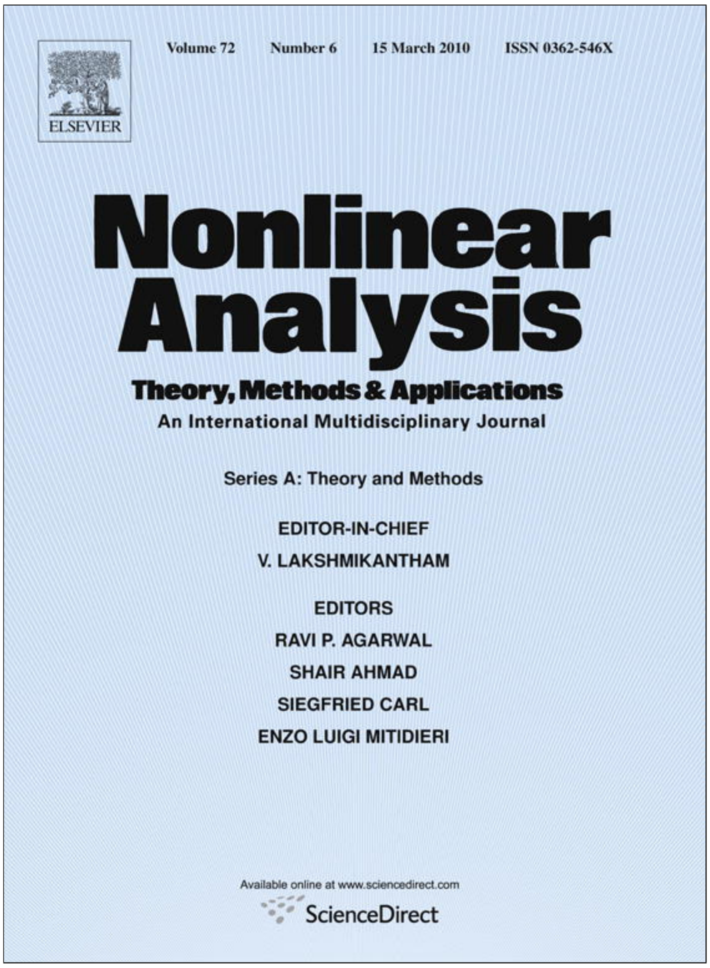
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Chebyshev type inequalities for pseudo-integrals

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Chebyshev type inequalities for two classes of pseudo-integrals are shown. One of them concerning the pseudo-integrals based on a function reduces on the g-integral where pseudo-operations are defined by a monotone and continuous function g. Another one concerns the pseudo-integrals based on a semiring $([a, b], \max, \odot)$, where $\odot$ is generated. Moreover, a strengthened version of Chebyshev's inequality for pseudo-integrals is proved.

1. Introduction

Sugeno [1] initiated the research on fuzzy measures and fuzzy integrals. Fuzzy measure is a generalization of the notion of measure in mathematical analysis (see, e.g., [2,3]). The Sugeno integral is analogous to the Lebesgue integral which has been studied by many authors, including Pap [2], Roman-Flores et al. [4] and, Wang and Klir [3], among others. In [4], Flores-Franulič and Román-Flores generalized the classical Chebyshev type inequality to the Sugeno integral. Ouyang et al. further generalized the fuzzy Chebyshev type inequalities [4] to the case of an arbitrary fuzzy measure-based Sugeno integral [5,6].

Note that the Sugeno integral is not an extension of the Lebesgue integral. The difference between the Sugeno integral and the Lebesgue integral is that addition and multiplication in the definition of the Lebesgue integral are replaced respectively by the operations "max" and "min" when the Sugeno integral is considered. Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers a semiring is defined on a real interval $[a, b] \subset [-\infty, +\infty]$ with pseudo-addition $\oplus$ and with pseudo-multiplication $\otimes$, [7,8,2,9,10]. Thus it would be an interesting topic to generalize an inequality from the framework of the classical analysis to that of some integrals which contain the classical analysis as special cases [11,12].

The paper is organized as follows. Section 2 consists of some preliminaries, such as pseudo-operations, pseudo-analysis and pseudo-additive measures and integrals. In Section 3 we prove two types of generalizations of the Chebyshev type inequality for pseudo-integrals. In Section 4, we construct a strengthened version of the Chebyshev type inequality for pseudo-integrals. Finally, a conclusion is given in Section 5.

2. Preliminaries

In this section, we are going to review some well-known results of pseudo-operations, pseudo-analysis and pseudo-additive measures and integrals. For details, we refer to [13–16,7,2,8,9,17,10]. For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper (see [15,17,11]).

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2.1. Pseudo-integrals

Let $[a, b]$ be a closed (in some cases can be considered semiclosed) subinterval of $[-\infty, \infty]$. The full order on $[a, b]$ will be denoted by $\preceq$.

Definition 2.1. The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a, b] \times [a, b] \to [a, b]$ which is commutative, non-decreasing (with respect to $\preceq$), associative and with a zero (neutral) element denoted by $\mathbf{0}$, i.e., for each $x \in [a, b]$, $\mathbf{0} \oplus x = x$ holds (usually $\mathbf{0}$ is either a or b).

Let $[a, b]_+ = \{x | x \in [a, b], \mathbf{0} \preceq x\}$.

Definition 2.2. The operation $\odot$ (pseudo-multiplication) is a function $\odot : [a, b] \times [a, b] \to [a, b]$ which is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a, b]_+$, associative and for which there exist a unit element $\mathbf{1} \in [a, b]$, i.e., for each $X \in [a, b]$, $\mathbf{1} \odot x = x$.

We assume also $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e., $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$. The structure $([a, b], \oplus, \odot)$is a semiring (see [13,2]). In this paper we will consider semirings with the following continuous operations:

Case I: The pseudo-addition is idempotent operation and the pseudo-multiplication is not.

(a) $x \oplus y = \sup(x, y)$, $\odot$ is arbitrary not idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0} = a$ and the idempotent operation sup induces a full order in the following way: $x \preceq y$ if and only if $\sup(x, y) = y$.

(b) $x \oplus y = \inf(x, y)$, $\odot$ is arbitrary not idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0} = b$ and the idempotent operation inf induces a full order in the following way: $x \preceq y$ if and only if $\inf(x, y) = y$.

Case II: The pseudo-operations are defined by a monotone and continuous function $g : [a, b] \to [0, \infty]$, i.e., pseudo-operations are given with $x \oplus y = g^{-1}(g(x) + g(y))$ and $x \odot y = g^{-1}(g(x)g(y))$. If the zero element for the pseudo-addition is a, we will consider increasing generators. Then $g(a) = 0$ and $g(b) = \infty$. If the zero element for the pseudo-addition is b, we will consider decreasing generators. Then $g(b) = 0$ and $g(a) = \infty$.

If the generator g is increasing (respectively decreasing), then the operation $\oplus$ induces the usual order (respectively opposite to the usual order) on the interval $[a, b]$ in the following way: $x \preceq y$ if and only if $g(x) \le g(y)$.

Case III: Both operations are idempotent. We have

(a) $x \oplus y = \sup(x, y)$, $x \odot y = \inf(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = a$ and $\mathbf{1} = b$. The idempotent operation sup induces the usual order ($x \preceq y$ if and only if $\sup(x, y) = y$).

(b) $x \oplus y = \inf(x, y)$, $x \odot y = \sup(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = b$ and $\mathbf{1} = a$. The idempotent operation inf induces an order opposite to the usual order ($x \preceq y$ if and only if $\inf(x, y) = y$).

Let X be a non-empty set. Let $\mathcal{A}$ be a σ-algebra of subsets of a set X.

Definition 2.3. A set function $m : \mathcal{A} \to [a, b]_+$ (or semiclosed interval) is a $\oplus$-measure if there holds:

(i) $m(\phi) = \mathbf{0}$ (if $\oplus$ is not idempotent);
(ii) m is σ-$\oplus$-(decomposable) measure, i.e.

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigoplus_{i=1}^{\infty} m(A_i)$$

holds for any sequence $\{A_i\}_{i \in N}$ of pairwise disjoint sets from $\mathcal{A}$.

We suppose that $([a, b], \oplus)$ and $([a, b], \odot)$ are complete lattice ordered semigroups. Further, suppose that $[a, b]$ is endowed with a metric d compatible with sup and inf, i.e. $\lim_{n\to\infty} \sup x_n = x$ and $\lim_{n\to\infty} \inf x_n = x$, imply $\lim_{n\to\infty} d(x_n, x) = 0$, and which satisfies at least one of the following conditions:

(a) $d(x \oplus y, x' \oplus y') \le d(x, x') + d(y, y')$,

(b) $d(x \oplus y, x' \oplus y') \le \max\{d(x, x'), d(y, y')\}$.

Both conditions (a) and (b) imply:

$d(x_n, y_n) \to 0 \Longrightarrow d(x_n \oplus z, y_n \oplus z) \to 0$.

Metric d is also monotonic, i.e.,

$x \le z \le y \Longrightarrow d(x, y) \ge \sup\{d(y, z), d(x, z)\}$.

Let f and g be two functions defined on X and with values in a semiring $([a, b], \oplus, \odot)$. Then for any $x \in X$ and for any $\lambda \in [a, b]$ we define $(f \oplus g)(x) = f(x) \oplus g(x)$, $(f \odot g)(x) = f(x) \odot g(x)$ and $(\lambda \odot f)(x) = \lambda \odot f(x)$.

Definition 2.4. The pseudo-characteristic function of a set A is:

$$\chi_A(x) = \begin{cases} 1 & x \in A, \\ 0 & x \notin A \end{cases}$$

where $\mathbf{0}$ is zero element for $\oplus$ and $\mathbf{1}$ is unit element for $\odot$.

Definition 2.5. An elementary (measurable) function is mapping $e : X \to [a, b]$ that has the following representation:

$$e_i = \bigoplus_{i=1}^{n} a_i \odot \chi_{A_i},$$

where $a_i \in [a, b]$ and sets $A_i \in \mathcal{A}$ are pairwise disjoint if $\oplus$ is nonidempotent.

Definition 2.6 ([15]). Let ε be a positive real number and $B \subset [a, b]$. A subset $\{l_i^\varepsilon\}$ of set B is a ε-net on B if for each $x \in B$ there exists l_i^ε such that $d(l_i^\varepsilon, x) \leq \varepsilon$. If we, also, have $l_i^\varepsilon \preceq x$, then we shall call $\{l_i^\varepsilon\}$ a lower ε-net. If $l_i^\varepsilon \preceq l_{i-1}^\varepsilon$ holds, then $\{l_i^\varepsilon\}$ is monotone.

Definition 2.7. Let $m : \mathcal{A} \to [a, b]$ be a $\oplus$-measure.

(i) The pseudo-integral of an elementary function $e : X \to [a, b]$ with respect to m is defined by

$$\int_X^{\oplus} e \odot \mathrm{d}m = \bigoplus_{i=1}^{n} a_i \odot m\left(A_i\right).$$

(ii) The pseudo-integral of a bounded measurable function $f : X \to [a, b]$, (if $\oplus$ is not idempotent we suppose that for each $\varepsilon > 0$ there exists a monotone ε-net in $f(X)$) is defined by

$$\int_X^{\oplus} e \odot \mathrm{d}m = \lim_{n\to\infty} \int_X^{\oplus} e_n\left(x\right) \odot \mathrm{d}m,$$

where $(e_n)_{n\in N}$ is a sequence of elementary functions such that $d(e_n(x), f(x)) \to 0$ uniformly as $n \to \infty$.

2.2. Explicit forms of special pseudo-integrals

We shall consider the semiring $([a, b], \oplus, \odot)$ for two important (with completely different behaviour) cases. First class is when pseudo-operations are generated by a monotone and continuous function $g : [a, b] \to [0, \infty]$, i.e., pseudo-operations are given with

$$x \oplus y = g^{-1}\left(g\left(x\right) + g\left(y\right)\right) \quad \text{and} \quad x \odot y = g^{-1}\left(g\left(x\right) g\left(y\right)\right).$$

Then the pseudo-integral for a function $f : [c, d] \to [a, b]$ reduces on the g-integral [16,7],

$$\int_{[c,d]}^{\oplus} f(x)\mathrm{d}x = g^{-1}\left(\int_c^d g\left(f(x)\right) \mathrm{d}x\right).$$

More on this structure as well as on corresponding measures and integrals can be found in [16,7].

The second class is when $x \oplus y = \max(x, y)$ and $x \odot y = g^{-1}\left(g\left(x\right) g\left(y\right)\right)$, the pseudo-integral for a function $f : \mathbb{R} \to [a, b]$ is given by

$$\int_{\mathbb{R}}^{\oplus} f \odot \mathrm{d}m = \sup_{x\in\mathbb{R}} \left(f(x) \odot \psi\left(x\right)\right),$$

where function ψ defines sup-measure m. Any sup-measure generated as essential supremum of a continuous density can be obtained as a limit of pseudo-additive measures with respect to generated pseudo-addition [15]. For any continuous function $f : [0, \infty] \to [0, \infty]$ the integral $\int^{\oplus} f \odot \mathrm{d}m$ can be obtained as a limit of g-integrals, [15].

We denote by μ the usual Lebesgue measure on $\mathbb{R}$. We have

$$m(A) = \operatorname{ess\,sup}_{\mu}\left(x \mid x \in A\right) = \sup\{a \mid \mu(\{x \mid x \in A, x > a\}) > 0\}.$$

Theorem 2.8 ([15]). *Let m be a sup-measure on $([0, \infty], \mathcal{B}([0, \infty]))$, where $\mathcal{B}([0, \infty])$ is the Borel σ-algebra on $[0, \infty]$, $m(A) = \operatorname{ess\,sup}_{\mu}\left(\psi\left(x\right) \mid x \in A\right)$, and $\psi : [0, \infty] \to [0, \infty]$ is a continuous density. Then for any pseudo-addition $\oplus$ with a generator g there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$-measure on $([0, \infty), \mathcal{B})$, where $\oplus_\lambda$ is generated by g^λ (the function g of the power λ), $\lambda \in (0, \infty)$, such that $\lim_{\lambda\to\infty} m_\lambda = m$.*

Theorem 2.9 ([15]). *Let $([0, \infty], \sup, \odot)$ be a semiring with $\odot$ with a generator g, i.e.,we have $x \odot y = g^{-1}(g(x)g(y))$ for every $x, y \in [a, b]$. Let m be the same as in Theorem 2.8. Then there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$-measures, where $\oplus_\lambda$ is generated by g^λ, $\lambda \in (0, \infty)$ such that for every continuous function $f : [0, \infty] \to [0, \infty]$*

$$\int^{\sup} f \odot \mathrm{d}m = \lim_{\lambda\to\infty} \int^{\oplus_\lambda} f \odot \mathrm{d}m_\lambda = \lim_{\lambda\to\infty} \left(g^\lambda\right)^{-1} \left(\int g^\lambda\left(f\left(x\right)\right) \mathrm{d}x\right).$$

For more details, we refer to [15,2,8,17,11].

Before stating our main result we need a definition.

Definition 2.10. Functions $f, g: X \to R$ are said to be comonotone if for all $x, y \in X$,

$$(f(x) - f(y))(g(x) - g(y)) \geq 0,$$

and f and g are said to be countermonotone if for all $x, y \in X$,

$$(f(x) - f(y))(g(x) - g(y)) \leq 0.$$

The comonotonicity of functions f and g is equivalent to the nonexistence of points $x, y \in X$ such that $f(x) < f(y)$ and $g(x) > g(y)$. Similarly, if f and g are countermonotone then $f(x) < f(y)$ and $g(x) < g(y)$ cannot happen. Observe that the concept of comonotonicity was first introduced in [18].

Now, our results can be stated as follows.

3. Chebyshev's inequality for pseudo-integrals

Classical Chebyshev's integral inequality is deeply connected with the study of positive dependence of random variables which are monotone functions of a common random variable (see [19]). Chebyshev type inequalities and their applications have been investigated by many authors (cf. [20–23]). The following is the Chebyshev integral inequality:

$$\int_0^1 fg\mathrm{d}\mu \geq \left(\int_0^1 f\mathrm{d}\mu\right)\left(\int_0^1 g\mathrm{d}\mu\right), \tag{3.1}$$

where μ is the Lebesgue measure on $\mathbb{R}$ and $f, g : [0, 1] \to [0, \infty)$ are comonotone functions and the reverse inequality holds whenever f and g are countermonotone functions. The aim of this section is to show the Chebyshev type inequality for pseudo-integrals.

Theorem 3.1. *Let $u, v : [0, 1] \to [a, b]$ be two measurable functions and let a generator $g : [a, b] \to [0, \infty)$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. If u and v are comonotone, then the inequality*

$$\int_{[0,1]}^{\oplus} (u \odot v)\,\mathrm{d}x \geq \left(\int_{[0,1]}^{\oplus} u\mathrm{d}x\right) \odot \left(\int_{[0,1]}^{\oplus} v\mathrm{d}x\right) \tag{3.2}$$

holds and the reverse inequality holds whenever u and v are countermonotone functions.

Proof. Observe that

$$\int_{[0,1]}^{\oplus} (u \odot v)\,\mathrm{d}x = g^{-1}\left(\int_0^1 g\,(u \odot v)\,\mathrm{d}x\right) = g^{-1}\left(\int_0^1 g\,(u)\,g\,(v)\,\mathrm{d}x\right). \tag{3.3}$$

If u and v are comonotone functions, then from (3.3) and using the Chebyshev integral inequality, we have

$$\begin{aligned}
\int_{[0,1]}^{\oplus} (u \odot v)\,\mathrm{d}x &= g^{-1}\left(\int_0^1 g\,(u \odot v)\,\mathrm{d}x\right) = g^{-1}\left(\int_0^1 g\,(u)\,g\,(v)\,\mathrm{d}x\right)\\
&\geq g^{-1}\left[\left(\int_0^1 g\,(u)\,\mathrm{d}x\right) \times \left(\int_0^1 g\,(v)\,\mathrm{d}x\right)\right]\\
&= g^{-1}\left[g\left(g^{-1}\left(\int_0^1 g\,(u)\,\mathrm{d}x\right)\right) \times g\left(g^{-1}\left(\int_0^1 g\,(v)\,\mathrm{d}x\right)\right)\right]\\
&= g^{-1}\left(g\left(\int_{[0,1]}^{\oplus} u\mathrm{d}x\right) \times g\left(\int_{[0,1]}^{\oplus} v\mathrm{d}x\right)\right) = \left(\int_{[0,1]}^{\oplus} u\mathrm{d}x\right) \odot \left(\int_{[0,1]}^{\oplus} v\mathrm{d}x\right).
\end{aligned}$$

Hence, (3.2) is valid. Similarly, if u and v are countermonotone functions, then we can prove the following:

$$\int_{[0,1]}^{\oplus} (u \odot v)\,\mathrm{d}x = g^{-1}\left(\int_0^1 g\,(u)\,g\,(v)\,\mathrm{d}x\right) \leq \left(\int_{[0,1]}^{\oplus} u\mathrm{d}x\right) \odot \left(\int_{[0,1]}^{\oplus} v\mathrm{d}x\right).$$

Thereby, the theorem is proved. □

Example 3.2. Let $g(x) = x^{\alpha}$ for some $\alpha \in (0, \infty)$. The corresponding pseudo-operations are $x \oplus y = (x^{\alpha} + y^{\alpha})^{\frac{1}{\alpha}}$ and $x \odot y = xy$. If f and g are comonotone, then it holds

$$\left(\int_0^1 (fg)^{\alpha}\,\mathrm{d}\mu\right)^{\frac{1}{\alpha}} \geq \left(\int_0^1 f^{\alpha}\mathrm{d}\mu\right)^{\frac{1}{\alpha}}\left(\int_0^1 g^{\alpha}\mathrm{d}\mu\right)^{\frac{1}{\alpha}}$$

and the reverse inequality holds whenever f and g are countermonotone functions.

Now, we generalize the Chebyshev type inequality by the semiring $([a, b], \max, \odot)$, where $\odot$ is generated.

Theorem 3.3. *Let $u, v : [0,1] \to [a,b]$ be two continuous functions and $\odot$ is represented by an increasing multiplicative generator g and m be the same as in Theorem 2.8. If u and v are comonotone, then the inequality*

$$\int_{[0,1]}^{\sup} (u \odot v) \odot \mathrm{d}m \geq \left(\int_{[0,1]}^{\sup} u \odot \mathrm{d}m\right) \odot \left(\int_{[0,1]}^{\sup} v \odot \mathrm{d}m\right) \tag{3.4}$$

holds and the reverse inequality holds whenever u and v are countermonotone functions.

Proof. Theorem 2.9 implies that

$$\int_{[0,1]}^{\sup} (u \odot v) \odot \mathrm{d}m = \lim_{\lambda\to\infty} \int_{[0,1]}^{\oplus_\lambda} (u \odot v) \odot \mathrm{d}m_\lambda = \lim_{\lambda\to\infty} \left(g^\lambda\right)^{-1} \left(\int_0^1 g^\lambda \left((u \odot v)(x)\right) \mathrm{d}x\right).$$

If u and v are comonotone functions, then using first Theorem 3.1 and then Theorem 2.9 we have

$$\begin{aligned}
\int_{[0,1]}^{\sup} (u \odot v) \odot \mathrm{d}m &= \lim_{\lambda\to\infty} \left(g^\lambda\right)^{-1} \left(\int_0^1 g^\lambda \left((u \odot v)(x)\right) \mathrm{d}x\right) \\
&\geq \lim_{\lambda\to\infty} \left(\left(g^\lambda\right)^{-1} \left(\int_0^1 g^\lambda (u(x))\, \mathrm{d}x\right) \odot \left(g^\lambda\right)^{-1} \left(\int_0^1 g^\lambda (v(x))\, \mathrm{d}x\right)\right) \\
&= \left(\lim_{\lambda\to\infty} \left(g^\lambda\right)^{-1} \left(\int_0^1 g^\lambda (u(x))\, \mathrm{d}x\right)\right) \odot \left(\lim_{\lambda\to\infty} \left(g^\lambda\right)^{-1} \left(\int_0^1 g^\lambda (v(x))\, \mathrm{d}x\right)\right) \\
&= \left(\int_{[0,1]}^{\sup} u \odot \mathrm{d}m\right) \odot \left(\int_{[0,1]}^{\sup} v \odot \mathrm{d}m\right).
\end{aligned}$$

Hence, (3.4) is valid. Similarly, if u and v are countermonotone functions, then by Theorems 3.1 and 2.9 we can prove the following:

$$\left(\int_{[0,1]}^{\sup} (u \odot v) \odot \mathrm{d}m\right) \leq \left(\int_{[0,1]}^{\sup} u \odot \mathrm{d}m\right) \odot \left(\int_{[0,1]}^{\sup} v \odot \mathrm{d}m\right).$$

Thereby, the theorem is proved. □

Example 3.4. Let $g^\lambda(x) = \mathrm{e}^{\lambda x}$ and $\psi(x)$ be from Theorem 2.8. Then

$$x \odot_\lambda y = x + y$$

and

$$\lim_{\lambda\to\infty} \left(\frac{1}{\lambda} \ln\left(\mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda y}\right)\right) = \max(x, y).$$

If u and v are comonotone, then the inequality

$$\sup\left(u(x) + v(x) + \psi(x)\right) \geq \sup\left(u(x) + \psi(x)\right) + \sup\left(v(x) + \psi(x)\right)$$

holds and the reverse inequality holds whenever u and v are countermonotone functions.

Note that the third important case $\oplus = \max$ and $\odot = \min$ has been studied in [4–6] and the pseudo-integral in such a case yields the Sugeno integral.

4. Further discussions

In this section, we provide a strengthened version of Chebyshev's inequality for pseudo-integrals.

Theorem 4.1. *Let $u, v : [0,1] \to [a,b]$ be two measurable functions and let a generator $g : [a,b] \to [0,\infty)$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function and $\varphi : [a,b] \to [a,b]$ be a continuous and strictly increasing function such that φ commutes with $\odot$. If u and v are comonotone, then the inequality*

$$\varphi^{-1}\left(\int_{[0,1]}^{\oplus} \varphi(u \odot v)\, \mathrm{d}x\right) \geq \left(\varphi^{-1}\left(\int_{[0,1]}^{\oplus} \varphi(u)\, \mathrm{d}x\right)\right) \odot \left(\varphi^{-1}\left(\int_{[0,1]}^{\oplus} \varphi(v)\, \mathrm{d}x\right)\right) \tag{4.1}$$

holds and the reverse inequality holds whenever u and v are countermonotone functions.

Proof. Since φ commutes with $\odot$, then we have

$$\int_{[0,1]}^{\oplus} \varphi(u \odot v)\, \mathrm{d}x = \int_{[0,1]}^{\oplus} \left(\varphi(u) \odot \varphi(v)\right) \mathrm{d}x. \tag{4.2}$$

If u and v are comonotone functions and φ is a continuous and strictly increasing function, then $\varphi(u)$ and $\varphi(v)$ are also comonotone. From (4.2) and using Theorem 3.1, we have

$$\begin{aligned}\int_{[0,1]}^{\oplus} \varphi\,(u \odot v)\,\mathrm{d}x &\geq \left(\int_{[0,1]}^{\oplus} \varphi(u)\mathrm{d}x\right) \odot \left(\int_{[0,1]}^{\oplus} \varphi(v)\mathrm{d}x\right)\\ &= \varphi\left[\left(\varphi^{-1}\left(\int_{[0,1]}^{\oplus} \varphi(u)\mathrm{d}x\right)\right) \odot \left(\varphi^{-1}\left(\int_{[0,1]}^{\oplus} \varphi(v)\mathrm{d}x\right)\right)\right],\end{aligned}$$

where φ commutes with $\odot$. Hence, (4.1) is valid. Similarly, if u and v are countermonotone functions and φ is a continuous and strictly increasing function, then $\varphi(u)$ and $\varphi(v)$ are also countermonotone. From (4.2) and using Theorem 3.1, we can prove the following:

$$\varphi^{-1}\left(\int_{[0,1]}^{\oplus} \varphi\,(u \odot v)\,\mathrm{d}x\right) \leq \left(\varphi^{-1}\left(\int_{[0,1]}^{\oplus} \varphi\,(u)\,\mathrm{d}x\right)\right) \odot \left(\varphi^{-1}\left(\int_{[0,1]}^{\oplus} \varphi\,(v)\,\mathrm{d}x\right)\right),$$

where φ commutes with $\odot$. Thereby, the theorem is proved. □

Corollary 4.2. *Let $u, v : [0, 1] \to [a, b]$ be two measurable functions and let a generator $g : [a, b] \to [0, \infty)$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be a increasing function. If u and v are comonotone, then the inequality*

$$\left(\int_{[0,1]}^{\oplus} (u \odot v)^s\,\mathrm{d}x\right)^{\frac{1}{s}} \geq \left(\int_{[0,1]}^{\oplus} u^s\mathrm{d}x\right)^{\frac{1}{s}} \odot \left(\int_{[0,1]}^{\oplus} v^s\mathrm{d}x\right)^{\frac{1}{s}} \tag{4.3}$$

holds for all $0 \leq s < \infty$ where $(.)^s$ commutes with $\odot$ and the reverse inequality holds whenever u and v are countermonotone functions.

Now we consider the second class, when $x \oplus y = \max(x, y)$ and $x \odot y = g^{-1}\left(g\,(x)\,g\,(y)\right)$.

Theorem 4.3. *Let $u, v : [0, 1] \to [a, b]$ be two continuous functions and $\odot$ is represented by an increasing multiplicative generator g and $\varphi : [a, b] \to [a, b]$ be a continuous and strictly increasing function such that φ commutes with $\odot$ and m be the same as in Theorem 2.8. If u and v are comonotone, it holds*

$$\varphi^{-1}\left(\int_{[0,1]}^{\sup} \varphi\,(u \odot v) \odot \mathrm{d}m\right) \geq \left(\varphi^{-1}\left(\int_{[0,1]}^{\sup} \varphi\,(u) \odot \mathrm{d}m\right)\right) \odot \left(\varphi^{-1}\left(\int_{[0,1]}^{\sup} \varphi\,(v) \odot \mathrm{d}m\right)\right) \tag{4.4}$$

and the reverse inequality holds whenever u and v are countermonotone functions.

Proof. Theorem 2.9 implies that

$$\int_{[0,1]}^{\sup} \varphi\,(u \odot v) \odot \mathrm{d}m = \lim_{\lambda\to\infty} \int_{[0,1]}^{\oplus_\lambda} \varphi\,(u \odot v) \odot \mathrm{d}m_\lambda = \lim_{\lambda\to\infty} \left(g^\lambda\right)^{-1} \left(\int_0^1 g^\lambda\left(\left(\varphi\,(u \odot v)\right)(x)\right)\mathrm{d}x\right).$$

If u and v are comonotone functions and φ is a continuous and strictly increasing function, then using first Theorem 4.1 and then Theorem 2.9 we have

$$\begin{aligned}\varphi^{-1}\left(\int_{[0,1]}^{\sup} \varphi\,(u \odot v) \odot \mathrm{d}m\right) &= \varphi^{-1}\left(\lim_{\lambda\to\infty} \left(g^\lambda\right)^{-1}\left(\int_0^1 g^\lambda\left(\varphi\,(u \odot v)\,(x)\right)\mathrm{d}x\right)\right)\\ &= \lim_{\lambda\to\infty}\left(\varphi^{-1}\left(\left(g^\lambda\right)^{-1}\left(\int_0^1 g^\lambda\left(\varphi\,(u \odot v)\,(x)\right)\mathrm{d}x\right)\right)\right)\\ &\geq \lim_{\lambda\to\infty}\left[\left(\left(g^\lambda\right)^{-1}\left(\varphi^{-1}\left(\int_0^1 g^\lambda\left(\varphi\,(u\,(x))\right)\mathrm{d}x\right)\right)\right) \odot \left(\left(g^\lambda\right)^{-1}\left(\varphi^{-1}\left(\int_0^1 g^\lambda\left(\varphi\,(v\,(x))\right)\mathrm{d}x\right)\right)\right)\right]\\ &= \lim_{\lambda\to\infty}\left(\left(g^\lambda\right)^{-1}\left(\varphi^{-1}\left(\int_0^1 g^\lambda\left(\varphi\,(u\,(x))\right)\mathrm{d}x\right)\right)\right) \odot \lim_{\lambda\to\infty}\left(\left(g^\lambda\right)^{-1}\left(\varphi^{-1}\left(\int_0^1 g^\lambda\left(\varphi\,(v\,(x))\right)\mathrm{d}x\right)\right)\right)\\ &= \left(\varphi^{-1}\left(\int_{[0,1]}^{\sup} \varphi\,(u) \odot \mathrm{d}m\right)\right) \odot \left(\varphi^{-1}\left(\int_{[0,1]}^{\sup} \varphi\,(v) \odot \mathrm{d}m\right)\right),\end{aligned}$$

where φ commutes with $\odot$. Hence, (4.4) is valid. Similarly, if u and v are countermonotone functions and φ is a continuous and strictly increasing function, then using first Theorem 4.1 and then Theorem 2.9 we can prove the following:

$$\varphi^{-1}\left(\int_{[0,1]}^{\sup} \varphi\,(u \odot v) \odot \mathrm{d}m\right) \leq \left(\varphi^{-1}\left(\int_{[0,1]}^{\sup} \varphi\,(u) \odot \mathrm{d}m\right)\right) \odot \left(\varphi^{-1}\left(\int_{[0,1]}^{\sup} \varphi\,(v) \odot \mathrm{d}m\right)\right),$$

where φ commutes with $\odot$. Thereby, the theorem is proved. □

Corollary 4.4. *Let $u, v : [0, 1] \to [a, b]$ be two continuous functions and $\odot$ is represented by an increasing multiplicative generator g and m be the same as in Theorem 2.8. If u and v are comonotone, it holds*

$$\left(\int_{[0,1]}^{\sup} (u \odot v)^s \odot \mathrm{d}m\right)^{\frac{1}{s}} \geq \left(\int_{[0,1]}^{\sup} u^s \odot \mathrm{d}m\right)^{\frac{1}{s}} \odot \left(\int_{[0,1]}^{\sup} v^s \odot \mathrm{d}m\right)^{\frac{1}{s}}$$

for all $0 \leq s < \infty$ where $(.)^s$ commutes with $\odot$ and the reverse inequality holds whenever u and v are countermonotone functions.

Note that the third important case $\oplus = \max$ and $\odot = \min$ has been studied in [24,25] and the pseudo-integral in such a case yields the Sugeno integral.

5. Conclusion

We have proved the Chebyshev type inequalities for pseudo-integrals. There are two classes of pseudo-integrals: the pseudo-integrals based on a function reduce on the g-integral, where pseudo-operations are defined by a monotone and continuous function g and the pseudo-integrals based on the semiring $([a, b], \max, \odot)$, where $\odot$ is generated. Moreover, a strengthened version of Chebyshev's inequality for pseudo-integrals is proved.

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