilde{f}(\mathbf{x})$.

Example 5: Now, let us consider $\mathbf{X} = (X_1, X_2, X_3)$, two pdfs $f_1(x_1, x_2),\ f_2(x_2, x_3)$, and weights ${}_1,\ {}_2$. Then, for $\tilde{f}(x_1, x_2, x_3)$ defined by 9 it holds

$$\tilde{f}(x_1, x_2, x_3) \in \operatorname*{argmin}_{f(x_1,x_2,x_3)} {}_1 \int f(x_1, x_2) \ln \frac{f(x_1|x_2)}{f_1(x_1|x_2)} \mathrm{d}x_1 \mathrm{d}x_2 + {}_2 \int f(x_3|x_2) \ln \frac{f(x_3|x_2)}{f_2(x_3|x_2)} \mathrm{d}x_3 \mathrm{d}x_2 + \\ + \int f(x_2) \ln \frac{f(x_2)}{f_1{}^{1}(x_2) f_2{}^{2}(x_2)} \mathrm{d}x_2 \quad .$$

For similar reasons as in Examples 2 and 4, $\tilde{f}(x_1, x_2, x_3)$ is an arbitrary pdf satisfying the conditions

$$\tilde{f}(x_1|x_2) = f_1(x_1|x_2), \quad \tilde{f}(x_3|x_2) = f_2(x_3|x_2), \quad \tilde{f}(x_2) = \frac{f_1{}^{1}(x_2) f_2{}^{2}(x_2)}{\int f_1{}^{1}(x_2) f_2{}^{2}(x_2) \mathrm{d}x_2}. \tag{10}$$

Like in the Example 2, the ambiguity of $\tilde{f}(x_1, x_2, x_3)$ can be suppressed by an additional requirement of minimal conditional mutual information between $\mathbf{x}_1$ and $\mathbf{x}_3$ when $\mathbf{x}_2$ given. Then, the optimal pdf is

$$\tilde{f}(x_1, x_2, x_3) = f_1(x_1|x_2) f_2(x_3|x_2) \frac{f_1{}^{1}(x_2) f_2{}^{2}(x_2)}{\int f_1{}^{1}(x_2) f_2{}^{2}(x_2) \mathrm{d}x_2}.$$

Example 6: Let us consider $\mathbf{X} = (X_1, X_2)$, three pdfs $f_1(x_1),\ f_2(x_2),\ f_3(x_1, x_2)$, and weights ${}_1,\ {}_2,\ {}_3$. Then, for $\tilde{f}(x_1, x_2)$ it holds

$$\tilde{f}(x_1, x_2) \in \arg\min_{f(x_1,x_2)} {}_1 \int f(x_1) \ln \frac{f(x_1)}{f_1(x_1)} \mathrm{d}x_1 + {}_2 \int f(x_2) \ln \frac{f(x_2)}{f_2(x_2)} \mathrm{d}x_2 + \\ + {}_3 \int f(x_1, x_2) \ln \frac{f(x_1, x_2)}{f_3(x_1, x_2)} \mathrm{d}x_1 \mathrm{d}x_2 \quad . \tag{11}$$

Again, finding the optimal pdf $\tilde{f}(x_1, x_2)$ is more difficult then in the two preceding examples.

For $f_1(x_1),\ f_2(x_2),\ f_3(x_1, x_2)$ from "reasonable" classes, it can be shown that $\tilde{f}(x_1, x_2)$ is the minimizer of 11 iff it holds

$$c\ (f_1(x_1)f(x_2|x_1))^{\ 1}\ (f_2(x_2)f(x_1|x_2))^{\ 2}\ (f_3(x_1, x_2))^{\ 3} = f(x_1, x_2), \tag{12}$$

where c is a normalization constant. Probably, a general analytical solution of 12 is not known. Nevertheless, the solution of 12 reduces to a solution of an algebraic equation for parameters of $\tilde{f}(x)$ if $f_1(x_1),\ f_2(x_2),\ f_3(x_1, x_2)$ are from a given exponential family.

Another possibility how to find an approximation of $\tilde{f}(x)$, is an iterative algorithm similar to that mentioned in the preceding paragraph. In this case, if $f_1(x_1),\ f_2(x_2),\ f_3(x_1, x_2)$ and the starting approximation are from a given exponential family, the iterative algorithm keeps all subsequent approximations in this family.

5.3 Bayesian estimation with incomplete data records

In this paragraph an approach completely different from the two preceding is presented. A different point of view is outlined rather then an algorithmic solution.

In many practical problems the pdfs $f_r(\mathbf{x}_r)$ are posterior pdfs obtained by sequential Bayesian estimation. Individual pdfs f_r are obtained from different data samples, in a sense of realizations of different random quantities. The idea presented in this paragraph is to create the pdf $\tilde{f}(\mathbf{x})$ as a posterior Bayesian estimate from all available data samples. However, this task differs from a common Bayesian estimation as the available data samples are incomplete.

The problem of Bayesian estimation with incomplete data is illustrated on the example of Sensor Fusion from the paragraph 2.2. As mentioned above, information provided by particular sensor, say r, corresponds with posterior pdfs $f_r(\mathbf{x}_r|\mathcal{P}_r)$, where $\mathcal{P}_r \quad (d_{r;t})_{t\in T_r^*}$ denotes data samples available to this sensor. To be able to compute the posterior pdf $f(\mathbf{x}_r|\mathcal{P}_r)$ we assume that the model $h_r(d_r|\mathbf{x}_r, \mathbf{y}_r)$ and prior pdf $f_r(\mathbf{x}_r, \mathbf{y}_r)$ are available. The model $h_r(d_r|\mathbf{x}_r, \mathbf{y}_r)$ describes how the data obtained by the sensor r depend on the quantity of our interest $(\mathbf{x}_r)$ and on the unknown parameter of the sensor itself $(\mathbf{y}_r)$. Then, the posterior pdf is

$$f_r(\mathbf{x}_r, \mathbf{y}_r|\mathcal{P}_r) = \frac{f_r(\mathbf{x}_r, \mathbf{y}_r)h(\mathcal{P}_r|\mathbf{x}_r, \mathbf{y}_r)}{\int f_r(\mathbf{x}_r, \mathbf{y}_r)h(\mathcal{P}_r|\mathbf{x}_r, \mathbf{y}_r)\mathrm{d}\mathbf{x}_r\mathrm{d}\mathbf{y}_r} = \frac{f_r(\mathbf{x}_r, \mathbf{y}_r)\prod_{t\in T_r^*} h(d_{r;t}|\mathbf{x}_r, \mathbf{y}_r)}{\int f_r(\mathbf{x}_r, \mathbf{y}_r)\prod_{t\in T_r^*} h(d_{r;t}|\mathbf{x}_r, \mathbf{y}_r)\mathrm{d}\mathbf{x}_r\mathrm{d}\mathbf{y}_r}, \tag{13}$$

however, only its marginal pdf $f_r(\mathbf{x}_r|\mathcal{P}_r)$ is in our interest.

In accordance with the procedures from the two preceding paragraphs, we would compute the particular pdfs $f(\mathbf{x}_r|\mathcal{P}_r)$ first and then construct the pdf $\tilde{f}(\mathbf{x})$ as a certain compromise among them. As an alternative to that, we can directly generalize the formula 13 so it provides a posterior pdf of all parameters. Thus, we obtain

$$f(\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_n|\mathcal{P}_1, \ldots, \mathcal{P}_n) = \frac{f(\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_n)\prod_{r=1}^{n}\prod_{t\in T_r^*} h(d_{r;t}|\mathbf{x}_r, \mathbf{y}_r)}{\int f(\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_n)\prod_{r=1}^{n}\prod_{t\in T_r^*} h(d_{r;t}|\mathbf{x}_r, \mathbf{y}_r)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}_1 \quad \mathrm{d}\mathbf{y}_n}, \tag{14}$$

where $f(\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_n)$ is a prior pdf (which is assumed to be given).

Considering the problem modelled, it is natural to expect that the data $d_{r;t}$ are conditionally independent of $\mathbf{x}_s, \mathbf{y}_s$, for $s \neq r$, given $\mathbf{x}_r, \mathbf{y}_r$. Thus, $h(d_r|\mathbf{x}_r, \mathbf{y}_r)$ in 14 can be taken as a marginal pdf of a "global" model $h(d_1, \ldots, d_n|\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_n)$ and each data sample $d_{r;t}$ as an incomplete record of $d_{1,\ldots,n;t}$.

If the models $h(d_r|\mathbf{x}_r, \mathbf{y}_r)$ belongs to a given exponential family the posterior pdf $f(\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_n|\mathcal{P}_1, \ldots, \mathcal{P}_n)$ holds a compact form (product of prior pdf and n appropriate conjugate pdfs) but computing the normalizing integral or even predictions will be a hard task. Nevertheless, the posterior pdf can be used at least for point estimation of parameters (maximum posterior estimation) or for numerical simulations.

However, much more interesting should be the relationship between the estimation with incomplete data and procedures outlined it the two preceding paragraphs (especially that in paragraph 5.1) applied to the posterior pdfs $f_r(\mathbf{x}_r, \mathbf{y}_r|\mathcal{P}_r)$ 13. It is possible to expect that the relationship is very close because Bayesian estimation is known to concentrate asymptotically the mass of the posterior pdf on parameters which minimize the KL divergence between the model and the "true" pdf according to which data are generated. Thus, the composition of particular posterior pdfs 13 via minimization of the KL divergences may be a suitable substitution for proper Bayesian estimation from incomplete data.

6 Concluding Remarks

The problem of composition of pdfs, in the form presented in the third section, has probably no fully suitable solution nowadays. Existing approaches, partially dealing with the problem, do not meet our requirements from various reasons like unclear relationship of information represented by original pdfs and resulting one, dependence on the order of processed pdfs, ambiguity of the resulting pdf, orientation towards discrete quantities only, etc.

Therefore, the problem of pdf composition has been formulated in a form which would satisfy our requirements – as a minimization of a weighted sum of Kullback–Leibler divergence. Two variants of this formulation differing in the order of arguments has been discussed. The variant with resulting pdf as a second argument of Kullback–Leibler divergence is the more suitable one. The general solution of this task is hard, if not impossible, to find, nevertheless, basic elements of an iterative algorithm providing an approximation of the optimal pdf has been outlined. The main disadvantage of this algorithm is the form of the approximations which is hard to handle in case of continuous quantities. For this reason it will be necessary to modify the algorithm to search the approximation in a certain reasonable class of pdfs – e.g. finite mixture of pdfs from a given exponential family.

The variant with resulting pdf as a first argument of Kullback–Leibler divergence seems to be less suitable from point of view of its interpretation. Nevertheless, it could be interesting at least for a comparison with the preceding variant. Furthermore, for pdfs from a given exponential family, the optimization reduces to a algebraic equation. Also, in this case the iterative algorithm keeps approximations in this exponential family.

In the last section a link between the preceding approaches and Bayesian estimation with incomplete data records has been outlined. Probably, the estimation with incomplete data cannot help to solve the problem of pdf composition because of technical circumstances (unsuitable forms of posterior pdfs). On the other hand, the solution of pdf composition may serve well in problems of estimation with incomplete data.

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COMPROMISE MAXMIN SOLUTIONS FOR MULTIPLE CRITERIA BARGAINING

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Abstract. In this paper we analyze bargaining models where each of the agents considers several criteria to value the results of the negotiation process. The problem needs a multi-criteria approach from the first stage, when deciding the set of outcomes from which agents will be willing to bargain. In this paper, Pareto-optimal guaranteed outcomes are proposed as disagreement points for the class of multicriteria bargaining games. The family of maxmin solutions is defined to formalize the range of solutions that rational agents will accept, and in order to select a final consensus agreement among the set of maximin solutions, compromise solutions are defined and analyzed.

Keywords: Bargaining games, Multi-criteria analysis, Disagreement points, Compromise solutions.

1 Introduction

Bargaining games are inherently multi-criteria problems. They model the interaction of the agents pursuing different goals and, therefore, several criteria have to be considered in order to arrive at consensus solutions. This multi-criteria nature is even more evident if each agent has to take into account several criteria when deciding which of the feasible solutions it is convenient to agree on. In this paper we show how many of the ideas and results of multi-criteria analysis can be applied to better understanding and development of these models.

A multicriteria bargaining problem is a generalization of the classical bargaining problem where each player has a set of criteria to value any decision. To analyze these situations we have to jointly consider two decision problems: one related to the preferences of the players with respect to their own criteria, and the other problem of selecting a solution that could be accepted by all the rational players.

Although the situation is often that the agents face multiple criteria in the bargaining process, there is not much literature on multi-criteria bargaining. A simple extension, suggested by (Hwang and Lin 1987), is to consider the multi-criteria bargaining game as a scalar game with the number of agents equal to the sum of the number of criteria of all the agents, but this approach does not take into account the fact that each agent has to control a group of criteria, and does not lead to characterize the set of solutions that the agents will be willing to bargain on. Another proposal (Krus and Bronisz 1993), (Krus 2001), basically consists of each agent solving its multi-criteria problem in order to obtain a compromise utopian outcome in an earlier previous stage, which allows consideration of an aggregated utility function for each agent and construction of an n-person conventional bargaining game. Also (Korhonen et al 1986) analyze multicriteria bargaining situations and develop an interactive approach in order to find a consensus solution. Recently, (Hinojosa et al. 2003) have proposed a solution concept for the case in which all the agents consider the same set of criteria to value the decisions.

In this paper we assume that the utility function of each of the players is not explicitly known. That is, our setting does not require any assumptions about the utility functions representing the preferences of the agents about the results of the negotiation, but deals directly with the whole set of non-dominated outcomes.

We show how, in a first stage of the analysis, the cooperative multiple criteria bargaining problem has to be analyzed as a non-cooperative multiple criteria game in order to obtain the set of outcomes from which the players will be willing to bargain. These disagreement outcomes serve as reference points to define a solution concept that seeks equalitarian levels in the agents joint achievement with respect to their criteria. Finally, we propose the compromise maxmin solutions, in particular, we focus on one of them, Chebychev maxmin outcomes, in order to identify a final outcome that all the agents are willing to accept.

The rest of the paper is organized as follows. In Section 2 we describe the bargaining game with multiple criteria. In Section 3 we deal with the extension to the multi-criteria case of the notion of maxmin disagreement points and extend the individual rationality axiom that must be inherent in any solution concept. In Section 4 we define and characterize the family of maxmin solutions and propose the concept of compromise maxmin

solution. In order to illustrate the concepts and results herein, Section 5 includes the analysis of a multicriteria bargaining situation.

2 Multiple criteria bargaining games

An n-person bargaining game is usually described by a pair (S, d), where S is interpreted as the set of all possible outcomes or utility vectors of the agents, and d is the disagreement point or *status quo* that is the result obtainable if the players fail to reach an agreement. When $S \subseteq I\!R^n$, (S, d) represents a conventional (single criterion) n-person bargaining game.

The extension of single criterion n-person games to the multicriteria case provides a more realistic model and permits more extensive applications. When modelling a situation where each of the agents considers different criteria, the possible agreement results in a point which represents the payoffs obtained by all the agents with respect to all the criteria.

If $N = \{1, 2, \ldots, n\}$ is the set of agents, and agent $i \in N$, values m_i criteria, the set of all feasible outcomes S is a subset of an $M = \sum_{i=1}^{n} m_i$ dimensional space, $S \subseteq \prod_{i=1}^{n} I\!R^{m_i}$. Therefore, if $Y = (y^1, \ldots, y^n) \in S$, there exists an agreement, that gives player i an outcome vector $y^i \in I\!R^{m_i}, \forall\, i = 1, \ldots, n$. We assume that S is convex and compact. The problem consists of supporting the agents in reaching solutions that may be interpreted as the fair or reasonable agreements.

Note that each agent can value a different number of criteria. An interesting type of multicriteria bargaining games arises when the payoffs for some single criterion game depend on a future uncertain event. Then the payoff under each possible outcome can be viewed as one criterion of a multicriteria game. Therefore, multiple scenario bargaining games and bargaining under uncertainty can be identified with this model with $m_i = m$, $\forall\, i = 1, \ldots, n$.

For our proposal in this paper, it is more convenient to describe the problem in the decision space. We consider a multi-criteria bargaining game as $\{N, \mathcal{X}^i, u^i\}$. For each $i \in N$, $\mathcal{X}^i$ is agent i's decision or strategy set, which is assumed to be a non-empty subset in some finite-dimensional euclidian space ($\mathcal{X}^i \subseteq I\!R^{L_i}$). If player i values m_i criteria, $u^i : \mathcal{X} = \prod_{i=1}^{n} \mathcal{X}^i \longrightarrow I\!R^{m_i}$ is player i's vector outcome which is a real m_i-dimensional vector valued function. A joint strategy or decision is $X \in \mathcal{X} = \prod_{i=1}^{n} \mathcal{X}^i$, $X = (x^1, x^2, \ldots, x^n)$ where $x^i \in \mathcal{X}^i$, and the joint vector of outcomes or utilities is $u(X) \in \prod_{i=1}^{n} I\!R^{m_i}$, $u(X) = \big(u^1(X), u^2(X), \ldots, u^n(X)\big)$, $u^i(X) \in I\!R^{m_i}$, $\forall i \in N$.

The set of all feasible outcomes is now $S = u(\mathcal{X}) \subseteq \prod_{i=1}^{n} I\!R^{m_i}$, thus S is a subset of an $M = \sum_{i=1}^{n} m_i$ dimensional space. If $Y = (y^1, \ldots, y^n) \in S$, there exists a joint decision, X, that gives player i an outcome vector $y^i, \forall\, i = 1, \ldots, n$, that is, the outcomes in S are obtained as a result of a joint decision of all the agents.

In the next section we will deal with the other element that defines a conventional bargaining game, the disagreement point. We investigate the appropriate way of establishing disagreement points for the multicriteria case taking into account that the agents can have different and conflicting criteria measuring their outcomes.

3 Pareto optimal guaranteed outcomes

The first difficulty that arises when analyzing multi-criteria bargaining games is the determination of the disagreement point. Although different ways of establishing this point have been proposed in conventional bargaining games, for instance the threat outcomes (Nash 1953), one obvious suggestion is to set this point as the maxmin values of the game, as they represent the outcomes that the agents can guarantee without cooperating with the others.

In order to extend the idea of the maxmin values to the multi-criteria bargaining problem, we want to determine which are the best outcomes that the agents can guarantee themselves.

For each joint decision, $X = (x^1, x^2, \ldots, x^n) \in \mathcal{X}$, where x^i is the decision of agent i, we will denote by X^{-i} the decision of the rest of the agents, that is

$$X^{-i} = (x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n) \in \mathcal{X}^{-i} = \prod_{k \neq i} \mathcal{X}^k.$$

Each decision of agent i is going to be valued by a vector in terms of the worst outcomes that could be obtained with respect to each one of the criteria.

Definition 1. The guaranteed outcome vector for each decision of agent $i \in N$, $x^i \in \mathcal{X}^i$, is:

$$V^i(x^i) = \left(\min_{X^{-i} \in \mathcal{X}^{-i}} u_1^i(x^i, X^{-i}), \ldots, \min_{X^{-i} \in \mathcal{X}^{-i}} u_{m_i}^i(x^i, X^{-i}) \right).$$

In order to formalize the idea of maximizing this vector of assured outcomes, we consider those decisions that maximize the guaranteed outcome in the multi-criteria sense, that is, those that are non-dominated in the sense that no other decision will provide guaranteed outcomes that are better in all its criteria[1].

Definition 2. $V^i = (V_1^i, \ldots, V_{m_i}^i)$ is a Pareto optimal guaranteed outcome for agent $i \in N$, if $V^i = V^i(x^i)$, where $x^i \in \mathcal{X}^i$, and there does not exist $y^i \in \mathcal{X}^i$ such that $V(y^i) \geq V(x^i)$.

We will denote by D^i, $i \in N$, the set of all Pareto optimal guaranteed outcomes for player i. Vectors in D^i provide lower bounds of the outcomes that the agents will obtain independently of the actions of the other agents, if they play the corresponding strategy. This set of assured outcomes for each of the players plays the role of the maxmin values in the conventional bargaining game, that is, when each agent values a unique criterion.

Note that the concept of Pareto optimal guaranteed outcome can be seen as a solution concept for non-cooperative bargaining models. It is also related to the notion of Pareto Optimal Security Strategies for n-person multi-criteria finite games explored in (Puerto et al. 1999), and in (Fernández and Puerto 1996).

From Definition 2, it follows that the set of Pareto optimal guaranteed outcomes for agent $i \in N$ can be obtained solving the following vector maximization problem:

$$\begin{array}{ll} \max & \{V_1^i(x^i), V_2^i(x^i), \ldots, V_m^i(x^i)\} \\ \text{s.t.:} & x^i \in \mathcal{X}^i \end{array} \qquad (P(i))$$

In general, problem $(P(i))$ is a multi-objective problem that can be difficult to deal with. Nevertheless, it can be solved under certain conditions on the strategy sets and on the functions u^i, $i = 1, \ldots, n$.

We will focus on an interesting special case, the class of multi-criteria bargaining games with linear criterion functions. The linear case is important because it permits to model a wide range of applications, also because n-person finite games have linear payoff functions when considered on the jointly randomized strategy space, (Mármol et al. 2003). Therefore multi-criteria cooperative n-person finite games may be analysed with the approach herein.

Let u^i, $i = 1, \ldots, n$ be linear vector valued functions defined on polyhedrical decision spaces $\mathcal{X}^i$. Let us denote by $x^{-i}(k)$, $k = 1, \ldots, p_i$ the p_i extreme points of $\mathcal{X}^{-i} = \prod_{k \neq i} \mathcal{X}^k$. The following result permits us to obtain the set of Pareto optimal guaranteed outcomes by solving a linear multi-objective problem for each agent.

Proposition 1. *$V^i = (V_1^i, \ldots, V_{m_i}^i)$ is a Pareto optimal guaranteed outcome for agent $i \in N$, if and only if $V^i = V^i(x^i)$, where x^i is an efficient solution of problem*

$$\begin{array}{ll} max & (V_1^i, V_2^i, \ldots, V_{m_i}^i) \\ s.t.: & u_1^i(x^i, X^{-i}(k)) \geq V_1^i \quad \forall\, k = 1, \ldots, p_i \\ & \vdots \qquad\qquad\qquad\quad \vdots \\ & u_{m_i}^i(x^i, X^{-i}(k)) \geq V_{m_i}^i \quad \forall\, k = 1, \ldots, p_i \\ & x^i \in \mathcal{X}^i \end{array}$$

Proof: Given a decision of agent $i \in N$, x^i, from the linearity of the functions $u^i = (u_1^i, \ldots, u_{m_i}^i)$, it follows that the values $V_k^i(x^i) = \min_{X^{-i} \in \mathcal{X}^{-i}} u_k^i(x^i, X^{-i})$ can be obtained as $V_k^i(x^i) = \min_{k=1,\ldots,p_i} u_k^i(x^i, X^{-i}(k))$, and therefore the problem of maximizing the guaranteed outcome vector, $(P(i))$, is equivalent to the stated linear multi-objective problem. □

Once the disagreement points have been determined, we have to establish an individual rationality condition. In (von Neumann and Morgenstern 1994) it is suggested that any solution of a cooperative game must be part of a set, $S(d^0)$, defined as $S(d^0) = \{y \in S,\ y \geq d^0\}$, where the point d^0 represents the maxmin values of the game. $S(d^0)$ is interpreted as the set of outcomes obtainable through agreement, because the agents will only negotiate above the values that can be obtained by themselves.

When extending this idea to the multi-criteria case, it is the whole set of guaranteed outcomes of all the players which plays the role of the maxmin values. We will call this set the *disagreement set*, $\mathcal{D} = \prod_{i=1}^n \mathcal{D}^i$,

[1] For $a, b \in \mathbb{R}^m$, we will use the following notation: $a < b$ if $a_j \leq b_j$, $\forall j = 1, \ldots, m$; $a \leq b$ if $a \leqq b$, and $a \neq b$.

and it is interpreted as the set of outcomes that the agents can achieve independently, and therefore cooperation should lead them to an improved outcome with respect to these points.

Note that the disagreement set, $\mathcal{D}$, may not be contained in the set of feasible outcomes S, nevertheless it is always in the comprehensive hull[2] of S.

In a multi-dimensional context the individual rationality axiom can be interpreted in different ways, that is, we can consider different relations that reflect the concept of improvement with respect to a set of points. We propose two different sets of of possible agreements which depend on the dominance relation that the agents accept.

3.1 Strict preference outcomes

The players may want to negotiate only on outcomes that are above all the possible outcomes guaranteed on all the criteria. In this case, the set of outcomes where agent $i \in N$ will be willing to negotiate is

$$S(D^i(max)) = \{y \in S,\, y^i \geq d^i,\, \forall\, d^i \in D^i\}.$$

For each agent $i \in N$ and each criterion $k = 1, \ldots, m_i$, let $d_k^{i^{max}}$ denote the maximum outcome the agent can guarantee independently of the other agents and of the other criteria, that is, $d_k^{i^{max}} = max_{x^i \in \mathcal{X}^i} V_k^i(x^i)$. Let $d^{i^{max}} = (d_1^{i^{max}}, \ldots, d_{m_i}^{i^{max}})$. The set $S(D^i(max))$ can also be described as $S(D^i(max)) = \{y \in S,\, y^i \geq d^{i^{max}}\}$. It is illustrated in Figure 1.

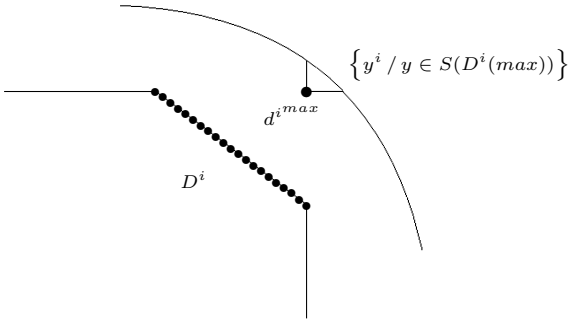


Figure 1. Strict preference outcomes

Definition 3. The set of strict preference outcomes for a multiple criteria bargaining game, $S(D(max))$, is defined as

$$S(D(max)) = \bigcap_{i \in N} S(D^i(max))$$

This negotiation set is equivalent to the set of individually rational points in the conventional bargaining problem obtained when considering $\sum_{i=1}^{n} m_i$ agents, with the disagreement point $d^{max} = (d^{1^{max}}, \ldots, d^{n^{max}})$, whose components represent the maxmin values of all the agents with respect to every criterion. Nevertheless, this point is not always useful for the multi-criteria problem, as far as players might not be able to attain their corresponding vector $d^{i^{max}}$, because different maxmin values are obtained from different decisions. It can happen that d^{max} is not included in the convex hull of S, and therefore the negotiation set $S(D^{max})$ could be empty. The existence of strict preference outcomes is related to the existence of decisions that simultaneously maximize the guaranteed outcomes for the agents [3].

The following result establishes a sufficient condition for the strict negotiation set being nonempty.

Proposition 2. *If for every agent, $i \in N$, the problem of maximizing the guaranteed outcome, $(P(i))$, has an ideal solution, then the strict preference negotiation set, $S(D(max))$, is not empty.*

[2] The comprehensive hull of S is the set $S - \mathbb{R}_+^M$, where $\mathbb{R}_+^M$ represents the positive M-dimensional octant in $\mathbb{R}^M$.

[3] $\bar{x}$ is an ideal solution of the vector maximization problem

$$\begin{array}{ll} \max & \{f_1(x), f_2(x), \ldots, f_m(x)\} \\ \text{s.t.:} & x \in X \end{array}$$

if $f_k(\bar{x}) = max_{x \in X} f_k(x),\, \forall\, k = 1, \ldots, m$.

Proof: If problem $P(i)$ has an ideal solution $\forall\, i \in N$, then the set of Pareto optimal guaranteed outcomes for agent i reduces to a singleton, $D^i = \{d^{i^{max}}\}$, with $d_k^{i^{max}} = max_{x^i \in X^i} V_k^i(x^i)\, k = 1, \ldots, m_i$, and there exists a decision of agent i, $\bar{x}^i$, for which the guaranteed outcomes are obtained $V^i(\bar{x}^i) = d^{i^{max}}$. It means that $u^i(\bar{x}^i, X^{-i}) \geq V^i(\bar{x}^i) = d^{i^{max}}$, $\forall X^{-i} \in \mathcal{X}^{-i}$, $\forall i \in N$. In particular, if $\bar{X} = (\bar{x}^1, \ldots, \bar{x}^n)$, then $u^i(\bar{X}) \geq d^{i^{max}}$, $\forall\, i \in N$, and therefore, at least $u(\bar{X}) \in S(D(max))$. □

Note that this is a very strong condition that does not hold frequently, therefore a less restrictive negotiation set has to be defined in order to analyze a wider class of problems.

3.2 Preference outcomes

A rational player will negotiate in the set of outcomes that are above at least one of the Pareto-optimal guaranteed outcomes. The set of outcomes where agent $i \in N$ will be willing to negotiate is now $S_{\geq}^i(D) = \{y \in S,\, y^i \geq d^i,\ \text{for some } d^i \in D^i\}$, (Figure 2). It coincides with the union when $d^i \in D^i$, of all the sets $S^i(d^i) = \{y \in S,\, y^i \geq d^i\}$,

$$S_{\geq}^i(\mathcal{D}) = \bigcup_{d^i \in D^i} S^i(d^i).$$

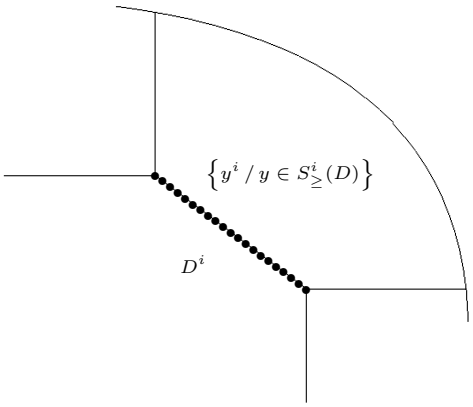


Figure 2. Preference outcomes

Definition 4. The set of preference outcomes for a multiple criteria bargaining game, $S_{\geq}(\mathcal{D})$, is defined as

$$S_{\geq}(\mathcal{D}) = \bigcap_{i \in N} S_{\geq}^i(D).$$

This set represents the feasible outcomes of the bargaining process such that all the agents obtain vector payoffs that are above some of their Pareto optimal guaranteed outcomes. The following result establishes that this set is always nonempty in a wide class of bargaining games.

Proposition 3. *If $\mathcal{X}$ is compact, the set of preference outcomes of the multicriteria bargaining problem is nonempty.*

Proof: If the agent's joint decision set, $\mathcal{X}$, is compact, there exists at least a decision of each player, $\bar{x}^i$, $i \in N$, associated with a Pareto optimal guaranteed outcome, $d^i \in D^i$. Consider $\bar{X} = (\bar{x}^1, \ldots, \bar{x}^n)$, $y^i = u^i(\bar{X}) \geq d^i$, $\forall\, i \in N$, and $y = u(\bar{X}) \in S_{\geq}(\mathcal{D})$. Therefore $S_{\geq}(\mathcal{D}) \neq \emptyset$. □

The set of preference outcomes can be considered as the set of individually rational points for a multiple criteria bargaining problem, since it consists of the range of acceptable proposals by all the players.

In what follows we will represent a multicriteria bargaining game (MCBG) by $(S, \mathcal{D})$, where S is the set of feasible outcomes and $\mathcal{D}$ the set of the agents' disagreement points, $S, \mathcal{D} \subset \prod_{i=1}^n I\!R^{m_i}$.

4 Compromise maxmin solutions for multicriteria bargaining games

The next task is to investigate what are the outcomes to which the agents will eventually arrive when playing the multi-criteria bargaining game. The solutions should reflect the preferences of the agents and should also fulfill some basic fairness rules. First of all, we will require that the outcomes a solution provides are individually rational, therefore we will define the solutions in the set of preference outcomes.

Definition 5. A solution concept for a multicriteria bargaining game is a correspondence that associates with each MCBG, $(S, \mathcal{D})$, a non-empty subset of $S_{\geq}(\mathcal{D})$.

In addition, the agreements in a solution must exhibit some property of efficiency or nondominance, in the sense that from them, further improvement must not be possible. Since there is only a partial order structure defined in the payoff space of each agent, the classical dominance relations among vectors do not suffice to compare the points in S, and it is necessary to consider more general dominance relations in order to select those agreements that the agents will be willing to accept.

We will use the terms Pareto-optimality and weak Pareto-optimality for the following extensions to points in the space $\prod_{i=1}^{n} I\!R^{m_i}$ of these concepts for vectors.

Definition 6. $Y \in S \subseteq \prod_{i=1}^{n} I\!R^{m_i}$ is Pareto-optimal in S, if there does not exist $Z \in S$, $Z \neq Y$, such that $z_j^i \geq y_j^i \; \forall i \in N$, $j = 1, \ldots, m_i$.

Definition 7. $Y \in S \subseteq \prod_{i=1}^{n} I\!R^{m_i}$ is weakly Pareto-optimal in S, if there does not exist $Z \in S$, such that $z_j^i > y_j^i, \forall i \in N$, $j = 1, \ldots, m_i$.

The solution concepts we propose in this paper are based on the agent's selection of one of his Pareto optimal guaranteed outcomes, and they consider also as a reference point the most optimistic expectations with respect to their criteria. Let $(d_1^i, \ldots, d_{m_i}^i) \in D^i$ be the disagreement point chosen by agent i, and let $I \in \prod_{i=1}^{n} I\!R^{m_i}$ be the ideal point of the agents with respect to their criteria, obtained as

$$I_k^i = \max_{y \in S_{\geq}(\mathcal{D})} y_k^i, \; i \in N \; k = 1, \ldots, m_i.$$

In order to define the concept of multicriteria maxmin solution, each feasible outcome is going to be valued by a n-dimensional vector in terms of the worst results the agents can achieve with respect to the disagreement values.

Definition 8. For each feasible outcome $Y \in S$, the minimum level vector is

$$t(Y) = (t_1(Y), \ldots, t_n(Y))$$

where $\forall i =, \ldots, n$, $t_i(Y) = min_{j=1,\ldots,m_i}\{\frac{y_j^i - d_j^i}{I_j^i - d_j^i}\}$ is the minimum level of agent i.

Remark that a minimum level vector $t \in I\!R^n$ can be obtained from different feasible outcomes $Y \in S$.

¿From a collective rationality point of view the solution must consist of a feasible outcome such that the associated minimum level vector is collectively as good as possible, that is to say, those outcomes that maximize, in the vector sense, the minimum level of the agents.

Definition 9. A vector $t \in I\!R^n$ is a minimum level vector in S if there exists $Y \in S$ such that $t(Y) = t$ and there does not exist $Z \in S$ such that $t(Z) \geq t$

Definition 10. A feasible outcome, $Y \in S$, is a multicriteria maxmin solution for the MCBG $(S, \mathcal{D})$ if its minimum level vector, $t(Y)$ is nondominated in $S_{\geq}(\mathcal{D})$.

In order to characterize the set of multicriteria maxmin solutions, consider the following multicriteria problem associated to the multicriteria bargaining game, that we will denote by (MP)

$$\begin{array}{lll} (MP): & max & t_1, \ldots, t_n \\ & s.t.: & \frac{y_j^1 - d_j^1}{I_j^1 - d_j^1} \geq t_1 \quad \forall j = 1, \ldots, m_1 \\ & & \vdots \\ & & \frac{y_j^n - d_j^n}{I_j^n - d_j^n} \geq t_n \; \forall j = 1, \ldots, m_n \\ & & Y \in S_{\geq}(\mathcal{D}) \end{array}$$

Proposition 4. *Let $(Y \;, t \;)$ be a nondominated solution of (MP), then Y is a multicriteria maxmin solution and t is its minimum level vector. Conversely, if Y is a multicriteria maxmin solution, then $(Y \;, t(Y \;))$ is a nondominated solution of (MP).*

Proof: From Definition 10, it follows that multicriteria maxmin solutions are the nondominated solutions of the vector maximization problem:

$$\begin{array}{ll} max & t_1(Y), \ldots, t_n(Y) \\ s.t.: & Y \in S_{\geq}(\mathcal{D}) \end{array}$$

which is equivalent to the problem (MP). □

In general, the multicriteria problem (MP) can be difficult to solve, but for the class of multicriteria linear bargaining games, where the feasible payoff set is a polyhedral set, the whole set of multicriteria maxmin solutions can be obtained by solving a linear multicriteria problem. Although the dimensionality of the problem can be high, existing software, for instance ADBASE (Steuer 1995), can be applied to solve this problem. ADBASE provides the nondominated minimum level vectors and the set of extreme points of the set $S_{\geq}(\mathcal{D})$ corresponding of each nondominated minimum level vector, making it possible to obtain the whole set of multicriteria maxmin solutions.

Next, we show that the condition of nondominance induced by the maxmin criterion implies weak Pareto optimality in the sense of 7. In fact, multicriteria maxmin solutions exhibit a stronger condition, as we will prove. First, we define the following dominance relation between points in the outcome space of the multicriteria bargaining game. Consider $X, Y \in \prod_{i=1}^{n} I\!R^{m_i}$.

Definition 11. $X \geq_m Y$ if $x^i \geqq y^i$, $\forall i \in N$, and there exists i, such that $x^i > y^i$.

That is to say an agreement X, dominates another Y, with respect to this relation, if at least one of the agents obtains a payoff with X strictly better for all the criteria, than the payoff obtained with Y. Remark that this dominance condition is stronger than $X \geq Y$ and weaker than $X > Y$.

Proposition 5. *Let Y be a multicriteria maxmin solution for $(S, \mathcal{D})$, then Y is nondominated with respect to $\geq_m$ in $S_{\geq}(\mathcal{D})$.*

Proof: Let Y be a multicriteria maxmin solution for $(S, \mathcal{D})$ and suppose that Y is dominated with respect to the relation $\geq_m$ in $S_{\geq}(\mathcal{D})$, then $\exists\, X \in S_{\geq}(\mathcal{D})$, such that $x^i \geqq y^i$, $\forall i \in N$, and there exists i, such that $x^i > y^i$. $t(X) \geqq t(Y)$, but $t_i(X) = \min_{1 \leq j \leq m_i}\{\frac{x_j^i - d_j^i}{I_j^i - d_j^i}\} > \min_{1 \leq j \leq m_i}\{\frac{{y_j}^i - d_j^i}{I_j^i - d_j^i}\} = t_i(Y)$. This is a contradiction to $t(Y)$ being a nondominated level vector. □

As a consequence, if Y is a generalized maxmin solution then Y is a weak Pareto-optimal solution in $S_{\geq}(\mathcal{D})$, in the sense defined in 7. In addition, we will prove that if the set of maxmin solutions associated to a nondominated minimum level vector is a singleton, then this solution is Pareto-optimal. We denote by $\mathcal{M}(t)$ the set of multicriteria maxmin solutions associated to the nondominated minimum level t.

Proposition 6. *If $\mathcal{M}(t) = \{Y\}$, then Y is Pareto-optimal in $S_{\geq}(\mathcal{D})$.*

Proof: Suppose Y is not Pareto-optimal, then $\exists Z \in S_{\geq}(\mathcal{D})$, $Z \geq Y$, that is $\exists i, j$ with $z_j^i > {y_i}^i$, but $z_j^i > {y_i}^i$ does not hold $\forall j = 1, \ldots, m_i$ as Y is nondominated with respect to $\geq_m$. Therefore, $\forall i \in N, t_i(Z) = \min_{1 \leq j \leq m_i}\{\frac{z_j^i - d_j^i}{I_j^i - d_j^i}\} = \min_{1 \leq j \leq m_i}\{\frac{{y_j}^i - d_j^i}{I_j^i - d_j^i}\} = t_i(Y)$, what contradicts $\mathcal{M}(t)$ being singleton. □

4.1 Compromise maxmin solutions

The family of multicriteria maxmin solutions provides a set of outcomes whose minimum levels can not be improved simultaneously. However, the range of achievement of the criteria for the different agents can vary significantly in this set, and the proposal of the whole set of maxmin agreements as a solution of the game may not be useful in order to finish the negotiation process. It is therefore, necessary to make further assumptions about the notion of consensus solution to apply, and to establish a procedure to select a reduced set of outcomes or even a unique agreement as the solution of the game. For one criterion bargaining games, (Yu 1973) introduced a family of solutions based on the idea of getting as close possible to satisfying every one. In what follows we extend this idea to define a solution concept for multicriteria bargaining games based on the distance of the outcomes to the ideal minimum level of the agents.

Definition 12. The ideal minimum level of the agents for the MCBG $(S, \mathcal{D})$ is $t = (t_1, \ldots, t_n)$ where $\forall i = 1, \ldots, n$, t_i is the optimal value of the following scalar optimization problem

$$(P_i): \quad \begin{array}{ll} max & t_i \\ s.t.: & \frac{y_j^i - d_j^i}{I_j^i - d_j^i} \geq t_i \quad \forall j = 1, \ldots, m_i \\ & Y \in S_{\geq}(\mathcal{D}) \end{array}$$

Definition 13. Y is a compromise maxmin solution for the MCBG $(S, \mathcal{D})$ if $(t(Y), Y)$ is an optimal solution of problem

$$(CP): \quad \begin{array}{ll} Min\ d(t, t) & \\ s.t.: & \frac{y_j^1 - d_j^1}{I_j^1 - d_j^1} \geq t_1 \quad \forall j = 1, \ldots, m_1 \\ & \vdots \\ & \frac{y_j^n - d_j^n}{I_j^n - d_j^n} \geq t_n \quad \forall j = 1, \ldots, m_n \\ & Y \in S_{\geq}(\mathcal{D}) \end{array}$$

where d is a distance in $\mathbb{R}^n$.

An interesting particular case is the compromise maxmin solution induced by the Chebyshev distance, that is, $d_\infty(t, t) = Max\{t_1 - t_1, \ldots, t_n - t_n\}$. This distance will yield an equitable solution, in the sense that it minimizes the distance to t of the agent whose distance is greater. The next result provides the characterization of the Chebyshev maxmin solution as the solution of a scalar optimization problem

Proposition 7. *Y is a Chebyshev maxmin solution for the MCBG $(S, \mathcal{D})$ if there exists $z \in \mathbb{R}$, and $t \in \mathbb{R}^n$, such that (z, t, Y) is an optimal solution for*

$$(CP_0): \quad \begin{array}{ll} Min\ z & \\ s.t.: z \geq t_i - t_i & \forall i =, \ldots, n \\ \frac{y_j^1 - d_j^1}{I_j^1 - d_j^1} \geq t_1 & \forall j = 1, \ldots, m \\ \vdots & \\ \frac{y_j^n - d_j^n}{I_j^n - d_j^n} \geq t_n & \forall j = 1, \ldots, m \\ Y \in S_{\geq}(\mathcal{D}) & \end{array}$$

Proof: The result follows from the fact that the problem of minimizing distance d_∞:

$$\begin{array}{ll} Min\ Max\{t_1 - t_1, \ldots, t_n - t_n\} & \\ s.t.: \quad \frac{y_j^1 - d_j^1}{I_j^1 - d_j^1} \geq t_1 & \forall j = 1, \ldots, m \\ \vdots & \\ \frac{y_j^n - d_j^n}{I_j^n - d_j^n} \geq t_n & \forall j = 1, \ldots, m \\ Y \in S_{\geq}(\mathcal{D}) & \end{array}$$

is equivalent to (CP_0). □

It is worth noting that problem (CP_0) may not have a unique optimal solution. However, among the optimal solutions of problem (CP_0), at least one provides an efficient solution for the multicriteria problem (MP), that is, a maxmin outcome. Such a solution can be found by a lexicographical procedure.

In the next section we illustrate the application of these ideas to analyze a two-agent multi-criteria bargaining game.

5 Example

Consider a situation with two agents facing a joint project to open a new factory. Agent 1 represents a private firm and its goal is to maximize the benefits derived from the activity of the factory. Agent 2 represents the public sector and its main goals are the creation of new jobs and the control of the pollution that the factory will generate. The private firm controls two decision variables: primary goods and number of workers to employ.

The public sector can decide how much has to be invested in technology and how many experts have to be employed to manage the new factory as efficiently as possible. Agents 1 and 2 have to arrive at an agreement about the levels of performance of their goals when the factory is open.

This bargaining problem is described as follows. Agent 1 has an unique criterion to value the alternatives, the benefit derived from the activity. Agent 2 has to value the alternatives taking into account two criteria: the total number of workers employed and the pollution level generated.

Primary goods are available up to a total of 50 units, and the firm can employ up to 20 workers. The decision set for agent 1 is $\mathcal{X}^1 = [0, 50] \times [0, 20]$. We denote by x_1 the quantity of primary goods to use, by x_2 the number of workers to employ and by X^1 the vector $X^1 = (x_1, x_2) \in \mathcal{X}^1$.

Agent 2 has a total amount of funds to invest in technology of 10 units and, depending on the way they are going to manage the factory, they can include in the staff a number of experts varying from 5 to 30. Its decision set is $\mathcal{X}^2 = [0, 10] \times [5, 30]$. We denote by x_3 the funds invested in technology, by x_4 the number of experts to employ and by X^2 the vector $X^2 = (x_3, x_4) \in \mathcal{X}^2$.

The joint space of the decisions is then $\mathcal{X} = \mathcal{X}^1 \times \mathcal{X}^2 \subset R^4$.

The utility function for agent 1, that values the benefit derived from a joint decision is $u^1(x_1, x_2, x_3, x_4) = 3x_1 + 10x_2 + 5x_3 + 0.5x_4$. The utility function for agent 2 is a vector-valued function $u^2 : \mathcal{X} \to R^2$, whose components are $u_1^2(x_1, x_2, x_3, x_4) = x_2 + x_4$ and $u_2^2(x_1, x_2, x_3, x_4) = -3x_1 - 8x_3 - 7(x_4 - 5)$. The joint utility function $u : \mathcal{X} \to R^3$, $u \equiv (u^1, u^2)$, values the feasible decisions in terms of the criteria of both agents. In order to simplify the exposition we have considered u_2^2 negative, as it represents a minimization criterion.

5.1 The guaranteed outcomes for the agents

Let us denote by $X^{-1}(k)$, $k = 1, \ldots, p_1$, the extreme points of the decision set for agent 2, $\mathcal{X}^2$. This set has 4 extreme points, $\{(0, 5), (0, 30), (10, 5), (10, 30)\}$, thus $p_1 = 4$. The Pareto optimal guaranteed outcomes for agent 1 are obtained by solving problem

$$\begin{array}{l} max\ V_1^1 \\ s.t. : u_1^1(x^1, X^{-1}(k)) \geq V_1^1\ \forall\, k = 1, \ldots, p_1 \\ \qquad x^1 \in \mathcal{X}^1 \end{array}$$

In this case the optimal guaranteed outcome is $V^1 = 352.5$, and therefore

$$D^1 = \{352.5\} \quad \text{and} \quad S(D^1(max)) = \{y \in S, \quad y^1 \geq 352.5\}.$$

The decision set of agent 1, $\mathcal{X}^1$, also has 4 extreme points, which are $\{(0, 0), (0, 20), (50, 0), (50, 20)\}$, $p_2 = 4$. The Pareto optimal guaranteed outcomes for agent 2 are obtained by solving the problem:

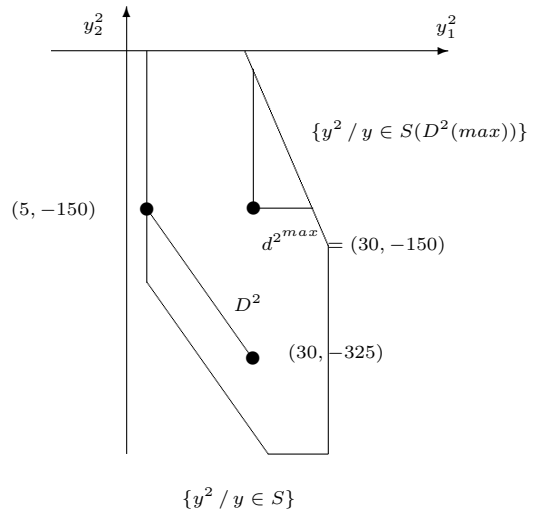
$$\begin{array}{l} max\ (V_1^2, V_2^2) \\ s.t. : u_1^2(X^{-2}(k), x^2) \geq V_1^2\ \forall\, k = 1, \ldots, p_2 \\ \qquad u_2^2(X^{-2}(k), x^2) \geq V_2^2\ \forall\, k = 1, \ldots, p_2 \\ \qquad x^2 \in X^2 \end{array}$$

where $X^{-2}(k)$ represents each of the extreme points of X^1.

This bi-objective linear problem has been solved with ADBASE (Steuer, 1995), obtaining two extreme efficient points. The corresponding guaranteed outcomes are $\{(30, -325), (5, -150)\}$. Therefore, the set of Pareto optimal guaranteed outcomes can be represented as

$$D^2 = \{d^2 = (d_1^2, d_2^2) \in I\!R^2 \,/\, (d_1^2, d_2^2) = \lambda(30, -325) + (1 - \lambda)(5, -150)\,, \quad 0 \leq \lambda \leq 1\}.$$

In the following picture we represent D^2 and $S(D^2(max))$ in the set of all the possible outcomes for agent 2, that is, in the set $\{y^2 \,/\, y = (y^1, y^2) \in S\}$.



Although there are strict preference outcomes for both agents, in this bargaining game the strict preference negotiation set is empty because

$$S(D(max)) = S(D^1(max)) \cap S(D^2(max)) = \emptyset.$$

Therefore, we will consider the solutions in the preference negotiation set

$$S_{\geq}\mathcal{D} = \{y \in S, y^1 \geq 352.5,\ y^2 \geq d^2 \text{ for some } d^2 \in D^2\}$$

5.2 Multicriteria Maxmin solutions.

The ideal point, $I = (I_1^1, I_1^2, I_2^2)$, that represents the agents' most optimistic expectations, is calculated as

$$I_k^i = \max_{y \in S_{\geq}(\mathcal{D})} y_k^i,\ i = 1, 2,\ k = 1, \ldots, m_i.$$

In this case $I = (415, 50, -150)$.

For each $D \in \mathcal{D}$, a set of maxmin solutions is obtained as the solutions problem

$$\begin{array}{l} max\ t_1, t_2 \\ s.a.: \dfrac{y_1^1 - d_1^1}{415 - d_1^1} \geq t_1 \\ \dfrac{y_1^2 - d_2^1}{50 - d_2^1} \geq t_2 \\ \dfrac{y_2^2 - d_2^2}{-150 - d_2^2} \geq t_2 \\ X \in S_{\geq}(\mathcal{D}) \end{array}$$

Consider the disagreement point $D = (352.5, 17.5, -237.5)$, which consists of the unique Pareto optimal guaranteed outcome of agent 1, and the middle point of the segment of Pareto optimal guaranteed outcomes of agent 2. ADBASE provides the extreme maxmin solutions represented in the following table.

$D = (352.5, 17.5, -237.5)$	
$(t_1,\ t_2)$	Y
$(0.81, 0.00)$	$(403.36, 26.07, -237.50)$
$(0.80, 0.09)$	$(402.50, 25, -230.00)$
$(0.673, 0.231)$	$(394.57, 25, -217.31)$
$(0.228, 0.385)$	$(366.78, 30, -203.85)$
$(0.208, 0.391)$	$(365.55, 30.22, -203.25)$
$(0.055, 0.444)$	$(355.97, 31.95, -198.61)$
$(0, 0.456)$	$(352.5, 32.32, -197.59)$

It is worth to remark that, as a consequence of Proposition 6, in this case, all these solutions are Pareto-optimal. However, the whole set of maxmin outcomes is unlikely to be useful as a solution of the bargaining game as the values obtained for the agents vary significantly in the different solutions.

5.3 The Chebyshev Maxmin solution

In order to determine a final consensus solution we will obtain the outcome whose Chebyshev distance to the ideal minimum level of the agents.

In this case $t = (0.81, 0.456)$, thus we solve the following problem

$$\begin{array}{l} Min\ z \\ s.a.: z \geq 0.81 - t_1 \\ \quad z \geq 0.456 - t_2 \\ \quad \dfrac{y_1^1 - 352.5}{415 - 352.5} \geq t_1 \\ \quad \dfrac{y_1^2 - 17.5}{50 - 17.5} \geq t_2 \\ \quad \dfrac{y_2^2 + 237.5}{-150 + 237.5} \geq t_2 \\ \quad X \in S_{\geq}(\mathcal{D}) \end{array}$$

The problem has a unique solution $z = 0.203$, $t_1 = 0.606$, $t_2 = 0.253$ $y^1 = 390.47$, $y_2^1 = 25.74$, $y_2^2 = -215.32$.

Therefore, The Chevysev Maxmin agreement for this multicriteria bargaining problem consists of: Agent 1, the private firm, achieves a level of benefit equal to 390.47. Agent 2, the public sector creates 25.74 new jobs and the level of pollution generated is 215.32. With this feasible outcome, agent 1 guarantees an improvement of a 60.6 percent with respect to his disagreement point in relation to the most optimistic expectation. Agent 2 guarantees a 25.3 with respect to his two criteria. In addition, the maximum regret of the agents is minimum.

6 Conclusions

In the paper we develop a mathematical background for the multicriteria analysis of decision problems described by bargaining cooperative games. There are two crucial problems related to the analysis: the selection of the disagreement point(s), and the selection of possible outcomes dominating the disagreement points, taking into account that the agents can have different and conflicting criteria measuring their outcomes.

With respect to the first question, we define the concept of Pareto optimal guaranteed outcome as an extension of the notion of maxmin values of the game and propose to replace the unique disagreement point of the classical game by a set of disagreement points, obtained from the set of Pareto optimal guaranteed outcomes. We prove that such points can be obtained solving the appropriate multicriteria programming problem.

Once the disagreement points have been determined, we propose a class of solutions consisting of those feasible outcomes such that the levels that each agent achieves with respect to the disagreement values in all the criteria can not be improved simultaneously. In particular, we focus on the class of compromise maxmin solutions that provides feasible outcomes minimizing the group regret and therefore, can be considered as final consensus solutions.

Acknowledgments

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Knowledge-Based Decision-Making Supported by Compositional Models

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Abstract. Complexity of practical decision-making problems requires that knowledge, if represented by a probability distribution, is represented by a distribution of a very high dimensionality. For this reason, several approaches enabling the user to process distributions of hundreds or even thousands of dimensions were developed in the last two decades of the 20th century. Most of them belong to the class of so called Graphical Markov Models, whose most popular representatives are Bayesian Networks.

The goal of this contribution is to illustrate that a new approach for multidimensional probability distribution representation is a proper tool for representing knowledge of various applicational areas. Namely, the approach is based on the idea of composing a multidimensional distribution from a great number of low-dimensional ones. It quite naturally corresponds to the fact that global knowledge about a field of practice can hardly be expressed. In textbooks and specialized monographs, it is always described by a number of facts that can be called pieces of knowledge, or local knowledge. Similarly, available data files usually make it possible to estimate only low-dimensional distributions. In this paper we explain, how these distributions, using special operators of composition, can create a multidimensional distributions manifesting the same properties as the distributions represented by Bayesian networks.

Keywords: Probability, multidimensional distribution, composition, Bayesian network

1 Introduction

In this paper, we will deal with a finite system of finite-valued random variables. Let N be an arbitrary finite index set, $N \neq \emptyset$. Each variable from $\{X_i\}_{i\in N}$ is assumed to have a finite (non-empty) set of values $\mathbb{X}_i$. Distributions of these variables will be denoted by Greek letters (usually $\pi, \kappa, \nu,$); thus for $K \subseteq N$, we can consider a distribution $\pi((X_i)_{i\in K})$. To make the formulae more lucid, the following simplified notation will be used. Symbol $\pi(x_K)$ will denote both a $|K|$-dimensional distribution and a value of a probability distribution π (when several distributions will be considered, we shall distinguish them by indices) for variables $(X_i)_{i\in K}$ at a combination of values x_K; x_K thus represents a $|K|$-dimensional vector of values of variables $\{X_i\}_{i\in K}$. Analogously, we shall also denote the set of all these vectors $\mathbb{X}_K$:

$$\mathbb{X}_K = \times_{i\in K}\mathbb{X}_i.$$

A distribution $\pi(x_K)$ is represented by a $|K|$-dimensional table containing numbers from the interval $[0, 1]$, and all the numbers of this distribution have to sum up to one:

$$\sum_{x_K\in\mathbb{X}_K} \pi(x_K) = 1.$$

For such a probabilistic distribution and $J \subset K$ we will often consider a *marginal distribution* $\pi(x_J)$ of distribution $\pi(x_K)$, which can be computed by

$$\pi(x_J) = \sum_{x_{K\setminus J}\in\mathbb{X}_{K\setminus J}} \pi(x_K) = \sum_{x_{K\setminus J}\in\mathbb{X}_{K\setminus J}} \pi(x_{K\setminus J}, x_J).$$

For computation of marginal distributions we need not exclude situations when $J = \emptyset$. In this case we get $\pi(x_\emptyset) = 1$.

In some situations, when we will want to stress that we are dealing with a marginal distribution of a distribution π, we will use symbol $\pi^{(J)}$ to denote the marginal distribution of π for variables $(X_i)_{i\in J}$. I.e., for $J \subseteq K$ and a distribution $\pi(x_K)$

$$\pi^{(J)} = \pi(x_J).$$

These uper indices will also be used when denoting the sets of all probability distributions defined for a given set of variables. The class of all distributions defined for all the considered variables will be denoted $\Pi^{(N)}$. Analogously, for $K \subset N$, symbol $\Pi^{(K)}$ denotes the class of all the distributions $\pi(x_K)$ defined for variables X_K.

The multidimensional models considered in this paper will be composed from a great number of low-dimensional ones. The basic idea of the approach is the same as that on which expert systems are based: it is beyond human capabilities to *represent/express/understand* global knowledge of an application area – one has always to work only with pieces of local knowledge. Such local knowledge can be, within probability theory, easily represented by a low-dimensional distribution. Therefore, in this paper we will assume that we have a system of low-dimensional distributions $\pi_1(x_{K_1}), \pi_2(x_{K_2}), \ldots, \pi_n(x_{K_n})$ representing pieces of local knowledge, from which we want to construct a multidimensioal distribution representing global knowledge. A great advantage of this type of local knowledge representation is the fact that in a majority of situations, low-dimensional distributions can be obtained from various data sources by classical statistical estimates. Let us stress that in such situations the dimensionality of the estimated distributions is strictly limited because of the size of available data. Whatever size of data is at our disposal we can hardly assume to obtain reliable estimates of probabilities of a 20-dimensional distribution (even for binary variables). Typically, one can assume that a dimensionality of the considered distributions $\pi_k \in \Pi^{(K_k)}$ (for all $k = 1, \ldots, n$) is 2 – 8. Therefore, we will call them oligodimensional distributions.

In artificial intelligence, application of the whole class of methods for knowledge modeling by multidimensional probability distributions – and here we have in mind distributions of hundreds rather than tens of variables – was catalyzed by success, which was achieved during the last twenty years in the field that is often called graphical Markov modeling (Lauritzen 1996). This term is used as a general term describing any of the approaches representing multidimensional probability distributions by means of graphs and systems of quantitative parameters. These parameters are usually oligodimensional, sometimes conditional, probability distributions. Therefore, graphical Markov modeling includes influence diagrams, decomposable and graphical models, chain graph models, and many others. Perhaps the most famous representative of these models, Bayesian networks (Jensen 1996), represent distributions with special dependence structures which are described by acyclic directed graphs. Some other models use for the dependence structure representation undirected graphs, or more complicated graphical tools like chain graphs, hypergraphs, or, annotated graphs.

The approach presented herein dispenses with the necessity to describe the dependence structure of a modeled distribution in a graph. In contrast to this, the presented technique of compositional models describes directly how the multidimensional distribution is computed – composed – from a system of oligodimensional distributions, and therefore need not represent the dependence structure explicitly. Thus, we start describing our model with an assumption that there are a (usually great) number of pieces of local knowledge represented by a system of oligodimensional distributions. The task we will address in this text resembles a jig-saw puzzle that has a great number of parts, each bearing a local piece of a picture, and the goal is to find how to assemble them in such a way that the global picture makes sense, reflecting all the individual small parts. In other words, we will look for an ordering of oligodimensional distributions in the way that, when composed together, the resulting multidimensional distribution optimally reflects all the local knowledge carried by the oligodimensional distributions.

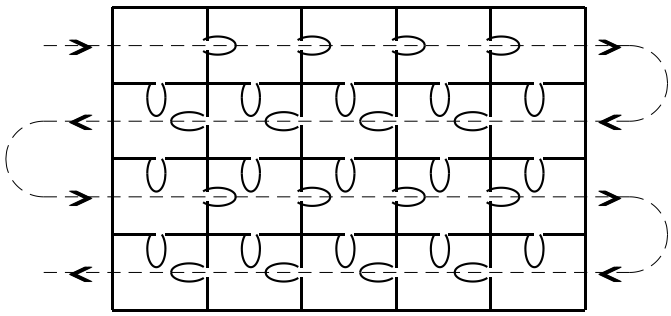


Figure 1. Ordering of pieces of jig-saw puzzle.

2 Foundations of Compositional Models

2.1 Conditional Distribution and Conditional Independence

Let us briefly summarize in this paragraph well-known notions of probability theory, which are necessary in the rest of the paper.

For a distribution $\pi(x_K)$ and two disjoint subsets $L_1, L_2 \subseteq K$ we will often speak about a *conditional distribution* $\pi(x_{L_1}|x_{L_2})$, which is, for each fixed $x_{L_2} \in \mathbb{X}_{L_2}$ an $|L_1|$-dimensional probability distribution, for which

$$\pi(x_{L_1}|x_{L_2})\pi(x_{L_2}) = \pi(x_{L_1 \cup L_2}).$$

(Notice that, the conditional distributions is always defined, even in case when $\pi(x_{L_2}) = 0$ for some combination(s) of values $x_{L_2} \in \mathbb{X}_{L_2}$, though in this case the definition is ambiguous.) The reader can immediately see that if $L_1 = \emptyset$ then

$$\pi(x_{L_1}|x_{L_2}) = 1,$$

and if $L_2 = \emptyset$ then

$$\pi(x_{L_1}|x_{L_2}) = \pi(x_{L_1}).$$

Considering three disjoint subsets $L_1, L_2, L_3 \subseteq K$ we can introduce a concept of *conditional independence* that generalizes well-known independence of variables.

Definition 1. We say that groups of variables X_{L_1} and X_{L_2} are *conditionally independent* given X_{L_3} (in symbol $X_{L_1} \perp\!\!\!\perp X_{L_2}|X_{L_3}[\pi]$) if

$$\pi(x_{L_1 \cup L_2 \cup L_3})\pi(x_{L_3}) = \pi(x_{L_1 \cup L_3})\pi(x_{L_2 \cup L_3}) \tag{1}$$

for all $x_{L_1 \cup L_2 \cup L_3} \in \mathbb{X}_{L_1 \cup L_2 \cup L_3}$.

Equality (1) certainly holds for all $x_{L_1 \cup L_2 \cup L_3} \in \mathbb{X}_{L_1 \cup L_2 \cup L_3}$, for which $\pi(x_{L_3}) = 0$. For those $x_{L_1 \cup L_2 \cup L_3}$, for which $\pi(x_{L_3}) > 0$, we can divide both sides of equality (1) by $\pi(x_{L_3})$, which gives us

$$\pi(x_{L_1 \cup L_2 \cup L_3}) = \pi(x_{L_1 \cup L_3})\pi(x_{L_2}|x_{L_3}). \tag{2}$$

Therefore, we could define conditional independence by requirement that equality (2) holds true for all $x_{L_1 \cup L_2 \cup L_3} \in \mathbb{X}_{L_1 \cup L_2 \cup L_3}$.

Another way how to define conditional independece is to require that the expression

$$\pi(x_{L_1 \cup L_2}|x_{L_3}) = \pi(x_{L_1}|x_{L_3})\pi(x_{L_2}|x_{L_3})$$

holds true for all vectors $x_{L_1 \cup L_2 \cup L_3}$, for which $\pi(x_{L_3}) > 0$. Moreover, since

$$\pi(x_{L_1 \cup L_2}|x_{L_3}) = \pi(x_{L_1}|x_{L_3})\pi(x_{L_2}|x_{L_1 \cup L_3})$$

is valid for all $x \in \mathbb{X}_{L_1 \cup L_2 \cup L_3}$, for which $\pi(x_{L_3})$ is positive, and for any distribution π (regardless it meets the property of conditional independence), we can see that the conditional independence can also be expressed in the following way:

$$X_{L_1} \perp\!\!\!\perp X_{L_2}|X_{L_3}[\pi] \iff \forall x \in \mathbb{X}_{L_1 \cup L_2 \cup L_3} : \pi(x_{L_3}) > 0 \quad (\pi(x_{L_2}|x_{L_3}) = \pi(x_{L_2}|x_{L_1 \cup L_3})).$$

The last formula is often used to explain the concept of conditional independence. It says that conditional probability of variables X_{L_2} given variables X_{L_3} is the same as conditional probability of these variables given variables $X_{L_1 \cup L_3}$. In other words, if we know values of variables X_{L_3}, the conditional probability of variables X_{L_2} does not change if we learn also values of variables X_{L_1}.

2.2 Extensions of distributions

Consider $K \subseteq L \subseteq N$ and a probability distribution $\pi(x_K)$. As already said above, by $\Pi^{(L)}$ we denote the set of all probability distributions defined for variables X_L. Similarly, $\Pi^{(L)}(\pi)$ will denote the system of all *extensions* of the distribution π to L-dimensional distributions:

$$\Pi^{(L)}(\pi) = \left\{ \kappa \in \Pi^{(L)} : \kappa(x_K) = \pi(x_K) \right\},$$

(recall that $\kappa(x_K)$ is the marginal distribution of κ for variables X_K). Having a system

$$\Xi = \{\pi_1(x_{K_1}), \pi_2(x_{K_2}), \ldots, \pi_n(x_{K_n})\},$$

of oligo-dimensional distributions ($K_1 \cup \ldots \cup K_n \subseteq L$), the symbol $\Pi^{(L)}(\Xi)$ denotes the system of distributions that are extensions of all the distributions from Ξ:

$$\Pi^{(L)}(\Xi) = \left\{\kappa \in \Pi^{(L)} : \kappa^{(K_i)} = \pi_i \ \ \forall i = 1, \ldots, n\right\} = \bigcap_{i=1}^{n} \Pi^{(L)}(\pi_i).$$

2.3 Operator of Composition

To be able to compose low-dimensional distributions to get a distribution of a higher dimension we will introduce an *operator of composition*.

To make it clear from the very beginning, let us stress that it is just a generalization of the idea of computing the 3-dimensional distribution from two 2-dimensional ones introducing the conditional independence:

$$\pi(x_1, x_2) \triangleright \kappa(x_2, x_3) = \frac{\pi(x_1, x_2)\kappa(x_2, x_3)}{\kappa(x_2)} = \pi(x_1, x_2)\kappa(x_3|x_2).$$

Definition 2. For arbitrary two distributions $\pi(x_K)$ and $\kappa(x_L)$ their *composition* is given by the formula:

$$\pi(x_K) \triangleright \kappa(x_L) = \begin{cases} \frac{\pi(x_K)\kappa(x_L)}{\kappa(x_{K\cap L})} & \text{if } \pi(x_{K\cap L}) \ll \kappa(x_{K\cap L}), \\ \text{undefined} & \text{otherwise,} \end{cases}$$

where $\pi(x_{K\cap L}) \ll \kappa(x_{K\cap L})$ means that $\kappa(x_{K\cap L})$ dominates $\pi(x_{K\cap L})$. In the finite case, which is considered in the paper, this simplifies to the condition that for all $x_{K\cap L}$

$$\kappa(x_{K\cap L}) = 0 \Longrightarrow \pi(x_{K\cap L}) = 0.$$

Notice that we do not impose any conditions on the relationship of the two sets of variables: X_K and X_L. Nevertheless, if these sets are not disjoint, it may happen that the composition $\pi \triangleright \kappa$ remains undefined (in this case the fomula need not define a probability distribution). It may happen only when for some $x_{K\cap L}$, $\pi(x_{K\cap L}) > 0$ and simultaneously $\kappa(x_{K\cap L}) = 0$. This situation is rather interesting from a theoretical point of view. In practical situations it usually does not occur, because $\pi(x_{K\cap L})$ and $\kappa(x_{K\cap L})$ are marginals of oligodimensional distributions, and therefore their dimensionality is very low. Moreover, when the oligodimensional distributions in question are received as estimations from one data file then, as a rule, all these distributions are pairwise consistent, i.e. $\pi(x_{K\cap L}) = \kappa(x_{K\cap L})$. In this case, if for any $x_{K\cap L}$ $\pi(x_{K\cap L}) = 0$ (and therefore also $\kappa(x_{K\cap L}) = 0$) then there is a product of two zeros in the numerator and we take, quite naturally,

$$\frac{0 \cdot 0}{0} = 0.$$

Notice also that if K and L are disjoint then $\kappa(x_{K\cap L}) = 1$ and the formula degenerates to a simple product $\pi \cdot \kappa$.

The following simple assertion, which summarizes two lemmata proven in (Jiroušek 1997), answers the question: what is the result of composition of two distributions?

Theorem 1. *If $\pi(x_K) \triangleright \kappa(x_L)$ is defined then it is a probability distribution of variables $X_{K\cup L}$ and its marginal distribution for variables X_K equals π:*

$$(\pi \triangleright \kappa)(x_K) = \pi(x_K).$$

Moreover, if π and κ are consistent (ie., $\pi(x_{K\cap L}) = \kappa(x_{K\cap L})$) then $\pi \triangleright \kappa = \kappa \triangleright \pi$ and this composition is the maximum entropy[1] extension of π and κ:

$$\pi \triangleright \kappa = \arg \max_{\nu \in \Pi^{(K\cup L)}(\pi,\kappa)} H(\nu).$$

[1] Here we consider classical Shannon entropy defined

$$H(\pi(x_K)) = -\sum_{x_K \in \mathbb{X}_K : \pi(x_K) > 0} \pi(x_K) \log \pi(x_K).$$

Since in this paper we are interested mainly in practical application of operators of composition, it is important for us just to remember that composing two distributions we are receiving a distribution whose dimension is larger than the dimensionality of both input distributions (generally $\pi \triangleright \kappa \neq \kappa \triangleright \pi$; operator $\triangleright$ is not commutative). In addition to this, π is always a marginal distribution of $\pi \triangleright \kappa$ (κ is a marginal of $\kappa \triangleright \pi$). The second part of the theorem says that if π and κ are consistent then $\pi \triangleright \kappa$ comprise all the information contained in π and κ. Moreover, the composition achieves the maximal entropy among all the distributions having this property. Such a distribution is considered by many authors to be the best representative of knowledge contained in π and κ (Cheeseman 1983, Cheeseman 1985).

2.4 Generating Sequences

In agreement with what has been said above, in the following discussion we will consider a system of n oligo-dimensional distributions $\pi_1(x_{K_1}), \pi_2(x_{K_2}), \ldots, \pi_n(x_{K_n})$. Therefore, whenever we will speak about a distribution π_k in this and the following sections, if not specified explicitely otherwise, the distribution π_k will always be assumed to be a distribuion from $\Pi^{(K_k)}$, which means it will be a distribution $\pi_k(x_{K_k})$.

Composition operator, when applied iteratively, constructs multidimensional distributions from sets of low-dimensional ones. Namely, formula $\pi_1 \triangleright \pi_2 \triangleright \ldots \triangleright \pi_n$, if it is defined, determines the distribution of variables $X_{K_1 \cup K_2 \cup \ldots \cup K_n}$. Regarding the fact that the operator $\triangleright$ is neither commutative nor associative, let us say that we always apply the operators from left to right; i.e.

$$\pi_1 \triangleright \pi_2 \triangleright \pi_3 \triangleright \ldots \triangleright \pi_n = (\ldots((\pi_1 \triangleright \pi_2) \triangleright \pi_3) \triangleright \ldots \triangleright \pi_n).$$

Therefore, in order to construct a multidimensional distribution it is sufficient to determine a sequence – we will call it a *generating sequence* – of oligo-dimensional distributions. A generating sequence $\pi_1, \pi_2, \ldots, \pi_n$ defines the multidimensional distribution

$$\pi_1 \triangleright \pi_2 \triangleright \ldots \triangleright \pi_n.$$

In agreement with what has just been said, for example, the generating sequence

$$\pi_1(x_1, x_3), \pi_2(x_3, x_5), \pi_3(x_1, x_4, x_5, x_6), \pi_4(x_2, x_5, x_6)$$

defines the distribution

$$\begin{aligned}(\pi_1 \triangleright \pi_2 \triangleright \pi_3 \triangleright \pi_4)(x_1, x_2, x_3, x_4, x_5, x_6) &= ((\pi_1(x_1, x_3) \triangleright \pi_2(x_3, x_5))\, \pi_3(x_1, x_4, x_5, x_6)) \triangleright \pi_4(x_2, x_5, x_6) \\ &= \pi_1(x_1, x_3)\pi_2(x_5|x_3)\pi_3(x_4, x_6|x_1, x_5)\pi_4(x_2|x_5, x_6).\end{aligned}$$

Not all generating sequences are equally efficient in their representations of multidimensional distributions. Among them, so-called perfect sequences hold an important position.

Definition 3. A generating sequence of probability distributions $\pi_1, \pi_2, \ldots, \pi_n$ is called *perfect* if $\pi_1 \triangleright \ldots \triangleright \pi_n$ is defined and

$$\begin{aligned}&\pi_1 \triangleright \pi_2 = \pi_2 \triangleright \pi_1, \\ &\pi_1 \triangleright \pi_2 \triangleright \pi_3 = \pi_3 \triangleright (\pi_1 \triangleright \pi_2), \\ &\quad\vdots \\ &\pi_1 \triangleright \pi_2 \triangleright \ldots \triangleright \pi_n = \pi_n \triangleright (\pi_1 \triangleright \pi_2 \triangleright \ldots \triangleright \pi_{n-1}).\end{aligned}$$

¿From this definition one can hardly see the importance of perfect sequences. This importance becomes clearer from the following characterization theorem.

Theorem 2. *A sequence of distributions $\pi_1, \pi_2, \ldots, \pi_n$ is perfect* iff *all the distributions from this sequence are marginals of the distribution $(\pi_1 \triangleright \pi_2 \triangleright \ldots \triangleright \pi_n)$.*

The above presented theorem claims that a model defined by a generating sequence preserves all the given marginals if and only if the model is defined by a perfect sequence. If the considered generating sequence is not perfect then there exists (at least one) $k \in \{1, \ldots, n\}$ such that the marginal distribution $(\pi_1 \triangleright \ldots \triangleright \pi_n)(x_{K_k})$ differs from the given $\pi_k(x_{K_k})$. It can easily be shown that non-perfect generating sequence need not preserve

even one-dimensional marginal distributions – even in case that the given oligodimensional distributions are pairwise consistent.

The message conveyed by the characterization Theorem 2 can also be interpreted in the following way. Considering that low-dimensional distributions π_k are carriers of local information, the constructed multidimensional distribution represents global information, faithfully reflecting all of the local input. This is why we will be so much interested in perfect sequence models.

The following important assertion shows that each generating sequence, for which $\pi_1 \triangleright \ldots \triangleright \pi_n$ is defined, can be transformed into a perfect sequence. This possibility is exploited in several computational procedures.

Theorem 3. *If $\pi_1 \triangleright \ldots \triangleright \pi_n$ is defined then the sequence $\kappa_1, \ldots, \kappa_n$ computed by the following process*

$$\begin{aligned}
\kappa_1(x_{K_1}) &= \pi_1(x_{K_1}),\\
\kappa_2(x_{K_2}) &= \kappa_1(x_{K_2\cap K_1}) \triangleright \pi_2(x_{K_2}),\\
\kappa_3(x_{K_3}) &= (\kappa_1 \triangleright \kappa_2)(x_{K_3\cap(K_1\cup K_2)}) \triangleright \pi_3(x_{K_3}),\\
&\vdots\\
\kappa_n(x_{K_n}) &= (\kappa_1 \triangleright \ldots \triangleright \kappa_{n-1})(x_{K_n\cap(K_1\cup\ldots K_{n-1})}) \triangleright \pi_n(x_{K_n})
\end{aligned}$$

is perfect and

$$\pi_1 \triangleright \ldots \triangleright \pi_n = \kappa_1 \triangleright \ldots \triangleright \kappa_n.$$

Remark. Notice that when defining perfect sequence, let alone generating sequence, we have not imposed any conditions on sets of variables, for which the distributions were defined. For example, considering a generating sequence where one distribution is defined for a subset of variables of another distribution (ie., $K_j \subset K_k$) is fully sensible and may carry an information about the distribution. If $\pi(x_1), \pi(x_2), \pi(x_1, x_2, x_3)$ is a perfect sequence modelling a 3-dimensional distribution, it is quite obvious that

$$\pi(x_1) \triangleright \pi(x_2) \triangleright \pi(x_1, x_2, x_3) = \pi(x_1, x_2, x_3)$$

(because all the elements of a perfect sequence are marginals of the resulting distribution and therefore $\pi(x_1, x_2, x_3)$ must be marginal to $\pi(x_1) \triangleright \pi(x_2) \triangleright \pi(x_1, x_2, x_3)$). Nevertheless, it may happen that for some reason or another it may be advantageous to work with the model defined by the perfect sequence than just with the distribution $\pi(x_1, x_2, x_3)$. From the model one can immediately see that variables X_1 and X_2 are independent, which, not knowing the numbers defining the distribution, one cannot say about distribution $\pi(x_1, x_2, x_3)$. (How to read all the conditional independence relations from a compositional model will be presented in Section 5.)

3 Relation of Compositional Models to Bayesian Networks

This section is intended mainly to the reader familier with Bayesain networks. To make it legible also to other readers let us say that, in principle, Bayesian networks can be introduced in two ways. One possibility is to define them as couples consisting of an acyclic directed graph and a corresponding system of conditional distributions, which defines a multidimensional distribution. The other possibility (from the theoretical point of view equivalent to the former one) is to say that a Bayesian network is a multidimensional distribution factorizing with respect to an acyclic directed graph. Let us adopt the latter possibility for the purpose of this section.

Definition 4. *Consider a system of variables X_N. Let $G = (N, E)$ be an acyclic directed graph with the set of verices N and a set of edges E that does not allow of directed cycles*[2]*. Denote for each $i \in N$*

$$pa(i) = \{j \in N : (j \to i) \in E\}.$$

We say that a probability distribution $\kappa(x_N)$ is a Bayesian network with graph *G if*

$$\kappa(x_N) = \prod_{i\in N} \kappa(x_i|x_{pa(i)}). \tag{3}$$

[2]Directed cycle is a sequence of edges $(i_0 \to i_1), (i_1 \to i_2), (i_2 \to i_3), \ldots, (i_{k-1} \to i_k)$, such that $i_k = i_0$.

From this definition one can immediatelly see that a Bayesian network $\kappa(x_{\ N})$ is uniquely defined by the system of conditional distributions

$$\{\kappa(x_i|x_{pa(i)})\}_{i\in N}.$$

Assume that we do not know the distribution κ, or we cannot handle it because of the size of N. If the dimensionality of the conditional distributions $\kappa(x_i|x_{pa(i)})$ is "reasonable", distributions $\pi(x_{fam(i)})$ $(fam(i) = \{i\} \cup pa(i))$ can easily be found, for which

$$\pi(x_i|x_{pa(i)}) = \kappa(x_i|x_{pa(i)}).$$

To do it, one can take, for example, a uniform distribution $(x_{pa(i)})$ and set

$$\pi(x_{fam(i)}) = \kappa(x_i|x_{pa(i)}) \ \ (x_{pa(i)}).$$

Now we immediately see that the considered Bayesian network can be expressed in the form

$$\kappa(x_N) = \pi_{i_1} \triangleright \pi_{i_2} \triangleright \ldots \triangleright \pi_{i_{|N|}},$$

if the permutation $i_1, i_2, \ldots, i_{|N|}$ is such that all parents of a node are allways before it[3]:

$$i_k \in pa(i_\ell) \implies k < \ell.$$

Therefore, each Bayesian network can be represented by a generating sequence, and due to Theorem 3, also by a perfect sequence. In the following we shall show how to find a Bayesian network representation for a distribution defined by a perfect sequence. Then we will see that the class of Bayesian networks is equivalent to the class of perfect sequences in the following sense:

1. If $\pi_1, \ldots, \pi_n$ is perfect, $K_1 \cup \ldots \cup K_n = N$, then there exists an acyclic directed graph $G = (N, E)$, such that the distribution $\pi_1 \triangleright \ldots \triangleright \pi_n$ is a Bayesian network with graph G and for each node $j \in N$ there exists (at least one) $k \in \{1, \ldots, n\}$ such that

   $$(\{j\} \cup pa(j)) \subset K_k.$$

   (It means that each conditional distribution appearing in the formula (3) can be computed directly from one of the distributions $\pi_1, \ldots, \pi_n$.)
2. For each Bayesian network κ one can construct a perfect sequence $\pi_1, \ldots, \pi_n$ such that $\kappa = \pi_1 \triangleright \ldots \triangleright \pi_n$ and for each distribution $\pi_k(x_{K_k})$ set K_k consists of a node and its parents of the graph of the original Bayesian network; i.e. there is a $j \in K_k$ for which

   $$K_k = \{j\} \cup pa(j).$$

Let us, now, present a simple procedure transforming an arbitrary perfect sequence $\pi_1, \ldots, \pi_n$ into a Bayesain network. What is important is the fact that the conditional distributions defining this network are not of greater dimensionality than distributions $\pi_1, \ldots, \pi_n$. Therefore, all the computations performed in the algorithm are "local", i.e. are performed within individual oligodimensional distributions.

The process of reconstruction of a Bayesian network from a perfect sequence must consist of a definition of a graph and in a computation of conditional distributions $\kappa(x_i|x_{pa(i)})$ defining the constructed Bayesian network. A possibility how to do it is described in the following algorithm.

[3] An existence of such an ordering follows from the fact that each finite acyclic directed graph has at least one node i for which $pa(i) = \emptyset$ (hint of a proof: If not, then each node has a parent and therefore a sequence $i_1, i_2, i_3, \ldots$ can be constructed, in which $i_{k+1} \in pa(i_k)$. If the sequence is long enough, and we can construct it as long as we need, it must contain a cycle.) This node can be placed at the first position of the permutation. Deleting this node we get again an acyclic directed graph, which has to possess a parentless node; this is placed at the second position of the permutation, and so on.

Algorithm

(a) Having a perfect sequence $\pi_1(x_{K_1}), \ldots, \pi_n(x_{K_n})$ we first order (in an arbitrary way) all the indices of the considered variables, i.e.

$$\{1, 2, 3, \ldots, |N|\} = K_1 \cup \ldots \cup K_n = N.$$

(b) A graph of the constructed belief network is defined in the following way:

1. the set of nodes is N;
2. there is an edge $(i \to j)$ if there exists a distribution π_k such that all the following three conditions hold:
   (i) $i, j \in K_k$,
   (ii) $j \notin K_1 \cup \ldots \cup K_{k-1}$
   (iii) either $i \in K_1 \cup \ldots \cup K_{k-1}$ or i is in the ordering defined in step **(a)** before j.

(c) For each j the requirement $j \in K_k$, $j \notin K_1 \cup \ldots \cup K_{k-1}$ is met exactly for one $k \in \{1, \ldots, s\}$. It means that all the parents of node X_j must be in the respective set X_{K_k} and therefore the necessary conditional distribution $\kappa_j(x_j|x_{pa(j)})$ can be easily computed from distribution $\pi_k(x_{K_k})$.

Remark. The reader familier with Bayesian networks knows that (with the exception of very specific situations) each probability distribution representable as a Bayesian network can be represented by several networks with different graphs. This possibility is reflected in the step (a) of the algorithm, where the nodes are ordered in an arbitrary way. As a rule it happens that different orderings lead to different graphs.

In the rest of the paper we shall illustrate the fact that representation of multidimensional distributions by perfect sequences is (at least) as efficient as their representation with the help of Bayesian networks.

4 Marginalization for Perfect Sequences

For the sake of simplicity, in this section we shall concentrate only on special cases. First, we will consider marginalization decreasing the dimensionality by one. For this purpose, we will use the simplified notation (recall that we still assume $\pi_k \in \Pi^{K_k}$):

$$\pi_k^{[\ell]} = \pi_k(x_{K_k \setminus \{\ell\}}) = \pi_k^{(K_k \setminus \{\ell\})},$$
$$(\pi_1 \triangleright \ldots \triangleright \pi_n)^{[\ell]} = (\pi_1 \triangleright \ldots \triangleright \pi_n)(x_{K_1 \cup \ldots \cup K_n \setminus \{\ell\}}),$$

for $\ell \in K_k$.

Using this notation we can immediately formulate the following simple but still interesting assertion. (The proofs of all the theorem presented in this section can be found in (Jiroušek 2000).)

Theorem 4. *If $\ell \in K_i$ for some $i \in \{1, 2, \ldots, n\}$ and $\ell \notin K_j$ for all $j \neq i$ then*

$$(\pi_1 \triangleright \pi_2 \triangleright \ldots \triangleright \pi_n)^{[\ell]} = \pi_1 \triangleright \ldots \triangleright \pi_{i-1} \triangleright \pi_i^{[\ell]} \triangleright \pi_{i+1} \triangleright \ldots \triangleright \pi_n.$$

However, situations, in which variable ℓ that is to be eliminated is contained in several distributions, are much more complicated (and, naturally, much more frequent). To present a solution to a general situation we shall need a very important property of the operator $\triangleright$. Namely, this operator can be substituted by an operator $\circledcirc_K$, depending on a set K, simultaneously with changing the ordering of operations.

Theorem 5. *If π_1, π_2 and π_3 are such that $\pi_1 \triangleright \pi_2 \triangleright \pi_3$ is defined then*

$$\pi_1 \triangleright \pi_2 \triangleright \pi_3 = \pi_1 \triangleright (\pi_2 \circledcirc_{K_1} \pi_3),$$

where

$$\pi_2 \circledcirc_{K_1} \pi_3 = (\pi_3^{((K_1 \setminus K_2) \cap K_3)} \pi_2) \triangleright \pi_3.$$

It should be highlighted here that the computational complexity of the composition $\pi_2 \circledcirc_{K_1} \pi_3$ does not differ substantially from the complexity of computation of $\pi_2 \triangleright \pi_3$. It follows, namely, from the fact that both of these distributions are of the same dimensionality; both are defined for variables $X_{K_2 \cup K_3}$. In other words, in both cases we have to compute the same number of probability values.

As the reader can see, the operator is parameterized by the index set K_1. The purpose of the operator is to compose the distributions (in our case distributions π_2 and π_3) but, simultaneously, to introduce the necessary independence of variables $X_{(K_1 \setminus K_2) \cap K_3}$ and X_{K_2} that would otherwise be omitted. If we want to compose distributions π_2 and π_3 before π_1 is considered, we have to "anticipate" the independence which was originally introduced by the previous operator. Therefore, operator $\circledcirc_K$ is called an *anticipating operator*.

Example. As said before, the specific purpose of the anticipating operator is to introduce the necessary conditional independence that would otherwise be omitted. To illustrate the point, let us consider the generating sequence $\pi_1(x_1), \pi_2(x_2), \pi_3(x_1, x_2)$ for which

$$\pi_1(x_1) \triangleright \pi_2(x_2) \triangleright \pi_3(x_1, x_2) = \pi_1(x_1)\pi_2(x_2).$$

If we used the operator $\triangleright$ instead of $\circledcirc_{K_1}$, we would get

$$\pi_1(x_1) \triangleright (\pi_2(x_2) \triangleright \pi_3(x_1, x_2)) = \frac{\pi_1(x_1)(\pi_2(x_2)\pi_3(x_1|x_2))}{\sum\limits_{x_1 \in \mathbb{X}_1} \pi_2(x_2)\pi_3(x_1|x_2)},$$

which evidently differs from $\pi_1(x_1)\pi_2(x_2)$ because $\pi_1 \triangleright (\pi_2 \triangleright \pi_3)$ inherits the dependence of variables X_1 and X_2 from π_3. Nevertheless, considering

$$\begin{aligned}\pi_1(x_1) \triangleright (\pi_2(x_2) \circledcirc_{\{1\}} \pi_3(x_1, x_2)) = \pi_1(x_1) \triangleright (\pi_3(x_1)\pi_2(x_2) \triangleright \pi_3(x_1, x_2)) = \pi_1(x_1) \triangleright \pi_3(x_1)\pi_2(x_2) \\ = \pi_1(x_1)\pi_2(x_2)\end{aligned}$$

we get the desired result.

Now, we are ready to present a solution to the general marginalization problem, which is given by the following theorem.

Theorem 6. *Let $\pi_1, \pi_2, \ldots, \pi_n$ be a generating sequence and $\ell \in K_{i_1} \cap K_{i_2} \cap \ldots \cap K_{i_m}$ for some*

$$\{i_1, i_2, \ldots, i_m\} \subseteq \{1, 2, \ldots, n\}$$

(assuming $(i_1 < i_2 < \ldots < i_m)$) such that $\ell \notin K_j$ for all $j \in \{1, 2, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_m\}$. Then

$$(\pi_1 \triangleright \pi_2 \triangleright \ldots \triangleright \pi_n)^{[\ell]} = \kappa_1 \triangleright \kappa_2 \triangleright \ldots \triangleright \kappa_n,$$

where

$$\begin{aligned}\kappa_j &= \pi_j \text{ for all } j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_m\}, \\ \kappa_{i_1} &= \pi_{i_1}^{[\ell]}, \\ \kappa_{i_2} &= (\pi_{i_1} \circledcirc_{L_{i_2-1}} \pi_{i_2})^{[\ell]}, \\ \kappa_{i_3} &= (\pi_{i_1} \circledcirc_{L_{i_2-1}} \pi_{i_2} \circledcirc_{L_{i_3-1}} \pi_{i_3})^{[\ell]}, \\ &\vdots \\ \kappa_{i_m} &= (\pi_{i_1} \circledcirc_{L_{i_2-1}} \pi_{i_2} \circledcirc_{L_{i_3-1}} \cdots \circledcirc_{L_{i_m-1}} \pi_{i_m})^{[\ell]},\end{aligned}$$

and $L_{i_k-1} = (K_1 \cup K_2 \cup \ldots \cup K_{i_k-1}) \setminus \{\ell\}$.

For benefit of the reader familiar with Shachter's method of Bayesian network marginalization based on *node deletion* and *edge reversal* procedures ((Shachter 1986)), let us point out an analogy. If a variable X_i appears among the arguments of only one distribution π_k (i.e. $i \in K_k$ and $i \notin K_1 \cup \ldots \cup K_{k-1} \cup K_{k+1} \cup \ldots \cup K_n$), then its elimination is very simple, analogous to elimination of a terminal node in Shachter's approach. In other cases we have to guarantee that dependences conveyed by the node being eliminated will not be suspended. These dependences are, in Shachter's approach, preserved by "parents' inheritance" in the *edge reversal* step, while in our approach they are retrieved in computations of $\pi_{i_1} \circledcirc_{L_{i_2-1}} \pi_{i_2} \circledcirc_{L_{i_3-1}} \cdots \circledcirc_{L_{i_r-1}} \pi_{i_r}$. A certain advantage of the presented approach consists in the fact that there are no "orientations of edges" and therefore all variables which can be assigned to a terminal node in any equivalent Bayesian network are deleted in an equally simple way. This is not true when Bayesian networks are considered, because there are usually several equivalent Bayesian networks and a variable can be deleted only when it is assigned to a terminal, otherwise one has to first reverse the respective edges.

The last assertion we are going to present in this section descibes rather general marginalization, which, however, cannot be always applied. In a way, it corresponds to situations when the considered multidimensional distribution can be *decomposed* into smaller parts.

Theorem 7. *Let $\pi_1, \pi_2, \ldots, \pi_n$ be a perfect sequence and $j_1, j_2, \ldots, j_{m_1}$ and $k_1, k_2, \ldots, k_{m_2}$ be two disjoint subsequences of $1, \ldots, n$ such that*

- $\{j_1, \ldots, j_{m_1}\} \cup \{k_1, \ldots, k_{m_2}\} = \{1, \ldots, n\}$,
- *there exists* $s \in \{1, \ldots, n\}$ *such that* $(K_{j_1} \cup \ldots \cup K_{j_{m_1}}) \cap (K_{k_1} \cup \ldots \cup K_{k_{m_2}}) \subseteq K_s$,
- $j_1 < j_2 < \ldots < j_{m_1}$ *and* $k_1 < k_2 < \ldots < k_{m_2}$ *(i.e. both subsequences preserve the ordering of the original sequence).*

Then, denoting $L = K_{j_1} \cup \ldots \cup K_{j_{m_1}}$,

$$(\pi_1 \triangleright \pi_2 \triangleright \ldots \triangleright \pi_n)^{(L)} = \kappa_1 \triangleright \kappa_2 \triangleright \ldots \triangleright \kappa_n,$$

where

$$\begin{array}{lll} \kappa_{j_r} = \pi_{j_r} & \text{for} & r = 1, 2, \ldots m_1, \\ \kappa_{k_r} = \pi_s^{(L \cap (K_{k_1} \cup \ldots \cup K_{k_r}))} & \text{for} & r = 1, 2, \ldots m_2. \end{array}$$

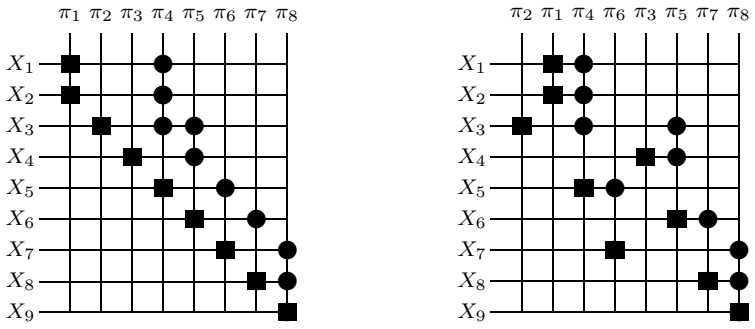
5 Conditional Independence Relations of Compositional Models

It is well-known that one can read conditional independence relations of a Bayesian network from its graph. This section is devoted to the task of detection of all the conditional independence relations that must hold for distributions represented by generating sequences (for proofs of the presented assertions see (Jiroušek 2002b, Jiroušek 2002a)). An appropriate tool for this will be a *persegram*[4].

Definition 5. *Persegram* of a generating sequence is a table in which rows correspond to variables (in an arbitrary order) and columns to low-dimensional distributions; ordering of the columns corresponds to the generating sequence ordering. A position in the table is marked if the respective distribution is defined for the corresponding variable. Markers for the first occurrence of each variable (i.e. the leftmost markers in rows) are squares (we will call them *box-markers*) and for other occurrences they are *bullets*.

In Figure 2(a) we can see a persegram for the sequence

$$\pi_1(x_1, x_2), \pi_2(x_3), \pi_3(x_4), \pi_4(x_1, x_2, x_3, x_5), \pi_5(x_3, x_4, x_6), \pi_6(x_5, x_7), \pi_7(x_6, x_8), \pi_8(x_7, x_8, x_9). \quad (4)$$



(a) (b)

Figure 2. Persegrams

Taking another generating sequence defining the same distribution

$$\pi_2 \triangleright \pi_1 \triangleright \pi_4 \triangleright \pi_6 \triangleright \pi_3 \triangleright \pi_5 \triangleright \pi_7 \triangleright \pi_8 = \pi_1 \triangleright \pi_2 \triangleright \pi_3 \triangleright \pi_4 \triangleright \pi_5 \triangleright \pi_6 \triangleright \pi_7 \triangleright \pi_8, \quad (5)$$

we get a different persegram given in Figure 2(b).

5.1 Unconditional independence

In this section we shall demonstrate how to read unconditional independence relations from a persegram representation of generating sequences. In fact, everything contained here is a special case of what will be presented in the next paragraph. The reason why this paragraph is included is just didactical; otherwise, the next section could be rather difficult to read.

[4] A word expressing that it is a graphical record of a PERfect SEquence.

Definition 6. For a generating sequence $\pi_1(x_{K_1}), \ldots, \pi_n(x_{K_n})$, a sequence of markers $m_0, m_1, \ldots, m_t$ in the corresponding persegram is called a *simple trail connecting* m_0 *and* m_t if it meets the following three conditions:

1. for each $s = 1, \ldots, t$, a couple (m_{s-1}, m_s) is in the same row (i.e. horizontal connection) or in the same column (vertical connection);
2. each vertical connection must be adjacent to a box-marker (more precisely, in the corresponding couple (m_{s-1}, m_s) either m_{s-1} or m_s must be a box-marker);
3. vertical and horizontal connections regularly alternate.

If a simple trail connects two box-markers corresponding to variables X_j and X_k, we also say that these *variables are connected by a simple trail*. This situation will be denoted by $X_j \rightsquigarrow_\emptyset X_k$.

Examples of simple trails can be seen in Figure 3. In Figure 3(a), the outlined simple trail $m_0 = X_3\pi_2, m_1 = X_3\pi_5, m_2 = X_6\pi_5, m_3 = X_6\pi_7, m_4 = X_8\pi_7$ connects variables X_3 and X_8 (in notation $X_3 \rightsquigarrow_\emptyset X_8$). Analogously, the simple trail in Figure 3(b) connects variables X_7 and X_8 ($X_7 \rightsquigarrow_\emptyset X_8$).

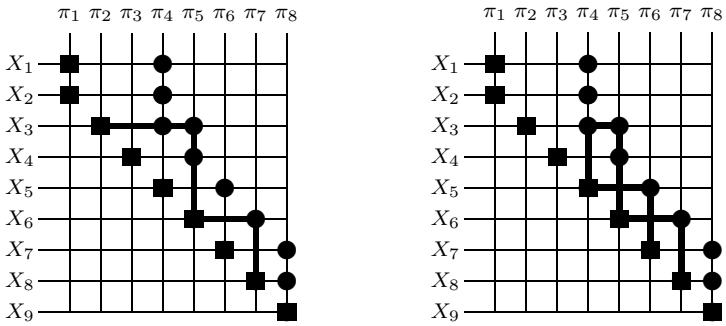


Figure 3. Persegrams with simple trails

Theorem 8. *Consider a generating sequence* $\pi_1(x_{K_1}), \ldots, \pi_n(x_{K_n})$*, and two disjoint nonempty subsets* $J_1, J_2 \subset K_1 \cup \ldots \cup K_n$*. If there does not exist a simple trail* $X_j \rightsquigarrow_\emptyset X_k$ *in the corresponding persegram such that* $j \in J_1$ *and* $k \in J_2$ *then the groups of variables* X_{J_1} *and* X_{J_2} *are independent under the distribution* $\pi_1 \triangleright \ldots \triangleright \pi_n$*:*

$$X_{J_1} \perp\!\!\!\perp X_{J_2}[\pi_1 \triangleright \ldots \triangleright \pi_n].$$

Remark. It is important to realize that (analogously to the situation when Bayesian networks are considered) the indicated independence relations are those about which one can be sure that must hold for any distribution being represented by a generating sequence with the given persegram. It can easily be shown that the set of indicated independence relations is maximal in the sense that if there exists a simple trail connecting variables X_i and X_j then there exists a distribution represented by a generating sequence with the given persegram, and variables X_i, X_j are dependent under this distribution. Nevertheless, for a specific situation it can happen that some variables are independent even if this independence cannot be read from the persegram.

Example. Let us find all the independence relations that must hold for any distribution represented by a generating sequence

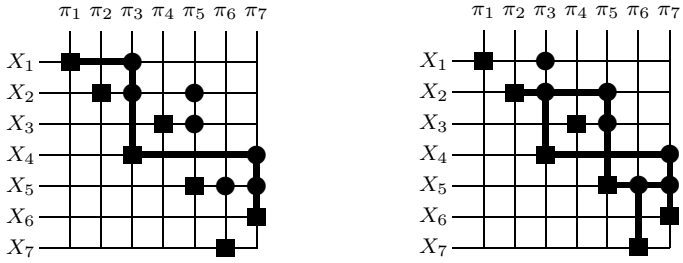
$$\pi_1(x_1), \pi_2(x_2), \pi_3(x_1, x_2, x_4), \pi_4(x_3), \pi_5(x_2, x_3, x_5), \pi_6(x_5, x_7), \pi_7(x_4, x_5, x_6). \tag{6}$$

Its persegram is in Figure 4.

Let us find all simple trails starting from box-marker $X_1\pi_1$. The only possibility is to go through $X_1\pi_3$ into $X_4\pi_3$. This simple trail connecting variables X_1 and X_4 can be prolonged only by adding two markers: $X_4\pi_7$, $X_6\pi_7$. Therefore, X_1 is connected by simple trails (which are in Figure 4(a)) only with X_4 and X_6 and therefore

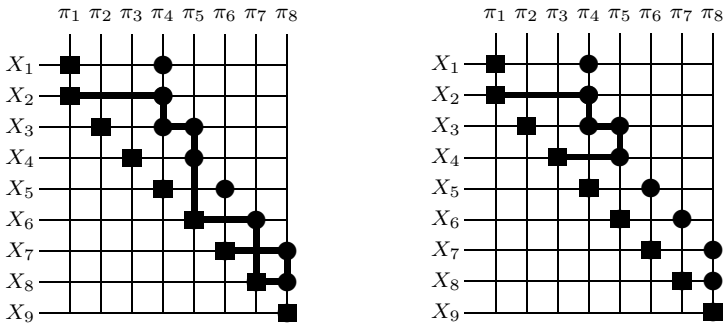
$$X_1 \perp\!\!\!\perp X_2, X_3, X_5, X_7.$$

Similarly, we shall find all simple trails starting from $X_2\pi_2$. ¿From this box-marker one can go through $X_2\pi_3$ into $X_4\pi_3$ and then to $X_4\pi_7$, $X_6\pi_7$. Thus, there are simple rails connecting X_2 with X_4 and X_6.



(a) (b)

Figure 4. Persegram of a sequence (6) with simple trails



trail avoiding $\{5, 9\}$ trail avoiding $\{5, 6\}$

(a) (b)

Figure 5. Trails avoiding subsets

However, form $X_2\pi_2$ one can go along two other simple trails either $X_2\pi_5$, $X_5\pi_5$, $X_5\pi_6$, $X_7\pi_6$, or $X_2\pi_5$, $X_5\pi_5$, $X_5\pi_7$, $X_6\pi_7$. Thus we got that there are simple trails connecting X_2 with X_4, X_5, X_6, X_7 (all of them are marked in Figure 4(a)). Therefore $X_2 \perp\!\!\!\perp X_1, X_3$.

Analogously, and this we left to the reader, it is simple to show that

$$X_3 \perp\!\!\!\perp X_1, X_2, X_4, \qquad X_4 \perp\!\!\!\perp X_3,$$
$$X_5 \perp\!\!\!\perp X_1, \qquad X_7 \perp\!\!\!\perp X_1.$$

5.2 Conditional independence

Definition 7. Consider a generating sequence $\pi_1(x_{K_1}), \ldots, \pi_n(x_{K_n})$, its corresponding persegram and a subset $L \subset K_1 \cup \ldots \cup K_n$. A sequence of markers $m_0, m_1, \ldots, m_t$ is called a *trail avoiding the subset* L that connects m_0 and m_t if it meets the following 4 conditions:

1. for all $s = 0, \ldots, t$ markers m_s do not correspond to any variable from X_L;
2. for each $s = 1, \ldots, t$ a couple (m_{s-1}, m_s) is in the same row (i.e. horizontal connection) or in the same column (vertical connection);
3. each vertical connection that does not correspond to a distribution with a box-marker of some variable from X_L, must be adjacent to a box-marker;
4. vertical and horizontal connections regularly alternate (though, there can be a vertical connection of "length 0", when it corresponds to a distribution with a box-marker of a variable from X_L.

If a trail avoiding L connects two box-markers corresponding to variables X_j and X_k, we also say that these *variables are connected by a trail avoiding* L. This situation will be denoted $X_j \rightsquigarrow_L X_k$.

The reader can immediately see that for $L = \emptyset$ this definition coincides with Definition 6. In this case, namely, the condition 3 (of Definition 7) requires that all vertical connections must be adjacent to a box-marker.

An example of a trail avoiding $L = \{5, 9\}$ is a trail in Figure 3(a) as it contains markers corresponding neither to X_5 nor X_9 and all vertical connections are adjacent to box-markers. Other examples can be seen in Figure 5. In Figure 5(a), the outlined trail connects X_2 and X_7 avoiding again $L = \{5, 9\}$. The vertical connection $(m_1 = X_2\pi_4, m_2 = X_3\pi_4)$ is admitted because it corresponds to distribution π_4 with the box-

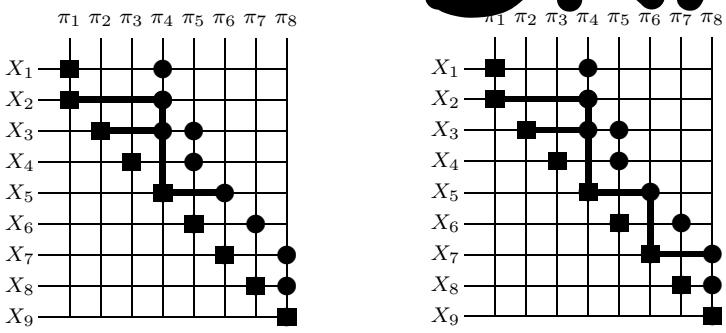


Figure 6. Trails avoiding subsets

marker of variable X_5 and $5 \in L$. Analogously, the vertical connection $(m_7 = X_8\pi_8, m_8 = X_7\pi_8)$ is admitted because of a box-marker $X_9\pi_8$. The trail presented in Figure 5(b) connects X_2 and X_4 avoiding $L = \{5, 6\}$:

$$X_2 \rightsquigarrow_{\{5,6\}} X_4.$$

Another examples of avoiding trails with vertical connections of "lenght 0" are in Figure 6.
In Figure 6(a) a trail avoiding $\{7\}$ is presented, which consists of the following sequence of markers:

$$X_2\pi_1, X_2\pi_4, X_5\pi_4, X_5\pi_6, X_5\pi_6, X_5\pi_4, X_3\pi_4, X_3\pi_2.$$

This trail contains a legal verical connection of "length 0" $(X_5\pi_6, X_5\pi_6)$ – marker $X_5\pi_6$ corresponds to a distribution π_6 with a box marker of variable X_7. An example of a trail realizing $X_2 \rightsquigarrow_{\{9\}} X_3$ is in Figure 6(b).

Theorem 9. *Consider a generating sequence $\pi_1(x_{K_1}), \ldots, \pi_n(x_{K_n})$, and three disjoint subsets $J_1, J_2, L \subset K_1 \cup \ldots \cup K_n$ such that $J_1 \neq \emptyset \neq J_2$. If there does not exist a trail $X_j \rightsquigarrow_L X_k$ in the corresponding persegram with $j \in J_1$ and $k \in J_2$ then the groups of variables X_{J_1} and X_{J_2} are conditionally independent given variables X_L under the distribution $\pi_1 \triangleright \ldots \triangleright \pi_n$:*

$$X_{J_1} \perp\!\!\!\perp X_{J_2} | X_L [\pi_1 \triangleright \ldots \triangleright \pi_n].$$

In analogy to the last Remark let us say that also in this case the conditional independence relations determined from a persegram are those, which are necessary for any distribution represented by a generating sequence with the given persegram. This system of conditional independence relations is also maximal in the sense that if there exists a trail $X_j \rightsquigarrow_L X_k$ then there exists a distribution represented by a generating sequence with the given persegram, and variables X_j, X_k are conditionally dependent given variables X_L under this distribution.

6 Conclusions

This paper is a survey of the main results concerning application of compositional models to knowledge-based decision-making. It should be stressed that, as we showed in Section 3, compositional models are, in a sense, equivalent to Bayesian networks. It means that they can be applied in all the situations when Bayesian networks are usually used. There is also no practical difference in the space requirements for these two models representation (compositional models need not code graphs but this can also be avoided even for Bayesian networks). The first achieved results concerning complexity of computational procedures give hint that computation with compositional models will not be more complex than computations with Bayesian networks, and in some special situations it seems to be even simpler (recall the mentioned example of Shachter's node deletion rule, which decreases the dimensionality of a distribution by one). So, let us conclude the paper by presenting two main reasons why we prefer compositional models to Bayesian networks.

- The first is that the apparatus of compositional models is developed exclusively within probability theory. It does not use graphs, which, although by some authors are considered to be intuitively well understood, are often misinterpreted. Especially, arrows of acyclic directed graphs of Bayesian networks are often

misinterpreted as causalities. Moreover, Bayesian networks, decomposable models and chain graph models, which can all be well described by the apparatus of composition operators, use different types of graphs.

- Generating sequences and their special subclass of perfect sequences were defined also in the framework of possibility theory (Jiroušek et al. 2002, Vejnarová 1998). In this framework the product, which in probability theory expresses "independence", is substituted by a continuous t-norm. In this way one gets a very wide class of models whose application is worthy of recommendation if either the assumption of additivity is not proper in the field of application, or if having a small amount of data one can hardly get reasonable estimates of probabilities.

Acknowledgments

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DRAFT

Some comparisons of the complexity of neural-network models and linear methods by means of metric entropy

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Abstract. This work extends the work in (Hlaváčková-Schindler and Sanguineti 2003) investigating a class of nonlinear models by presenting their further advantages in comparison to the class of linear models. The class of nonlinear models has the structure of combinations of simple, parametrized basis functions; it includes widespread neural networks in which the basis functions correspond to the computational units of a type of networks. Bounds on the complexity of such models are derived in terms of the number of adjustable parameters necessary for a given modelling accuracy. These bounds guarantee a more advantageous tradeoff between modelling accuracy and model complexity than linear methods: the number of parameters may increase much more slowly, in some cases only polynomially, with the dimensionality of the input space in modelling tasks. Polynomial bounds on complexity allow one to cope with the so-called "curse of dimensionality", which often makes linear methods either inaccurate or computationally unfeasible. The presented results let one gain a deeper theoretical insight into the effectiveness of neural-network architectures for complex modelling applications.

Keywords: linear model, nonlinear model, neural-network model, upper bound, lower bound, n-width, deviation

1 Introduction

In many applications, in which strong nonlinearities are involved, linear models show their limitations (see, e.g., (Kanellakopoulos et al. 1991)). Recently, learning-based nonlinear modelling has been applied as a promising alternative to implement identification and control structures for adaptive control of nonlinear dynamical systems (see, e.g., (Chen and Liu 1994, Chen and Khalil 1995)). Parametrized classes of nonlinear input-output mappings with powerful approximating capabilities, such as neural networks, have become attractive tools for modelling complex systems on the basis of input-output data (see. e.g., (Narendra and Mukhopadhyay 1997, Narendra et al. 1995)), thanks to their computationally effectiveness and to the possibility of achieving a desired modelling accuracy by means of much fewer parameters than in the case of linear methods. Besides their powerful approximation capabilities, another feature making neural-network models particularly attractive is their intrinsic parallelism. The availability of parallel VLSI realizations will make such models even more interesting for solving of complex tasks.

The *density* property (in the neural-network terminology also called *universal approximation* property), garanteeing the possibility of approximating arbitrarily well all "reasonable" functions encountered in applications, has been proven for neural networks with various types of architectures and computational units (e.g., radial-basis-functions and perceptrons; see (Pinkus 1999, Kůrková 2002, Park and Sandberg 1993, Leshno et al. 1993) and the references therein) and is a necessary condition to guarantee an arbitrary modelling accuracy.

However, density does not ensure feasibility: in complex modelling tasks, classes of models are required to allow a good compromise between the accuracy of approximation and the "complexity" necessary to achieve such an accuracy. For linear and neural-network methods, the complexity can be expressed by the size of a properly defined parameter vector (e.g., the degree of a polynomial or the number of knots of a fixed-knots spline in the linear case, the number of hidden units of a neural network in the nonlinear context, etc.). In linear methods, the adjustable parameters are the coefficients of the linear combinations of fixed basis functions, whereas in neural-network architectures these combinations also include the weights in computational units.

High-dimensional modelling is often unfeasible because of the *curse of dimensionality* (Bellman 1957): the number of parameters required to obtain a desired modelling accuracy grows exponentially with the number d of variables of the nonlinear mappings to be modelled. On the other hand, applications have shown that the neural networks with a moderate number of parameters can be effectively used in approximation and optimization tasks dependent on a large number of variables, for which traditional linear methods are inefficient (see e.g. (Burr 1988, Sejnowski and Rosenberg 1987) and the references therein). These results have motivated the

development of a theoretical framework to investigate the capabilities of neural architectures and to compare them to classical linear methods (see, e.g., (Barron 1993, Kůrková and Sanguineti 2002) and the references therein).

This paper has two objectives. The first objective is to show how the methods from approximation theory can be applied to bound the complexity of various models by introducing a general formulation including linear methods and a variety of widespread nonlinear ones (neural networks, radial basis networks, wavelets, hinging hyperplanes, fuzzy models, etc.). We show that they have the same geometrical structure, and we point out the basic differences between them and linear methods. The second objective is to make some comparisons between linear and neural-network models by showing of some advantages of the latter, using an a-priori knowledge to define suitable smoothness classes for the nonlinearities to be modelled.

We formalize modelling tasks in terms of approximation theory in functional spaces. A given modelling task determines the choice of an *ambient functional space*, the a-priori assumptions about the systems to be modelled identify a *hypothesis class*, and the choice of a class of models corresponds to the choice of an *approximation scheme*. Multiple models can be used for the same system in different operating environments, which identify different hypothesis classes.

In (Hlaváčková-Schindler and Sanguineti 2003), bounds on the complexity of linear and nonlinear models of the neural-network type were given for different hypothesis classes (e.g., corresponding to different operating environments for the same system) and in different ambient spaces, each associated with a certain modelling task. Bounds on model complexity were determined by estimating the rate of decrease in the modelling error as a function of the number of adjustable parameters. Here we present these results in a shortened way and the new results concerning the application of knowledge of entropy or covering numbers we discuss more in detail. The provided lower bounds on the worst-case modelling error by linear methods are larger than the upper bounds on the same error by nonlinear models of the neural-network type. In all such cases, network modelling architectures outperform linear methods. We also exhibit neural-network models characterized by moderate complexity, in the sense that the number of parameters necessary for a desired accuracy grows at most polynomially with the dimension of the input space, in certain modelling tasks for which the use of linear methods is computationally unfeasible. Our estimates show that the use of neural-network models instead of linear ones often allows a significant reduction in the number of parameters required for a given modelling accuracy.

This paper is organized as follows. Section 2 deals with basic notations and definitions and introduces a mathematical framework for description of the network models. Section 3 shows how tools from approximation theory can be used to develop a procedure of complexity-based model selection. Section 4 presents a variety neural-network models with moderate complexity. Section 5 provides lower bounds on linear methods that imply their poor efficiency in certain modelling tasks. Lower bounds on linear approximation derived from the knowledge of entropy or covering numbers are discussed in Section 6, where also an application of the above discussed approach, a comparison between linear and neural-network models (represented by perceptron networks) is presented. Section 7 contains some concluding remarks.

2 Definition of linear and neural-network models

Both linear and neural-network models can be represented as sets of parametrized mappings in which the parameters have to be updated in such a way that the modelling error is decreased (e.g., by gradient-type descent methods like back-propagation for neural networks). In classical linear models, the adjustable parameters are the coefficients of the linear combinations of fixed basis functions, whereas in neural-network architectures the parameters also include the weights in computational units. In this section, we describe a mathematical framework to represent these two different kinds of parametrizations.

Linear models are based on the use of linear combinations of a set of fixed basis functions. Such linear combinations "span" a finite-dimensional subspace, of dimension equal to n; the dimension is n when the fixed basis functions are linearly independent. Thus, the free parameters are the n coefficients of the linear combinations of the fixed basis functions. For example, the subspace of all polynomials of order at most $n-1$ is spanned by the first n elements of the set $\{x^{i-1} : \ i \in \mathcal{N}_+\}$, where $\mathcal{N}_+$ denotes the set of positive integers. As this corresponds to the use of linear combinations of a certain number of fixed basis functions, the term *fixed-basis models* is used, too.

In recent years, there has been growing interest in nonlinear models such as wavelets, radial basis networks, kernel estimators, B-splines, sigmoidal neural networks, hinging hyperplanes, etc. (see the references in (Sjöberg et al. 1995)). All these models share a common feature: they are built as combinations of a number

of nonlinearities corresponding to parametrized basis functions with a fixed structure in which a certain number of "free" parameters has to be adjusted. In the following, we introduce a general class of nonlinear models that encompasses all such modelling techniques.

We call ϕ-*network models*, with computational units $\phi : \mathcal{R}^p \times \mathcal{R}^d \to \mathcal{R}$ ($\mathcal{R}$ denotes the set of real numbers) and a single linear output unit, the nonlinear mappings $\sum_{i=1}^{n} w_i \phi(a_i, .)$, where $a_i \in A \subseteq \mathcal{R}^p$, and p and d represent the dimensions of a *parameter space* and an *input space*, respectively. As in applications all inputs are bounded, the set A is a bounded subset of $\mathcal{R}^d$; without loss of generality, we consider A to be the unit cube $[0,1]^d$. Widespread types of ϕ-network models are perceptron networks and radial basis functions networks.

A perceptron with an activation function $\psi : \mathcal{R} \to \mathcal{R}$ computes functions of the form $\phi((v,b),x) = \psi(v \cdot x + b) : \mathcal{R}^{d+1} \times \mathcal{R}^d \to \mathcal{R}$, where $v \in \mathcal{R}^d$ is an *input weight* vector and $b \in \mathcal{R}$ is a *bias*. The most common activation functions in perceptrons are *sigmoidals*, i.e., bounded measurable functions $\quad : \mathcal{R} \to [0,1]$ such that $\lim_{t \to -\infty} \quad (t) = 0$ and $\lim_{t \to +\infty} \quad (t) = 1$. One can use both continuous sigmoidals (like the logistic sigmoid $1/(1+e^{-t})$ or the hyperbolic tangent) and the discontinuous Heaviside function ϑ, defined as $\vartheta(t) = 0$ for $t < 0$ and $\vartheta(t) = 1$ for $t \geq 0$. By $P_d(\ ) = \{f : [0,1]^d \to \mathcal{R} : f(x) = \quad (v \cdot x + b),\ v \in \mathcal{R}^d,\ b \in \mathcal{R}\}$ we denote the set of functions on $[0,1]^d$ computable by $\quad$-perceptrons. A *radial basis function* (RBF) unit with a radial (even) function $\psi : \mathcal{R} \to \mathcal{R}_+$ ($\mathcal{R}_+$ is the set of positive real numbers) computes $\phi((v,b),x) = \psi(b\|x - v\|)$, where $v \in \mathcal{R}^d$ is a *centroid*, $b \in \mathcal{R}_+$ is a *width* and $\|.\|$ is a norm on $\mathcal{R}^d$. By $F_d(\psi) = \{f : [0,1]^d \to \mathcal{R}; f(x) = \sum_{i=1}^{n} w_i \psi(b_i \|x - v_i\|), v_i \in \mathcal{R}^d, w_i, b_i \in \mathcal{R}, n \in \mathcal{N}_+\}$ we denote the set of functions on $[0,1]^d$ computable by ψ-RBF networks. The standard choice of a radial function is the Gaussian function.

Models having the structure of ϕ-networks can be mathematically formalized as linear combinations of at most n elements of the set $G_\phi = \{\phi(a,\cdot); a \in A \subseteq \mathcal{R}^p\}$, representing the set of functions computable by hidden network units. The set of input/output mappings computed by a ϕ-network with a single linear output unit and n hidden units, each computing the function ϕ, is given by $span_n G_\phi = \{\sum_{i=1}^{n} w_i g_i; w_i \in \mathcal{R}, g_i \in G_\phi\}$. Such a set has the geometrical structure of the union of finite-dimensional subspaces generated by all n-tuples of elements of G_ϕ (Kůrková and Sanguineti 2002). Thus, nonlinear models of the neural-network type correspond to the unions of finite-dimensional subspaces generated by hidden unit functions. As such functions play the role of a variable basis, in contrast to the fixed basis corresponding to the linear case, the term *variable-basis models* is used, too. Commonly used RBF and perceptron models can be formalized as follows: the sets $span_n\, P_d(\ )$ and $span_n\, F_d(\psi)$ represent the sets of functions on $[0,1]^d$ computable by ψ-perceptron models and ψ-RBF models, respectively, with n hidden units; $span\, P_d(\ )$ and $span\, F_d(\psi)$ denote the set of functions on $[0,1]^d$ computable by such networks, resp., with any number of hidden units.

Note that variable-basis models include not only neural networks and radial basis functions, but also many other nonlinear models (see the references in (Kůrková and Sanguineti 2002, Sjöberg et al. 1995)), such as trigonometric polynomials with free frequencies, free-node splines, etc. Multilayer networks with a single linear output unit and n units in the last hidden layer can also be represented by this class of models; they compute functions from $span_n G$, with G dependent on the numbers of units in the previous hidden layers.

3 Task and complexity of the model

Modelling can be formulated as an *approximation task* in a functional space (we call it the *ambient space*), which is defined by the application: for example, the space of continuous functions with the supremum norm (when asymptotic properties of the model have to be fulfilled, like stability or convergence of estimation errors), the space of square-integrable functions with the $\mathcal{L}_2$ norm, etc. In general, by $(X, \|.\|)$ we denote a normed linear space corresponding to the ambient space considered (when the norm is clear from the context, we write X only). We recall that a *Banach space* is a normed linear space that is complete and a *Hilbert space* is a Banach space with a norm induced by an inner product, i.e., $\|f\| = \sqrt{f \cdot f}$ (Friedman 1982, p. 126 and p. 201). Let M, Y be two subsets of $(X, \|.\|)$: M represents a class of models and Y is a set of mappings to be modelled. The error made in modelling the mapping $f \in Y$ by the model $g \in M$ is measured by the distance $\|f - g\|$.

When linear and ϕ-network models are used, *model complexity* can be expressed by the size of a properly defined parameter vector: the degree of a polynomial or the number of knots of a fixed-knots spline in the linear case, the number of computational units of a neural network, etc. One naturally expects that the larger the number of parameters used to describe a model, the more accurate the description. However, implementation feasibility requires a tradeoff between approximation accuracy and model complexity (i.e., the number of parameters), particularly for high-dimensional modelling tasks. Classes of models should guarantee an ar-

bitrarily high modelling accuracy when one uses a moderate number of parameters, i.e., models that have a *limited complexity*. *Complexity-based multiple models selection* entails the following steps:

1) Identifying different *operating environments* (corresponding, for instance, to the failures of one or more subsystems, to the presence or absence of disturbances, to the change of certain parameters).

2) Exploiting the a-priori assumptions to restrict the sets of mappings to be approximated. Various *hypothesis classes* can be defined, one for each operating environment. As different environments are often represented by nonlinear mappings with different properties, a single model may not be adequate enough to identify the changes in a dynamic system. In such cases, the use of multiple models is mandatory.

Then, for each hypothesis class, the following tasks should be accomplished:

3) Estimating the *number of parameters* required by different modelling networks to achieve the desired modelling accuracy.

4) Choosing the structure (e.g., linear, nonlinear of the neural-network type, etc.) that allows the desired modelling accuracy with the smallest number of parameters. In particular, in tasks involving high-dimensional input spaces the curse of dimensionality has to be avoided.

5) Tuning the parameters.

In the remainder of this section, we introduce a mathematical framework to formalize steps 3) and 4) and to compare the complexities of linear and neural-network models in terms of approximation theory.

As both linear and neural-network models are obtained by combining elements from a set of (fixed or variable, resp.) basis functions, the number of such functions can be considered as a measure of model complexity. Classes of models with increasing complexity can be represented as the union of a nested sequence $\{M_n : n \in \mathcal{N}_+\}$ of sets of functions spanned by an increasing number n of basis functions. In applications, the rate of decrease in the modelling error in dependence of the number of basis functions has to be high enough so that models with a moderate complexity may guarantee a sufficient accuracy.

This can be studied with tools from approximation theory by introducing the concept of *worst-case modelling error* from a class M of models, used to model elements from a hypothesis class Y. Such an error is mathematically formalized by the *deviation of Y from M*, defined as

$$\delta(Y, M) = \sup_{f \in Y} \|f - M\| = \sup_{f \in Y} \inf_{g \in M} \|f - g\|.$$

Thus we are interested in the rate of decrease of the deviations $\delta(Y, M_n)$ for increasing values of n. If $\bigcup_{n \in \mathcal{N}} M_n$ is dense in X, then, for any $f \in X$, the sequence $\{\|f - M_n\| : n \in \mathcal{N}_+\}$ converges to 0. But, in practical applications, this convergence has to be sufficiently fast to guarantee any desired modelling accuracy for n small enough, to allow models from M_n to have moderate numbers of parameters. For functions of d variables, it often happens that deviations are of order $O(1/n^{1/d})$. In such a case, to achieve a modelling accuracy within ϵ, approximating functions with complexity of order $O\left(1/\epsilon^d\right)$ are needed. Such exponential dependence of complexity on the number of variables determines the so-called *curse of dimensionality* (Bellman 1957).

Recalling that linear methods correspond to the use of finite-dimensional subspaces X_n of the ambient space X (i.e., $M_n = X_n$), to compare neural networks with linear techniques we shall compare the deviation $\delta(Y, span_n\, G)$ with the deviation of Y from the "best" approximating linear subspace X_n of X. This is formalized by the concept of *n-width* of Y in X, defined as (see, e.g., (Pinkus 1986, pp. 1-2))

$$d_n(Y) = \inf_{X_n} \delta\big(Y, X_n\big) = \inf_{X_n} \sup_{f \in Y} \inf_{g \in X_n} \|f - g\|,$$

where the left-most infimum is taken over all n-dimensional subspaces X_n of X, spanned by n linearly independent basis functions.

Estimating deviation and n-width in dependence of the number n of basis functions and the dimension d of the input space (i.e., the number of variables) is essential to the choice of a model with limited complexity. Approximation theory provides lower bounds on n-width that often result in poor performances of linear methods (see, e.g., (Kůrková and Sanguineti 2002, Pinkus 1986)). When the parameters of the basis functions are continuously adjusted to the function to be approximated, it is possible to extend the lower bounds on n-width from linear methods to nonlinear models. This was done in (DeVore et al. 1989) by defining a proper *continuous nonlinear n-width*. However, such lower bounds cannot be applied *a priori* to neural networks, as it has recently been shown (Kainen et al. 1999) that, for most standard types of such networks (e.g., Gaussian radial-basis-functions and Heaviside perceptrons) the best approximation cannot be achieved in a continuous way. Roughly speaking, neural-network models might exploit the nonlinearity-in-parameters to achieve the same modelling accuracy as attained by linear methods but with a reduced complexity. Indeed, experimental

results have shown that neural-network adaptive architectures with relatively few adjustable parameters can obtain surprisingly good performances.

Next sections give a theoretical insight into the effectiveness of neural networks in complex modelling tasks, in which traditional linear models are often unable to perform satisfactorily. We make comparisons between estimates of $d_n(Y)$ and $\delta(Y, span_n\, G)$, i.e., between the best achievable modelling accuracy of mappings from a hypothesis class Y by linear models with n basis functions and the best nonlinear modelling accuracy of the same mappings by elements of network models represented by $span_n\, G$.

4 Upper bounds on the complexity of neural-network models

From a qualitative point of view, the main limitation of linear models consists in the use of linear combinations of certain *fixed* functions: being fixed, such functions cannot be adapted to mappings representing different systems to be modelled or different operating environments of the same system. In contrast to this, nonlinear approximators of the variable-basis type such as neural networks take advantage of their adjustable parameters (e.g., the centres and the weight matrices in radial basis functions, the frequencies and phases in trigonometric basis functions, the thresholds in sigmoidal basis functions, etc.), which allow one to choose the basis functions adaptively, i.e., depending on the input-output mapping to be approximated. The description of various sets of d-variable functions that do not exhibit the curse of dimensionality in variable-basis approximation can be derived by exploiting the following result of nonlinear approximation theory, obtained by Maurey (see (Pisier 1980)), Jones (Jones 1992) and Barron (Barron 1993)).

Theorem 1. ((Pisier 1980, Jones 1992, Barron 1993)). Let $(X, \|.\|)$ be a Hilbert space, G its bounded subset, $s_G = \sup_{g\in G} \|g\|$, and $f \in cl\, conv\, G$. Then, for every positive integer n, $\|f - conv_n\, G\| \leq \sqrt{\frac{s_G^2 - \|f\|^2}{n}}$.

This theorem gives an upper bound of order $O(n^{-1/2})$, independently of the number d of variables of functions in X. As $conv_n\, G \subseteq span_n\, G$, the upper bound from Theorem 1 also applies to rates of approximation by $span_n\, G$. However, when G is not closed under multiplication by scalars, $cl\, conv\, G$ is a proper subset of $cl\, span\, G$. Thus, the density of $span\, G$ in $(X, \|.\|)$ does not guarantee that Theorem 1 can be applied to all elements of X. By replacing G with $G(c) = \{wg; w \in \mathcal{R}, |w| \leq c, g \in G\}$, for any $c > 0$ one gets $conv_n\, G(c) \subseteq span_n G(c) = span_n G$, and so one can apply Theorem 1 to all elements of $\cup_{c\in\mathcal{R}_+} cl\, conv\, G(c)$. This approach was formulated in (Kůrková 1997) in terms of a norm "tailored" to a set G, called G-*variation* (variation with respect to G) and defined (Kůrková 1997) as the Minkowski functional of the set $cl\, conv\, (G \cup -G)$, i.e., $\|f\|_G = \inf\{c \in \mathcal{R}_+; f/c \in cl\, conv\, (G \cup -G)\}$.

G-variation is a norm on $\{f \in X : \|f\|_G < \infty\} \subseteq X$. We write only $\|.\|_G$ whenever it is clear with respect to which norm G-variation is defined. Intuitively, $\|f\|_G$ shows, how much the set G should be dilated, so that f may be in the closure of the convex symmetric hull of the dilated set. G-variation is a generalization of two concepts: the notion of total variation used in integration theory (for functions of one variable, it coincides up to a constant with total variation) and l_1 norm (see (Kůrková and Sanguineti 2002) for details). The following upper bound is a reformulation of Theorem 1 in terms of G-variation (Kůrková 1997).

Theorem 2. (Kůrková 1997) Let $(X, \|.\|)$ be a Hilbert space, G its bounded subset, $s_G = \sup_{g\in G} \|g\|$. Then, for every $f \in X$ and every positive integer n, $\|f - span_n\, G\| \leq \sqrt{\frac{(s_G\, \|f\|_G)^2 - \|f\|^2}{n}}$.

Let $B_r(\|.\|')$ denote the ball of radius $r > 0$ with respect to a norm $\|.\|'$, i.e. $B_r(\|.\|') = \{f \in (X, \|.\|); \|f\|' \leq r\}$. As $conv\, (G \cup -G) = conv\, G(1)$, we have $\|f\|_G = \inf\{c \in \mathcal{R}_+ : \ f \in cl\, conv\, G(c)\}$ and the unit ball $B_1(\|.\|_G)$ in G-variation is equal to $cl\, conv\, (G \cup -G)$. Thus Theorem 2 implies the following upper bound on deviation of balls in G-variation. For a subset G of X, $G^0 = \{g^0 = g/\|g\| : 0 \neq g \in G\}$ denotes the set of its normalized elements.

Corollary 1. (Kůrková and Sanguineti 2002) Let $(X, \|.\|)$ be a Hilbert space, G its bounded subset, and $s_G = \sup_{g\in G} \|g\|$. Then, for every positive integer n and every $r > 0$, $\delta(B_r(\|.\|_G), span_n\, G) \leq \frac{r s_G}{\sqrt{n}}$, and $\delta(B_r(\|.\|_{G^0}), span_n\, G) \leq \frac{r}{\sqrt{n}}$.

Corollary 1 can be used as explained in the following to bound the complexity of variable-basis models. Suppose, for example, that a system-theoretic analysis guarantees the existence of a nonlinear input-output mapping or of a mapping between measurements and controls. If one can prove that such nonlinearities belong to balls of fixed radius in G_ϕ-variation, for a certain ϕ-network, then any desired modelling accuracy ϵ can be

obtained by using ϕ-network models with $n \leq r^2/\epsilon^2$ basis functions (i.e., at most n adjustable network units), independently of the number d of variables of the mappings to be modelled. This is particularly interesting in high-dimensional problems, in which linear methods are often computationally unfeasible as too many parameters are required, and assumes major importance when the input space is of large dimension, such as in the case of high-order plants (Narendra et al. 1995, p. 43).

When G corresponds to the set G_ϕ of functions computable by hidden units, Corollary 1 applies to neural-network models. The bound of order $O(n^{-1/2})$, when combined with the availability of gradient-like training algorithms (based on the output error of the network and well-suited to parallel implementation) for adjusting parameters, makes ϕ-network models very attractive. Further theoretical results on upper bounds on the complexity of ϕ-network models can be found in (Kůrková et al. 1998). The practical importance of Corollary 1 relies on the possibility of embedding a ball of some radius in G-variation in the set representing the hypothesis class of the modelling problem at hand. In other words, Corollary 1 becomes useful only if sets $B_r(\|.\|_{G_\phi})$, where G_ϕ corresponds to a ϕ-network model, contain functions of interest in applications.

In (Hlaváčková-Schindler and Sanguineti 2003) we gave some examples, for various hypothesis classes defined by smoothness conditions in terms of the Fourier transform and for ϕ-networks with various types of basis functions (for sigmoidal perceptron models, ramp perceptron models, trigonometric polynomials with free frequencies, Sobolev spaces as hypothesis classes).

It is worth noting that the upper bound on the modelling error by $span_n\, G_\phi$ given by Theorems 1 and 2 and Corollary 1 can be applied not only to neural-network models but also to various other classes of widespread nonlinear models, such as free-node splines, splines with free nodes, trigonometric polynomials with free frequencies, etc. (Kůrková and Sanguineti 2002).

5 Lower bounds on the complexity of linear models

The first result pointing out the advantages of neural-network approximation over linear methods is the theoretical comparison in (Barron 1993) of the worst-case errors in linear and neural-network approximations. Applying Theorem 1, classes of multi-variable functions were described in (Barron 1993); for such classes the $\mathcal{L}_2$ approximation error by one-hidden-layer sigmoidal perceptron networks (deviation) is bounded from above by a quantity of order $O(1/\sqrt{n})$, where n is the number of network hidden units. There was also proven that the $\mathcal{L}_2$ error of the best linear approximator is bounded from below by a quantity of order $O(1/(d\sqrt[d]{n}))$, where n is the dimension of the linear approximating subspace and d is the number of variables of the functions to be approximated.

Hypothesis classes for which such dependence is linear or, more generally, polynomial, are described in (Barron 1993). For such classes, which include various sets of functions of interest in applications, the lower bound on Kolmogorov n-width implies the curse of dimensionality, whereas the upper bound on the above defined approximation grows only polynomially with increasing dimension d. In these cases, the advantageous behaviour of sigmoidal neural-network models contrasts with the unfeasibility of linear methods in high-dimensional modelling tasks. It was shown in (Kůrková and Sanguineti 2002) that, for linear models, lower bounds of order $O(1/(d\sqrt[d]{n}))$ can be derived for more general classes, for which the upper bound of order $O(1/\sqrt{n})$ on modelling accuracy by neural networks holds. Moreover, a general framework was developed in (Kůrková and Sanguineti 2002) for the description of sets of functions the approximation of which by linear methods may exhibit the curse of dimensionality, whereas the rate of approximation by $span_n\, G_\phi$ depends polynomially on d for suitable classes of ϕ-networks.

In this section we turn to the investigation of lower bounds on linear methods. We start with some results proving large lower bounds on the modelling error by linear methods, for certain balls in G-variation. As neural models have small upper bounds for such sets (see Theorems 1 and 2 and Corollary 1), they outperform linear methods. We first give some general estimates based on the geometry of Banach and Hilbert spaces, then we consider more concrete hypothesis classes to be modelled.

Let $B_1^{n+1}(\|.\|) = B_1(\|.\|) \cap X_{n+1}$ denote the unit ball of an $(n+1)$–dimensional subspace X_{n+1} of a Hilbert space $(X, \|.\|)$.

Proposition 1. *Let $(X, \|.\|)$ be a Hilbert space, G its subset, n a positive integer, X_{n+1} an $(n+1)$–dimensional subspace of X and $r \leq \sqrt{n}$ such that $B_r(\|.\|_{G^\circ}) \supset B_1^{n+1}(\|.\|)$. Then $\delta(B_r(\|.\|_{G^\circ}), span_n\, G) \leq 1$, while $d_n(B_r(\|.\|_{G^\circ})) > 1$.*

Proof. If we restrict to $B_1^{n+1}(\|.\|)$, then $\|.\|_{G^\circ}$ and $\|.\|$ are equivalent, as $B_1^{n+1}(\|.\|))$ is finite-dimensional. Then, for a given $n \in \mathcal{N}_+$, there exists $r > 0$ such that $B_r(\|.\|_{G^\circ}) \supset B_1^{n+1}(\|.\|)$. According to

(Pinkus 1986, Theorem 1.5), for every integer $n \geq 0$, $d_k(B_1^{n+1}(\|.\|)) = 1$, $k = 0, 1, \ldots, n$. Thus, $d_n(B_r(\|.\|_{G^\circ})) > 1$. On the other hand, Corollary 1 gives $\delta(B_r(\|.\|_{G^\circ}), span_n\, G) \leq \frac{r}{\sqrt{n}}$ and $r \leq \sqrt{n}$ implies $\delta(B_r(\|.\|_{G^\circ}), span_n\, G) \leq 1$. $\diamond$

Another insight into the comparison between n-width and deviation from $span_n\, G$ of certain balls in G-variation intersected with finite-dimensional subspaces can be obtained using the characterization of n-width from (Pinkus 1986)(Theorem 2.1, p. 16). Given $X^{n+1} \subset X$ let us consider, for $c > 0$, the balls $B_c^{n+1}(\|.\|)$ and $B_1^{n+1}(\|.\|_G)$, and define $c_n(X^{n+1}) = \sup\{c;\, B_c^{n+1}(\|.\|) \subseteq B_1^{n+1}(\|.\|_G)\}$. (Note that $c_n(X^{n+1}) < \infty$ for every $n \in \mathcal{N}_+$, as all norms are equivalent in a finite-dimensional space). According to (Pinkus 1986)(Theorem 2.1, p. 16), if Y is a closed, convex, centrally symmetric, proper subset of an $(n+1)-$dimensional subspace X_{n+1} of a normed linear space $(X, \|.\|)$ and ∂Y denotes the boundary of Y, then $d_n(Y) = \inf\{\|f\| : f \in \partial Y\}$. Applying this to $B_1^{n+1}(\|.\|_G)$, one has $d_n(B_1^{n+1}(\|.\|_G)) = c_n(X^{n+1})$. On the other hand, Corollary 1 gives $\delta(B_1^{n+1}(\|.\|_G), span_n\, G) \leq \frac{s_G}{\sqrt{n}}$. Defining $c_n = \sup_{X^{n+1} \subset X} c_n(X^{n+1})$, it follows that, for every ball $B_1^{n+1}(\|.\|_G)$, the upper bound on nonlinear models by a parametrized set G of functions corresponding to a type of network computational unit is smaller than the lower bound on linear approximation if $c_n \geq s_G/\sqrt{n}$. A necessary condition for this is that, for $n \to \infty$, $c_n \to 0$ more slowly than $s_G/\sqrt{n}$: in other words, the radius of the largest ball in $\|.\|$ contained in the unit ball in $\|.\|_G$ has to be "large enough". In this case, for functions in the intersection of the unit ball in G-variation with any $(n+1)$-dimensional subspace linear methods are a weak tool with respect to neural-network models.

Now we consider a result from Mityagin (Mityagin 1961), which is a simplification of a statement proven by Krein et al. (Krein et al. 1948). We derive from this proposition two corollaries for comparison of deviation from $span_n G$ to Kolmogorov n-width.

Theorem 3. (Mityagin 1961) *Let B be the unit ball in the Hilbert space X and G be some subset of X. Suppose that for some $(n+1)$-dimensional subspace X^{n+1} holds $\alpha B^{n+1} \subset G_{n+1}$, where $G_{n+1} = G \cap X^{n+1}$, $B^{n+1} = B \cap X^{n+1}$. Then $\alpha \leq d_n(G, B)$.*

For $G = X^{n+1}$ we get the following results.

Corollary 2. *Let G be a set in the Hilbert space $(X, \|.\|)$. and let $\|f\|_G$ denote the $G-$ variation of function $F \in (X, \|.\|)$. If $\|f\|_G . B_{\|f\|_G}^{n+1} \subset X^{n+1}$ then holds $\|f\|_G \leq d_n(B_{\|f\|_G})$.*

Comparing this bound to Theorem 1, we get the following proposition.

Proposition 2. *Let G be a set in the Hilbert space $(X, \|.\|)$. and let $\|f\|_G$ denote the $G-$ variation of function $F \in (X, \|.\|)$. If $\|f\|_G . B_{\|f\|_G}^{n+1} \subset X^{n+1}$ then $\delta_{G,n}(B_{\|f\|_G}) \leq d_n(B_{\|f\|_G})$.*

In other words, for balls with radius satisfying the above condition nonlinear approximation of function f is better than the linear one.

6 Lower bounds based on covering numbers and ϵ-metric entropy

Using tools from metric entropy theory, one can derive lower bounds on the worst-case modelling error of various classes of functions of interest in applications. We will show that linear models are outperformed by neural-network models for such hypothesis classes.

Recall that, for $\epsilon > 0$, the *ϵ-covering number* of a subset K of a normed linear space $(X, \|.\|)$ is defined as $cov_\epsilon(K, \|.\|) = \min\{m \in \mathcal{N}_+ : K \subseteq \cup_{i=1}^m B_\epsilon(f_i, \|.\|),\ f_i \in X,\ i = 1, \ldots, m\}$ if the set over which the minimum is taken is nonempty, otherwise $cov_\epsilon(K, \|.\|) = +\infty$. The *$\epsilon$-covering number of a subset K* is the minimal cardinality of an ϵ-net in K. Roughly speaking, the asymptotic behavior of $cov_\epsilon(K, \|.\|)$ characterizes the 'degree of compactness'. It can be shown that K is compact if and only if $\lim_{\epsilon \to \infty}(K)$ is finite.

The *ϵ-metric entropy* of K is defined as $H_\epsilon(K, \|.\|) = log_2\, cov_\epsilon(K, \|.\|)$. When it is clear from the context which norm is considered, we will simply write $cov_\epsilon(K)$ and $H_\epsilon(K)$ instead of $cov_\epsilon(K, \|.\|)$ and $H_\epsilon(K, \|.\|)$, respectively. It is easy to prove that the covering numbers and consequently metric entropy have properties of monotonicity and additivity (i.e. (i) monotonicity: $cov_{\epsilon_1}(K) \leq cov_{\epsilon_2}(K) \leq \ldots$, where $\epsilon_1 \geq \epsilon_2 \ldots$ and $\lim_{n \to \infty} \epsilon_n = 0$; (ii) additivity: $cov_\epsilon(K \cup L) \leq cov_\epsilon(K \cap L) + cov_\epsilon(K - L) + cov_\epsilon(L - K) \leq cov_\epsilon(K) + cov_\epsilon(L)$ for K, L nonempty sets; The second equality holds for K, L disjoint).
The connections between metric entropy and approximation have been investigated by various authors, for example (Lorentz 1963), (Mityagin 1961), (Clements 1963), (Vvosburg 1966), from the more recent works,

e.g. (Temljakov 1996), (Bartlett at all 1997). As pointed out by Lorentz (Lorentz 1986, p. 150 and p. 163), (Lorentz et al. 1996), both the Kolmogorov widths of a set and its metric entropy measure the "size" of that set, although in a different way. The n-width of a set is "small" if the set can be "squeezed" within a small neighbourhood of some n-dimensional subspace, while it has a "small" metric entropy when it can be covered by a "small" number of balls in the norm with respect to which the neighbourhood is defined. Both measures of size are in terms of balls of radius ε. In the case of Kolmogorov width, the interest is in the lowest dimensional subspace on which the centers lie while covering the set. Metric entropy is the minimum number of such balls necessary to cover it, with no restriction on the positions of their centers.

In general, in an infinite-dimensional Banach space, there exist sets, the ϵ-entropy of which increases arbitrarily fast for $\epsilon \to 0$ (Timan 1993, 6.8.3). This indicates the need for the introduction of a scale of increase of entropy $H_\epsilon(G)$ as $\epsilon \to 0$. By $\mathcal{M}_k^\beta(X)$ let us denote the class of sets $G \subset X$ for which

$$a(\log_2 \frac{1}{\epsilon})^\beta \leq (\log_2)_k H_\epsilon(G) \leq b(\log_2 \frac{1}{\epsilon})^\beta$$

where $(\log_2)_k H_\epsilon(G)$ is the iterated algorithm of order k, $(k = 0, 1, 2, \ldots)$ of $H_\epsilon(G)$, b and a are some positive constants and $\beta > 0$. It was shown in (Pontrjagin and Shnirel'man 1932) that $\inf_\epsilon \frac{\log_2 \frac{1}{\epsilon}}{H_\epsilon(G)} > 0$ holds only in the case that the lower bound of $d_n(G) = 0$ for all sufficiently large n. However, when a set of functions to be approximated is characterized by suitable bounds on the rate of increase of its entropy, some lower bounds on the Kolmogorov n-width can be derived. We will apply the following reformulations of results on Kolmogorov's n-width from Timan (Timan 1993, Theorems 6.8.31 to 6.8.34) to compare to the upper bounds based on deviation which were presented in the previous section.

Proposition 3. ((Timan 1993)). *Let G be a subset of an infinite dimensional Banach space $(X, \|.\|)$.*

(i) If $G \in \mathcal{M}_0^\beta(X)$ $(\beta > 1)$, then for all sufficiently large n

$$d_n(G) \geq \rho^{n^{\frac{1}{\beta-1}}},$$

where $\rho < 1$ is a positive constant independent of n.

(ii) If $G \in \mathcal{M}_1^\beta(X)$ $(0 < \beta \leq 1)$, then for all sufficiently large n

$$d_n(G) \geq \Phi_G[C(\log_2 n)^{\frac{1}{\beta}}],$$

where C is a positive constant independent of n and Φ_G is the function inverse to the ϵ-entropy $H_\epsilon(G)$, i.e. the upper bound of all ϵ for which $H_\epsilon(G) \geq n + 1$.

(iii) If $G \in \mathcal{M}_1^\beta(X)$ $(\beta > 1)$, then for all sufficiently large n

$$d_n(G) \geq \rho^{(\ln n)^{\frac{1}{\beta}}},$$

where $\rho < 1$ is a constant independent of n.

(iv) If $G \in \mathcal{M}_k^\beta(X)$, $k \geq 2$, then for all sufficiently large n

$$d_n(G) \geq \Phi_G(n^2),$$

where Φ_G is the function inverse to the ϵ-entropy $H_\epsilon(G)$.

Roughly said, when suitable estimates of ϵ-entropy are available, linear methods imply limited accuracy. Before we present our main proposition of this section, we mention an important lemma from Lorentz (Lorentz 1963).

Lemma 1. ((Lorentz 1963)).
For all balls B_r of an $n-$ dimensional metric space X^n and $0 < \epsilon \leq r$ holds

$$\frac{1}{2^n}(\frac{r}{\epsilon})^n \leq cov_\epsilon(B_r) \leq 3^n(\frac{r}{\epsilon})^n.$$

From this lemma, one gets in a straightforward way lower and upper bounds on entropy by logarithming:

$$\log_2(\frac{r}{2})^n + \log_2(\frac{1}{\epsilon})^n \leq H_\epsilon(G) \leq \log_2(3r)^n + \log_2(\frac{1}{\epsilon})^n.$$

Theorem 4. *Let G be a bounded subset of an infinite dimensional Banach space $(X, \|.\|)$. Let $B_r(\|.\|_G)$ denote the ball of diameter $r > 0$ in X^n, a n-dimensional subspace of X and let $H_\epsilon(B_r)$ be its entropy for $0 \leq \epsilon \leq r$. Let $s_G = \sup_{g\in G} \|g\|$. If $B_r \in \mathcal{M}_1^1(X)$ for some $a, b > 0$ and $0 < \epsilon < 1$ so that also $\frac{\epsilon^a - \epsilon^b}{\epsilon^{a+b}} \geq n.(\log_2 3 + 1)$, then for Kolmogorov width holds*

$$d_n(G) \geq \Phi_G[C(\log_2 n)],$$

where C is a positive constant independent of n and Φ_G is the function inverse to the ϵ-entropy $H_\epsilon(G)$, while for deviation holds

$$\delta(B_r(\|.\|_G), span_n G) \leq \frac{r s_G}{\sqrt{n}}.$$

Proof. We compare the lower bound of the condition for $\mathcal{M}_1^1(X)$ to the lower bound of Lorentz lemma, similarly we do it for the upper bounds.
The condition given by $\mathcal{M}_1^1(X)$ for entropy (by exponenting) is rewritten to $[2^{\log_2 \frac{1}{\epsilon}}]^a \leq H_\epsilon(G) \leq [2^{\log_2 \frac{1}{\epsilon}}]^b$, where a, b are positive constants.
I. We look for conditions on a, b, ϵ, r so that $[2^{\log_2 \frac{1}{\epsilon}}]^a \leq \log_2(\frac{r}{2})^n + \log_2(\frac{1}{\epsilon})^n$. This is equivalent to $(\frac{1}{\epsilon})^a \leq n \log_2 \frac{r}{2} + n \log_2 \frac{1}{\epsilon}$ and this to $\epsilon^{-a} + n \log_2 \epsilon \leq n \log_2 \frac{r}{2}$. Lets call the last formula condition (L) for ϵ and r.
II. The condition (U) for the upper bounds will be obtained by the following comparison of the upper bound given by $\mathcal{M}_1^1(X)$ and the upper bound from Lorentz lemma: $\log_2(3r)^n + \log_2(\frac{1}{\epsilon})^n \leq (\frac{1}{\epsilon})^b$. We look for ϵ and r satisfying conditions (L) and (U), i.e. $\epsilon^{-a} + n \log_2 \epsilon \leq n \log_2 \frac{r}{2}$ and $\log_2(3r)^n - n \log_2 \epsilon \leq (\frac{1}{\epsilon})^b$. By adding (L) to (U), we get $\epsilon^{-a} - \epsilon^{-b} \leq n(\log_2 \frac{r}{2} - \log_2(3r)) = n(\log_2 3 + 1)$. It is equivalent to $\frac{\epsilon^a - \epsilon^b}{\epsilon^{a+b}} \geq n(\log_2 3 + 1)$. Such ϵ, a, b, satisfying the last formula exist, namely such, for which $0 < \epsilon < 1$ and $a < b$. ◇

One would logically apply the same method of proof to the remaining statements of Proposition 3. However, the conditions for $\mathcal{M}_k^\beta(X)$ for $\beta \neq 1$ or $k > 1$ compared tothe Lorentz lemma lead to very complicated conditions for ϵ which one cannot expect to be applicable. In the following, we present some results on the lower bounds on n-width in terms of metric entropy of the class of functions with bounded variation.

Recall that the *total variation* of a function $h : J \to \mathcal{R}$ of bounded variation on $J = [t_0, t_n]$ is defined as $V(h, J) = \sup \sum_{i=1}^n \left| h(t_i) - h(t_{i-1}) \right|$, where the supremum is taken over all finite partitions $t_0 < \ldots < t_n$ of J ((Kolmogorov and Fomin 1999, p. 328)). Tight bounds on the metric entropy of the class of real-valued functions with bounded variation and metric $\rho = \mathcal{L}_1(dP)$, where P is a probability distribution, have been derived in (Bartlett at all 1997).

Theorem 5. ((Bartlett at all 1997)). *Let $(X, \mathcal{L}_1)$ be a normed space and $\mathcal{G}$ be a compact class of functions $f : [0, T] \to [-\frac{V}{2}, \frac{V}{2}]$, with total variation at most $V > 0$ and $T > 0$. Then for all $\epsilon \leq \frac{VT}{12}$*

$$(\log_2 e)\frac{VT}{54\epsilon} \leq H_\epsilon(G) \leq \frac{13VT}{\epsilon}.$$

The following proposition was derived from Theorem 5 and Proposition 3.

Proposition 4. *Let $(X, \mathcal{L}_1)$ be a (inf. dim.) Banach space and let $G \subset X$ be a set of all functions on $f : [0, T] \to [\frac{-V}{2}, \frac{V}{2}]$ with total variation at most $V > 0$. Let $\epsilon \leq \frac{VT}{12}$.*

(i) Assume that a) $0 < \epsilon < 1$ and $\beta > 1$ is a real number or b) $\epsilon > 1$ and $\beta > 1$ is an even integer or $\beta = \frac{b_1}{b_2}$, where b_1, b_2 are some positive integers and b_2 is even. Then there exists integer $n_1 > 0$ such that for all $n \geq n_1$

$$d_n(G) \geq \tau_1^{n^{\frac{1}{\beta-1}}}$$

where τ_1 is a constant independent of n and $0 < \tau_1 < 1$.

(ii) Let $\epsilon < \frac{VT}{54}(\log_2 e)$. Assume that a) $0 < \epsilon < 1$ and $\beta > 1$ is a real number or b) $\epsilon > 1$ and $\beta = \frac{b_1}{b_2} > 1$, where b_1, b_2 are some positive integers and b_2 is even. Then there exists $n_2 > 0$ such that for all $n \geq n_2$

$$d_n(G) \geq \tau_2^{\ln n^{\frac{1}{\beta}}}$$

where τ_2 is a constant independent of n and $0 < \tau_2 < 1$.

(iii)Let $\epsilon > \frac{VT}{54}(\log_2 e)$ *and* $\epsilon > 1$. *Let* $\beta = \frac{b_1}{b_2} > 1$, *where* b_1, b_2 *are some positive integers and* b_2 *is odd. Then there exists* $n_3 > 0$ *such that for all* $n \geq n_3$

$$d_n(G) \geq \tau_2^{\ln n^{\frac{1}{\beta}}}$$

where τ_3 *is a constant independent of* n *and* $0 < \tau_3 < 1$.

Proof. It is clear that G is compact since it is bounded and closed. (i): From Theorem 5 for all $0 < \epsilon \leq \frac{VT}{12}$ holds $(\log_2 e)\frac{VT}{54\epsilon} \leq H_\epsilon(G) \leq \frac{13VT}{\epsilon}$. Some $a_1, z_1 > 0$ so that the condition (i) in Proposition 3 is satisfied can be found if the conditions for ϵ of Proposition 4(i) hold. In other words, for these $a_1, z_1 > 0$ holds $0 < a_1(\log_2 \frac{1}{\epsilon})^\beta \leq (\log_2 e)\frac{VT}{54\epsilon} \leq H_\epsilon(G) \leq \frac{13VT}{\epsilon} \leq z_1(\log_2 \frac{1}{\epsilon})^\beta$. (ii) and (iii) : The proofs are similar as (i) but the logarithmic form of Theorem 5 is considered: $\log_2(\log_2 e) + \log_2 \frac{VT}{54\epsilon} \leq \log_2 H_\epsilon(G) \leq \log_2 \frac{13VT}{\epsilon}$ for $\epsilon \leq \frac{VT}{12}$. ◇

We still exploit Proposition 3 to prove limitations of linear methods for certain sets of analytic functions, in terms of lower bounds on their n-width. The use of linear models based on various types of orthogonal basis functions was studied, for example, in (Wahlberg 1994) in the case where the systems to be modelled are analytic. It was proven in (Wahlberg 1994) that Laguerre basis functions and Kautz basis functions are convenient for well-damped and lightly damped systems, resp. However, this requires information about the dominating poles of a plant, which might not be available. We denote by $A^d(S)$ the set of all real-valued analytic functions of d variables defined on a bounded set S in the d-dimensional complex space. $A^d(S)$ is a d-dimensional linear space over the complex numbers. If $\mathcal{F}$ is a compact subset of S, we denote by $A = A^d(\mathcal{F}, S, M)$ the metric space of all functions $f \in A^d(S)$ that satisfy $|f(z)| \leq M$ for $z \in S$; A is metrized by the supremum norm on $\mathcal{F}$.

The following proposition was proven in (Hlaváčková-Schindler and Sanguineti 2003).

Proposition 5. ((Hlaváčková-Schindler and Sanguineti 2003)). *Let* d *be a positive integer,* $M > 0$, S *a bounded set in the* d*-dimensional complex space,* $\mathcal{F}$ *its compact subset with nonempty interior. Then there exists an integer* n_1 *such that in the supremum norm, for all* $n \geq n_1$,

$$d_n(A^d(\mathcal{F}, S; M)) \geq \rho^{n^{\frac{1}{d}}},$$

where ρ *is a constant independent of* n *and* $0 < \rho < 1$.

We now turn our attention to lower bounds on the n-width of balls in G-variation, for which we know from Corollary 1 that nonlinear models of neural-network type achieve a modeling error of the order of $O(1/\sqrt{n})$. Let us first introduce some terms. In some cases of interest in applications (see below], the unit ball in G variation contains an orthogonal subset having "sufficiently many" elements with norms larger than or equal to $1/k$, for each positive integer k. In (Kůrková and Sanguineti 2002) there has been introduced the concept of a set *not quickly vanishing with respect to* a positive integer d, as a subset A of a normed linear space $(X, \|.\|)$ such that $A = \cup_{k \in \mathcal{N}_+} A_k$, where, for each $k \in \mathcal{N}_+$, $card\, A_k \geq k^d$ and for each $h \in A_k$, $\|h\| \geq 1/k$. The following proposition gives a lower bound on the n-width $d_n(B_1(\|.\|_G))$. $\mathcal{H}(p) = -p\log_2(p) - (1-p)\log_2(1-p)$, for $0 < p < 1$, denotes the binary entropy function and $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to a real number x. The proofs of the following proposition and corollary can be found in ((Hlaváčková-Schindler and Sanguineti 2003)).

Proposition 6. ((Hlaváčková-Schindler and Sanguineti 2003)). *Let* $(X, \|.\|)$ *be a Banach space,* G *be its subset containing an orthogonal set not quickly vanishing with respect to* d, *and* $\varepsilon > 0$. *Then, for every positive integer* n,

$$d_n(B_1(\|.\|_G)) < \varepsilon \quad \text{implies} \quad n \geq \frac{b\left\lfloor \left(\frac{1}{4\varepsilon}\right)^{\frac{2d}{d+2}} \right\rfloor - 1}{3 + \log_2(1 + \frac{s_G}{2\varepsilon})}, \text{ where } s_G = \sup_{g \in G} \|g\| \text{ and } b = 1 - \mathcal{H}\left(\tfrac{1}{4}\right) \simeq 0.085.$$

The lower bound from Proposition 6 implies a "slow" decrease with respect to n of the n-width of balls in G-variation, for G containing an orthogonal set not quickly vanishing with respect to d. Such a slow decrease implies a poor modelling accuracy by linear methods of functions from balls in G-variation. The following corollary describes an application of Proposition 6 to a widespread class of ϕ-network models, namely neural networks with perceptrons as hidden units.

Corollary 3. ((Hlaváčková-Schindler and Sanguineti 2003))**Application to perceptron-network models**. *Let* d *and* n *be positive integers and* $: \mathcal{R} \to \mathcal{R}$ *be an increasing sigmoidal function.*

(i) Then in $(\mathcal{L}^2([0,1]^d), \|.\|_2)$*, for every positive integer* n *and every* $r > 0$ $\delta(B_1(\|.\|_{P_d(\sigma)}), span_n\, G) \leq \frac{1}{\sqrt{n}}$.

(ii) Then there exists $\epsilon_0 > 0$ *such that for* $0 \leq \epsilon \leq \epsilon_0$ *and every positive integer* n*,*

in $(\mathcal{L}^2([0,1]^d), \|.\|_2)$ *we have* $d_n(B_1(\|.\|_{P_d(\sigma)})) < \epsilon$ implies $n > \frac{b\left\lfloor \left(\frac{1}{4\epsilon}\right)^{\frac{2d}{d+2}} \right\rfloor - 1}{3+\log_2(1+\frac{1}{2\epsilon})}$ *where* $b = 1 - \mathcal{H}\left(\frac{1}{4}\right)$.

As discussed in Section 4, many functional classes that are important in applications are contained in balls $B_r(\|.\|_{P_d(\sigma)})$ (see also (Barron 1993)); in all such cases, one can apply the upper bound from Corollary 3 on the number of basis functions in sigmoidal perceptron networks models and the lower bound from Corollary 3 on the number of fixed basis functions in linear methods.

7 Conclusions

We studied the class of nonlinear models including neural networks, in terms of approximation rates of nonlinear schemes having the structure of linear combinations of parametrized basis functions, corresponding to a type of network computational unit. The mappings to be modelled were represented as hypothesis classes defined by smoothness conditions (corresponding to different operating environments) in functional spaces dependent on the specific modelling task. Using tools from approximation theory, we derived estimates of the complexity of linear and neural-network models; these estimates are expressed as bounds on the number of adjustable parameters necessary for a given modelling accuracy.

We discussed classes of nonlinear models with desirable computational capabilities and bounded complexity, guaranteeing that the number of adjustable parameters does not increase too fast (e.g., only polynomially) with the dimensionalities of certain modelling tasks and is often much smaller than the number of parameters in linear methods. Our theoretical analysis supports the intuition that the nonlinear methods, such as neural-networks, in comparison to linear ones, 'better' use adjustable parameters and exploit smoothness hypotheses on the sets of multivariable mappings to be modelled.

Besides their powerful approximation capabilities, another feature making neural-network models particularly attractive is their intrinsic parallelism. Although until now such networks have mostly been simulated on classical serial computers, the availability of parallel VLSI realizations will make such models even more interesting for solving of complex tasks.

From an experimental point of view, the use of nonlinear models such as neural networks is enforced by their efficiency in complex adaptive tasks; from a theoretical perspective, their use is made particularly interesting by the possibility of achieving a desired modelling accuracy by means of much fewer parameters than in the case of linear methods. Thus, experience is supplemented with the design criteria based on a mathematical analysis: theoretical results explain why neural-network models should be preferred, and allow a deeper insight into the choice of a network architecture for a given task. (Barron 1993)

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DRAFT

Local traffic control of a microregion

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Abstract. The advanced traffic control systems can provide new levels of the efficiency with an increasing sophistication in the detection and the real-time optimization. Those systems use the on-line traffic flow data to set the dynamic parameters of the signal timing plans in response to the random traffic intensity fluctuation. The traffic control system is supposed to be decentralized and hierarchical. One of the control levels is the local control of a microregion. The function of this sectional controller is to deduce the optimal stage green times for each junction in the microregion and their intervals of the possible change.
We formulate the problem of the microregion traffic control as a task of linear programming. But it assumes the complete knowledge of the microregion elements parameters and that is usually impossible at all. So we approximate the non-detected states by Kalman filter.

Keywords: traffic control, linear programming, Kalman filter

1 Introduction

A feature of the majority of urban networks is the high density of the streets with large amount of junctions. The networks were formed during the tens or hundreds years in the past but the traffic demand has continually increased. So in respect to the traffic flow fluency, the existing structures of the streets can't usually accommodate such big volume of the traffic participants and it is either very expensive or even impossible to reconstruct the deficient street network. Due to these facts, efficient traffic control mechanisms are urgently required to reform this situation, particularly if congestion is to be contained (Hong 2002, Hong 2001, Přibyl 2001).

Optimal traffic control of urban street networks is an illustration of the problems of multiple participant decision making. We can comprehend the urban network as the set of microregions, formed by the lesser number of streets and their junctions. It is generally known that the optimal control of each microregion does not necessarily lead to the optimal control of the whole urban network. The resultant traffic flow behaviour of a certain microregion can radically change the traffic situation in the adjacent microregions and consecutively also the global traffic conditions. That is why the traffic flow control task must be formulated and solved consistently.

Traffic signals have become the most widely used form of the traffic flow control in this context (Přibyl 2001). They are commonly used at the road junctions to control pedestrians and vehicular movements to reduce the traffic congestions and to improve road safety. Now, they can also perform dynamically and the signalized junctions within the computer-controlled UTC systems (urban traffic control systems) are increasingly at the heart of the traffic control in cities all over the world (Hong 2001). During the last twenty years, a lot of various computer assisted traffic control schemes were developed that reflected the changes in the optimization criteria evolving consecutively in line with gained experience (Kratochvílová 2003).

The early traffic control systems essentially performed the static optimization of green "waves" on the specific city routes. But the optimization of the more complex city traffic requires taking into account other characters of the traffic flow as a queue length on the junction arms, the number of vehicle stops during the journey, a total delay, environmental impacts as emissions and so on (Heydecker 1997, Whittaker 1997).

In the recent time, the advanced traffic control systems use the on-line traffic flow data to set the dynamic parameters of the signal timing plans in response to the random traffic intensity fluctuation (Abu-Lebdeh 2002, Hong 1999, Wong 2002). So these control systems can quickly react to the changed traffic conditions at the optimized area in which a functional and effective framework of vehicle detectors (e.g. the loop detectors frequently) exists. Thus those advanced control systems can provide new levels of the efficiency with an increasing sophistication in the detection and the real-time optimization.

However, the problems of the optimal traffic control is still not solved. It is true that there exist a lot of

the optimization techniques, successfully finished and tested on the real traffic area. Some of those control systems became the object of the commercial sphere and it is possible to buy them as a software package, e.g. SCOOT, MOTION, etc. (Kratochvílová 2003, Přibyl 2001). They are usually quite expensive and supplied as a ”black box”. For the integration of such system to the real area control, there is then necessary the assistance of the experts to set the parameters of it. It is not guaranteed that such system will be suitable for another traffic control situation, although it has been successfully applied elsewhere (Kratochvílová 2003).

In the recent time, the precise global concept of the traffic control system is known well (Heydecker 1996, Kratochvílová 2003) but the specific optimal methodology is still not clear. There are developed a lot of different approaches which would be possible to use for this purpose (Kárný 1998, Nagy 2003, Valečková 1998). Some of them are very promising (SMART NETS 2002) and they show that the prospective solution can fall back on the known and current methods (Beck 1982, Kárný 1985).

2 Aim of the Work

We consider the optimal traffic control as the task of linear programming because of the assumption of the linear character of the basic traffic flow relations (for example traffic flow changes in addition to the length of the green time) in case of the signal traffic control. The decisive restrictions on the parameters are considered also to be linear and known.

It is obvious that the linear relationship holds well in the ideal conditions where all traffic flow inputs and outputs are detected. In reality, there exist such inputs and outputs that can never be accurately detected: temporary or longtime parking, non-controlled junctions or the streets without traffic detectors. It is shown that such inputs and outputs can reach the high value and it is not possible to ignore them. As it is not possible to measure them permanently, these values must be estimated. Moreover, we intend to describe the state of the junction by the lengths of the queues. This quantity is also not measurable in practice (at least, it is not easy to measure it). We assume to use the Kalman filter for this purpose which can respect such non-detected inputs and outputs or unmeasurable states.

The approach mentioned seems to be quite simple and quick enough. We expect that it will follow the traffic control step by step: microregion by microregion, junction by junction. Moreover, it will respect traffic flow defects with a possibility of prompt information about oncoming traffic flow changes.

3 Traffic Systems

We intend to control the urban traffic region consisting of several microregions. Each microregion is formed by several streets that cross at the junctions which may or may not be signal-controlled. Generally, its shape and extent are not limited (except the computational time limit). The streets and junctions, which pertain to the certain microregion, are usually chosen on the basis of the physical microregion structure and control needs.

For practical reasons, the microregion is usually created by some main traffic line and its adjoining junctions. The limiting amount of the included junctions can pertinently eventuate from the computational possibilities of the proposed algorithm.

3.1 Basic Notions

The urban junctions consist of a set of *approaches* and a *common crossing area* into which these approaches lead. An approach is the part of a street where all vehicles go in the same direction. The approaches consist of one *lane* at least. Some lanes can be reserved only for vehicles turning left, turning right or keeping the straight direction in case of the enough space. The lane can also be determined for allowable combination of turning or straight going vehicles.

In case of two identical lanes, two straight lanes for example, it is assumed that incoming vehicles from both lane queues evenly. In case of more than two different lanes for the approach, the way of vehicle inclusion to the lanes depends on their drive direction in which will continue after coming through the junction. The traffic on equivalent lanes has the right of way simultaneously and the vehicle can expect to pass the signal at roughly the same time whichever lane it chooses.

This structural configuration of lanes is determined in respect to the compound of the traffic flow (i.e. to the proportions of individual or public transportation, etc.) and the prospective traffic intensities obtained from the traffic survey.

The traffic at the junction is divided into *streams* that are portions of the input traffic considered. The stream is formed by all vehicles that cross the junction from the same approach and go in the same direction. Two streams are called *compatible* when they can safely cross the junction simultaneously, otherwise they are called *conflicting*. The aim of the signal control is to separate the streams as much as possible and thus to minimize the conflict situations.

Some of the junctions in the microregion are supposed to be controlled by the *flare* (signal) *equipment*. For such junctions the *traffic signal scheme* is composed on the basis of the traffic solution of the junction.

The *cycle* corresponds to a single repetition of the basic series of signal combinations at the junction, it means that all lanes must get their right of way and must have time, at least the minimal, for passing the junction.

The knowledge of the traffic situation of the junction leads to the defining of the *stage*. It is the part of the cycle time, during which some set of the streams receives their right of way. The stage describes the sequence of the signals during the cycle time. It has three parts: the green, the inter-green and the red time. The duration of the *green* signal may be variable in dependence of the immediate traffic demand. The length of the *inter-green* signal is fixed. The inter-green is interposed before the start of the green signal in order to avoid an interference between the conflict streams of the consecutive stages. The *red* signal is the addition of those parameters appointed to the cycle time.

Thus, for each junction the number of stages and their sequence is specified. The *optimal sequence of the stages* minimize the *total loss time* in the passage through the junction, it means the income and outcome times and the inter-green times. The *optimal cycle time* and the minimal times of green and red signals are deduced for each junction and its stages.

The *offsets* are time differences between the cycle times for successive junctions that may give rise to the ”green” waves along the arterial (i.e. no interruption of the drive) in condition that the drivers keep the suggested speeds. Those parameters must be proposed so that the *capacity* of lanes (i.e. the total amount of the vehicles that can drive through the junction during the green time) are higher than their traffic intensities.

Measurement of the current traffic variables are performed by *detectors*. They are usually placed on each arm of the junction with the flare equipment control. There are two possibilities for placing the detector: sometimes they can be placed on the stop-line or 20 - 30 meters far from it (so called far-away detectors). But there are also special detectors that are called strategic and that are placed on the stage between the two junctions. The *strategic detectors* give the more accurate information about the global traffic situation because the detectors near the junction are influenced by queueing vehicles and their slowing down and moving off. The detectors can provide the following traffic variables:

- *Intensity*: the amount of passing vehicles in unit vehicles per hour [uv/h],
- *Occupation*: the proportion of the sample period when the detector is being activated by vehicles [%],
- *Speed*: the point speed of the passing vehicles [km/h],
- *Density*: the amount of the vehicles per kilometer [uv/h].

3.2 Signal Traffic Control

Practically, there are three possible ways to influence the traffic flow by using the traffic signals on condition that the optimal number of stages and their constitution has been specified (typically solved off-line by a traffic engineer). The controlled parameters are the times of the greens, the cycles and the offsets:

- *Green time (split)*: the green duration of each stage should be optimized according to the actual demand of the involved streams. The most preferably, it should be given the optimal value of the green time and the allowable interval of its change. The longer green time allows the more cars, on the corresponding approaches, to drive through the junction. On the other hand, it causes the longer waiting times for the vehicles queueing on the remainder of the approaches. We consider to optimize the split, i.e. the relative duration of the green time, as the portion of the cycle time. Then the sum of all stage splits and inter-greens is equal one.
- *Cycle time*: The total capacity of the junction can be enlarged by the longer cycle time because this change enlarges the capacities of all approaches at once. The proportion of the inter-green times becomes smaller and conversely, the proportion of the total green time is bigger. The change of the cycle time is considered for the control of several microregions in case of the congestion of the whole traffic region. We consider the fixed cycle time for the local control of one microregion.

- *Offset*: The specification of the offsets allows to create green waves. Ideally, it takes into account the possible existence of the queues. We consider the offset control as a part of the higher level of traffic control where exceptional and irregular events as accidents, closures, the preference of special vehicles and public transport are supposed to be solved.

4 General Design of the Traffic Control System

It is possible to deduce how such traffic control system could be structured. First, it is obvious that the decentralized system is more efficient. The distributed, traffic adaptive and dynamic computer control offers some advantages over centralized and pre-timed approaches:

1. It is obvious that any central controller cannot easily consider the sudden changes in the local traffic flow conditions. The spatial and temporal local conditions must be respected in order to make the relevant decision.
2. Even the high quality communication and computational equipment cannot guarantee the transfer that is faultless and fast enough for the centrally controlled environment.
3. The decentralized system allows the parallel data processing, which permits to make the relevant decisions more quickly. There is also saved the time that would be necessary for data communicating to the individual traffic signals.
4. The changes of local area of the street structure, permanent or temporary, can be realized more easily and inexpensively in case of the distributed controllers.
5. In case of the local computational blackout, the whole system will be not paralyzed at all. It will probably work only with reduced quality and somewhat slowly.

Finally, it should be said that any proposed system must be designed in a way that is open to the further development in the future. The system should be sufficiently general for its application in the different areas, with small modifications dependent on the local traffic conditions.

The system must be able to optimize the traffic signal plan parameters as the cycle time, the offsets and the splits for all junctions in the network. From the network position, the delays and the total stopped time should be also optimized. The intelligent control system could not only react to immediate data inputs but it must be able to forecast impended changes of the traffic flows.

Obviously, it is important to choose the data detection and collection, the way of parameter estimation and the optimization criterion, etc. very carefully. The system should respect the experience acquired (the traffic volume dependence on the time and the local conditions, for example). The user requirements and the public transport requirements above all should be taken into account in case of the urban traffic control. The system must be open to the telematics environment.

4.1 Hierarchy of the Traffic Control System

The system should definitely have two control levels at least - the junction control level and the network one; in other words, the decomposition approach should be applied. Our proposed system has three levels of control:

1. Intersection controller
2. Microregion (local) controller
3. Region controller

The first one works on the level of a single junction. It is supposed that the cycle time, the offsets and the green times of each stage are set from the higher levels and they are fix at the moment. The interval of the maximal change of all the green times of the stages are also recommended. The single junction controller can only change the respective stage green times that means to spin out or to cut them according to the recommended time interval and current traffic conditions that are observed from the loop detectors in the junction. The standard junction controllers handle such operations.

The group of several single junctions forms the microregion that is necessary to control in view of the traffic flow maximization. The function of the microregion controller is to deduce the optimal stage green times for each junction and their intervals of the change. The criterion is the minimal total (weighted) queue length. The queue length directly relates to the waiting time so the minimization of the queue lengths leads also to the minimization of the total loss time. We also consider the penalization of the split changes. The controller

has the current measured traffic data at its disposal and the microregion cycle time that is set from the highest level.

It is obvious that the attainment of the optimal solutions in every microregion doesn't necessarily mean the global optimal solution for whole region. The improvement of the traffic conditions of the one microregion can often cause the deterioration of other microregion situation. That is why the higher, the region controller is needed. This controller is supposed to set the cycle times for each microregion in the way that the traffic flow through the urban net is maximized. The special cases as the public transportation preference, the emergency vehicles preference or vehicles' bumps and accidents are supposed to be solved on this level of the control.

5 Local Control of a Microregion

It is necessary to design a suitable model of the traffic flows in the area. Such model must be the macroscopic one. The model of this concept avoids the computational complexity associated with the microscopic models. For example, it is quite sufficient to know the percentages of the cars turning or keeping the same direction at the junction and single junction capacity.

In case of complete knowledge of the microregion element parameters, i.e. parameters of each single junction, that is shown that the relations between the parameters and restrictions on them are linear. The optimization criterion is also possible to construct as linear. From this reason, the linear programming is supposed to use for the solving of this problem. Generally, the traffic control problem then can be described by the following equations:

- State vector: could be formed by the inputs and outputs of the traffic flows, stage splits, queue lengths, etc.

$$s' = [s_1, s_2, \ldots, s_n] \tag{1}$$

where

s_i is the input or output of traffic flow in uv/h (i.e. unit vehicles per hour) or it represents the stage green time in s.

- Optimization criterion: is defined as the product of the state and weight vectors.

$$J = w's = \sum_{i=1}^{n} w_i s_i \tag{2}$$

where

$w' = [w_1, w_2, \ldots, w_n]$ is a vector of the relative weights.

- Restriction conditions: in the matrix notation for the state vector.

$$Qs \leq p \tag{3}$$

where

s is the state vector,
Q is the $n \times n$ matrix of restriction parameters,
p is the $1 \times n$ vector of the right sides.

Unfortunately, not all junctions of the microregion are equipped by detectors. Mostly, the detectors are placed only on those junctions that are controlled by the flare equipment. Thus there is a number of non-detected inputs and outputs of the traffic flow that can be significant and so they can't be ignored in the modelling. Such inputs and outputs can be interpreted as the parking of vehicles, arrivals and departures through the junction arms with no detection, etc. Then the precondition, that the output of one controlled junction is equally the input of the following controlled junction, is not fulfilled.

The traffic detectors are usually placed on each arm of junction with the flare equipment control. But there are also special detectors that are called strategic and that are placed on the stage between the two junctions. Owing to this fact, it is possible to count the mentioned defects of the traffic flow retrospectively. We suppose the approximating of the states and the defect values and subsequent specification of the linear programming parameters by the Kalman filter. Then the benefits of the supposed method approach (specially the low time strenuousness of the computation) can still be released on condition of the parallel parameters' approximating.

6 State Equations

6.1 Junction Diagram

In this paper, we will consider a simple three-arm junction (see fig. 1) with simple (one lane) approaches, just two queues, one external non-detected input and other special traffic conditions (one-way flows). This special junction will be also used for the experiments. The final state model can be easily extended for general traffic flow conditions (multidirectional flows). All quantities are supposed to be related to a fix sampling period.

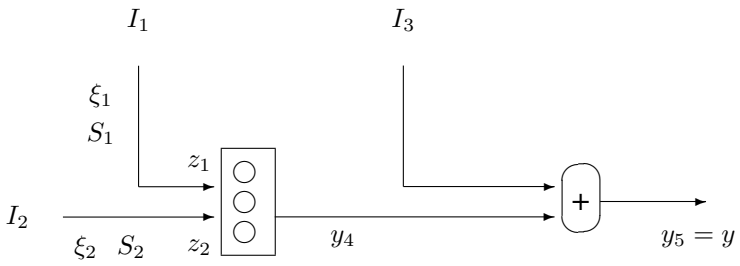

Fig.1: Junction design

where

1, 2, 3, 4, 5	marks of the arms, only No. 3 non-detected,
$I_{1;t}, I_{2;t}, I_{3;t}$	input intensities in time t $[uv/T_s]$,
$\xi_{1;t}, \xi_{2;t}$	queue lengths in time t $[uv/T_s]$,
S_1, S_2	saturation flows ($[uv/T_s]$),
T_s	sample time period,
$z_{1;t}, z_{2;t}$	splits in time t $[\%]$,
y_4, y	output intensities in time t $[uv/T_s]$.

6.2 Model Equations

There exist more possibilities for choosing quantity to form the state vector. It can be for example formed by values of the output flows, the length of green times or splits, the queue length, etc. In order to obtain the observable and stable linear model and to use all information, we consider following form of the junction state vector:

$$x_{t+1} = \left[\, \xi_{1;t+1};\ \xi_{1;t};\ \xi_{2;t+1};\ \xi_{2;t};\ \tilde{I}_{1;t};\ \tilde{I}_{2;t};\ \tilde{O}_{1;t};\ \tilde{O}_{2;t};\ \tilde{I}_{3;t} \,\right]' \tag{4}$$

where

$\xi_{i;t}, \xi_{i;t}$	queue lengths, $i = 1, 2$,
$\tilde{I}_{i;t}, \tilde{I}_{i;t}$	intensity deviations from its typical daily course, $i = 1, 2$,
$\tilde{O}_{i;t}, \tilde{O}_{i;t}$	density deviations from its typical daily course, $i = 1, 2$,
$\tilde{I}_{3;t}$	non-detected intensity deviations from its typical daily course.

The form of the state vector is clear now. For deducing the appropriate model, it is necessary to specify the relations between the state vector elements. We suppose the following equations to expres the time course of the queues:

$$\begin{aligned} \xi_{1;t+1} &= \delta_{1;t}\xi_{1;t} + \delta_{1;t}I_{1;t} - \delta_{1;t}z_{1;t}S_1 \\ \xi_{2;t+1} &= \delta_{2;t}\xi_{2;t} + \delta_{2;t}I_{2;t} - \delta_{2;t}z_{2;t}S_2 \end{aligned} \tag{5}$$

where

$\delta_{i;t}$	indicates the possibility of queue creation, alternatively the type of the passage.

That is to say if the queue length and also the input intensity is "small" enough ($\delta_{i;t} = 0$), then all vehicles (queueing or incoming) can drive through the junction. Logically, there is no queue in the end of the green. In the other case, if the input intensity is too "big" ($\delta_{i;t} = 1$), only the approach capacity can pass the junction and a queue is made.

We assume the traffic conditions (intensities and splits) don't change radically during the two following periods (or cycles). It is also known that the vehicles start to queue during the red and inter-green signal of

cycles, except the exceptional or irregular cases. Then the conditions of "big" or "small" intensities and the appropriate passage can be written consecutively:

$$\begin{aligned} I_{i;t} \leq K_{i;t} = S_i z_{i;t} \quad &\Rightarrow \quad P_{ij;t} = \alpha_{ij}(I_{i;t} z_{i;t} + \xi_{i;t}), \\ I_{i;t} > K_{i;t} = S_i z_{i;t} \quad &\Rightarrow \quad P_{ij;t} = \alpha_{ij} K_{i;t}, \end{aligned} \tag{6}$$

where

$K_{i;t}$	approach capacity $[uv/h]$,
$P_{ij;t}$	passage, i.e. the amount of the vehicles passing the junction from the approach i to j $[uv/h]$,
α_{ij}	directional relations, i.e. the proportions of vehicles driving from approach i to j to total approach intensity.

Now, we specify the time course of the intensity and density deviations. As we said, we consider the typical daily courses of intensities and densities. These courses can be derived by averaging apriori measured data. The deviations are defined as $\tilde{I}_{k;t+1} = I_{k;t+1} - \bar{I}_{k;t+1}$, respectively $\tilde{O}_{k;t+1} = O_{k;t+1} - \bar{O}_{k;t+1}$ where $I_{k;t+1}$,$O_{k;t+1}$ are quantities measured by detectors, $\bar{I}_{k;t+1}$ and $\bar{O}_{k;t+1}$ are their typical daily courses. We assume that the deviations of the intensity and density depend not only on their previous values but also on the length of corresponding approach queue.

In the case of non-detected intensities, we can't determine the typical daily course from the measured data. We can derive this course as the proportion of the typical one of measured intensities or as the weighted sum of all of them. The non-detected inputs have no direct relation with signal controlled junction and so we don't assume any dependence on measured or calculated quantities of the junction. The searched equations are following:

$$\begin{aligned} \tilde{I}_{1;t+1} &= \beta^I_{1;t} \tilde{I}_{1;t} + \kappa^I_{1;t} \xi_{1;t} + \omega^I_{1;t} \tilde{O}_{1;t} \\ \tilde{I}_{2;t+1} &= \beta^I_{2;t} \tilde{I}_{2;t} + \kappa^I_{2;t} \xi_{2;t} + \omega^I_{2;t} \tilde{O}_{2;t} \\ \tilde{O}_{1;t+1} &= \beta^O_{1;t} \tilde{O}_{1;t} + \kappa^O_{1;t} \xi_{1;t} + \omega^O_{1;t} \tilde{I}_{1;t} \\ \tilde{O}_{2;t+1} &= \beta^O_{2;t} \tilde{O}_{2;t} + \kappa^O_{2;t} \xi_{2;t} + \omega^O_{2;t} \tilde{I}_{2;t} \\ \tilde{I}_{3;t+1} &= \beta^I_{3;t} \tilde{I}_{3;t} + \bar{I}_{3;t} \end{aligned} \tag{7}$$

Note: All parameters of the previous equations are assumed unknown and they have to be estimated. Moreover, the coefficients κ enter the model as products with the state ξ, which is also estimated. That is why their estimation is non-linear.

The resultant state equation is

$$x_{t+1} = A_t x_t + B_t z_t + F_t + e_t, \tag{8}$$

where

A	represents the matrix 9×9 of parameters,
B	matrix 9×2 reflects the passage dependence on the splits,
z_t	is the vector of control variables, $z_t = [z_{1;t}; \frac{C - t_m}{C} - z_{1;t}]$, t_m is the total time of the inter-green signal
F	vector 9×1 adds the typical courses of intensities,
e	is the state noise.

The output state equation is defined as

$$y_t = C\, x_{t+1} + G_t + \epsilon_t, \tag{9}$$

where

y_t	vector of all measured quantities, $y_t = [I_{1;t+1}; I_{2;t+1}; O_{1;t+1}; O_{2;t+1}; y_{5,t}]'$,
y_t	is the sum of the passages from the preceding arms to the arm No. 5,
C	matrix 5×9 reflects directional relations of the junction,
G_t	vector 5×1 adds the typical courses of intensities and densities,
ϵ_t	is the output noise.

7 Kalman Filter

The state model 8, 9 was used for the common version of the Kalman filter to estimate values of the state vector from the measured data. In our case, the state model is partially based on the relations that are supposed to be known, i.e. given by the physical nature of the transportation system (they are (5)) and partially on the relations that are unknown and have to be estimated (they are (7)). So before the Kalman filter can run, the estimation has to be performed. For testing, we used standard procedure from the Identification package of MATLAB. This estimation is performed in off-line mode, from prior data sample, lately it can repeated with further data and thus the parameter estimates are set more precise. After estimation, the unknown parameters of the state space model are replaced by their estimates and the Kalman filtration can run. During it, the state is identified, i.e. both the queue length and the unmeasured input intensities are estimated. In this way, we have all we need for the linear optimization.

8 Formulation of the Linear Task

What we need here, is to adapt the previously constructed state space model (8), (9) into the form (1), (2), (3) suitable for straightforward application of linear programming. First we have to compose the vector (1), which enters both, the criterion (2) and the system of restrictions (3). The widely used criterion in the traffic control problems is the total delay of vehicles, passing through the microregion. As the majority of the time spent in the microregion is due to standing in queues, we will specify the criterion as a sum of lengths of all queues in the microregion. The restrictions in our case are given by the state space model (8), where the state vector x occurs. Finally, we want to minimize the criterion by adjusting the input variables z. From this follows, that the vector s from (1) must contain (i) the queue lengths, (ii) the state vector x (which is in accordance with (i)) and (iii) the input variables z. So, the following vector s_t at time t must be constructed

$$s_t = [x'_{t+1}, z'_t]'. \tag{10}$$

Then the criterion is

$$J = \xi_{1;t+1} + \xi_{2;t+1} = [1, 0, 1, 0, \ldots, 0]s_t \tag{11}$$

and the major part of restriction is given by (8)

$$x_{t+1} - Bz_t = Ps_t = A\hat{x}_t + F, \tag{12}$$

where

I	a unit matrix of appropriate dimension and $P = [I, -B]$,
$\hat{x}_t$	point estimates of the last state,
noise	is substituted by its mean value, which is zero.

As the right-hand side of the above equations (12) are constants, the equations represent the restrictions in the form of the equalities. Other restrictions of the same type are laid on the input variables. The relative green times with respect to the cycle (the splits) must be some constant smaller then 1, due to the intergreens. The condition can be constructed as follows

$$[0, \ldots, 0, 1, 1]s_t = 1$$

and can be added to the previous restrictions.

Finally, all queue lengths and splits must be nonnegative, so the last restriction is in the form of inequality

$$\mathrm{diag}([1, 1, 1, 1, -\infty, \ldots, -\infty, 1, 1])s_t > 0,$$

where

diag(.)	a diagonal matrix, with the specified diagonal,
0	a vector of zeros of appropriate dimension.

With this denotation, the task is prepared for standard use of linear programming.

9 Experiments

For experiments, the junction structure defined above (see 6.1) was used. The typical daily courses of modelled intensities and densities were derived from the real data measured in Prague. The derived typical courses of the measured variables are shown on the figures 1.

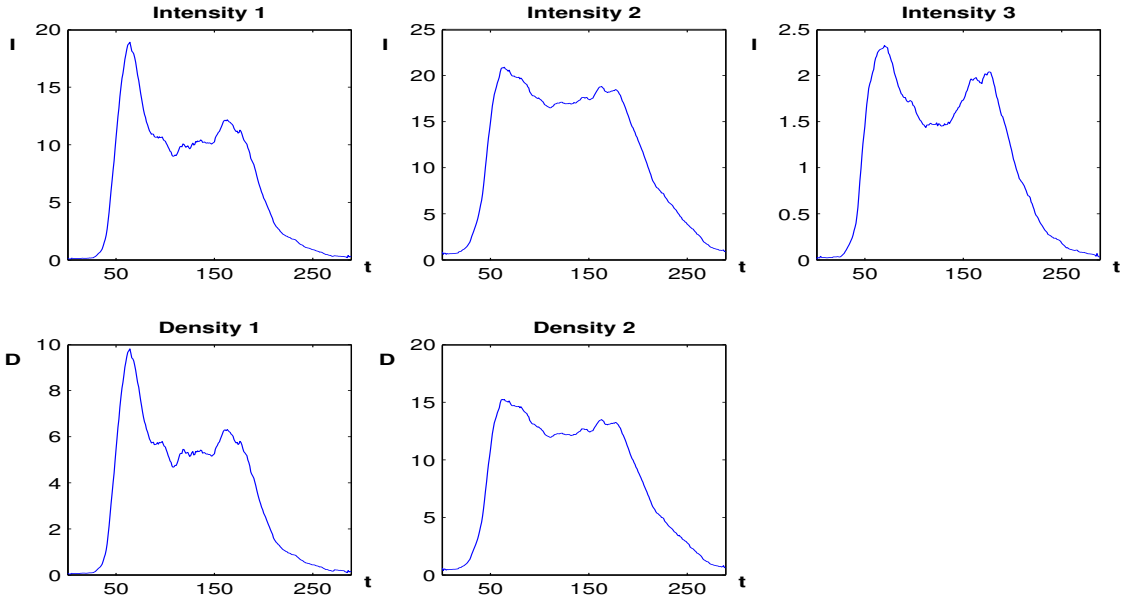

Figure 1: Typical daily course of intensities and densities

This typical daily courses and the model structure (8, 9) with fixed parameters were used for the simulation and also the same structure for the estimation by the Kalman filter. Moreover, a disturbance was added to the first state, i.e. queue length 1. We simulate and estimate the courses and states during one day which represents 288 measurements (in 5 minutes sample period). The simulated queue lengths and their estimates are plotted in the following figure 2.

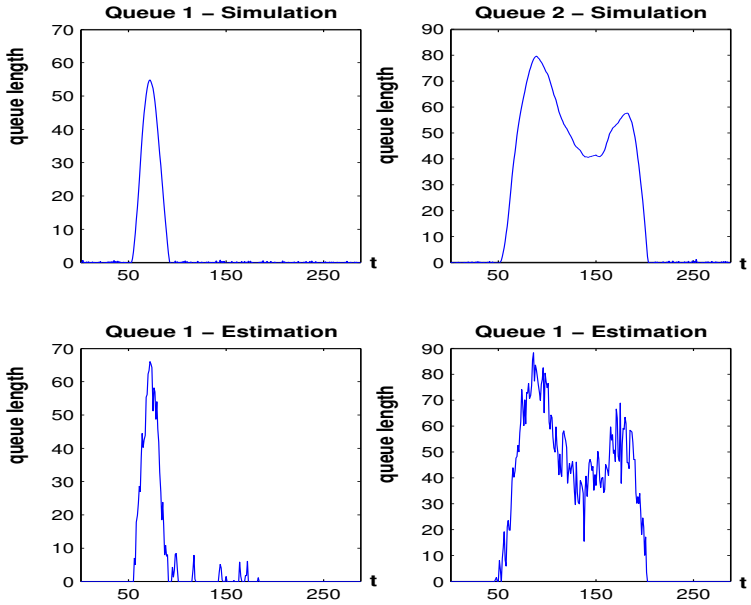

Figure 2: Courses of simulated and estimated queue lengths

As it can be seen, the estimated and simulated curves match well, despite the estimation disturbance of the first queue. The most conclusive experiment, which proofs the efficiency of the proposed optimization method, is the comparison of the automatically and manually controlled junction. We made a control during one day. First, we control the junction optimally by junction parameters but the setting of control variables is constant all the time. Second, we control automatically - the setting of control parameters is changing owing to actual data values. The optimization criterion is the minimal sum of both queue length.

The queue length courses are shown for both cases in the following figures 3 nad 4. Solid lines represent the queue length, the splits are plotted by crosses.

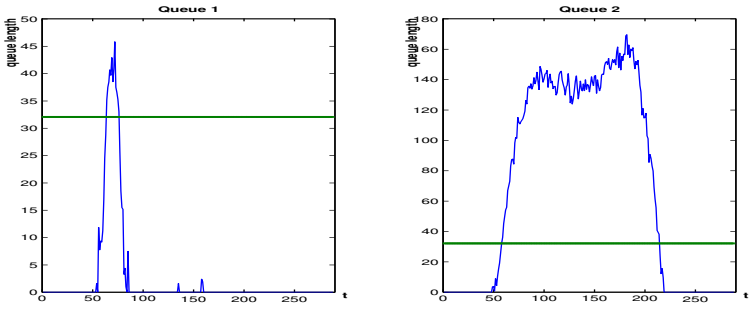

Figure 3: Manually controlled junction

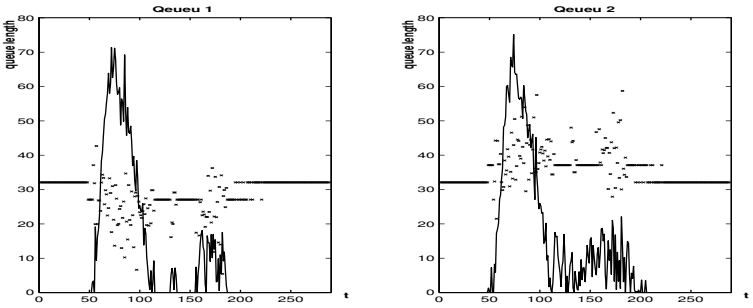

Figure 4: Automatically controlled junction

10 Conclusions

The object of our research is the control of the urban network and the control of the microregion is the first step towards it. In the recent time, we have found the description of microregion by the state model. We have to estimate some of the parameters to get the observable and stable model but this off-line estimation runs without any problem and, moreover, we obtained the feasible form of the model. This form allows to add any new relation between the arbitrary model variables with respect to new measurements. The experiments gave the good results (see figures) and so the proposed method seams to be very promising.

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Analytic Hierarchy Process, a Psychometric Approach

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Abstract. The Analytic Hierarchy Process, or AHP for short, is a decision method that is often used in Management and Economics. In this paper, we study real data that we obtained from a simple experiment. Then, from the analysis of these data, we point out that some pathological behaviours of the AHP correspond to phenomena already encountered in Psychometric Choice Theory.

Keywords: Analytic Hierarchy Process, Psychometrics

1 Introduction and Presentation of the Experiment

The Analytic Hierarchy Process (AHP) is a method often used in Management and Economics for the ranking of a set of alternatives and the choice of a preferred one. The AHP has been applied to measure performances of productive systems (Rangone 1996), to the ABC classification of warehouse items (Partovi et al. 1993), to strategic planning (Armel and Orgler 1990), to evaluate priorities in customers' needs (Armacost et al. 1994) and to investment analysis in innovative technologies [(Stout et al. 1991), (Weber 1993), (Wicks and Boucher 1993)]; other applications are in (Saaty 1994). Moreover, software products implementing this technique are available on the market, such as Expert Choice, HIPRE3+, Criterium, AUTOMAN [see (Stout et al. 1991), (Weber 1993)] and NCIC (Wicks and Boucher 1993). The AHP is connected with Psychometric Choice Theory, which is inspired from results of psychophysical experiments. In these experiments, a subject has to compare the intensities of physical stimuli of different magnitude, in order to rank them. The Fechnerian interpretation consists in associating sensations to stimuli: sensations caused by external stimuli allow the individual to discriminate between stimuli. In particular the Weber-Fechner law states that sensations are not linearly related to stimuli, but the relation is increasing and convex (like the logarithm).[1] The Thurstonian theory[2] in Psychometrics (Thurstone 1927) lends itself to this idea, under the additional hypothesis that sensation is the realization of a random variable: this implies that the individual is not able to perfectly discriminate between stimuli. Hence, the comparison of two stimuli amounts to the comparison of the realizations of two random variables representing the sensations induced by the two stimuli.

These random variables are called *discriminal processes* and this principle is called *law of comparative judgement*. As (Yellot 1977) states:

> The "law of comparative judgement" postulates an imaginary psychological process, underlying observable behaviour, wherein choice objects are represented by random variables ("discriminal processes") which the subject compares in order to arrive at a decision.

The application of the AHP involves 3 steps [for a more comprehensive introduction see (Saaty 1986)]:

1. The problem is reduced to a hierarchy of levels (see Figure 1): the highest level corresponds to the overall objective (say, the convenience of a workplace). The lowest level is formed by a set of alternatives (say, a set of companies among which to choose). The intermediary levels are composed by attributes (called *criteria*) of the upper levels (as an example, for a workplace, the stability, the economic treatment and so on).
2. The elements of any level are subject to a series of paired comparisons (on the Saaty's scale, ranging from $1/9$ to 9) and a pairwise comparison matrix is built. If ψ_i is the relevance of criterion i, a_{ij} is intended to be an estimate of ψ_i/ψ_j: the matrix $\mathbf{A}$ has therefore the characteristic that $a_{ij} = 1/a_{ji}$. The eigenvector associated with the greatest eigenvalue of $\mathbf{A}$ is considered an estimator of $\psi = [\psi_i]$. The extent with which the matrix $\mathbf{A}$ reflects these measures is called *implicit judgements consistency index* and is defined as: $\alpha = (\lambda_{\max}(\mathbf{A}) - n)/(n-1)$ where $\lambda_{\max}(\mathbf{A})$ is the greatest eigenvalue of $\mathbf{A}$ and n is the number of criteria in the level.

[1] The Fechner approach is widely present also in the first works on utility theory.
[2] A good introduction is in (Nunnally 1978), pp. 54-62.

3. The estimated weights are then aggregated in an overall score that makes it possible to order the alternatives.

From a *descriptive* point of view, the AHP yields a framework in which the usual decisional process of the individual can be interpreted through a decomposition in elementary units. The bases on which the AHP is grounded are essentially linked to the theory of preference, that is to Economics, and to the theory of judgement measurement, that is to Psychometrics.

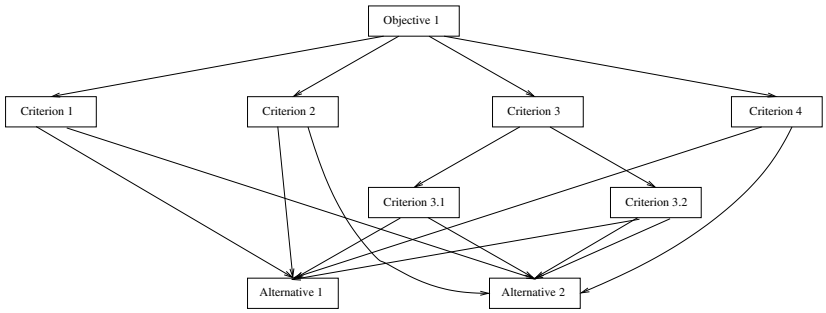

Figure 1. Structure of the AHP.

From a *normative* point of view, the AHP tries to replicate, clarify and correct the mental process that carries the decision maker to prefer an alternative to the others: it gives, without any doubt, a useful instrument of decision support. It is not clear, on the other hand, whether the order of the alternatives that emerges from the AHP is the correct one and if the chosen alternative is really better than the others.

To illustrate this, we have tried to analyze whether the AHP method based on pairwise comparisons of alternatives on Saaty scale is able to recover the importances of different alternatives. We do not tackle the related (but much more complex) problem of aggregating preferences in tree value diagrams.

We have asked six people to locate on Saaty's scale $(1 - 9)$ the ratio between the distance from Milan to some city A and the distance from Milan to some city B. The cities of the sample are Turin, Venice, Rome, Naples and Bari; the distances to Milan are respectively 140, 270, 575, 785 and 880 kilometres, and are sufficiently regular to avoid the presence of outliers. Moreover, their positions are well known to the interviewed people. In what follows, people answering the questions will be called Decision Makers (DM).

The paper will be organized as follows. In Section 2.1 we show that the pairwise comparisons of an individual can be grossly biased: in particular, the relation between the real ratios and the elicited ones seems to be concave and increasing; this means that the elements of the matrix $\mathbf{A}$ chosen by the decision maker can be badly biased estimates of the real ones. In Section 2.2, we consider the performances of the eigenvector method when the elements of the matrix $\mathbf{A}$ are given by the real ratios distorted by deterministic functions: this shows the relevance of the scale chosen and the distorsions caused by the inability of the decision maker to locate ratios on the scale. Section 2.3 displays the weights (in our case, the distances) implicitly chosen by the decision makers, in order to show that some distorsions can indeed arise in practice.

2 Remarks

In this Section, we analyse the data obtained from our experiment. In the same spirit as Neural Networks or Genetic Algorithms, the AHP is biomimicking: this method indeed breaks up the decision process into elementary units, just as the decision maker is supposed to be doing mentally. But, the most critical aspect of that method, which we are about to detail, appears in the ability of the subjects to estimate pairwise preferences expressed on a ratio scale. The analysis of our empirical results shows that the very limits of the AHP can be interpreted through a psychometric viewpoint.

2.1 Linearity of the Individual Judgements

The first problem concerns the capacity of the individual to evaluate stimuli of varying intensities: in particular, it is to be hoped that the individual doubles its response when confronted to a stimulus doubling. The responses of Decision Maker 1 are reported in Table 1: if we consider only the weights greater than 1 (since the others are characterized by the reciprocity property), we get a relation of logarithmic type, as shown in Figure 2.

Table 1. Matrix **A** for Decision Maker 1.

	Turin	Venice	Rome	Naples	Bari
Turin	1	0.333	0.143	0.125	0.125
Venice	3	1	0.2	0.167	0.143
Rome	7	5	1	0.333	0.25
Naples	8	6	3	1	0.333
Bari	8	7	4	3	1

The logarithmic regression of the stimuli on the responses, given by:

$$a_{ij} = 1.4267 + 3.963 \cdot \log\left(\frac{d_i}{d_j}\right)$$

has an excellent fit and a determination coefficient (the so-called R^2) equal to 94.22%. A linear regression on the same data gives a determination coefficient equal to 83.19%, which is clearly worse.

Nonlinear and nonnegative relations showing a saturation effect are rather frequent in psychology. When the stimuli increase in strength, it often becomes hard for the individual to evaluate properly the differences among the stimuli: therefore, and above all in presence of alternatives characterized by very different levels of magnitude, the individual may not be able to evaluate correctly the relative levels.

A relation of that type appeared for the first time in the solution given by Daniel Bernoulli to the so-called Paradox of Saint-Petersburg: he supposed that the *moral fortune* (ancestor of the modern utility) was a logarithmic function of the *physical fortune* (the gain obtained from the game in his case); for discussion on the Paradox, see (Boyer 1968). Moreover, for human sensations the so-called *Weber-Fechner law* holds: on the basis of empirical observation, it states that there is a logarithm-like relationship between the stimulus on the sense organs and the sensation.

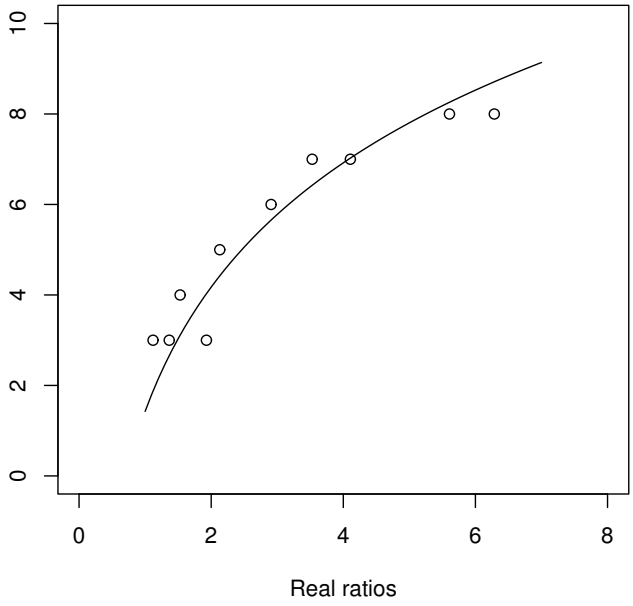

Figure 2. Relation between the real ratios (d_i/d_j on the X axis) and the estimted ones (a_{ij} on the Y axis).

2.2 Eigenvector under Alternative Distortion Scales

The problem pointed out in the former paragraph is a sort of *bias* or *systematic error* that the Decision Maker makes when he evaluates the ratio d_i/d_j through his personal estimate a_{ij}: the underlying idea is that the individual confronted to stimuli responds according to a scale that can be a linear or nonlinear function of the real stimuli. We wonder whether this phenomenon leads to important distortions in weight estimation. To evaluate the robustness of that model with respect to scale transformations, we adapt the former example to see whether a scale change (i.e. using a logarithmic or linear scale, $1-5$, $1-9$, etc.) yields notably different results. To separate the analysis of systematic bias from the study of judgement subjectivity, we use real data d_i/d_j, proceeding as follows:

1. we consider the values of the ratio d_i/d_j greater than 1 and we apply the following increasing transformations:

$$\begin{array}{ll} \text{Series 1} & a_{ij} = \exp\left(d_i/d_j - 1\right) \\ \text{Series 2} & a_{ij} = \log\left(d_i/d_j\right) + 1 \\ \text{Series 3} & a_{ij} = \left(d_i/d_j\right)^2 \\ \text{Series 4} & a_{ij} = d_i/d_j \end{array}$$

   Moreover we apply some linear transformations (identified with Series $5-9$) so as to rescale the values on a scale going from 1 to M where $M = 1.9,\ 5,\ 9,\ 15,\ 35$;
2. we calculate the matrix $\mathbf{A}$ through the reciprocity property;
3. for any matrix we obtain the eigenvector associated with the maximal eigenvalue.

The results are shown in Figure 3: from the graph it is evident that the choice of a scale has a strong impact on the calculation of the final weights ψ_i's. As an example, the exponential scale (Series 1) and the logarithmic one (Series 2) have very different behaviours, since the former gives less importance to small observations, while the latter gives more importance to small observations. On the other hand, linear scales behave better, but in the present case the best one appears to be the $1-5$ scale, since it reflects the range of the data (that is from 1 to 6.285714).

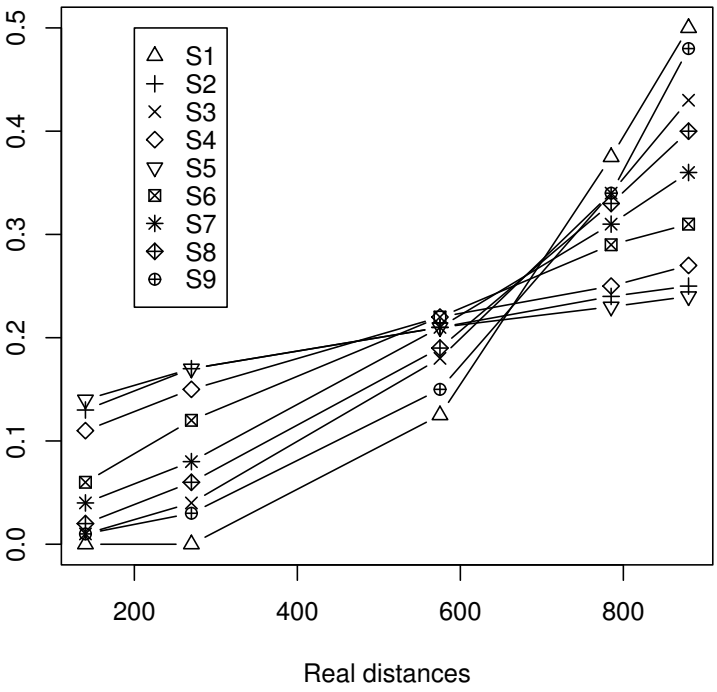

Figure 3. Relation between the real distances (d_i on the X axis) and the estimated ones (ψ_i on the Y axis).

Therefore, the choice of the upper limit of the scale can have a nonnegligible impact on the weights obtained from the judgements of the decision maker: (Weber 1993) remarks that two alternatives (say 1, 2) one of which (1) is weakly preferred to the other (2), would yield on Saaty's scale a ratio and final weights equal to $(\psi_1, \psi_2) = (0.75, 0.25)$: this does not coincide with the usual notion of weak preference. The choice of the upper limit of the scale is then crucial since it provides the decision maker with a benchmark with respect to which the alternatives are evaluated.

The weights obtained from the eigenvector method, the correlation with the real values and the consistency index are displayed in Table 2.

Table 2. Eigenvectors for the 9 series, correlation with the true values, greatest eigenvalue and Saaty's α.

	Data	Series 1	Series 2	Series 3	Series 4
Eigenvalue	140	0.0059	0.1339	0.0108	0.1076
	270	0.0360	0.1651	0.0402	0.1495
	575	0.1628	0.2138	0.1823	0.2181
	785	0.3246	0.2390	0.3398	0.2549
	880	0.4707	0.2885	0.4270	0.2698
Correlation		0.9663	0.9928	0.9833	0.9948
Eigenvalue		5.5822	5.0009	5	5
Alpha		0.1455	0.0002	0	0
	Series 5	Series 6	Series 7	Series 8	Series 9
Eigenvalue	0.1368	0.0650	0.0374	0.0220	0.0084
	0.1723	0.1155	0.0821	0.0583	0.0310
	0.2142	0.2200	0.2088	0.1903	0.1497
	0.2342	0.2854	0.3112	0.3271	0.3358
	0.2425	0.3143	0.3606	0.4024	0.4750
Correlation	0.9840	0.9994	0.9984	0.9900	0.9607
Eigenvalue	5.0039	5.0010	5.0048	5.0431	5.2829
Alpha	0.0010	0.0003	0.0012	0.0108	0.0707

From this analysis, it is clear that, if the decision maker is able to express linearly the relative relevances of the alternatives, the scale has only a limited impact that sometimes can be relevant. On the other hand, if the perceptions of the decision maker are nonlinearly related to the stimuli, then the calculated weights can yield a very misleading picture of the real ones. In particular exponential or logarithmic scales lead to serious distortions.

2.3 Impact of Individual Variability

In this paragraph, we present the answers of the 6 Decision Makers, to show the impact of individual subjectivity on weight calculation and to argue that the systematic distortion of the weights is a typical feature.

In order to investigate the sensitivity of the AHP as concerns the choice of the decision maker, we reproduce in Table 3 the weights calculated for the Decision Makers, together with the values of the consistency index and the true values.

It is not difficult to remark that the weights calculated in this way are quite far from the expected linear relation: the weight of the most important criterion (or alternative) would be overestimated and the contrary would hold for the weight of the less important. Figures 4 and 5 show that the behavior of elicited weights is almost exponential.

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Table 3. Relevance of the Decision Maker.

	Turin	Venice	Rome	Naples	Bari	Alpha	Corr.
Data	140	270	575	785	880		
D.M. 1	0.03087	0.05451	0.15974	0.27657	0.47831	0.09613	0.93233
D.M. 2	0.02468	0.04303	0.12304	0.35439	0.45831	0.19793	0.94535
D.M. 3	0.02811	0.04289	0.10223	0.20972	0.45485	0.18504	0.80755
D.M. 4	0.02775	0.06416	0.15304	0.23303	0.61705	0.09742	0.8792
D.M. 5	0.0438	0.0685	0.1495	0.26615	0.52202	0.09675	0.9148
D.M. 6	0.0261	0.0564	0.13014	0.22660	0.56070	0.19215	0.8506

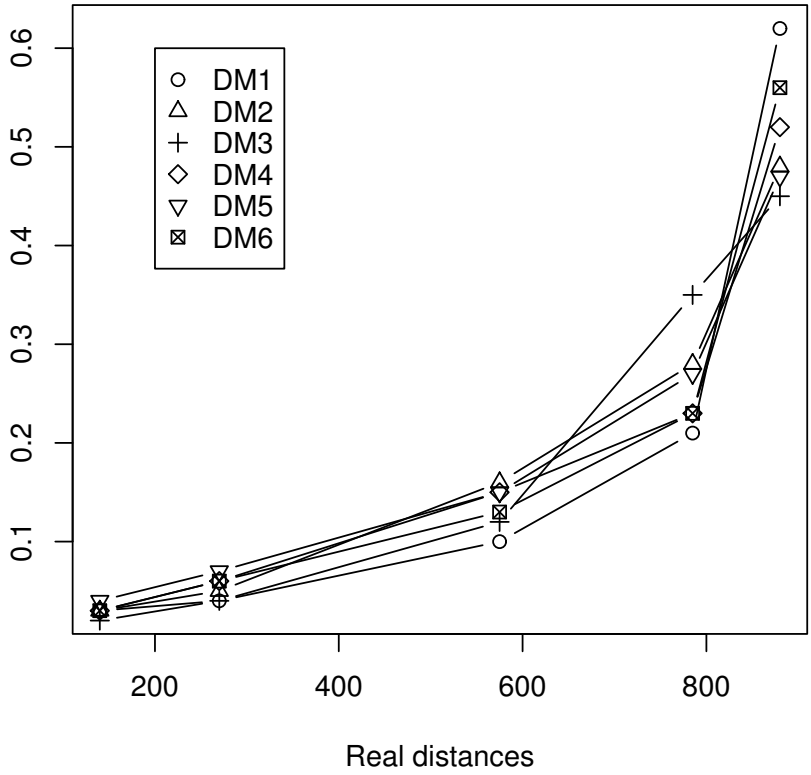

Figure 4. Estimated distances (ψ_i, Y axis) and real ones (d_i, X axis) for the different decision makers.

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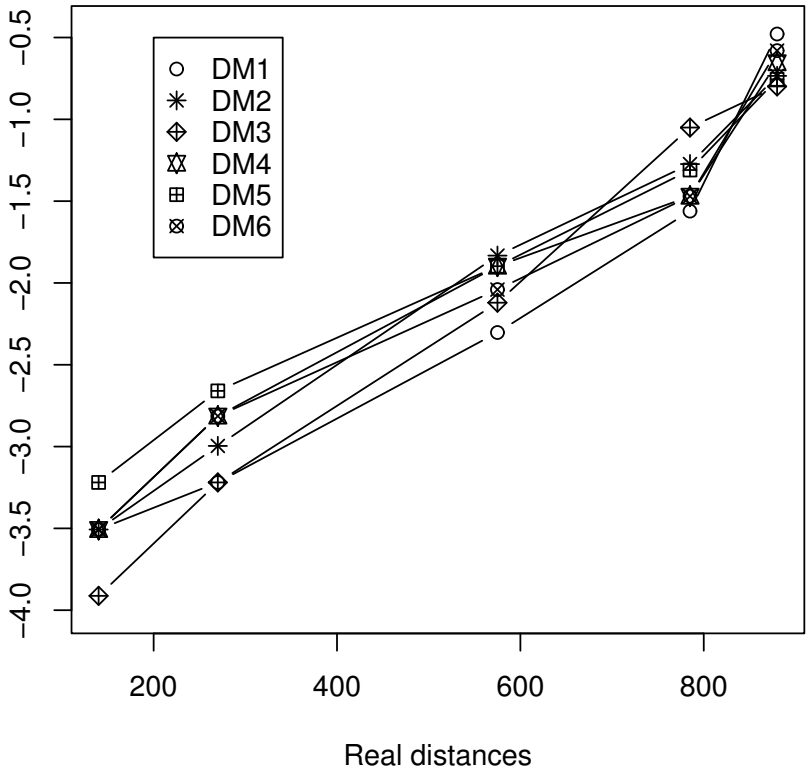

Figure 5. Logarithm of the estimated distances ($\log(\psi_i)$, Y axis) and real ones (d_i, X axis) for the different decision makers.

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