Equivalence problem in Compositional models

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Abstract

Structure of each Compositional model can be visualized by a tool called persegram. Every persegram over a finite non-empty set of variables N induces an independence model over N, which is a list of conditional independence statements over N. The Equivalence problem is how to characterize (in graphical terms) whether all independence statements in the model induced by persegram P are in the model induced by a second persegram P' and vice versa. Three different operations preserving independence model were introduced in previous papers. If combined, one is able to generate the (whole) class of equivalent persegrams. This characterization is indirect: Two persegrams P,P' are equivalent if there exists a sequence of persegrams from P,P' such that only so called IE-operations are performed to get next persegram in the sequence.

In this paper we give the motivation and introduction for direct characterization of equivalence. We have found some invariants among equivalent persegrams that have to be remained. In spite of that, the final simple direct characterization is not given. Instead we give several properties of equivalent persegrams that could be helpful.

Structure of each Compositional model can be visualized by a tool called persegram. Every persegram over a finite non-empty set of variables N induces an independence model over N, which is a list of conditional independence statements over N. The *equivalence problem* is how to characterize (in graphical terms) whether all independence statements in the model induced by persegram $\mathcal{P}$ are in the model induced by a second persegram $\mathcal{P}'$ and vice versa. In the previous paper [5] indirect characterization of equivalence was done. We introduced three different operations on persegrams remaining independence model which combined together are able to generate the (whole) class of equivalent persegrams. That characterization is indirect in the following sense: Two persegrams $\mathcal{P}$, $\mathcal{P}'$ are equivalent if there exists a sequence of persegrams from $\mathcal{P}$ to $\mathcal{P}'$ such that only so called IE-operations are performed to get next persegram in the sequence.

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lent persegrams that have to be remained. In spite of that, the final simple direct characterization is not given. Instead we give several properties of equivalent persegrams that could be helpful.

1 Introduction

The ability to represent and process multidimensional probability distributions is a necessary condition for the application of probabilistic methods in Artificial Intelligence. Among the most popular approaches are the methods based on Graphical Markov Models, e.g., Bayesian Networks. The Compositional models are an alternative approach to Graphical Markov Models. These models are generated by a sequence (generating sequence) of low-dimensional distributions, which, composed together, create a distribution - the so called *Compositional model*. Moreover, while a model is composed together, a system of (un)conditional independencies is simultaneously introduced by the structure of the generating sequence.

The structure can be visualized by a tool called *persegram* and one can read induced independencies directly using this tool. That is why we can say that every persegram over a finite non-empty set of variables N induces an *independence model* over N - a list of conditional independence statements over N. The *equivalence problem* is how to characterize (in graphical terms) whether all independence statements in the model induced by persegram $\mathcal{P}$ are also in the independence model induced by a second persegram $\mathcal{P}'$ and vice versa.

2 Compositional Models

A Bayesian network may be defined as a multidimensional distribution factorizing with respect to an acyclic directed graph. Alternatively, it may be defined by its graph and an appropriate system of low-dimensional (oligodimensional) *conditional distributions*. Contrary, Compositional models are defined as a multidimensional distribution assembled from a sequence of oligodimensional *unconditional distributions*, with the help of operators of composition. The main advantage of both approaches lies in the fact that oligodimensional distributions could be easily stored in a computer memory. However, computing with a multidimensional distribution that is split into many pieces may be exceptionally complicated. The advantage of Compositional models in comparison with Bayesian networks consists in the fact that compositional models explicitly express some marginals, whose computation in a Bayesian network may be demanding. Compositional model is assembled ,in contrast to Bayesian network, from unconditional distributions.

2.1 Notation and Basic Properties

Throughout the paper the symbol N will denote a non-empty set of finite-valued *variables*. From the next chapter on, variables will be represented by markers of a persegram. All probability distributions of this variables will be denoted by Greek letters (usually π, κ); thus for $K \subset N$, we consider a distribution (a probability measure over K) $\pi(K)$ which is defined for variables K. When several distributions will be considered, we shall distinguish them by indices. For a probability distribution $\pi(K)$ and $U \subset K$ we will consider a *marginal distribution* $\pi(U)$.

The following conventions will be used throughout the paper. Given sets $K, L \subset N$ the juxtaposition KL will denote their union $K \cup L$. The following symbols will be reserved for special subsets of N: K, R, S. The symbol U, V, W, Z will be used for general subsets of N. The symbol $|U|$ will be used to denote the number of elements of a finite set U, that is, its *cardinality.* u, v, w, z denotes variables as well as singletons $\{x\}, \ldots$

Independence and dependence statements over N correspond to special *disjoint triples* over N. Thy sumbol $\langle U, V|Z\rangle$ denotes a triplet of pirwise disjoint subsets U, V, Z of N. This notations anticipates the intended meaning: the set of variables U is conditionally independent or dependent of the set of variables V given the set of variables Z. This is why the third set Z is separated by a straight line: it has a special meaning of the conditioning set. The symbol $\mathcal{T}(N)$ will denote the class of all disjoint triplets over N:

$$\mathcal{T}(N) = \{\langle U, V|Z\rangle; U, V, Z \subseteq N \quad U \cap V = V \cap Z = Z \cap U = \emptyset\}$$

To describe how to compose low-dimensional distributions to get a distribution of a higher dimension we use the following operator of composition.

Definition 2.1. For arbitrary two distributions $\pi(K)$ and $\kappa(L)$ their *composition* is given by the formula

$$\pi(K) \triangleright \kappa(L) = \begin{cases} \frac{\pi(K)\kappa(L)}{\kappa(K\cap L)} & \text{if } \pi^{\downarrow K\cap L} \ll \kappa^{\downarrow K\cap L}, \\ \text{undefined} & \text{otherwise,} \end{cases}$$

where the symbol $\pi(M) \ll \kappa(M)$ denotes that $\pi(M)$ is *dominated* by $\kappa(M)$, which means (in the considered finite setting)

$$\forall x \in \times_{j\in M}\mathbf{X}_j; (\kappa(x) = 0 \Longrightarrow \pi(x) = 0).$$

The result of the composition (if defined) is a new distribution. We can iteratively repeat the process of composition to obtain a multidimensional distribution - a model approximating the original distribution with corresponding marginals. That is why these multidimensional distributions (and the whole theory as well) are called *Compositional models*. To describe such a model it is sufficient to introduce an ordered system of low-dimensional distributions $\pi_1, \pi_2, \ldots, \pi_n$. If all compositions are defined, we call this ordered system a *generating sequence*. To get a distribution represented by this sequence one has to apply the operators from left to right:

$$\pi_1 \triangleright \pi_2 \triangleright \pi_3 \triangleright \ldots \triangleright \pi_{n-1} \triangleright \pi_n = (\ldots((\pi_1 \triangleright \pi_2) \triangleright \pi_3) \triangleright \ldots \triangleright \pi_{n-1}) \triangleright \pi_n.$$

From now on, we consider generating sequence $\pi_1(K_1), \pi_2(K_2), \ldots, \pi_n(K_n)$ which defines a distribution

$$\pi_1(K_1) \triangleright \pi_2(K_2) \triangleright \ldots \triangleright \pi_n(K_n).$$

Therefore, whenever distribution π_i is used, we assume it is defined for variables K_i. In addition, each set K_i can be divided into two disjoint parts. We denote them R_i and S_i with the following sense:

$$R_i = K_i \backslash (K_1 \cup \ldots \cup K_{i-1}), S_i = K_i \cap (K_1 \cup \ldots \cup K_{i-1})$$

.

R_i denotes variables from K_i with the first appeared with respect to the sequence (meaning from left to right). S_i denotes the already used.

2.2 Graphical concepts

It is well-known that one can read conditional independence relations of a Bayesian network from its graph. A similar technique is used in compositional models. An appropriate tool for this is a *persegram*. Persegram is used to visualize the structure of a compositional model and is defined bellow.

Definition 2.2. Persegram $\mathcal{P}$ *of a generating sequence is a table in which rows correspond to variables (in an arbitrary order) and columns to low-dimensional distributions; ordering of the columns corresponds to the generating sequence ordering. A position in the table is marked if the respective distribution is defined for the corresponding variable. Markers for the first occurrence of each variable (i.e., the leftmost markers in rows) are squares (we call them* box-markers*) and for other occurrences there are* bullets*.*

Persegram $\mathcal{P}$ is a table of markers. Since the markers in the i-th column highlight variables for which generating sequence is defined, we denote markers in i-th column as K_i. Box-markers in i-th column of $\mathcal{P}$ are denoted like R_i and bullets like S_i. $K_i = R_i \cup S_i$. This notation is purposely in accordance with notation of variable sets in generating sequences to simplify readability and lucidity of the text.

Persegrams are usually denoted by $\mathcal{P}$ and if it is not specified otherwise $\mathcal{P}$ corresponds to the generating sequence $\pi_1(K_1), \ldots, \pi_n(K_n)$ where $K_1 \cup \ldots \cup K_n = N$. We say that $\mathcal{P}$ is defined over N. (i.e. $\mathcal{P}$ over N has n columns with markers $K_1, \ldots, K_n$ where $K_1 \cup \ldots \cup K_n = N$.)

To simplify the notation we will use the following symbol: Let $\mathcal{P}$ be a persegram over N. We introduce a function $][_{\mathcal{P}}$: $N \to \mathbb{N}$, which for every variable $u \in N$ returns the index of set K_i with the first appearance of u in the persegram $\mathcal{P}$. Due to the previously established notation can be said that $K_{]u[_{\mathcal{P}}}$ is a column K_i where $u \in R_i$. In other words: $]u[_{\mathcal{P}} = i : u \in R_i$.

Definition 2.3. *Let $\mathcal{P}$ be a persegram over N and $\preceq_{\mathcal{P}}$ a binary relation. For arbitrary $u, v \in N$ $u \preceq_{\mathcal{P}} v$ if $]u[_{\mathcal{P}} \leq]v[_{\mathcal{P}}$. Moreover we introduce the relation $\prec_{\mathcal{P}}$: $u \prec_{\mathcal{P}} v \Leftrightarrow u \preceq_{\mathcal{P}} v$ AND $v \not\preceq_{\mathcal{P}} u$.*

The following convention will be used throughout the paper: Given variables $u, v, w \in N$ and $\mathcal{P}$ over N, the term $u, v \prec_{\mathcal{P}} w$ denotes that $u \prec_{\mathcal{P}} w$ and $v \prec_{\mathcal{P}} w$. The symbol $\mathcal{P}$ may be omitted, if the content is clear.

2.3 Conditional independence

Conditional independence statements over N induced by the structure of Compositional model can be read from its persegram. Such independence is indicated by the absence of a *trail connecting or avoiding relevant markers*. It is defined below.

Definition 2.4. *Consider a persegram over N and a subset $Z \subset N$. A sequence of markers $m_0, \ldots, m_t$ is called a* Z-avoiding trail *that connects m_0 and m_t if it meets the following 4 conditions:*

1. *for each $s = 1, \ldots, t$ a couple (m_{s-1}, m_s) is in the same row (i.e., horizontal connection) or in the same column (vertical connection);*
2. *each vertical connection must be adjacent to a box-marker (one of the markers is a box-marker);*
3. *no horizontal connection corresponds to a variable from Z;*
4. *vertical and horizontal connections regularly alternate with the following possible exception: two vertical connections may be in direct succession if their common adjacent marker is a box-marker of a variable from Z;*

If a Z-avoiding trail connects two-box markers corresponding to variables u and v, we also say that these variables are connected by a Z-avoiding trail. *Suppose $\langle U, V|Z\rangle \in \mathcal{T}(N)$ is a disjoint triplet over N. One says that U and V are conditionally dependent by Z, written $U \not\perp\!\!\!\perp V|Z[\mathcal{P}]$, if there exists a Z-avoiding trail between variable $u \in U$ and variable $v \in V$ in $\mathcal{P}$. In the opposite case one says that U and V are conditionally independent by Z in $\mathcal{P}$, written $U \perp\!\!\!\perp V|Z[\mathcal{P}]$. We also say that $\langle U, V|Z\rangle$ is* represented *in $\mathcal{P}$. The induced independence model $\mathcal{I}(\mathcal{P})$ and the induced dependence model $\mathcal{D}(\mathcal{P})$ are defined as follows:*

$$\mathcal{I}_{\mathcal{P}} = \{\langle U, V|Z\rangle \in \mathcal{T}(N); U \perp\!\!\!\perp V|Z[\mathcal{P}]\}$$

$$\mathcal{D}_{\mathcal{P}} = \{\langle U, V|Z\rangle \in \mathcal{T}(N); U \not\perp\!\!\!\perp V|Z[\mathcal{P}]\}$$

Example 2.5. *Consider persegram from Figures 1 and 2.*

In Figure 1 a $\emptyset$-avoiding trail is depicted. Therefore $u \not\perp\!\!\!\perp z|\emptyset$. Moreover, one can replace $\emptyset$ by any subset of $\{v, w, x, y\}$ which is avoiding as well. In Figure 2 is depicted another trail connecting u and x this time. Therefore $u \not\perp\!\!\!\perp x|z$. On the contrary to Figure 1 one can not replace z by any other variable except of v. Otherwise the condition 3. from the Definition 2.4 will be corrupted. (i.e. $u \perp\!\!\!\perp x|y[\mathcal{P}]$ for example)

From the previous Definition 2.4 one can almost immediately get an interesting fact about variables appeared for the first time in the last column.

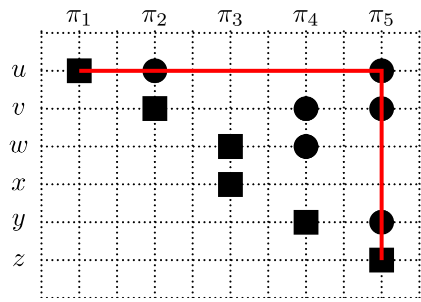


Figure 1: $\mathcal{P} : u \not\perp\!\!\!\perp z|\emptyset, u \not\perp\!\!\!\perp z|v$

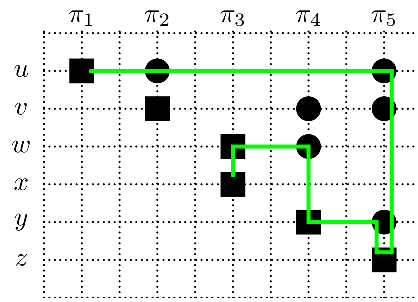


Figure 2: $\mathcal{P} : u \not\perp\!\!\!\perp x|z$

Lemma 2.6. *Consider a persegram $\mathcal{P}$ with n columns $K_1, \ldots, K_n$ and distinct variables $u, v \in K_1 \cup \ldots \cup K_n$ such that $u \notin K_n$ and $v \in R_n$. Then $u \perp\!\!\!\perp v|S_n[\mathcal{P}]$.*

Proof. Since v belongs to the last column of $\mathcal{P}$ only and u do not, every trail to v has to contain a horizontal connection to n-th column corresponding to some variable from S_n. By condition 3. of the Definition 2.4: No horizontal connection can correspond to variable from S_n. Then a S_n-avoiding trail between u and v can not exist. □

The following theorem shows an important parallel between independence read from compositional model and from its persegram. This theorem is given without proof, one can find it in [1].

Theorem 2.7. *Consider a generating sequence $\pi_1(K_1), \ldots, \pi_n(K_n)$, its corresponding persegram $\mathcal{P}$, and three disjoint subset $U, V, Z \subset K_1 \cup \ldots \cup K_n$ such that $U \neq \emptyset \neq V$. Then:*

$$U \perp\!\!\!\perp V|Z[\mathcal{P}] \Rightarrow U \perp\!\!\!\perp V|Z[\pi_1 \triangleright \ldots \triangleright \pi_n].$$

Notice that in definition 2.4 there is no condition concerning the order of rows in persegrams. This is not surprising because there is no rows ordering in definition 2.2 either.

To simplify proofs done by induction on the number of columns we introduce the concept of the *subpersegram* induced by subset of variables U. Unlike the subgraph which contains exactly those variables that induce it, subpersegram induced by a set U may be defined for some superset of U.

Definition 2.8. *Let $\mathcal{P}$ be a persegram over N. $U \subseteq N$. A* subpersegram *$\mathcal{P}[U]$ induced by U is the minimal left part of $\mathcal{P}$ containing all box-markers corresponding to U.*

Example 2.9. *Let $\mathcal{P}$ be the persegram represented in Example 2.5. Then the corresponding induced subpersegram $\mathcal{P}[z]$ is in Figure 3 and induced subpersegram $\mathcal{P}[w]$ is in Figure 4.*

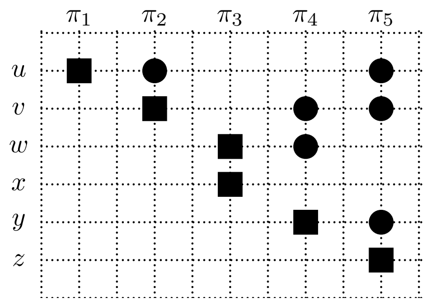


Figure 3: $\mathcal{P} \equiv \mathcal{P}[z]$

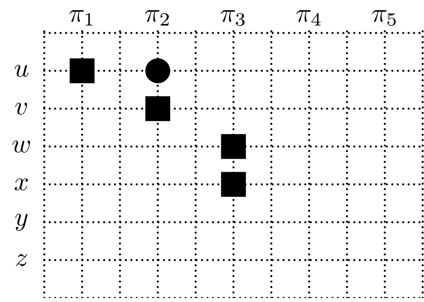


Figure 4: $\mathcal{P}[w]$

Lemma 2.10. *Let $\mathcal{P}$ be a persegram over N, and $u \not\perp\!\!\!\perp v|Z[\mathcal{P}]$. Then all Z-avoiding trails connecting u with v are in subpersegram $\mathcal{P}[u \cup v \cup Z]$ too.*

Proof. Suppose that $u \not\perp\!\!\!\perp v|Z[\mathcal{P}]$ and that there is a trail with a connection in $\mathcal{P}$ but not in $\mathcal{P}[u \cup v \cup Z]$. Let m is the first marker on the trail from u to v which belongs to such a column. Because this marker is the first one in such a column, a horizontal connection was used and therefore m is a bullet. Now one has to continue with a vertical connection(down) to box-marker. This box-marker does not correspond to any variable from Z (this column is not in $\mathcal{P}[u \cup v \cup Z]$). Therefore one has to continue with horizontal connection (to the right, this is a box-marker - there is nothing on left in the same row) to a bullet. Then down to a box-marker which does not correspond to any variable from Z etc. From such a trail is no return. Therefore such a trail can not exist. □

This lemma basically means, that if we are interested in relation $u \perp\!\!\!\perp v|Z[\mathcal{P}]$ we may focus on the subpersegram $\mathcal{P}[u \cup v \cup Z]$ only. This observation is summarized in the following corollary.

Corollary 2.11. *Let $\mathcal{P}$ be a persegram over N and $u, v \in N, Z \subset N \setminus \{u, v\}$. Then $u \perp\!\!\!\perp v|Z[\mathcal{P}[u \cup v \cup Z]] \Leftrightarrow u \perp\!\!\!\perp v|Z[\mathcal{P}]$.*

The following specific notation for certain composite dependence statements will be useful. Given a persegram $\mathcal{P}$ over N, distinct variables $u, v \in N$ and disjoint set $U \subseteq N \setminus \{u, v\}$ the symbol $u \not\perp\!\!\!\perp v| + U[\mathcal{P}]$ will be interpreted as the condition

$$\forall W \text{ such that } U \subseteq W \subseteq N \setminus \{u, v\} \text{ one has } u \not\perp\!\!\!\perp v|W[\mathcal{P}].$$

In words, u and v are (conditionally) dependent in $\mathcal{P}$ given any superset of U. If U is empty we write $*$ instead of $+U$. In particular, the following two symbols will be sometimes used

$$u \not\perp\!\!\!\perp v| * [\mathcal{P}] \equiv \forall W \text{ such that } W \subseteq N \setminus \{u, v\} \; u \not\perp\!\!\!\perp v|W[\mathcal{P}].$$

for distinct nodes $u, v \in N$, and

$$u \not\perp\!\!\!\perp v|+w[\mathcal{P}] \equiv \forall W \text{ such that } \{w\} \subseteq W \subseteq N \setminus \{u,v\} \text{ one has } u \not\perp\!\!\!\perp v|W[\mathcal{P}].$$

for distinct nodes $u, v, w \in N$. We give a certain graphical characterization of composite dependence statements of this kind below.

3 Equivalence problem

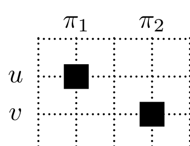
By the equivalence problem we understand the problem how to recognize whether two given persegrams $\mathcal{P}_1, \mathcal{P}_2$ over N induce the same independence model ($\mathcal{I}_{\mathcal{P}_1} = \mathcal{I}_{\mathcal{P}_2}$). It is of special importance to have an easy rule to recognize that two persegrams are equivalent in this sense and an easy way to convert $\mathcal{P}_1$ into $\mathcal{P}_2$ in terms of some elementary operations on persegrams. Another very important aspect is the ability to generate all persegrams which are equivalent to a given persegram.

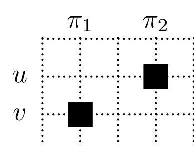
Definition 3.1. *Persegrams $\mathcal{P}_1, \mathcal{P}_2$ (over the same variable set N) are called* independence equivalent*, if they induce the same independence model $\mathcal{I}_{\mathcal{P}_1} = \mathcal{I}_{\mathcal{P}_2}$.*

Remark 3.2. *One may easily see that the above mentioned definition could be formulated with the term of dependence model.* Persegrams $\mathcal{P}_1, \mathcal{P}_2$ (over the same variable set N) are independence equivalent, iff $\mathcal{D}_{\mathcal{P}_1} = \mathcal{D}_{\mathcal{P}_2}$. *This alternative is used in most proofs primarily.*

Like in Bayesian networks, it may happen that different persegrams induce the same independence model.

Example 3.3. 1. *The following example is simple: $N = \{u, v\}$ and the following two persegrams $\mathcal{P}_1, \mathcal{P}_2$:*

$\mathcal{P}_1$: 

$\mathcal{P}_2$: 

$\mathcal{I}_{\mathcal{P}_1} = \mathcal{I}_{\mathcal{P}_2} = \{\langle u, v|\emptyset\rangle\}$ *in this case.*

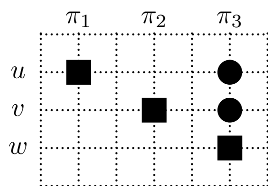
2. *On the other hand, the persegrams which have the same variable sets in columns in different order do not have to be equivalent. Let $N = \{u, v, w\}$ and consider the following persegrams:*

   $u \perp\!\!\!\perp v|\emptyset[\mathcal{P}_1]$ *but* $u \not\perp\!\!\!\perp v|\emptyset[\mathcal{P}_2]$. *On the contrary,* $u \not\perp\!\!\!\perp v|w[\mathcal{P}_1]$ *and* $u \not\perp\!\!\!\perp v|w[\mathcal{P}_2]$. *The order of the columns in persegram is important.*

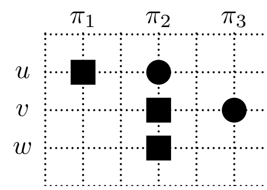
3.1 Direct characterization

The solution of equivalence problem can be done in several ways. Some kind of *indirect characterization* of equivalence follows was done in the paper [5] where

$\mathcal{P}_1$:



$\mathcal{P}_2$:



four special operations on persegrams were introduced. These operations are called *IE operations* (Independence equivalent) and they preserve independence statements induced by a persegram. These operations give us a tool to equivalence recognition: If two persegrams can be transformed from one to the other by a sequence of IE operations, then the persegrams are independence equivalent. Anyway, this characterization is indirect in the sense that, if two persegrams over same set of variables are given, then searching of such a sequence can be time demanding or even impossible. However, indirect characterization offers a method to generate a class of equivalent persegrams.

We are more interested in some type of *direct characterization* which allows us to decide on equivalence "immediately". This characterization should be based on some independence equivalence invariants.

Definition 3.4. *Let $\mathcal{P}$ be a persegram over N and $u, v \in N$ be two distinct variables. u, v are* connected *in $\mathcal{P}$ ($u \leftrightarrow v[\mathcal{P}]$) if there is a column in $\mathcal{P}$ containing markers of both variables and where at least one of them is a box-marker. Otherwise u, v are* disconnected *($u \not\leftrightarrow v$).*

The following convention will be used thorough the paper: Given variables $u, v, w \in N$ and $\mathcal{P}$ over N, the term $u, v \leftrightarrow w$ denotes that $u \leftrightarrow w$ and $v \leftrightarrow w$.

For the purpose of the following text one should realize the obvious parallel between relation $u \leftrightarrow v$ and columns order and content. This parallel is summarized in the following remark.

Remark 3.5. *Let u, v are two different variables in $\mathcal{P}$ and $u \preceq_{\mathcal{P}} v$. Then $u \leftrightarrow v[\mathcal{P}] \Leftrightarrow u \in K_{]v[}$.*

Lemma 3.6. *Let $\mathcal{P}$ be a persegram over N and $u, v \in N$ are distinct variables, $u \preceq_{\mathcal{P}} v$. Then*

$$u \perp\!\!\!\perp v | S_{]v[}[\mathcal{P}] \Leftrightarrow u \not\leftrightarrow v[\mathcal{P}]$$

Proof. $\Rightarrow$ Suppose $u \perp\!\!\!\perp v | S_{]v[}[\mathcal{P}]$ and $u \leftrightarrow v[\mathcal{P}]$. Since $u \preceq v$ then by Remark 3.5 $u \in S_{]v[}$. This however contradicts with the fact that sets involved in independence statements are not disjoint.

$\Leftarrow$ Suppose $u \not\leftrightarrow v$ and $u \not\perp\!\!\!\perp v | S_{]v[}[\mathcal{P}]$. Since $u \preceq v$, and $S_{]v[} \prec v$ then by Lemma 2.6 $u \perp\!\!\!\perp v | S_{]v[}[\mathcal{P}[v]]$. By corollary 2.11 $u \perp\!\!\!\perp v | S_{]v[}[\mathcal{P}]$, which contradicts with assumptions.

□

Anyway, please realize that because of using an induced subpersegram $\mathcal{P}[v]$ in the proof, the equation $u \preceq v; u \not\leftrightarrow v \Leftrightarrow u \perp\!\!\!\perp v | + S_{]v[}[\mathcal{P}]$ generally does not hold.

With the help of the previous lemma one can prove the following important assertion.

Lemma 3.7. *Let $\mathcal{P}$ be a persegram over N and $u, v \in N$ are distinct variables. Then*

$$u \leftrightarrow v[\mathcal{P}] \Leftrightarrow u \not\perp\!\!\!\perp v | * [\mathcal{P}].$$

Proof. $\Rightarrow$ Let $u \leftrightarrow v$ and $u \perp\!\!\!\perp v | w$ where $w \in N \setminus \{u, v\}$. Because $u \leftrightarrow v$ then the trail $u \rightsquigarrow_{\emptyset} v$ consists of one vertical and perhaps one horizontal connection and avoid any $w \in N \setminus \{u, v\}$. It contradicts the fact $a \perp\!\!\!\perp v | w$.

$\Leftarrow$ Suppose $u \not\perp\!\!\!\perp v | *$ and $u \not\leftrightarrow v$; one can assume without loss of generality that $u \prec v$. Then $u \perp\!\!\!\perp v | S_{]v[}$ according to the lemma 3.6.

□

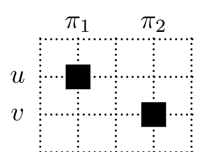
The previous two lemmata shows an interesting invariant of independence equivalence. Two persegrams, if equivalent, have the same set of connections.

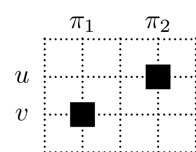
Definition 3.8. *Let $\mathcal{P}$ be a persegram over N. A* connection set *$\mathcal{E}(\mathcal{P})$ is a set of all pairs $\langle u, v \rangle : u, v \in N$, where $u \leftrightarrow v[\mathcal{P}]$. $\mathcal{E}(\mathcal{P}) = \{\langle u, v \rangle : u, v \in N, u \leftrightarrow v[\mathcal{P}]\}$*

Corollary 3.9. *Let $\mathcal{P}, \mathcal{P}'$ are persegrams over N. If $\mathcal{I}_{\mathcal{P}} = \mathcal{I}_{\mathcal{P}'}$ then $\mathcal{E}(\mathcal{P}) = \mathcal{E}(\mathcal{P}')$.*

Example 3.10. *In the Example 3.3 four different persegrams are shown. The first two are equivalent, the second two are not. Let us show this example again with knowledge of the previous lemma.*

1. *Let $\mathcal{P}_1, \mathcal{P}_2$ are the following simple persegrams over $N = \{u, v\}$: One can*

   $\mathcal{P}_1$:
   

   $\mathcal{P}_2$:
   

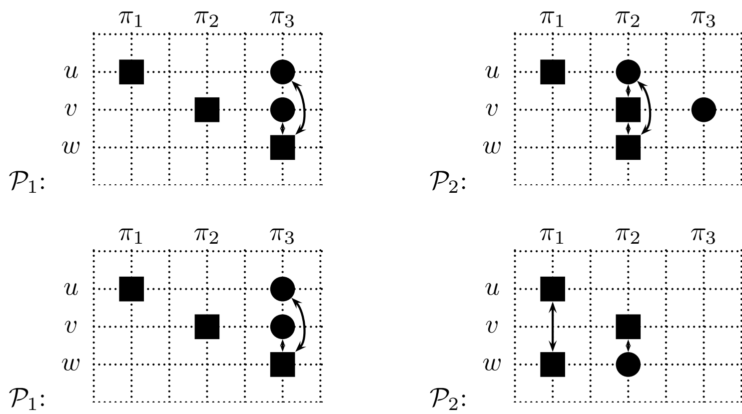
   easily see that $\mathcal{E}(\mathcal{P}_1) = \mathcal{E}(\mathcal{P}_2) = \emptyset$. The claim $\mathcal{I}_{\mathcal{P}_1} = \mathcal{I}_{\mathcal{P}_2} = \{\langle u, v | \emptyset \rangle\}$ is known from the Example 3.3.

2. *On the other hand, consider the following persegrams over $N = \{u, v, w\}$. Connections between variables are highlighted by arrows.*

   Thanks to Example 3.3 one knows that $\mathcal{I}_{\mathcal{P}_1} \neq \mathcal{I}_{\mathcal{P}_2}$. Since $\mathcal{E}(\mathcal{P}_1) = \{\langle u, w \rangle, \langle v, w \rangle\}$ but $\mathcal{E}(\mathcal{P}_2) = \mathcal{E}(\mathcal{P}_1) \cup \{\langle u, v \rangle\}$, non-equivalence is obvious now.

3. *Anyway, there exist persegrams $\mathcal{P}_1, \mathcal{P}_2$ where $\mathcal{E}(\mathcal{P}_1) = \mathcal{E}(\mathcal{P}_2)$ but $\mathcal{I}_{\mathcal{P}_1} \neq \mathcal{I}_{\mathcal{P}_2}$.*

   $\mathcal{E}(\mathcal{P}_1) = \{\langle u, w \rangle, \langle v, w \rangle\} = \mathcal{E}(\mathcal{P}_2)$. However $\mathcal{I}_{\mathcal{P}_1} \neq \mathcal{I}_{\mathcal{P}_2}$ since $u \not\perp\!\!\!\perp v | w[\mathcal{P}_1]$ and $u \perp\!\!\!\perp v | w[\mathcal{P}_2]$.



It follows from the previous example, that the previous invariant is not strong enough to ensure the equivalence. It is necessary to try to find an another invariant.

When one consider a relation $\preceq_{\mathcal{P}}$, then every persegram satisfy some partial variables ordering. For example, $u \prec v \prec w$ in persegram $\mathcal{P}_1$ but $u \preceq w \prec v$ in persegram $\mathcal{P}_2$ in the third part of the previous Example 3.10. Is it possible that the order of the variables will be some kind of invariant? It will be definitely not in that simple way. It can be easily seen in the first part of the previous Example 3.10, where $u \prec v$ in $\mathcal{P}_1$ but $v \prec u$ in $\mathcal{P}_2$.

Two equivalent persegrams may have different ordering of variables. If, however, we are interested in the ordering of several specially connected variables only, then we obtain an another invariant of independence equivalence. It is based on *Ordering conditions* defined bellow.

Definition 3.11. *Let $\mathcal{P}$ be a persegram over N. An* Ordering condition *is a triplet of variables $u, v, w \in N$ where $u, v \prec w$; $u, v \leftrightarrow w$; and $u \not\leftrightarrow v$ in $\mathcal{P}$. Such an ordering condition is denoted by $[u, v] \prec w[\mathcal{P}]$.*

An example of an ordering condition can be found it the second and third part of the Example 3.10 in $\mathcal{P}_1$. $[u, v] \prec w[\mathcal{P}_1]$ in that case. Persegrams $\mathcal{P}_2$ from both those parts of that Example do not contain any ordering condition.

Lemma 3.12. *Let $\mathcal{P}$ be a persegram over N, $u, v, w \in N$ distinct nodes. Then*

$$[u, v] \prec w \Leftrightarrow u \not\perp\!\!\!\perp v | + w[\mathcal{P}].$$

Proof. Suppose $[u, v] \prec w[\mathcal{P}]$. By Remark 3.5 $u, v \in S_n$. Then Let $W \subset N \setminus \{u, v\}$ such that $w \in W$. As one can see on the Figure 5 W-avoiding trail composed from u, w, v connects u, v for every W. Hence, $u \not\perp\!\!\!\perp v | W[\mathcal{P}]$ for every W (denoted by $u \not\perp\!\!\!\perp v | + w[\mathcal{P}]$). To prove sufficiency suppose by contradiction that $u \succ w \vee v \succ w$. (Since $u \not\leftrightarrow v$, one does not have to consider equalities in $u \succeq w, v \succeq w$) Suppose $u \prec v$, then by remark 3.5 $w \in S_{]v[}$. Since $u \not\leftrightarrow v$ the lemma 3.6 leads to contradiction. □

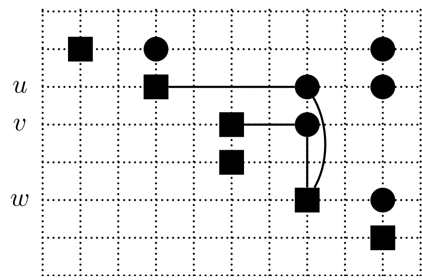


Figure 5: $u \not\perp\!\!\!\perp v| + w$

The above mentioned invariant can be easily concluded into the following implication.

Corollary 3.13. *Let $\mathcal{P}, \mathcal{P}'$ be two persegrams over N. If $\mathcal{I}_{\mathcal{P}} = \mathcal{I}_{\mathcal{P}'}$ then $\mathcal{E}(\mathcal{P}) = \mathcal{E}(\mathcal{P}')$ and they induce the same set of ordering conditions.*

The question is: Does this implication hold also in the opposite direction? I.e. If two persegrams $\mathcal{P}, \mathcal{P}'$ over the same set N induce the same ordering conditions and $\mathcal{E}(\mathcal{P}) = \mathcal{E}(\mathcal{P}')$, are $\mathcal{P}, \mathcal{P}'$ independence equivalent? The answer for this question is still unknown. Despite the fact that all experiments confirm this theory, the formal proof has not been finished yet.

4 Conclusion

In this paper we gave a short introduction into equivalence problem. This problem includes several sub-problems where one of them is how to simply recognize whether two given persegrams are equivalent. One can say, how to recognize equivalence "on the first sight". The solution to this problem is a direct characterization involving some invariants sufficient for equivalence decision.

Two invariants we introduced: *Connections set* and *Ordering conditions*. Are these invariants sufficient to decide whether two given persegrams are equivalent? This question remains open.

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