Price Tails in the Smith and Farmer's Model

Martin Šmíd*

Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic 182 08 Prague, Pod Vodárenskou věží 4, Czech Republic

e-mail: smid@utia.cas.cz

Abstract. We analytically prove that the tails of the price increments in the model by Smith et al. [2003] are fat with the tail exponent one if the initial order books are empty; however, they become thin if an initial call auction is held before the start of the trading. This way, our results point out to the stabilizing role of the initial call auction.

Keywords: limit order market, continuous double auction, price increments, fat tails, tail exponent

JEL classification: C51,G10
AMS classification: 91B26, 91B70

1. Introduction

In our paper, we deal with the zero intelligence model of a limit order market introduced by Smith et al. [2003]. Using the recent paper by Šmíd [2008], where the stochastic properties of a general model whose special case the Smith and Farmer's model is, are rigorously described, we deal with the tails of the price increments in the Smith & Farmer's model - we prove the tails to be fat given the empty initial order books but thin if the initial order book is infinite and dense enough. The latter situation happens, for instance, if an initial call auction is held before the start of the continuous trading.

2. The S.& F. Model

By their model, Smith et al. [2003] describe a limit order market with discrete equidistant (log)prices and unit order sizes (for a description of the functioning of limit order markets, see Smith et al. [2003] or Šmíd [2008]). It is assumed that there exist constants $\eta > 0$, $\kappa > 0$ and $\iota > 0$ such that

- the sell market orders arrive with the rate η
- the buy market orders arrive with the rate η
- sell limit orders with a limit price $p \in \mathbb{Z}$, greater than the best bid, arrive with the rate κ

*This work is supported by the grants no. 402/06/1417 and 402/07/1113 of the Czech Science Foundation.

- buy limit orders with a limit price p, less than the best ask, arrive with the rate κ
- the rate of a cancelation of each (buy or sell) waiting limit order is ι

For details and the description of the dynamics of the model, see Smith et al. [2003] or Šmíd [2008], Example 3.

3. Notation

For each $p \in \mathbb{Z}$ and $t \geq 0$, denote $A_t(p)$ and $B_t(p)$ the number of the unsatisfied sell limit orders, buy limit orders respectively, with the limit price p at the time t. As demonstrated by Šmíd [2008] (see Proposition 1 therein), the (infinitely dimensional) process

$$\Xi_t = (A_t(\bullet), B_t(\bullet)), \qquad t \geq 0,$$

is Markov.

Denote

$$a_t = \min\{p : A_t(p) > 0\}, \qquad b_t = \max\{p : B_t(p) > 0\},$$

the best ask, best bid respectively, and introduce

$$\xi_t = (a_t, A_t(a_t), b_t, B_t(b_t))$$

the process of the best quotes and offered volumes. Demote $0 = \tau_0 < \tau_1 < \tau_2 < \dots$ the jump times of ξ.

4. Initial Call Auction

Suppose further that there is a period before the start of the continuous time trading when, instead of immediate processing, the orders are accumulated and, at the end of the period, the (initial call) auction is held. The auction is made as follows: first, a market price p_0 is determined and, consequently, all the feasible orders (i.e. those for which the price p_0 is satisfactory) are traded at the price p_0. In practice, usually, one of the prices maximizing the traded volume is taken as the market price; in the present work, we shall assume that p_0 is set to one those prices, in particular we shall assume that p_0 is such that $b_0 < p_0 < a_0$ where

$$a_0 = \min\left\{p \in \mathbb{Z} : \sum_{\pi=a_-}^{p} A_-(\pi) > \sum_{\pi=p}^{b_-} B_-(\pi)\right\},$$

$$b_0 = \max\left\{p \in \mathbb{Z} : \sum_{\pi=a_-}^{p} A_-(\pi) < \sum_{\pi=p}^{b_-} B_-(\pi)\right\}.$$

Here, a_- and b_- are some constants (maximal sell limit price, minimal buy limit price respectively) and, for each $\pi \in \mathbb{Z}$, $A_-(\pi)$ and $B_-(\pi)$ are the numbers of sell orders, buy orders respectively, with the limit price π having arrived during the accumulation period (note that the demand exactly equals the supply for such p_0).

The (limit) order flow during the accumulation period is assumed to be the same as that during the continuous trading, i.e. with the intensity κ per price; however, instead of the current best ask and bid, the limit prices are bounded by a_- and b_- during the accumulation period.

For simplicity, we suppose that no market orders come and no cancelations take place during the accumulation period (our further results would be preserved up to constants after the inclusion of market orders and cancelations). Quite naturally, we assume the order flow during the initial period to be independent of the flow of the orders starting from time zero.

Even if the joint distribution of (B_0, A_0) could be determined quite easily, the knowledge of the distribution of $(A_0(a_0+1), A_0(a_0+2), \dots)$ will suffice for us in the sequel:

Proposition 1. *Denote $\lambda = \kappa\theta$ where $\theta \geq 0$ is the length of the accumulation period. If $\theta > 0$ then*

$$A_0(a_0+1), A_0(a_0+2), \dots$$

where, for each $\pi \in \mathbb{Z}$, $A_0(\pi)$ is the number of limit orders with the limit price π, unsatisfied by the call auction, is a sequence of i.i.d. Poisson variables with intensity λ.

Proof. Since $A_-(a_-), A_-(a_-+1), \dots$ are i.i.d. Poisson with intensity λ, it may be assumed that $A_-(a_- + p) = u_{p+1} - u_p$, $p \in \mathbb{N}_0$, where u is a Poisson process with intensity λ. Given such a redefinition, $a_0 = a_- + \tau - 1$ where $\tau = \lceil \min\{t : u_t > \beta_{\lfloor t \rfloor}\} \rceil$, $\beta_t = \sum_{\pi=t}^{b_-} B_-(\pi)$. If B_- was deterministic then τ would clearly be $\sigma(u_t)$-optional time hence $u_{\tau+\bullet}$ would be Poisson process with intensity λ so the assertion would clearly hold. Therefore and since $A_- \perp\!\!\!\perp B_-$, the assertion holds even given the true distribution of (A_-, B_-) by Hoffmann-Jørgensen [1994], 4.5.2.
$\square$

In accordance with reality, we shall assume that the orders, unsatisfied by the initial auction, "advance" to the continuous trading, i.e. that the initial (sell) order book of the continuous trading is $A_0(\bullet)$.

In case that $\theta = 0$, i.e. there is no initial auction, we shall assume the initial order books to consist each of a single (deterministic startup) order with limit prices a_-, b_- respectively.

5. The Tail Index of Price Increments

5.1. Tails of the i-th jump

In the present subsection, we shall deal with the right tail of a at the time of the i-th jump of ξ (the case of the left tail of b is symmetric and, because a and b never jump simultaneously, the increments of the midpoint price $p = (a+b)/2$ inherit the tail behavior of a and b).

In particular, we show the tails of a are thin given an initial auction but they are fat with the tail index one if no initial auction is held:

Theorem 2. *If $\lambda = 0$ then*

$$\lim_{p\to\infty} \frac{\mathbb{P}(a_{t_i} - a_0 > p)}{p^n} = 0, \qquad i \in \mathbb{N}, \qquad n \in \mathbb{N}.$$

If $\lambda > 0$ then

$$0 < \lim_{p\to\infty} \frac{\mathbb{P}(a_{t_i} - a_0 > p)}{p} < \infty, \qquad i \in \mathbb{N}.$$

Proof. First, let us note that each jump *up* of a has to happen at one of the times

$$t_1, t_2, \dots$$

where, for each $i \in \mathbb{N}$, t_i is either an arrival of a buy market order or a cancelation of the sell order with limit price $a_{t_i^-}$ having the lowest limit price in the underlying continuous model (see Šmíd [2008], Sec. 3 and Example 3) - clearly, $\Delta t_1, \Delta t_2, \dots$ are i.i.d. exponential with intensity $\eta + \iota$. Note also that each t_i, $i \in \mathbb{N}$, causes a jump of ξ.

It follows from the definition of the dynamics of Ξ that

$$a_{t_i} \le d_i, \qquad i \in \mathbb{N}, \tag{1}$$

where $d_0 = a_0$ and, for each $i \in \mathbb{N}$,

$$d_i = \min\{p \in \mathbb{Z}, p > \bar{a}_i : A_{t_i^-}(p) > 0\}, \qquad \bar{a}_i = \max\{a_0, a_{t_1}, \dots, a_{t_{i-1}}\}.$$

Let $i \in \mathbb{N}$. Denote $\Theta_i = (a_0, t_1, a_{t_1}, \dots, t_{i-1}, a_{t_{i-1}}, t_i)$. The following follows from the Markov property of Ξ and from Šmíd [2008], Proposition 9:

Auxiliary assertion 1

$$A_{t_i^-}(\bar{a}_i + p)|\Theta_i \sim \text{Poisson}\left(\Lambda_{t_i}\right), \qquad \Lambda_t = \frac{\kappa}{\iota} + e^{-\iota t}\left[\lambda - \frac{\kappa}{\iota}\right], \qquad p > 0, \tag{2}$$

and

$$A_{t_i^-}(\bar{a}_i + 1), A_{t_i^-}(\bar{a}_i + 2), \dots$$

are conditionally independent given Θ_i.

Proof of a. a. 1. *Note that $t_1, t_2, \dots$ may be determined from $\underline{\xi}$, where $\underline{\xi}$ is the process of the best quotes of the underlying model, i.e. $t_1, \dots, t_i$ are $\sigma(\underline{\xi}_{[0,\underline{\tau}_J]})$ measurable where $\underline{\tau}_i$ is the i-th jump of $\underline{\xi}$ and $J = \max\{\nu : \underline{\tau}_\nu \le t_i\}$. Applying the Local property [Kallenberg, 2002, Lemma 6.2] to the sets $S_j = [J = j]$, $j \in \mathbb{N}$, and using the fact that $\mathcal{L}\left(A_{t_i^-}(\bar{a}_i + \bullet)\,\middle|\,\underline{\xi}_{[0,\underline{\tau}_J]}\right)$ depends only on $t_1, \dots, t_i$, we get that $\underline{A}_{t_i^-}(\bar{a}_i + \bullet)|t_1, \dots, t_i$ is a Poisson process with intensity Λ_{t_i} (here, $\underline{A}$ is the sell order book of the underlying model, note also that $\bar{a}_i = \bar{a}_0^{t_i}$, see Šmíd [2008] for the definition of $\bar{a}_0^{\bullet}$). The distribution of $A_{t_i^-}(\bar{a}_i + \bullet)$ my be easily obtained by rounding (see Šmíd [2008], Example 3).* $\square$

Clearly, by (2),

$$\mathbb{P}(d_i - \bar{a}_i > p|\Theta_i) = \exp\left\{-\Lambda_{t_i} p\right\}, \qquad p \in \mathbb{N}_0.$$

Using the fact that $\mathbb{P}(d_i - \bar{a}_i \in \bullet|\Theta_i)$ does not depend on $a_0, a_{t_1}, \dots, a_{t_{i-1}}$ hence it coincides with $\mathbb{P}(d_i - \bar{a}_i \in \bullet|t_1, \dots, t_i)$, we are getting

$$\mathbb{P}(d_i - \bar{a}_i > p) = \int_0^\infty \dots \int_0^\infty \mathbb{P}(d_i - \bar{a}_i > p|s_1, \dots, s_i) d\mathbb{P}_{t_1}(s_1) \dots d\mathbb{P}_{t_i}(s_i)$$

$$= \int\limits_0^\infty \dots \int\limits_0^\infty \exp\left\{-\left(\frac{\kappa}{\iota} + e^{-\iota \sum_{j=1}^i s_j}\left[\lambda - \frac{\kappa}{\iota}\right]\right) p\right\} \prod_{j=1}^i \left[(\iota + \eta) e^{-(\iota+\eta)s_j}\right] ds_1 \dots ds_i$$

$$\stackrel{s_j = -\frac{\ln u_i}{\iota}}{=} \left(1 + \frac{\eta}{\iota}\right)^i \exp\left\{-\frac{\kappa}{\iota} p\right\} \int\limits_0^1 \dots \int\limits_0^1 \exp\left\{\left[\frac{\kappa}{\iota} - \lambda\right] p \prod_{j=1}^i u_j\right\} \prod_{j=1}^i u_j^{\kappa/\iota} du_1 \dots du_i$$

$$= \left(1 + \frac{\eta}{\iota}\right)^i \exp\left\{-\frac{\kappa}{\iota} p\right\} \sum_{\nu=0}^\infty \frac{\left(\left[\frac{\kappa}{\iota} - \lambda\right] p\right)^\nu}{(\nu + 1 + \kappa/\iota)^i \nu!}$$

(we got the last "=" by integrating the Taylor expansion of the integrand at the previous term). By an easy calculation we get that, for any $1 \le k \le i$,

$$l_k \le \frac{\nu + k}{\nu + 1 + \kappa/\iota} \le h_k, \qquad l_k = 1 \wedge \frac{k}{1 - \kappa/\iota} \qquad h_k = 1 \vee \frac{k}{1 - \kappa/\iota}$$

(indeed, for $1 - \kappa/\iota \le k$ it holds that $1 \le \frac{\nu+k}{\nu+1+\kappa/\iota} = 1 + \frac{k-(1+\kappa/\iota)}{\nu+1+\kappa/\iota} \le 1 + \frac{k-(1+\kappa/\iota)}{k} = \frac{k}{1+\kappa/\iota}$, similarly for $1 - \kappa/\iota > k$) hence

$$\sum_{\nu=0}^\infty \frac{\left(\left[\frac{\kappa}{\iota} - \lambda\right] p\right)^\nu}{(\nu + 1 + \kappa/\iota)^i \nu!} \quad \begin{Bmatrix} \ge & L_i \\ \le & H_i \end{Bmatrix} \cdot \sum_{\nu=0}^\infty \frac{\left(\left[\frac{\kappa}{\iota} - \lambda\right] p\right)^\nu}{(\nu + i)!}$$

$$= \begin{Bmatrix} L_i \\ H_i \end{Bmatrix} \cdot \frac{1}{\left[\frac{\kappa}{\iota} - \lambda\right]^i p^i} \left(\exp\left\{\left[\frac{\kappa}{\iota} - \lambda\right] p\right\} - \sum_{k=0}^{i-1} \frac{1}{k! \left[\frac{\kappa}{\iota} - \lambda\right]^k p^k}\right)$$

where

$$L_i = [\min\{l_1, l_2, \dots, l_i\}]^i, \qquad H_i = [\max\{h_1, h_2, \dots, h_i\}]^i$$

(we have used the formula for the Taylor expansion of the exponential at the last "=") implying that

$$L_i \le \frac{\mathbb{P}(d_i - \bar{a}_i > p)}{\exp\{\lambda p\} \frac{c_i}{p^i} - \exp\left\{\frac{\kappa}{\iota} p\right\} \sum_{k=0}^{i-1} \frac{c_k}{p^k}} \le H_i$$

for some positive $c_0, \dots, c_i$, further yielding

$$0 < \lim_{p \to \infty} \frac{\mathbb{P}(d_i - \bar{a}_i > p)}{\exp\{\lambda p\} p^i} < \infty.$$

Using (1) and the fact that

$$\mathbb{P}(d_i - a_0 > p) \le \mathbb{P}([\Delta d_1 > p/i] \cup [\Delta d_2 > p/i] \cup \dots \cup [\Delta d_i > p/i])$$

$$\le \sum_{j=1}^i i\mathbb{P}(\Delta d_j > p/i) \le \sum_{j=1}^i i\mathbb{P}(d_j - a_{t_{i-1}} > p/i),$$

we are getting

$$\lim_{p\to\infty} \frac{\mathbb{P}(d_i - a_0 > p)}{\exp\left\{\frac{\lambda}{i}p\right\}p} < \infty.$$

Moreover, since $\tau_i \leq t_i$, we have that $a_{\tau_i} \leq d_I$ for some $I \leq i$ hence it has to be $a_{\tau_i} \leq d_i$ and, consequently

$$\lim_{p\to\infty} \frac{\mathbb{P}(a_{\tau_i} - a_0 > p)}{\exp\left\{\frac{\lambda}{i}p\right\}p} < \infty$$

proving that the tail exponent of a_{τ_i} is ∞ if $\lambda > 0$ and it is at least one if $\lambda = 0$.

It remains to show that the exponent is exactly one if $\lambda = 0$.

Let $p > 0$ be a constant divisible by 2^{i-1}. Denote ϑ the minimum of

- the time of the first sell market order arrival
- the time of the cancelation of the initial sell order
- the time of the arrival of the first sell limit order with a limit price π fulfilling $b_0 < \pi \leq a_0$.

It is clear that

$$\vartheta_1 > t_1 \Rightarrow a_1 = d_1. \tag{3}$$

Further, define a random event

$$E = [\vartheta > t_i, \Delta d_1 > 2p, \Delta\theta_i - t_1 < \Delta t_2, \sigma > \Delta t_2, \zeta_2 = 1] \wedge E_2 \wedge \cdots \wedge E_i$$

where

$E_\nu = [\pi_\nu \in (a_0 + 2p - 2^{-\nu+2}p, a_0 + 2p - 2^{-\nu+1}p]]\,, \quad 2 \leq \nu \leq i,$

$\theta_1 = t_1,$

$\theta_2, \theta_3, \ldots$ are the times of the arrivals of (buy or sell) limit orders with limit prices belonging to $(a_0, a_0 + 2p)$

π_ν is the limit price of the order having arrived at θ_ν, $\nu > 1$,

$\zeta_2 = 1$ if and only if the order having arrived at θ_2 is the buy one

σ is the lifetime of the order having arrived at θ_2.

Before going on, let us prove the following auxiliary result:

Auxiliary assertion 2.

$$\mathbb{P}(a_{t_i} - a_0 > p) \geq \mathbb{P}(E).$$

Proof of a. a. 2. *Let $\omega \in E$ and agree to write X instead of $X(\omega)$ during the proof of a.a.1. The fact, being true on E, that $\vartheta > t_2$ guarantees, in addition to (3), that the only possible jumps of a from t_1 until t_2 are the arrivals of limit orders with limit prices π, $a_0 < \pi < a_{t_1}$ or cancelations of best quotes. Further, thanks to E_2, the* *limit price of the order coming at θ_2 (which is a buy one because $\zeta_2 = 1$) is greater than a_0+p; therefore and since this order becomes the new bid, we have $b_{\theta_2} > a_0+p$. Furthermore, since $\sigma > \Delta t_2$, the order having arrived at θ_2 stays at the market until t_2 which guarantees that $b_{\theta_\nu} \geq b_{\theta_2}$ with the consequence that $a_{t_\nu} > b_{\theta_2}$, $2 \leq \nu \leq i$. Moreover, since the only possible events changing a between $\theta_{\nu-1}$ and θ_ν, $2 \leq \nu \leq i$, are limit orders with limit prices greater than $a_0 + 2p$ or cancelations of the best ask, none of those events causing any jump of a below b_{θ_2}, we are getting $a_t > a_0 + p$, $t_1 \leq t \leq t_2$. Finally, since $b_{\theta_\nu^-} \leq \pi_\nu < a_{\theta_\nu^-}$ we may be sure that each θ_ν, $3 \leq \nu \leq i$, causes a jump of ξ hence $\tau_i \leq t_2$. Putting all this together, we get that $a_{\tau_i} - a_0 > p$ on E which immediately implies the auxiliary assertion.* $\square$

Let $T > 0$ be an arbitrarily chosen constant. It follows from the definition of E that

$$\mathbb{P}(E) \geq P(F_1 \cap F_2 \cap F_2' \cap E_2 \cap \cdots \cap E_i), \qquad F_1 = [t_1 < T, \Delta d_1 > 2p],$$
$$F_2 = [\vartheta > T + \Delta t_2, \Delta\theta_2 + \cdots + \Delta\theta_i < p^{-1}\Delta t_2, \sigma > \Delta t_2], \qquad F_2' = [\zeta_2 = 1].$$

It holds that

$$\begin{aligned}
\mathbb{P}(F_1) = \mathbb{P}(d_1 - a_0 > p, t_1 \leq T) &= \left(\left(1 + \frac{\kappa}{\iota}\right) \exp\left\{-\frac{\kappa}{\iota}p\right\}\right) \int_{e^{-\iota t}}^{1} \exp\left\{\frac{\kappa}{\iota}pu\right\} u^{\kappa/\iota} du \\
&= \left(\left(1 + \frac{\kappa}{\iota}\right) \exp\left\{-\frac{\kappa}{\iota}p\right\}\right) \sum_{\nu=0}^{\infty} \frac{\left(\frac{\kappa}{\iota}p\right)^\nu}{(\nu + 1 + \kappa/\iota)\nu!} (1 - \exp\{-(\iota\nu + \iota + \kappa)T\}) \\
&= \left(\left(1 + \frac{\kappa}{\iota}\right) \exp\left\{-\frac{\kappa}{\iota}p\right\}\right) \sum_{\nu=0}^{\infty} \frac{\left(\frac{\kappa}{\iota}p\right)^\nu - \exp\{-(\iota + \kappa)T\} \left(\exp\{-\iota T\}\frac{\kappa}{\iota}p\right)^\nu}{(\nu + 1 + \kappa/\iota)\nu!} \\
&\geq L\frac{1}{p} - H\frac{1}{p} \exp\left\{-e^{-\iota T}\frac{\kappa}{\iota}p\right\} \qquad (4)
\end{aligned}$$

where $H > 0$ and $L > 0$ are constants. Moreover the following holds true:

Auxiliary assertion 3. $F_1, F_2, F_2', E_2, \ldots, E_i$ are mutually independent.

Proof of the a. a. 3. *Let $\ldots, u_{-1}, u_0, u_1, \ldots,\ \ldots, v_{-1}, v_0, v_1, \ldots,$ be distinct Poisson mutually independent processes with intensity κ, all independent of the flow of market orders and of the orders' lifetimes. Clearly, the distribution of Ξ would not change if the arrivals of sell/buy limit orders with a limit price p greater/less then the best bid/ask were, in fact, jumps of u_p/v_p. Using this redefinition, we easily get that $\Delta t_1, \Delta t_2, \zeta_2, \vartheta, \sigma, \theta_1, \theta_2, \ldots$ are mutually independent - the independence we are proving easily follows.* $\square$

Since $\mathbb{P}(F_2') = 1/2$ and $\mathbb{P}(E_\nu) = 2^{-\nu+1}$, $2 < \nu$, and thanks to the fact that $\mathbb{P}(F_2)$ does not depend on p (to see it, multiply the second inequality defining F_2 by p and note that $p\Delta\theta_\nu \sim \mathrm{Exp}\,(4\kappa)$, $2 \leq \nu \leq i$), we are finally getting, with a help of a. a. 3 and (4), the existence of a constant $\zeta > 0$ such that

$$\lim_{p\to\infty} \frac{\mathbb{P}(a_{t_i} - a_0 > p)}{p} \geq \lim_{p\to\infty} \left(L - H \exp\left\{-e^{-\iota T}\frac{\kappa}{\iota}p\right\}\right) \pi = L\pi > 0, \qquad i \in \mathbb{N}.$$

i.e. the Theorem has been completely proved.

$\square$

5.2. Tails at a Fixed Horizon

To confirm our findings under "changed conditions", let us examine, in addition to the "tick time", the price increment at a fixed horizon $T > 0$.

Theorem 3. *If $\lambda > 0$ then*

$$\lim_{p\to\infty} \frac{\mathbb{P}(a_T - a_0 > p)}{p^n} = 0,$$

for each $n \in \mathbb{N}$. If $\lambda = 0$ then

$$\lim_{p\to\infty} \frac{\mathbb{P}(a_T - a_0 > p)}{p} > 0.$$

Proof. Assume $\lambda > 0$ first. Clearly,

$$a_{t_i} \leq \delta_i, \qquad i \in \mathbb{N}, \tag{5}$$

where $\delta_0 = a_0$ and,

$$\delta_i = \min\{p \in \mathbb{Z}, p > \delta_{i-1} : A_{t_i^-}(p) > 0\}, \qquad i \in \mathbb{N},$$

It is easy to show that

Auxiliary assertion 4

$$\mathbb{P}(\Delta\delta_i > p|\delta_1, \delta_2, \dots \delta_{i-1}, t_1, t_2, \dots, t_{i+1}) = \exp\{-\Lambda_{t_i} p\}. \tag{6}$$

Proof of a. a. 4. *Let $\nu \in \mathbb{Z}$. The distribution of Ξ won't change if $A_t(\pi) = v_t^\pi$, $0 < t < t_i$, $\pi > \nu$, where $\dots, v^{-1}, v^0, v^1, \dots$ are independent immigration and death processes with immigration rate κ and death rate ι starting at $A_0(\pi)$, and if $V = (v_{\nu+1}, v_{\nu+2}, \dots)$ are independent on the remaining order flows (represented, for instance, by Poisson processes of arrivals and by the orders' lifetimes).* □

On the set $S_\nu = [\delta_{i-1} = \nu]$, clearly

$$\delta_j = D_j, \quad i < j, \qquad \Delta\delta_i = d,$$

where $D_j = \delta_j \wedge \nu$, $j < i$, and $d = \min\{p > \nu : v_{t_i^-}(\nu + p) > 0\}$. If all the order flows except V were deterministic, then also $D_1, \dots, D_{i-1}, t_1, t_2, \dots, t_{i+1}$ would be deterministic and it would easily hold that

$$\mathbb{P}(d > p|D_1, D_2, \dots D_{i-1}, t_1, t_2, \dots, t_{i+1}) = \exp\{-\Lambda_{t_i} p\} \tag{7}$$

(see the informal proof in Sec. 2.3. of Šmíd [2008]); however, since V is independent of the remaining flows, (7) holds even given the true distribution of the underlying elements (by Šmíd [2008], Lemma 1 (i)). Further, since $\mathbf{1}_{S_\nu}$ may be clearly determined from both the conditioning random elements in (6) and (7), (6) holds on each S_ν, $\nu \in \mathbb{Z}$, by the Local Property [Kallenberg, 2002, Lemma 6.2]. Finally, since $S_\nu, \nu \in \mathbb{Z}$ clearly exhaust all the possibilities, (6) is proved.

Thanks to the a.a. and since, for each $t \geq 0$,

$$\Lambda_t \geq \gamma, \qquad \gamma = \lambda \wedge \frac{\kappa}{\iota},$$

it follows that

$$\mathbb{P}(\delta_i - a_0 > p | t_1, t_2, \ldots, t_{i+1}) \leq \int_p^\infty \epsilon_i(p) dp$$

where

$$\epsilon_i(p) = \frac{\gamma^i p^{i-1} e^{-\gamma p}}{(i-1)!}$$

is the p.d.f. of the Erlang distribution with parameters i and γ (to see it, note, that i.i.d. exponential variables $s_1, s_2, \ldots$ with intensity γ may be constructed (by transformations of $\Delta\delta_i$ by the superpositions of their conditional c.d.f.'s and the quantiles of $\mathrm{Exp}\,(\gamma)$) such that $\Delta\delta_\nu \leq s_\nu$, $\nu \in \mathbb{N}$, and recall that the distribution of a sum of i.i.d. exponential variables is Erlang).

Because $I = \max\{i : t_i \leq T\}$ is Poisson with intensity $\eta + \iota$ and since I is $\sigma(t_1, t_2, \ldots, t_{i+1})$-measurable, we have, for any $p > 1$,

$$\mathbb{P}(\delta_I - a_0 > p) = \sum_{i=1}^\infty \mathbb{P}(\delta_i - a_0 > p | I = i) \mathbb{P}(I = i)$$

$$= \sum_{i=1}^\infty \mathbb{P}(\delta_i - a_0 > p | t_1, t_2, \ldots, t_{i+1}) \mathbb{P}(I = i) \leq \int_p^\infty e^{-\eta-\iota-\gamma z} \sum_{i=1}^\infty \frac{(\eta+\iota)^i \gamma^i z^{i-1}}{i!(i-1)!} dz$$

$$\leq \int_p^\infty e^{-\eta-\iota-\gamma z} \left(\sum_{i=1}^{2\lceil \eta+\iota \rceil} \tilde{c}_i z^i + \tilde{C} \sum_{i=0}^\infty \frac{(\gamma/2)^i z^i}{i!} \right) dz$$

$$\leq \sum_{i=1}^{2\lceil \eta+\iota \rceil} \int_p^\infty e^{-\eta-\iota-\gamma z} \tilde{c}_i z^i dz + \tilde{C} \int_p^\infty e^{-\eta-\iota-(\gamma/2) z} dz$$

for some positive $\tilde{C}, \tilde{c}_1, \ldots, \tilde{c}_{2\lceil \eta+\iota \rceil}$ - since all the summands at the r.h.s. vanish at the exponential rate, we are getting that the tails of $a_T - a_0$ are thin.

Finally, assume $\lambda = 0$. Since $a_T - a_0 > p$ if $\Delta d_1 > 2p$, $t_1 \leq T$, $t_2 \geq T$, $\vartheta \geq T$ and $\theta_2 \geq T$ we get, similarly to the previous subsection, that

$$\mathbb{P}(a_T > p) \geq \left(\frac{\tilde{L}}{p} - \frac{\tilde{H}}{\exp\{\tilde{b}p\} p} \right) \beta$$

for some $\tilde{H} > 0, \tilde{L} > 0, \tilde{b} > 0, \beta > 0$ independent of p (see also (4)) i.e.

$$\lim_{p \to \infty} \frac{\mathbb{P}(a_T > p)}{p} > 0$$

which is what we wanted to prove.

□

6. Conclusion

We studied the behavior of the price increments of in the model by Smith et al. [2003]. We found the tail exponent to be one if the initial order book is empty but infinity if an initial call auction is held at the start of the trading.

Concluding the paper, let us stress that we are not in any contradiction with papers finding tail exponents greater then one at this or similar models (e.g. Slanina [2001]) because, contrary to them, we do not study stationary distributions; since, by our computations, the weights of tails of order one decrease with the increasing time, our results in fact support the hypothesis that the tails of stationary price increments are lighter than one. In this light, our "fat-tailed" result does not seem to be any revolutionary one. What appears quite surprising is the demonstrated stabilizing role of the initial call auction.

References

J. Hoffmann-Jørgensen. *Probability with a View Towards to Statistics I.* Chapman and Hall, New York, 1994.

O. Kallenberg. *Foundations of Modern Probability.* Springer, New York, second edition, 2002.

F. Slanina. Mean-field approximation for a limit order driven market. *Phys. Rev.*, 64:056136, 2001. Preprint at `http://xxx.sissa.it/abs/cond-mat/0104547`.

M. Šmíd. Probabilistic properties of the continuous double auction, 2008. Submitted. Preprint at SSRN (ID=1107114) or at `http://martinsmid.naokraj.cz/mat/probprop.pdf`.

E. Smith, J. D. Farmer, L. Gillemot, and S. Krishnamurthy. Statistical theory of the continuous double auction. *Quantitative finance*, 3(6):481–514, 2003. `http://www.santafe.edu/~baes/jdf/papers/doubleauctionfinal.pdf`.