Available online at www.sciencedirect.com
ScienceDirect
Fuzzy Sets and Systems 160 (2009) 816–831
FUZZY sets and systems
www.elsevier.com/locate/fss

Parametric characterization of aggregation functions

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Available online 14 August 2008

Abstract

The aim of this contribution is to find a common framework for parametric characterization of aggregation functions exploiting the notions of unipolar and bipolar parametric characterization, and also unify the ideas of global and local parametric characterization. We revise the known approaches to classification of aggregation functions in special classes in this framework, and also propose some new parameters in classes of averaging, conjunctive and disjunctive aggregation functions.

Keywords: Aggregation function; Conjunction measure; Disjunction measure; Idempotency measure; Global/local parametric characterization

1. Introduction

The choice of an appropriate function for aggregating input data in practical applications can be a complicated task. Usually, the possible choice is constrained by the character of the discussed problem. For example, for modeling conjunctive problems (proposals of conjunctors in many-valued logics, intersections of fuzzy sets, building joint distributions from marginal ones, etc.) some special types of conjunctors are required, such as triangular norms, copulas, etc. Expected properties of an aggregation function [3,15], as, for example, anonymity (symmetry), unanimity (idempotency), conjunctive, disjunctive or averaging behavior, etc., determine the framework we are working in.

In order to choose an appropriate aggregation function, we should be able to measure the degree of the required properties or their defect. In that case one assigns to aggregation functions some values—parameters—characterizing the properties of aggregation functions. The process of assigning parameters to aggregation functions will be called parametric characterization of aggregation functions. Note that characterization of an aggregation function A by means of some value $v(A)$ can philosophically have two meanings. Either $v(A)$ is given a priori, and then it is a parameter of A (e.g., we estimate $v(A)$ from real data and then fit A to given data to preserve the computed estimation $v(A)$), or $v(A)$ is computed from some formula exploiting the knowledge of A. Both approaches are frequently exploited, e.g., in mathematical statistics when dealing with expected value or variance of random variables, and then these characterizing values are called parameters. Therefore we also adopt this terminology and throughout the paper we

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doi:10.1016/j.fss.2008.08.002

will discuss the parameters of aggregation functions, though sometimes it would be more appropriate to call them descriptors (or characteristics) of aggregation functions.

If an aggregation function A possesses the property $\mathcal{P}$, the defect of $\mathcal{P}$ should be $d_{\mathcal{P}}(A) = 0$, and if the property $\mathcal{P}$ is completely missing we will ask $d_{\mathcal{P}}(A) = 1$, compare [2]. For example, the defect of the idempotency of an aggregation function A (i.e., the property $A(x, \ldots, x) = x$ for each $x \in [0, 1]$) from a given class of aggregation functions can be expressed by the formula

$$d_{\mathcal{I}}(A) = k \sup_{x\in[0,1]} |x - A(x, \ldots, x)|,$$

where k is an appropriate normalizing constant.

The degree of a property $\mathcal{P}$ for an aggregation function A is often determined as the defect of the "dual" property $\mathcal{P}^*$. For example, the degree of asymmetry $m_{\mathcal{A}s}(A)$ of an aggregation function A can be expressed as its defect of symmetry. E.g., for binary aggregation functions we can introduce the degree of asymmetry by the formula

$$m_{\mathcal{A}s}(A) = k \sup_{(x,y)\in[0,1]^2} |A(x, y) - A(y, x)|,$$

where k is again an appropriate normalizing constant depending on the class of aggregation functions we are working with. Recall that for the class $\mathcal{C}_{(2)}$ of all binary copulas the constant k has the value $k = 3$ because $\sup_{C\in\mathcal{C}_{(2)}} \sup_{(x,y)\in[0,1]^2} |C(x, y) - C(y, x)| = \frac{1}{3}$, see [14]. Evidently, in the case of all binary aggregation functions it holds $k = 1$. Observe that the constant k has the value $k = 1$ also in the class of all (binary) conjunctors, though as a consequence of partial symmetry forced by the neutral element $e = 1$ there is no conjunctor K satisfying $m_{\mathcal{A}s}(K) = 1$.

In the above mentioned situations the degree or the defect of a property $\mathcal{P}$ was introduced as the satisfaction degree expressing to what extent an aggregation function satisfies the definition of the considered property.

If a class $\mathcal{A}$ of aggregation functions contains an extremal element E (with respect to the ordering of aggregation functions) possessing a property $\mathcal{P}$, then this extremal element can be chosen as the prototype, and all other members of the class can be compared to it. The degree of the property $\mathcal{P}$ of the members of the class can be measured by means of a *unipolar measure of similarity*, i.e., a function $\mu : \mathcal{A} \to [0, 1]$, such that $\mu(E) = 1$ (0 is attained by the other extremal element of $\mathcal{A}$ if such an element exists) and with the property $|E - A| \leqslant |E - B| \Rightarrow \mu(A) \geqslant \mu(B)$ for all $A,\ B \in \mathcal{A}$. The values of μ express similarity between aggregation functions of the considered class and the prototype with respect to property $\mathcal{P}$. An example of a unipolar measure of similarity (in the class of averaging aggregation functions) is, e.g., the measure of the degree of orness [27] with prototype *Max*, or, in the dual case, the measure of the degree of andness with prototype *Min*.

In classes of aggregation functions with the greatest and smallest elements $\overline{A}$ and $\underline{A}$, respectively, which possess a property $\mathcal{P}$ in two dual forms (e.g., increasing and decreasing functional dependence) and where exists an element O representing total absence of that property (e.g., independence), a function measuring the difference of aggregation functions from the element O (a central element of the class) can be defined. The range of such function is the interval $[-1, 1]$; for the central element O its value is zero, the values -1 and 1 are usually attained for the extremal elements of the class only. The functions of this type will be called *bipolar measures of dissimilarity*. A non-trivial example of a bipolar measure of dissimilarity (with respect to the product copula Π as the central element) is, e.g., the Spearman rho, a well-known measure of association introduced in statistics [23].

Although in special classes of aggregation functions, for example, in the class of OWA operators, root-power operators, triangular norms, copulas, etc., certain parameters expressing the degree of an investigated property were already introduced, a systematic approach to the parametric characterization of aggregation functions is missing. The aim of this contribution is to find a common framework for parametric characterization of aggregation functions, revise the known approaches to classification of special types of aggregation functions in this framework, and propose some new parameters in classes of conjunctive, disjunctive and averaging aggregation functions or their subclasses. Briefly, this paper brings the state-of-art overview of parametric characterization of aggregation functions.

The paper is organized as follows. In the next section we review several known approaches to the classification of aggregation functions in special classes of aggregation functions. In Section 3 we will deal with unipolar parametric characterization. Main attention will be paid to the classes of averaging, conjunctive and disjunctive aggregation functions, we introduce two types of idempotency measures for conjunctive (disjunctive) aggregation functions. In Section 4 a new approach to the global and local parametric characterization of aggregation functions is presented, as well as its generalization—a mixed approach. In Section 5 we briefly discuss bipolar parametric characterization of aggregation functions. Finally, some concluding remarks are given.

2. A review of some known approaches

As mentioned above, in special classes of aggregation functions certain parameters expressing the degree of some property $\mathcal{P}$ have already been introduced. For example, for OWA operators Yager [27] defined the measure of the degree of orness/andness, expressing the possibility of an OWA operator to stand as an operator for disjunction/conjunction. Recall that an n-ary OWA operator ($n \geqslant 2$) is the function $M'_{\mathbf{w}} : [0,1]^n \to [0,1]$ given by

$$M'_{\mathbf{w}}(x_1, \dots, x_n) = \sum_{i=1}^{n} w_i\, x'_i,$$

where $\mathbf{w} = (w_1, \dots, w_n)$ is a weighting vector with $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$, and $(x'_1, \dots, x'_n)$ is a non-decreasing permutation of the input n-tuple $(x_1, \dots, x_n)$. The measure of the degree of orness of $M'_{\mathbf{w}}$ was defined by

$$m_{\mathrm{or}}(M'_{\mathbf{w}}) = \sum_{i=1}^{n} \frac{i-1}{n-1}\, w_i. \tag{1}$$

Aggregation functions *Min* and *Max* are OWA operators with weighting vectors $\mathbf{w}_{Min} = (1, 0, \dots, 0)$, $\mathbf{w}_{Max} = (0, \dots, 0, 1)$. Evidently, $m_{\mathrm{or}}(Min) = 0$, $m_{\mathrm{or}}(Max) = 1$, and for any other OWA operator $m_{\mathrm{or}}(M'_{\mathbf{w}}) \in]0,1[$. Note that to a given value $m_{\mathrm{or}}(M'_{\mathbf{w}}) \in]0,1[$ an OWA operator can be assigned uniquely only for $n = 2$. For distinguishing n-ary OWA operators with $n \geqslant 3$ an additional parameter, the entropy, was introduced. Usually, for a given degree of orness, an operator with maximal entropy is chosen [27]. The measure of the degree of andness is a dual notion, and it holds $m_{\mathrm{and}}(M'_{\mathbf{w}}) = 1 - m_{\mathrm{or}}(M'_{\mathbf{w}})$.

Another distinguished class of aggregation functions is the class of root-power operators $(A_p)_{p \in [-\infty,\infty]}$. For a fixed $n \in \mathbb{N}$, and $p \in]-\infty, 0[\cup]0, \infty[$, the n-ary root-power operator $A_p : [0,1]^n \to [0,1]$ is defined by

$$A_p(x_1, \dots, x_n) = \left(\frac{1}{n} \sum_{i=1}^{n} x_i^p \right)^{1/p},$$

and the limit operators are $A_0 = G$ (the geometric mean), $A_{-\infty} = Min$ and $A_{\infty} = Max$. Evidently, A_1 is the arithmetic mean M.

For measuring the degree of disjunctive/conjunctive behavior of these operators Dujmović [5] proposed the concepts of local orness/andness and mean local orness/andness. E.g., the mean local orness of A_p was defined by

$$\int_{[0,1]^n} \frac{A_p(\mathbf{x}) - Min(\mathbf{x})}{Max(\mathbf{x}) - Min(\mathbf{x})}\, \mathrm{d}\mathbf{x},$$

and this value was later again studied as the orness average value in the class of all averaging (mean) operators by Salido and Murakami [24], see also [8,20–22].

In [6] Dujmović characterized root-power operators by the *mean value*, defined for an n-ary operator A_p by

$$m(A_p) = \int_{[0,1]^n} A_p(x_1, \dots, x_n)\, \mathrm{d}x_1 \dots \mathrm{d}x_n, \tag{2}$$

and he explicitly showed how to compute some of the values $m(A_p)$. For example, $m(A_1) = \frac{1}{2}$, $m(A_{-\infty}) = 1/(n+1)$, $m(A_\infty) = n/(n+1)$ and $m(A_0) = (n/(n+1))^n$. The family $(A_p)_{p \in [-\infty,\infty]}$ is increasing and continuous with respect to the parameter p, and for any fixed $n \in \mathbb{N}$ and $\alpha \in [1/(n+1), n/(n+1)]$, there is a unique parameter $p \in [-\infty, \infty]$ such that $m(A_p) = \alpha$. Therefore, the mean value for root-power operators is a strong parameter. Based on the mean value, Dujmović [7] introduced the global orness/andness, initially called the disjunction/conjunction degree. For example, the global orness was defined by

$$\omega_g(A_p) = \frac{m(A_p) - m(Min)}{m(Max) - m(Min)} = \frac{(n+1)m(A_p) - 1}{n-1}. \tag{3}$$

Another known parametric characterization concerns triangular norms (t-norms for short). For the definition of a t-norm and basic properties of t-norms we refer, e.g., to [13]. In [29] Yager et al. studied possible generalizations of "and" operator for conjunction in fuzzy logics. They proposed to classify t-norms with respect to their value at the "fuzziest" point (0.5, 0.5), i.e., with respect to the value $T(0.5, 0.5)$, and to compare them to the t-norm T_M, $T_M(x, y) = \min\{x, y\}$. It is clear that the normed value $\tau(T) = T(0.5, 0.5)/T_M(0.5, 0.5) = 2\,T(0.5, 0.5)$ is equal to 1 not only for $T = T_M$, but also for all t-norms T for which the point 0.5 is the idempotent element. And, on the other hand, the value $\tau(T)$ is equal to zero for all t-norms with $T(0.5, 0.5) = 0$, as, for example, for the Łukasiewicz t-norm T_L or the drastic product T_D, thus $\tau(T)$ is a weak parameter. This method of comparing t-norms is not enough sensitive; t-norms with the same value $\tau(T)$ can be essentially different. However, in parametrized families of strict t-norms, for example, in the Hamacher, Frank, Aczél–Alsina or Dombi families, parameters $\tau(T)$ distinguish all single members of the family [29].

3. Unipolar parametric characterization of aggregation functions

Consider an n-ary aggregation function A, $n \geqslant 2$, i.e., a function $A : [0, 1]^n \to [0, 1]$ with the properties

(A1) $A(0, \ldots, 0) = 0$, $A(1, \ldots, 1) = 1$;
(A2) $A(x_1, \ldots, x_n) \leqslant A(y_1, \ldots, y_n)$ for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in [0, 1]^n$ such that $x_i \leqslant y_i$, $i \in \{1, \ldots, n\}$.

From the monotonicity of A and boundary conditions (A1) it follows that the *mean value* $m(A)$,

$$m(A) = \int_{[0,1]^n} A(x_1, \ldots, x_n)\, \mathrm{d}x_1 \ldots \mathrm{d}x_n, \tag{4}$$

can be introduced for each Borel measurable n-ary aggregation function A. Since now, we will assume that each discussed aggregation function is Borel measurable (and we will not mention this fact explicitly), though there exist peculiar non-measurable aggregation functions, see, e.g., [12].

The integral in (4) is computed over the whole domain, thus the value $m(A)$ is a kind of a global parameter assigned to A. For the weakest and strongest aggregation functions A_w and A_s, respectively, which are defined by

$$A_w(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} = (1, \ldots, 1), \\ 0 & \text{otherwise}, \end{cases} \qquad A_s(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} = (0, \ldots, 0), \\ 1 & \text{otherwise}, \end{cases}$$

we have $m(A_w) = 0$ and $m(A_s) = 1$. Obviously, $m(A) = 0$ if, and only if, $A = A_w$ a.e., and $m(A) = 1$ if, and only if, $A = A_s$ a.e.. Fixing the dimension n, for each weighted arithmetic mean $M_\mathbf{w}$ it holds $m(M_\mathbf{w}) = \frac{1}{2}$, further, $m(Min) = 1/(n+1)$, $m(Max) = n/(n+1)$, the mean value for the drastic product T_D (the weakest t-norm) is $m(T_D) = 0$, and for the drastic sum S_D (the strongest t-conorm) $m(S_D) = 1$. Moreover, for the product t-norm T_P and the Łukasiewicz t-norm T_L it holds $m(T_P) = 1/2^n$, $m(T_L) = 1/(n+1)!$

Note that we will respect the notations of functions usual in special classes of aggregation functions. For example, the function whose value at any point $(x, y) \in [0, 1]^2$ is $\min\{x, y\}$, is denoted by T_M if it is considered as a t-norm, and by *Min* if it is considered in a more general framework as an aggregation function. Similarly, the notation of the function defined by $\max\{x, y\}$ is either S_M (t-conorm) or *Max* (aggregation function), and the notation of the function defined by $\max\{x + y - 1, 0\}$ is either T_L (the Łukasiewicz t-norm) or W (copula). Moreover, we will not indicate the arity of arguments under considerations explicitly up to special cases when we stress the actual value of n. Though the usual notation for associative functions is related to their binary form, we will use the same notation also for their n-ary forms.

Formula (3) can be further generalized. In each subclass $\mathcal{A}$ of n-ary aggregation functions with the smallest and greatest elements $\underline{A}$ and $\overline{A}$, respectively, one can introduce a global parameter as the *normalized mean value* by

$$\widetilde{m}(A) = \frac{m(A) - m(\underline{A})}{m(\overline{A}) - m(\underline{A})} \tag{5}$$

or by

$$\widetilde{m}^*(A) = \frac{m(\overline{A}) - m(A)}{m(\overline{A}) - m(\underline{A})}. \tag{6}$$

Note that if in $\mathcal{A}$ there is only one extremal element then the previous formulae can be modified by using inf or sup of the set $\{m(B)|B \in \mathcal{A}\}$ instead of $m(\underline{A})$ and $m(\overline{A})$, respectively.

It holds $\widetilde{m}(\overline{A}) = 1$, $\widetilde{m}(\underline{A}) = 0$, and conversely for $\widetilde{m}^*$. The functions $\widetilde{m}, \widetilde{m}^* : \mathcal{A} \to [0,1]$ defined by (5) and (6) are unipolar measures of similarity, mentioned in Introduction, with prototypes $\overline{A}$ and $\underline{A}$, respectively. The values $\widetilde{m}(A)$ and $\widetilde{m}^*(A)$ are global parameters assigned to A. Evidently, they are complementary, $\widetilde{m}(A)+\widetilde{m}^*(A) = 1$. If A^{d} is the standard dual aggregation function to $A \in \mathcal{A}$, i.e., $A^{\mathrm{d}}(x_1,\dots,x_n) = 1 - A(1-x_1,\dots,1-x_n)$ for all $(x_1,\dots,x_n) \in [0,1]^n$, then $m(A^{\mathrm{d}}) = 1 - m(A)$, and if $\mathcal{A}$ is closed under duality, then it holds

$$\widetilde{m}(A^{\mathrm{d}}) = 1 - \widetilde{m}(A) = \widetilde{m}^*(A).$$

Clearly, if $A^{\mathrm{d}} = A$, that is, if A is a self-dual function (symmetric sum [25]), then necessarily $m(A) = \widetilde{m}(A) = \widetilde{m}^*(A) = 0.5$.

In the next section, we will discuss the normalized mean value of aggregation functions given by (5) in classes $\mathcal{A}_{\mathrm{av}}, \mathcal{A}_{\mathrm{c}}, \mathcal{A}_{\mathrm{d}}$, i.e., in classes of averaging, conjunctive and disjunctive aggregation functions, respectively, and give the interpretation of the obtained global parameters in each case.

3.1. Unipolar parametric characterization in the class of averaging aggregation functions

In the class $\mathcal{A}_{\mathrm{av}}$ of *averaging aggregation functions*, i.e., aggregation functions characterized by the property $Min \leqslant A \leqslant Max$, formula (5) leads to

$$\widetilde{m}_{\mathrm{av}}(A) = \frac{m(A) - m(Min)}{m(Max) - m(Min)} = \frac{(n+1)\,m(A) - 1}{n-1}. \tag{7}$$

This characterization of averaging aggregation functions provides the comparison of $A \in \mathcal{A}_{\mathrm{av}}$ to the aggregation function *Max*, and expresses the degree of similarity between A and *Max*, the basic operator for disjunction. In the class $\mathcal{A}_{\mathrm{av}}$ the function $\widetilde{m}_{\mathrm{av}} : \mathcal{A}_{\mathrm{av}} \to [0,1]$ is a unipolar measure of similarity with prototype *Max*. It will be called the *global disjunction measure* and in what follows denoted by *GDM*, i.e.,

$$GDM(A) = \frac{(n+1)\int_{[0,1]^n} A(x_1,\dots,x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_n - 1}{n-1}, \quad A \in \mathcal{A}_{\mathrm{av}}, \tag{8}$$

compare with [7,19,24].

Remark 1. Though formulae (3) and (7) are essentially the same, and in the case of OWA operators (7) reduces to (1), formulae (1) and (3) were introduced for special classes of idempotent aggregation functions only. Marichal [18,19] has adopted Dujmović's formula to the Choquet integrals. In general, formula (7) for averaging aggregation functions was first considered by Salido and Murakami [24].

Note that the complementary function $\widetilde{m}^*_{\mathrm{av}}$ in the class $\mathcal{A}_{\mathrm{av}}$ is a unipolar measure of similarity with prototype *Min*. It will be called the *global conjunction measure* and denoted by *GCM*,

$$GCM(A) = \frac{n - (n+1)\int_{[0,1]^n} A(x_1,\dots,x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_n}{n-1}, \quad A \in \mathcal{A}_{\mathrm{av}}.$$

Again, for OWA operators it holds $GCM(A) = m_{\mathrm{and}}(A)$.

Example 1. An interesting family of averaging aggregation functions is formed by k-medians [4,9,10], for $k \in [0,1]$. The k-medians as n-ary aggregation functions are defined by

$$Med_k(x_1,\dots,x_n) = \mathrm{med}(x_1,k,x_2,k,\dots,k,x_n).$$

The global disjunctive measure of n-ary k-median has the value

$$GDM(Med_k) = \frac{(1-k)^{n+1} - k^{n+1} + (n+1)k - 1}{n-1}.$$

To compute the mean value $m(Med_k)$ recall that

$$Med_k(\mathbf{x}) = \begin{cases} Max(\mathbf{x}) & \text{if } \mathbf{x} \in D_1, \\ Min(\mathbf{x}) & \text{if } \mathbf{x} \in D_2, \\ k & \text{if } \mathbf{x} \in D_3, \end{cases}$$

where $D_1 = [0, k]^n$, $D_2 = [k, 1]^n \setminus \{(k, \dots, k)\}$ and $D_3 = [0, 1]^n \setminus (D_1 \cup D_2)$. Then

$$\int_{D_1} Med_k(\mathbf{x})\,\mathrm{d}\mathbf{x} = k^{n+1}\frac{n}{n+1}, \quad \int_{D_2} Med_k(\mathbf{x})\,\mathrm{d}\mathbf{x} = k(1-k)^n + (1-k)^{n+1}\frac{1}{n+1},$$

$$\int_{D_3} Med_k(\mathbf{x})\,\mathrm{d}\mathbf{x} = k(1-k^n-(1-k)^n),$$

thus

$$m(Med_k) = k^{n+1}\frac{n}{n+1} + (1-k)^{n+1}\frac{1}{n+1} + k(1-k^n).$$

Consequently,

$$GDM(Med_k) = \frac{nk^{n+1} + (1-k)^{n+1} + (n+1)k(1-k^n) - 1}{n-1} = \frac{(1-k)^{n+1} - k^{n+1} + (n+1)k - 1}{n-1},$$

which is the mentioned formula. For example, for $n = 2$ we obtain

$$GDM(Med_k) = k^2(3-2k).$$

Note that $Med_0 = Min$, $Med_1 = Max$ and that $Med_{0.5}$ is a symmetric sum, see [25].

Observe that weighted means are symmetric sums and hence the global disjunctive (conjunctive) measure of any weighted mean is equal to 0.5.

3.2. Unipolar parametric characterization in the class of conjunctive (disjunctive) aggregation functions

A natural requirement often put on an aggregation function is *idempotency*, i.e., the property

$$A(x, \dots, x) = x \quad \text{for all } x \in [0, 1].$$

For aggregation functions the idempotency of A is equivalent to the property $Min \leqslant A \leqslant Max$, which means that all averaging aggregation functions are idempotent. In the class $\mathcal{A}_c$ of all *conjunctive aggregation functions*, i.e., aggregation functions bounded from above by Min, the only idempotent function is Min. Similarly, in the class $\mathcal{A}_d$ of all *disjunctive aggregation functions*, i.e., aggregation functions bounded from below by Max, the only idempotent function is just Max. To obtain the degree of idempotency, one can compare conjunctive (disjunctive) aggregation functions to Min (Max).

In the class $\mathcal{A}_c$ the smallest element is the function A_w, the greatest Min. Since $m(Min) = 1/(n+1)$ and $m(A_w) = 0$, the normalized mean value of a conjunctive aggregation function A is given by

$$\widetilde{m}_c(A) = \frac{m(A) - m(A_w)}{m(Min) - m(A_w)} = (n+1)\,m(A). \tag{9}$$

It holds $\widetilde{m}_c(A) = 1$ if, and only if, $A = Min$, i.e., a conjunctive aggregation function A is idempotent if, and only if, the value $\widetilde{m}_c(A) = 1$. The number $\widetilde{m}_c(A)$ expresses the degree of similarity between a conjunctive aggregation function A and Min, and can be interpreted as the degree of the idempotency of A. Therefore we define the *global idempotency measure of a conjunctive aggregation function A*, with notation $GIM_c(A)$, by

$$GIM_c(A) = (n+1)\int_{[0,1]^n} A(x_1, \dots, x_n)\,\mathrm{d}x_1 \dots \mathrm{d}x_n, \quad A \in \mathcal{A}_c. \tag{10}$$

In fuzzy logics, and consequently, in fuzzy set theory, important conjunctive aggregation functions are t-norms. In the class $\mathcal{T}$ of all t-norms (with extremal elements T_M and T_D) formula (5) also leads to the global idempotency measure given by (10). For t-norms this parameter has already been introduced and studied in [17], compare also [26,28]. This measure is not injective, but in each distinguished parametric family of t-norms discussed in [13] distinguishes all single members of the family.

Example 2. (i) Let $n = 2$. It can be shown that the global idempotency measure of the product t-norm T_P is $GIM_c(T_P) = 0.75$, for the Łukasiewicz t-norm T_L we obtain $GIM_c(T_L) = 0.5$ and for the nilpotent minimum T^{nM} we have $GIM_c(T^{nM}) = 0.75$.

(ii) For the Sugeno–Weber family of t-norms $(T_\lambda^{SW})_{\lambda\in[-1,\infty]}$, where

$$T_\lambda^{SW}(x,y) = \begin{cases} \max\left\{0, \dfrac{x+y-1+\lambda xy}{1+\lambda}\right\} & \text{for } \lambda \in]-1,\infty[, \\ T_D & \text{for } \lambda = -1, \\ T_P & \text{for } \lambda = \infty, \end{cases}$$

we have

$$GIM_c(T_\lambda^{SW}) = \frac{3}{4} - \frac{3}{4\lambda} - \frac{3}{2\lambda^2} + \frac{3\lambda+3}{\lambda^2}\log(1+\lambda) \quad \text{for } \lambda \in]-1,0[\,\cup\,]0,\infty[$$

and

$$GIM_c(T_{-1}^{SW}) = 0, \quad GIM_c(T_0^{SW}) = 0.5, \quad GIM_c(T_\infty^{SW}) = 0.75.$$

For ordinal sums of t-norms [13] we have the following result.

Proposition 1. *Let $n = 2$ and let $T = (\langle a_k, b_k, T_k\rangle)_{k\in\mathcal{K}}$ be an ordinal sum of* t-*norms. Then*

$$GIM_c(T) = 1 - \sum_{k\in\mathcal{K}} (b_k - a_k)^3 (1 - GIM_c(T_k)).$$

Proof. Recall that if $T = (\langle a_k, b_k, T_k\rangle)_{k\in\mathcal{K}}$, then

$$T(x,y) = \begin{cases} a_k + (b_k - a_k) T_k\left(\dfrac{x-a_k}{b_k-a_k}, \dfrac{y-a_k}{b_k-a_k}\right) & \text{if } (x,y) \in [a_k, b_k[^2, \\ T_M(x,y) & \text{else,} \end{cases}$$

and thus

$$\begin{aligned} m(T) &= m(T_M) - \sum_{k\in\mathcal{K}} \int_{[a_k,b_k[^2} (b_k - a_k)\left(T_M\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right) - T_k\left(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}\right)\right) \mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{3} - \sum_{k\in\mathcal{K}} (b_k - a_k)^3 \int_{[0,1[^2} (T_M(u,v) - T_k(u,v))\,\mathrm{d}u\,\mathrm{d}v = \frac{1}{3} - \sum_{k\in\mathcal{K}} (b_k - a_k)^3 \left(\frac{1}{3} - m(T_k)\right). \end{aligned}$$

Taking into account that in binary case $GIM_c(T) = 3m(T)$, we obtain the result. □

Since idempotency is in fact the property of an aggregation function A on the diagonal of the unit cube $[0,1]^n$, it has sense to define the *diagonal idempotency measure of a conjunctive aggregation function* $A \in \mathcal{A}_c$ by

$$DIM_c(A) = 2\int_{[0,1]} \delta_A(x)\,\mathrm{d}x,$$

where $\delta_A(x) = A(x, \ldots, x)$. This formula can naturally be derived from (5), where instead of the mean value m on $[0, 1]^n$ the mean value Δ on the diagonal is considered, i.e., $\Delta(A) = \int_{[0,1]} A(x, \ldots, x)\,dx$, $\Delta(Min) = \frac{1}{2}$ and $\Delta(A_w) = 0$. Clearly, $DIM_c(Min) = 1$, $DIM_c(A_w) = 0$. The function DIM_c is a unipolar measure of similarity with Min as the prototype.

Example 3. For $n = 2$ the values of the diagonal idempotency measure of t-norms T_D, T_L, T^{nM} and T_P are

$$DIM_c(T_D) = 0, \quad DIM_c(T_L) = 0.5, \quad DIM_c(T^{nM}) = 0.75 \quad \text{but } DIM_c(T_P) = \tfrac{2}{3},$$

compare with the previous example.

The coefficient $\tau(T)$ introduced for t-norms by Yager et al. [29] can also be explained as a measure of similarity with prototype T_M. Computing the mean values in (5) with respect to a one point set, we obtain

$$\tau(T) = \frac{T(0.5, 0.5) - T_D(0.5, 0.5)}{T_M(0.5, 0.5) - T_D(0.5, 0.5)} = 2T(0.5, 0.5).$$

The *global idempotency measure of a disjunctive aggregation function* $A \in \mathcal{A}_d$ can be defined directly following (6) by

$$GIM_d(A) = \frac{m(A_s) - m(A)}{m(A_s) - m(Max)} = (n+1)(1 - m(A)) = (n+1)\left(1 - \int_{[0,1]^n} A(\mathbf{x})\,d\mathbf{x}\right),$$

or, as the global idempotency measure of the corresponding dual conjunctive aggregation function, $GIM_d(A) = GIM_c(A^d)$.

4. General approach to unipolar parametric characterization of aggregation functions

4.1. Global and local parametric characterization

A common property of unipolar measures of similarity is that they measure the degree of some property of aggregation functions by comparing with a prototype. Therefore a genuine requirement for a unipolar measure of similarity μ is its monotonicity. Let us consider the case that for all $A, B \in \mathcal{A}$, $A \leqslant B \Rightarrow \mu(A) \leqslant \mu(B)$. Similarly, in convex classes of aggregation functions a genuine requirement for all convex combinations of aggregation functions is the validity of the property

$$\mu\left(\sum_i \lambda_i A_i\right) = \sum_i \lambda_i \mu(A_i).$$

Two solutions with expected properties are the global and local parametric characterizations given for n-ary aggregation functions by

$$\mu_G(A) = f\left(\int_{[0,1]^n} A(\mathbf{x})\,dP(\mathbf{x})\right), \tag{11}$$

$$\mu_L(A) = \int_{[0,1]^n} f_{\mathbf{x}}(A(\mathbf{x}))\,dP(\mathbf{x}), \tag{12}$$

respectively, where P is a probability measure on the Borel subsets of $[0, 1]^n$ and $f : \mathbb{R} \to \mathbb{R}$ is an increasing function with the property

$$f\left(\sum_i \lambda_i x_i\right) = \sum_i \lambda_i f(x_i)$$

for all convex combinations of arguments. By [1] the only functions of this property are those of the form $f(u) = b+cu$, where $b \in \mathbb{R}$, $c > 0$. The same holds for functions $f_{\mathbf{x}}$, see below.

- First, let us consider a global parametric characterization μ_{G} of aggregation functions from the class $\mathcal{A}$,

$$\mu_{\mathrm{G}}(A) = b + c \int_{[0,1]^n} A(\mathbf{x})\, \mathrm{d}P(\mathbf{x}), \tag{13}$$

with P distinguishing the smallest and the greatest member of $\mathcal{A}$.

Proposition 2. *Let $\mathcal{A}$ be a convex class of aggregation functions with the smallest and greatest elements $\underline{A}$ and $\overline{A}$, respectively. Let P be a probability measure on $\mathcal{B}([0,1]^n)$ such that $\int_{[0,1]^n} \underline{A}(\mathbf{x})\, \mathrm{d}P(\mathbf{x}) \neq \int_{[0,1]^n} \overline{A}(\mathbf{x})\, \mathrm{d}P(\mathbf{x})$. Then the global parametric characterization $\mu_{\mathrm{G}} : \mathcal{A} \to [0,1]$ introduced by (13) is given by*

$$\mu_{\mathrm{G}}(A) = \frac{\int_{[0,1]^n} A(\mathbf{x})\, \mathrm{d}P(\mathbf{x}) - \int_{[0,1]^n} \underline{A}(\mathbf{x})\, \mathrm{d}P(\mathbf{x})}{\int_{[0,1]^n} \overline{A}(\mathbf{x})\, \mathrm{d}P(\mathbf{x}) - \int_{[0,1]^n} \underline{A}(\mathbf{x})\, \mathrm{d}P(\mathbf{x})}. \tag{14}$$

Proof. To determine the constants b and c in (13), it is enough to solve the system of equations $\mu_{\mathrm{G}}(\overline{A}) = 1$ and $\mu_{\mathrm{G}}(\underline{A}) = 0$. The result follows immediately. □

According to the chosen probability measure P on $\mathcal{B}([0,1]^n)$ and the considered class of aggregation functions we obtain various types of measures of similarity. Note that each parametric characterization introduced in a convex class $\mathcal{A}$ of aggregation functions with the smallest and greatest elements $\underline{A}$ and $\overline{A}$, respectively, can be restricted to any subclass $\mathcal{B} \subset \mathcal{A}$ possessing the same extremal elements (possibly as limit members, in the sense a.e.). We give several examples.

(i) Let $\mathcal{A} = \mathcal{A}_{\mathrm{av}}$ and let P be the probability measure uniformly distributed on $[0,1]^n$. From (14), taking into account

$$\int_{[0,1]^n} \mathit{Max}(\mathbf{x})\, \mathrm{d}\mathbf{x} = \frac{n}{n+1}, \quad \int_{[0,1]^n} \mathit{Min}(\mathbf{x})\, \mathrm{d}\mathbf{x} = \frac{1}{n+1},$$

we obtain

$$\mu_{\mathrm{G}}(A) = \frac{(n+1)\int_{[0,1]^n} A(\mathbf{x})\, \mathrm{d}\mathbf{x} - 1}{n-1}, \quad A \in \mathcal{A}_{\mathrm{av}}, \tag{15}$$

which coincides with the formula for the global disjunction measure in the class $\mathcal{A}_{\mathrm{av}}$, i.e., $\mu_{\mathrm{G}} = \mathit{GDM}$.

(ii) Let $\mathcal{A} = \mathcal{A}_{\mathrm{c}}$ and let P be the probability measure uniformly distributed over the diagonal of the unit cube $[0,1]^n$. Since $\int_{[0,1]} \mathit{Min}(x, \ldots, x)\, \mathrm{d}x = \frac{1}{2}$ and $\int_{[0,1]} A_{\mathrm{w}}(x, \ldots, x)\, \mathrm{d}x = 0$, formula (14) results in

$$\mu_{\mathrm{G}}(A) = 2\int_{[0,1]} A(x, \ldots, x)\, \mathrm{d}x, \quad A \in \mathcal{A}_{\mathrm{c}},$$

which is the formula for the diagonal idempotency measure $\mathit{DIM}_{\mathrm{c}}$ for conjunctive aggregation functions introduced in Section 3.

(iii) Finally, let P be the Dirac measure distributed over the singleton $\{(\frac{1}{2}, \frac{1}{2})\}$. Then in the class $\mathcal{T}$ of all t-norms (with extremal elements T_{M} and T_{D}) formula (14) gives

$$\mu_{\mathrm{G}}(T) = 2T(\tfrac{1}{2}, \tfrac{1}{2}),$$

i.e., the values $\mu_{\mathrm{G}}(T)$ are just parameters $\tau(T)$ introduced for t-norms by Yager et al. [29]. Observe that $\mathcal{T}$ is not a convex class, but is a subclass of the convex class of all conjunctors.

- Now, let us consider a local parametric characterization given by (12), where $f_{\mathbf{x}}(u) = b_{\mathbf{x}} + c_{\mathbf{x}}u$ with local constants $b_{\mathbf{x}} \in \mathbb{R}$, $c_{\mathbf{x}} > 0$ corresponding to the points $\mathbf{x} \in [0,1]^n$. In a class $\mathcal{A}$ of aggregation functions with the greatest element

$\overline{A}$ as the prototype and the smallest element $\underline{A}$, a pointwise fitting gives the following equations for computing $b_{\mathbf{x}}, c_{\mathbf{x}}$:

$$b_{\mathbf{x}} + c_{\mathbf{x}}\,\underline{A}(\mathbf{x}) = 0, \quad b_{\mathbf{x}} + c_{\mathbf{x}}\,\overline{A}(\mathbf{x}) = 1,$$

leading to

$$b_{\mathbf{x}} = -\frac{\underline{A}(\mathbf{x})}{\overline{A}(\mathbf{x}) - \underline{A}(\mathbf{x})}, \quad c_{\mathbf{x}} = \frac{1}{\overline{A}(\mathbf{x}) - \underline{A}(\mathbf{x})}$$

for all points $\mathbf{x} \in [0,1]^n$ for which $\overline{A}(\mathbf{x}) \neq \underline{A}(\mathbf{x})$. In that case $f_{\mathbf{x}}(A(\mathbf{x})) = (A(\mathbf{x}) - \underline{A}(\mathbf{x}))/(\overline{A}(\mathbf{x}) - \underline{A}(\mathbf{x}))$. Note that if for an $\mathbf{x} \in [0,1]^n$, $\overline{A}(\mathbf{x}) = \underline{A}(\mathbf{x})$ then $A(\mathbf{x}) = \overline{A}(\mathbf{x}) = \underline{A}(\mathbf{x})$ for all $A \in \mathcal{A}$. Applying the convention $\frac{0}{0} = 0$ and supposing that $\underline{A} \neq \overline{A}$ P-a.e., we will write

$$\mu_{\mathrm{L}}(A) = \int_{[0,1]^n} \frac{A(\mathbf{x}) - \underline{A}(\mathbf{x})}{\overline{A}(\mathbf{x}) - \underline{A}(\mathbf{x})}\,\mathrm{d}P(\mathbf{x}), \quad A \in \mathcal{A}. \tag{16}$$

If $\mathcal{A} = \mathcal{A}_{\mathrm{av}}$ and P is the probability measure uniformly distributed on $\mathcal{B}([0,1]^n)$, the right-hand side of the previous formula is of the form

$$\int_{[0,1]^n} \frac{A(\mathbf{x}) - Min(\mathbf{x})}{Max(\mathbf{x}) - Min(\mathbf{x})}\,\mathrm{d}\mathbf{x}, \tag{17}$$

and defines the *mean local disjunction measure* of A, in notation $LDM(A)$. This value coincides with mean local orness of Dujmović [5] and orness average value studied by Salido and Murakami [24].

In [24] the authors proved that for all OWA operators $GDM(A) = LDM(A)$. Marichal [22] extended this result for any discrete Choquet integral. As the following example shows the class of aggregation functions fulfilling this property is certainly larger.

Example 4. Let $e \in]0,1[$. Consider the function $U_e : [0,1]^2 \to [0,1]$ given by

$$U_e(x,y) = \begin{cases} \min\{x,y\} & \text{if } y \leqslant f_e(x), \\ \max\{x,y\} & \text{if } y > f_e(x), \end{cases}$$

where

$$f_e(x) = \begin{cases} 1 - \dfrac{1-e}{e}x & \text{if } x \in [0,e], \\ \dfrac{e}{1-e}(1-x) & \text{if } x \in]e,1]. \end{cases}$$

The function f_e is piecewise linear, its graph links the points $(0,1)$, (e,e) and $(1,0)$. For each $e \in]0,1[$ the function U_e is a non-trivial conjunctive uninorm. It is only a matter of computation to show that the mean value

$$m(U_e) = \int_{[0,1]^2} U_e(x,y)\,\mathrm{d}x\,\mathrm{d}y = \frac{2-e}{3},$$

and consequently, the global disjunction measure of U_e is

$$GDM(U_e) = 3m(U_e) - 1 = 1 - e.$$

On the other hand, since for all $(x,y) \in [0,1]^2$, $x \neq y$,

$$\frac{U_e(x,y) - Min(x,y)}{Max(x,y) - Min(x,y)} = \begin{cases} 0 & \text{if } x \in [0,1],\, y \leqslant f_e(x), \\ 1 & \text{if } x \in [0,1],\, y > f_e(x), \end{cases}$$

the mean local disjunction measure of U_e computed by formula (17) is $LDM(U_e) = 1 - e$, i.e.,

$$GDM(U_e) = LDM(U_e).$$

Example 5. Let $a \in [0,1]$ and let $A_a : [0,1]^2 \to [0,1]$ be defined by

$$A_a(x,y) = \begin{cases} \min\{x,y\} & \text{if } x \leqslant a, \\ \max\{x,y\} & \text{if } x > a. \end{cases}$$

The mean value of the function A_a is

$$m(A_a) = \int_{[0,1]^2} A_a(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^a \mathrm{d}x \int_0^x y\,\mathrm{d}y + \int_0^a \mathrm{d}x \int_x^1 x\,\mathrm{d}y + \int_a^1 \mathrm{d}x \int_0^x x\,\mathrm{d}y + \int_a^1 \mathrm{d}x \int_x^1 y\,\mathrm{d}y$$
$$= \frac{1}{6}(-2\,a^3 + 3\,a^2 - 3\,a + 4),$$

and the global disjunction measure is

$$GDM(A_a) = 3\,m(A_a) - 1 = -a^3 + \tfrac{3}{2}\,a^2 - \tfrac{3}{2}\,a + 1.$$

As for all $(x,y) \in [0,1]^2$, $x \neq y$,

$$\frac{A_a(x,y) - Min(x,y)}{Max(x,y) - Min(x,y)} = \begin{cases} 0 & \text{if } x \leqslant a, \\ 1 & \text{if } x > a, \end{cases}$$

the mean local disjunction measure of A_a is $LDM(A_a) = 1 - a$. It holds

$$GDM(A_a) \;=\; LDM(A_a) \;\Leftrightarrow\; -a^3 + \tfrac{3}{2}\,a^2 - \tfrac{3}{2}\,a + 1 = 1 - a \;\Leftrightarrow\; a(-a^2 - \tfrac{3}{2}\,a - \tfrac{1}{2}) = 0$$
$$\Leftrightarrow\; a \in \{0, \tfrac{1}{2}, 1\}.$$

Note that for $a = 0$ we have $A_0 = Max$ and $GDM(A_0) = LDM(A_0) = 1$, and for $a = 1$ we have $A_1 = Min$, with $GDM(A_1) = LDM(A_1) = 0$.

This example shows that the piecewise linearity of aggregation functions (Example 4) is not a sufficient condition for the equality of GDM and LDM. The question for which types of averaging aggregation functions GDM and LDM coincide is still open.

If we consider the class $\mathcal{A}_c$ of all conjunctive aggregation functions and the probability uniformly distributed over $[0,1]^n$, the local parametric characterization (16) gives the value

$$\int_{[0,1]^n} \frac{A(\mathbf{x})}{Min(\mathbf{x})}\,\mathrm{d}\mathbf{x}, \quad A \in \mathcal{A}_c, \tag{18}$$

which expresses the mean local idempotency measure of a conjunctive aggregation function A. It will be denoted by $LIM_c(A)$, i.e., $LIM_c(A) = \int_{[0,1]^n} A(\mathbf{x})/(Min(\mathbf{x}))\,\mathrm{d}\mathbf{x}$. The parameter $LIM_c(A)$ coincides with the idempotency average value of a conjunctive aggregation function A introduced by Marichal, see [22]. For example, by (18) for the product t-norm T_P (binary form) we have

$$LIM_c(T_P) = \int_0^1 \mathrm{d}x \int_0^x x\,\mathrm{d}y + \int_0^1 \mathrm{d}x \int_x^1 y\,\mathrm{d}y = \frac{2}{3}.$$

As mentioned above, $GIM_c(T_P) = \frac{3}{4}$ and $DIM_c(T_P) = \frac{2}{3}$. In general, for n-ary form of the product t-norm it holds $GIM_c(T_P) = (n+1)/2^n$, $DIM_c(T_P) = 2/(n+1)$ and by [22], $LIM_c(T_P) = 2^{n-1}/\binom{2n-1}{n}$.

4.2. A mixed approach

The concepts of local and global characterizations can be unified into a more general mixed approach assigning to n-ary aggregation functions parameters defined by

$$\mu(A) = f\left(\int_{[0,1]^n} f_{\mathbf{x}}(A(\mathbf{x}))\,\mathrm{d}P(\mathbf{x})\right), \tag{19}$$

with the same general requirements on f, $f_{\mathbf{x}}$ and P as in the previous part. Evidently, if f is the identity function, μ coincides with μ_L and μ_G is a special case of μ for $f_{\mathbf{x}} = id$ (independently of $\mathbf{x}$).

Next, using the linear form of functions f and $f_{\mathbf{x}}$ we obtain

$$\mu(A) = b + c\int_{[0,1]^n} (b_{\mathbf{x}} + c_{\mathbf{x}}A(\mathbf{x}))\,\mathrm{d}P(\mathbf{x}) = b + c\int_{[0,1]^n} b_{\mathbf{x}}\,\mathrm{d}P(\mathbf{x}) + c\int_{[0,1]^n} c_{\mathbf{x}}A(\mathbf{x})\,\mathrm{d}P(\mathbf{x})$$
$$= k + c\int_{[0,1]^n} c_{\mathbf{x}}A(\mathbf{x})\,\mathrm{d}P(\mathbf{x}).$$

The requirements $\mu(\underline{A}) = 0$ and $\mu(\overline{A}) = 1$, lead to

$$c = \frac{1}{\int_{[0,1]^n} c_{\mathbf{x}}\overline{A}(\mathbf{x})\,\mathrm{d}P(\mathbf{x}) - \int_{[0,1]^n} c_{\mathbf{x}}\underline{A}(\mathbf{x})\,\mathrm{d}P(\mathbf{x})} \quad \text{and} \quad k = -c\int_{[0,1]^n} c_{\mathbf{x}}\underline{A}(\mathbf{x})\,\mathrm{d}P(\mathbf{x}),$$

thus

$$\mu(A) = \frac{\int_{[0,1]^n} c_{\mathbf{x}}A(\mathbf{x})\,\mathrm{d}P(\mathbf{x}) - \int_{[0,1]^n} c_{\mathbf{x}}\underline{A}(\mathbf{x})\,\mathrm{d}P(\mathbf{x})}{\int_{[0,1]^n} c_{\mathbf{x}}\overline{A}(\mathbf{x})\,\mathrm{d}P(\mathbf{x}) - \int_{[0,1]^n} c_{\mathbf{x}}\underline{A}(\mathbf{x})\,\mathrm{d}P(\mathbf{x})}. \tag{20}$$

For example, in the class of averaging aggregation functions the previous formula is of the form

$$\mu(A) = \frac{\int_{[0,1]^n} c_{\mathbf{x}}A(\mathbf{x})\,\mathrm{d}P(\mathbf{x}) - \int_{[0,1]^n} c_{\mathbf{x}}Min(\mathbf{x})\,\mathrm{d}P(\mathbf{x})}{\int_{[0,1]^n} c_{\mathbf{x}}Max(\mathbf{x})\,\mathrm{d}P(\mathbf{x}) - \int_{[0,1]^n} c_{\mathbf{x}}Min(\mathbf{x})\,\mathrm{d}P(\mathbf{x})}. \tag{21}$$

Not to prefer any of the coordinates, we will always suppose that $c_{\mathbf{x}}$ is symmetric.

Remark 2. (i) Formula (20) can be written in the form

$$\mu(A) = \frac{1}{M([0,1]^n)}\int_{[0,1]^n} \frac{A(\mathbf{x}) - \underline{A}(\mathbf{x})}{\overline{A}(\mathbf{x}) - \underline{A}(\mathbf{x})}\,\mathrm{d}M(\mathbf{x}),$$

where M is a σ-additive measure with Radon–Nikodym derivative $\mathrm{d}M/\mathrm{d}P(\mathbf{x}) = c_{\mathbf{x}}(\overline{A}(\mathbf{x}) - \underline{A}(\mathbf{x}))$. However, then $M/(M([0,1]^n))$ is a probability measure on the Borel subsets of $[0,1]^n$ (supposing the finiteness of M), i.e., formulae (20) and (16) coincide. However, formula (20) can be easier to compute and therefore we will deal with it. Moreover, the above comments lead to a new fact: the global parametric characterization μ_{G} given by (14) is a special case of the mean local parametric characterization given by (16). For example, for $n = 2$, formula (15) (or (7)) can be written in the form (16), with P described by a joint distribution function (restricted to $[0,1]^2$) $F : [0,1]^2 \to [0,1]$ given by

$$F(x, y) = \frac{Min(x, y)}{2}((x - y)^2 + (x + y)Min(x, y)).$$

(ii) An important advantage of all introduced parametric characterizations of aggregation functions is their convexity. For example, the Choquet integral based on a given fuzzy measure v can always be expressed as a convex combination of lattice polynomials (i.e., Choquet integrals with respect to $\{0, 1\}$-valued fuzzy measures), and thus in such a case it is enough to compute the considered parametric characterization for lattice polynomials only.

Example 6. In the class of averaging aggregation functions let us introduce a moment parametric characterization $\mu^{(p)}$, $p \in]-1, \infty[$, based on the weighting function $c_{\mathbf{x}}^{(p)} = (\prod_{i=1}^n x_i)^p$ and on the uniform probability P on Borel subsets of $[0,1]^n$ by

$$\mu^{(p)}(A) = \frac{\int_{[0,1]^n}(\prod_{i=1}^n x_i)^p A(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_{[0,1]^n}(\prod_{i=1}^n x_i)^p Min(\mathbf{x})\,\mathrm{d}\mathbf{x}}{\int_{[0,1]^n}(\prod_{i=1}^n x_i)^p Max(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_{[0,1]^n}(\prod_{i=1}^n x_i)^p Min(\mathbf{x})\,\mathrm{d}\mathbf{x}}. \tag{22}$$

Then, for example, for $n = 3$ we have

$$\int_{[0,1]^3} x^p y^p z^p \min\{x, y, z\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 6\int_0^1\int_0^x\int_0^y x^p y^p z^{p+1}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{6}{(p+2)(2p+3)(3p+4)},$$

$$\int_{[0,1]^3} x^p y^p z^p \max\{x, y, z\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 6\int_0^1\int_0^x\int_0^y x^{p+1} y^p z^{p}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{6}{(p+1)(2p+2)(3p+4)},$$

$$\int_{[0,1]^3} x^p y^p z^p \mathrm{med}\{x, y, z\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 6\int_0^1\int_0^x\int_0^y x^p y^{p+1} z^{p}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{6}{(p+1)(2p+3)(3p+4)},$$

and substituting these values into (22) we obtain the parametric characterization of median,

$$\mu^{(p)}(Med) = \frac{2p+2}{3p+4}.$$

Therefore, for ternary OWA operators which are convex combination of *Min*, *Max* and *Med*, whose parameters are $\mu^{(p)}(Min) = 0$, $\mu^{(p)}(Max) = 1$ and $\mu^{(p)}(Med) = (2p+2)/(3p+4)$, it holds

$$\mu^{(p)}(OWA) = w_2\frac{2p+2}{3p+4} + w_3.$$

For the arithmetic mean M it can easily be computed that

$$\int_0^1\int_0^1\int_0^1 x^p y^p z^p \frac{x+y+z}{3}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{1}{(p+1)^2(p+2)}$$

and next, by (22),

$$\mu^{(p)}(M) = \frac{5p+6}{9p+12}.$$

Observe that M can be seen as an OWA operator with weighting vector $\mathbf{w} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, thus

$$\mu^{(p)}(M) = \frac{1}{3}\frac{2p+2}{3p+4} + \frac{1}{3} = \frac{5p+6}{9p+12},$$

which confirms the previous result.

For the geometric mean G it holds

$$\int_0^1\int_0^1\int_0^1 x^p y^p z^p \sqrt[3]{xyz}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{27}{(3p+4)^3},$$

thus by (22)

$$\mu^{(p)}(G) = \frac{(p+1)^2(15p+22)}{(3p+4)^3}.$$

Note that for $p = 1$ it holds

$$\mu^{(1)}(OWA) = \tfrac{4}{7}w_2 + w_3, \quad \mu^{(1)}(M) = \tfrac{11}{21}, \quad \mu^{(1)}(G) = \tfrac{148}{343},$$

and the limit values are

$$\lim_{p\to -1}\mu^{(p)}(OWA) = w_3, \quad \lim_{p\to -1}\mu^{(p)}(M) = \frac{1}{3}, \quad \lim_{p\to -1}\mu^{(p)}(G) = 0,$$

and

$$\lim_{p\to \infty}\mu^{(p)}(OWA) = \frac{2}{3}w_2 + w_3, \quad \lim_{p\to \infty}\mu^{(p)}(M) = \frac{5}{9}, \quad \lim_{p\to \infty}\mu^{(p)}(G) = \frac{5}{9}.$$

Example 7. For $n = 2$, there are four lattice polynomials: two projections, *Min* and *Max*. For the moment parametric characterization $\mu^{(p)}$, $p \in]-1, \infty[$, we have

$$\int_{[0,1]^2} x^p y^p Min(x,y)\,dx\,dy = \frac{2}{(p+2)(2p+3)}, \quad \int_{[0,1]^2} x^p y^p Max(x,y)\,dx\,dy = \frac{2}{(p+1)(2p+3)},$$

$$\int_{[0,1]^2} x^p y^p P_F(x,y)\,dx\,dy = \int_{[0,1]^2} x^p y^p P_L(x,y)\,dx\,dy = \frac{1}{(p+1)(p+2)}.$$

Therefore $\mu^{(p)}(Min) = 0$, $\mu^{(p)}(Max) = 1$, $\mu^{(p)}(P_F) = \mu^{(p)}(P_L) = \frac{1}{2}$.

Let the Choquet integral based aggregation function $Ch_{a,b} : [0,1]^2 \to [0,1]$ be determined by the corner points values a, b, $Ch_{a,b}(1,0) = a$, $Ch_{a,b}(0,1) = b$ (for more details concerning the Choquet integral based aggregation functions we recommend [11]). Then the convex representation of $Ch_{a,b}$ by means of lattice polynomials is

$$Ch_{a,b}(x,y) = \begin{cases} aP_F(x,y) + bP_L(x,y) + (1-a-b)Min(x,y) & \text{if } a+b \leqslant 1, \\ (1-b)P_F(x,y) + (1-a)P_L(x,y) + (a+b-1)Max(x,y) & \text{otherwise,} \end{cases}$$

and thus $\mu^{(p)}(Ch_{a,b}) = (a+b)/2$, independently of p.

5. Bipolar parametric characterization of aggregation functions

Let $\mathcal{A}$ be a class of aggregation functions with extremal elements $\overline{A}$, $\underline{A}$ and let $\mathcal{A}$ be closed under some kind of duality preserving convex sums, exchanging elements $\overline{A}$ and $\underline{A}$. Then instead of the measures of similarity $\widetilde{m}$ and $\widetilde{m}^*$ with prototypes $\overline{A}$ and $\underline{A}$, respectively, and ranges [0, 1], we can define a function v measuring the difference of aggregation functions from any self-dual element O, which is taken as a central element. As a central element one can take, e.g., the element $O = \frac{1}{2}(\underline{A} + \overline{A})$. Note that then its mean value is $m(O) = \frac{1}{2}(m(\underline{A}) + m(\overline{A}))$.

For each $A \in \mathcal{A}$ we put

$$v(A) = 2\frac{m(A) - m(O)}{m(\overline{A}) - m(\underline{A})}, \tag{23}$$

which is the same as $v(A) = \widetilde{m}(A) - \widetilde{m}^*(A) = 2\,\widetilde{m}(A) - 1$.

The range of v is the interval $[-1, 1]$, $v(\overline{A}) = 1$, $v(\underline{A}) = -1$ and $v(O) = 0$. In this way we can "catch" two dual properties $\mathcal{P}$ and $\mathcal{P}^*$ by one parameter [16]. The function v will be called a *bipolar measure of dissimilarity.*

If we consider the class of all n-ary aggregation functions, with $\overline{A} = A_s$, $\underline{A} = A_w$ and the standard duality, then from (23) we obtain

$$v(A) = 2m(A) - 1.$$

It holds $v(A) = -1$ if, and only if, $A = A_w$ a.e., $v(A) = 1$ if, and only if, $A = A_s$ a.e., and $v(O) = 0$. In general, $v(A^d) = -v(A)$. For example, v attains the next values for distinguished binary aggregation functions:

$$v(T_L) = -\tfrac{2}{3}, \quad v(T_P) = -\tfrac{1}{2}, \quad v(Min) = -\tfrac{1}{3}, \quad v(G) = -\tfrac{1}{9}, \quad v(Max) = \tfrac{1}{3}, \quad v(S_L) = \tfrac{2}{3}.$$

In a special subclass $\mathcal{A}_{av}$ of averaging aggregation functions, where $\underline{A} = Min$ and $\overline{A} = Max$, if we consider the standard duality of aggregation functions and use as the central element O the arithmetic mean M, we obtain the bipolar measure of dissimilarity given by

$$v_{av} : \mathcal{A}_{av} \to [-1, 1], \quad v_{av}(A) = \frac{(n+1)}{n-1}(2m(A) - 1). \tag{24}$$

The property $\mathcal{P}$ is "A has disjunctive behavior", $\mathcal{P}^*$: "A has conjunctive behavior", therefore v_{av} can be interpreted as the conjunction–disjunction measure. Note that from a mathematical point of view, any weighted mean could be taken as the central element of $\mathcal{A}_{av}$. Evidently, $v_{av}(Min) = -1$, $v_{av}(Max) = 1$ and for the arithmetic mean M it holds $v_{av}(M) = 0$. For example, for the geometric mean G the value is $v_{av}(G) = ((n+1)/(n-1))(2(n/(n+1))^n - 1) < 0$, which confirms that G is more similar to *Min* than to *Max*.

Note that for the class $\mathcal{A}_{av}$ the concept of bipolar measure of dissimilarity coincides with the concept of symmetric global orness/andness for GCD functions (generalized conjunctions/disjunctions) introduced independently by Dujmović [8].

A non-trivial example of a bipolar measure of dissimilarity is, e.g., the Spearman rho, a well-known measure of association introduced in statistics. It is based on the notion of concordance, and for a pair (X, Y) of random variables it was defined as the difference of the probability of concordance and the probability of discordance, see, e.g., [23]. However, for continuous random variables X and Y linked by a copula C the parameter $\rho_{X,Y} = \rho_C$ can be computed [23] by the formula

$$\rho_C = 12\iint_{[0,1]^2} C(x, y)\,\mathrm{d}x\,\mathrm{d}y - 3 = 12\iint_{[0,1]^2} (C(x, y) - x\,y)\,\mathrm{d}x\,\mathrm{d}y.$$

The last formula can be interpreted as a measure of "average distance" between the copula C representing the dependence structure of the joint distribution of X and Y and the copula Π representing independence. The range of Spearman's rho is $[-1, 1]$; for the product copula Π we have $\rho_\Pi = 0$, $\rho_C = 1$ iff $C = Min$ and $\rho_C = -1$ iff $C = W$ (recall that formally $W = T_{\mathrm{L}}$). Note that ρ_C can be seen as a coefficient of functional dependence between random variables: $\rho_{X,Y} = 1$ if and only if Y is an increasing transformation of X, a.e., and $\rho_{X,Y} = -1$ if and only if Y is a decreasing transformation of X, a.e. If X and Y are independent then $\rho_{X,Y} = 0$.

In the class of copulas Min and W are extremal elements, for each copula C it holds $W \leqslant C \leqslant Min$. The mapping $\widehat{\ }: \mathcal{C}_{(2)} \to \mathcal{C}_{(2)}$, defined by $\widehat{C}(x, y) = x - C(x, 1 - y)$ is a duality in $\mathcal{C}_{(2)}$, exchanging $Min \leftrightarrow W$ and preserving convex sums. The copula Π is a self-dual element under this duality. Note that if $C = \widehat{C}$ then

$$m(C) = \int_0^1 \int_0^1 (x - C(x, 1 - y))\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2} - m(C),$$

hence $m(C) = \frac{1}{4}$ for any self-dual copula C.
As for binary functions $m(Min) = \frac{1}{3}$, $m(W) = 1/3! = \frac{1}{6}$, and $m(\Pi) = \frac{1}{4}$, in the class $\mathcal{C}_{(2)}$ formula (23) leads to

$$v_{\mathrm{cop}}(C) = 2\frac{m(C) - \frac{1}{4}}{\frac{1}{3} - \frac{1}{6}} = 12m(C) - 3,$$

that is, $v_{\mathrm{cop}}(C) = \rho_C$, so the mapping $C \mapsto \rho_C$ is a kind of bipolar measure of dissimilarity. Note that the latest formula can also be applied in the class of all binary quasi-copulas without additional changes.

Another bipolar parametric characterization (again with central element Π) in the class of copulas can be obtained if the mean value is computed with respect to the measure concentrated at the singleton $\{(\frac{1}{2}, \frac{1}{2})\}$. Substituting $\Pi(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$, $Min(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$ and $W(\frac{1}{2}, \frac{1}{2}) = 0$ into (23), we obtain the value $\beta_C = 4C(\frac{1}{2}, \frac{1}{2}) - 1$, which is known in statistics as Blomqvist's beta [23].

6. Conclusion

We have discussed a framework for parametric characterization of aggregation functions from distinguished classes of aggregation functions. As special cases of our general approach, several well-known parametric characterizations were recovered, such as the orness/andness measure in the class of averaging aggregation functions, or Spearman's rho and Blomqvist's beta in the class of copulas. Moreover, new types of parametric characterizations were introduced. We expect that these parametric characterizations will be of use in fitting aggregation functions to real data in several areas, especially in multicriteria decision aid. Recall, e.g., that OWA operators are often fitted to a prescribed global orness $m_{\mathrm{or}}(OWA) = \alpha$, minimizing the corresponding dispersion (or entropy) of the assigned weighted vector $(w_1, \ldots, w_n)$. Using the notation of Example 6 (then $m_{\mathrm{or}} = \mu^{(0)}$) one can choose the best fitting OWA satisfying $\mu^{(0)}(OWA) = \alpha$ and minimizing $\mu^{(1)}(OWA)$.

Acknowledgments

The first author kindly acknowledges the support of the Grant VEGA 1/3012/06 and the second author the support of the Grants VEGA 1/4209/07 and GAČR 402/08/0618. Both authors also acknowledge the support of Science and

Technology Assistance Agency under the Contract No. APVV-0375-06. Our gratitude also goes to anonymous referees whose comments helped us to improve significantly the original version of this paper.

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