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There are Combinations and Compositions in Dempster-Shafer Theory of Evidence

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Abstract

It is a generally accepted fact that the Dempster's rule of combination plays a key role in Dempster-Shafer Theory of Evidence. In this paper the authors compare this combination rule with another one, which is called composition, and which was designed to create multidimensional basic assignments from a system of low-dimensional ones. The goal of this paper is to show that though the mentioned methods of combination were designed for totally different reasons, they manifest some similar formal properties and under very special conditions they even coincide.

1 Introduction

Dempster's rule of combination is often used as a method of fusion of several sources of information: combining two subjective evaluations of beliefs one can get a "summarized" evaluation expressing knowledge from both the considered sources (e.g. [6, 1, 4]).

It is not the goal of this paper to bring arguments for or against the above mentioned way of interpretation of the Dempster's rule of combination. Our goal is to compare this rule of combination with another combining tool, so called *operator of composition*, proposed for construction of multidimensional models from a number of low-dimensional ones. Here we do not consider fusion in its proper meaning. The purpose why the operator of composition was designed was not to fuse imprecise descriptions about the same object but to compose a number of descriptions each of them describing different properties of the object to get its global description. Using the terminology of AI, operator of composition was proposed to construct a model of global knowledge from

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a system of pieces of local knowledge. So, it corresponds to the process of *knowledge integration.*

Keeping this in mind, it is quite natural that we do not want to compare the mentioned two ways of combination to show that one of them is better than the other. Having been inspired by an anonymous referee of [3], we want to compare them from the formal point of view, because, though they were designed for different purposes, they manifest some similar properties, and they even coincide under some very special situations.

2 Notation and basic notions

2.1 Set notation

In the whole paper we will deal with a finite number of variables $X_1, X_2, \ldots, X_n$ each of which is specified by a finite set $\mathbf{X}_i$ of its values. So, we will consider multidimensional space of discernment

$$\mathbf{X}_N = \mathbf{X}_1 \times \mathbf{X}_2 \times \ldots \times \mathbf{X}_n,$$

and its *subspaces*. For $K \subset N = \{1, 2, \ldots, n\}$, $\mathbf{X}_K$ denotes a Cartesian product of those $\mathbf{X}_i$, for which $i \in K$:

$$\mathbf{X}_K = \times_{i \in K} \mathbf{X}_i.$$

A *projection* of $x = (x_1, x_2, \ldots, x_n) \in \mathbf{X}_N$ into $\mathbf{X}_K$ will be denoted $x^{\downarrow K}$, i.e. for $K = \{i_1, i_2, \ldots, i_\ell\}$

$$x^{\downarrow K} = (x_{i_1}, x_{i_2}, \ldots, x_{i_\ell}) \in \mathbf{X}_K.$$

Analogously, for $K \subset L \subseteq N$ and $A \subset \mathbf{X}_L$, $A^{\downarrow K}$ will denote a *projection* of A into $\mathbf{X}_K$:

$$A^{\downarrow K} = \{y \in \mathbf{X}_K : \exists x \in A \ \ (y = x^{\downarrow K})\}.$$

Let us remark that we do not exclude situations when $K = \emptyset$. In this case $A^{\downarrow \emptyset} = \emptyset$.

In addition to the projection, in this text we will need also the opposite operation which will be called join. By a *join* of two sets $A \subseteq \mathbf{X}_K$ and $B \subseteq \mathbf{X}_L$ we will understand a set

$$A \otimes B = \{x \in \mathbf{X}_{K \cup L} : x^{\downarrow K} \in A \ \ \& \ \ x^{\downarrow L} \in B\}.$$

Notice that if K and L are disjoint then their join is just their Cartesian product

$$A \otimes B = A \times B.$$

If $K = L$ then

$$A \otimes B = A \cap B. \tag{1}$$

If $K \cap L \neq \emptyset$ and $A^{\downarrow K \cap L} \cap B^{\downarrow K \cap L} = \emptyset$ then also $A \otimes B = \emptyset$. Generally,

$$A \otimes B = (A \times \mathbf{X}_{L \setminus K}) \cap (B \times \mathbf{X}_{K \setminus L}). \tag{2}$$

2.2 Basic assignment notation

The role of a probability distribution from a probability theory is in Dempster-Shafer theory played by a basic (*probability or belief*) assignment. In this paper we shall use exclusively normalized basic assignments.

A *basic assignment* m on $\mathbf{X}_K$ is a function

$$m : \mathcal{P}(\mathbf{X}_K) \longrightarrow [0,1],$$

for which $m(\emptyset) = 0$ and

$$\sum_{A \subseteq \mathbf{X}_K} m(A) = 1.$$

A basic assignment on $\mathbf{X}_K$ is called *vacuous* if $m(\mathbf{X}_K) = 1$, and it is called *simple* basic assignment *focused* on A (for $\emptyset \neq A \subset \mathbf{X}_K$) if $m(A) = a$ for $a > 0$ and $m(\mathbf{X}_K) = 1 - a$.

If $m(A) > 0$, then A is said to be a *focal element* of m. If all the focal elements of m are singletons (i.e. $m(A) > 0$ implies that $|A| = 1$) then we say that m is *Bayesian.*

For $L \subset K$ and basic assignment m on $\mathbf{X}_K$ one gets its *marginal basic assignment* $m^{\downarrow L}$ by computing for each $B \subseteq \mathbf{X}_L$:

$$m^{\downarrow L}(B) = \sum_{A \subseteq \mathbf{X}_K : A^{\downarrow L} = B} m(A).$$

Conversely, let m be a basic assignment on $\mathbf{X}_L$. Its *vacuous extension* on $\mathbf{X}_K$ is defined for all $A \subseteq \mathbf{X}_K$ in the following way

$$m^{\uparrow K}(A) = \begin{cases} m(A^{\downarrow L}) & \text{if } A = A^{\downarrow L} \times \mathbf{X}_{K \setminus L}, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

2.3 Dempster's rule of combination

Dempster's rule of combination is usually defined for two basic assignments m_1, m_2 defined on the same frame of discernment (say $\mathbf{X}_K$) by the formula

$$(m_1 \oplus m_2)(C) = \frac{\sum_{A,B \subseteq \mathbf{X}_K A \cap B = C} m_1(A) m_2(B)}{1 - \sum_{A,B \subseteq \mathbf{X}_K : A \cap B = \emptyset} m_1(A) m_2(B)}, \tag{4}$$

for each $C \subseteq \mathbf{X}_K$. For the purpose of this paper we need its generalization to cover situations when one wants to combine two basic assignments, which are not defined on the same frame of discernment. Regarding equality (1), the natural generalization, which will be used in this paper, is the one introduced in the following definition.

Definition 1. For two arbitrary basic assignments m_1 on $\mathbf{X}_K$ and m_2 on $\mathbf{X}_L$ ($K \neq \emptyset \neq L$) their *combination* is computed according to the formula (for all

$C \subseteq \mathbf{X}_{K\cup L})$[1]:

$$(m_1 \oplus m_2)(C) = \frac{\sum\limits_{A\subseteq \mathbf{X}_K} \sum\limits_{B\subseteq \mathbf{X}_L : A\otimes B = C} m_1(A)m_2(B)}{1 - \sum\limits_{A\subseteq \mathbf{X}_K} \sum\limits_{B\subseteq \mathbf{X}_L : A\otimes B = \emptyset} m_1(A)m_2(B)}.$$

Substituting vacuous extensions of m_1 and m_2 on $\mathbf{X}_{K\cup L}$ into formula (4), one gets

$$\begin{aligned}(m_1 \oplus m_2)(C) &= \frac{\sum\limits_{A,B\subseteq \mathbf{X}_{K\cup L} A\cap B = C} m_1^{\uparrow K\cup L}(A) m_2^{\uparrow K\cup L}(B)}{1 - \sum\limits_{A,B\subseteq \mathbf{X}_{K\cup L} : A\cap B=\emptyset} m_1^{\uparrow K\cup L}(A) m_2^{\uparrow K\cup L}(B)} \\ &= \frac{\sum\limits_{D\subseteq \mathbf{X}_K} \sum\limits_{E\subseteq \mathbf{X}_L : (D\times \mathbf{X}_{L\setminus K})\cap(E\times \mathbf{X}_{K\setminus L}) = C} m_1(D)m_2(E)}{1 - \sum\limits_{D\subseteq \mathbf{X}_K} \sum\limits_{E\subseteq \mathbf{X}_L : (D\times \mathbf{X}_{L\setminus K})\cap(E\times \mathbf{X}_{K\setminus L}) = \emptyset} m_1(D)m_2(E)},\end{aligned}$$

which is equivalent (taking into account expression (2)) the formula in Definition 1.

It is well known [5] that the following basic properties hold true for Dempster's rule of combination.

Lemma 1. *Let $K, L, M \subseteq N$. For arbitrary basic assignments m_1, m_2, m_3 defined on $\mathbf{X}_K, \mathbf{X}_L \mathbf{X}_M$, respectively:*

(i) *$m_1 \oplus m_2$ is a basic assignment on $\mathbf{X}_{K\cup L}$;*

(ii) *$m_1 \oplus m_2 = m_2 \oplus m_1$;*

(iii) *$(m_1 \oplus m_2) \oplus m_3 = m_1 \oplus (m_2 \oplus m_3)$.*

2.4 Operator of composition

An operator of composition was for basic assignments defined in [2] by the following definition.

Definition 2. For two arbitrary basic assignments m_1 on $\mathbf{X}_K$ and m_2 on $\mathbf{X}_L$ $(K \neq \emptyset \neq L)$ a *composition* $m_1 \triangleright m_2$ is defined for each $C \subseteq \mathbf{X}_{K\cup L}$ by one of the following expressions:

[a] if $m_2^{\downarrow K\cap L}(C^{\downarrow K\cap L}) > 0$ and $C = C^{\downarrow K} \otimes C^{\downarrow L}$ then

$$(m_1 \triangleright m_2)(C) = \frac{m_1(C^{\downarrow K}) \cdot m_2(C^{\downarrow L})}{m_2^{\downarrow K\cap L}(C^{\downarrow K\cap L})};$$

[1] For the purpose of this paper we do not consider situations when

$$\sum_{A\subseteq \mathbf{X}_K} \sum_{B\subseteq \mathbf{X}_L : A\otimes B=\emptyset} m_1(A)m_2(B) = 1.$$

[b] if $m_2^{\downarrow K\cap L}(C^{\downarrow K\cap L}) = 0$ and $C = C^{\downarrow K} \times \mathbf{X}_{L\setminus K}$ then

$$(m_1 \triangleright m_2)(C) = m_1(C^{\downarrow K});$$

[c] in all other cases $(m_1 \triangleright m_2)(C) = 0$.

Example 1. Let $\mathbf{X}_1 = \{a, \bar{a}\}, \mathbf{X}_2 = \{b, \bar{b}\}$ and $\mathbf{X}_3 = \{c, \bar{c}\}$ be three frames of discernment and let us consider the following two simple basic assignments m_1 and m_2 defined on $\mathbf{X}_1 \times \mathbf{X}_2$ and $\mathbf{X}_2 \times \mathbf{X}_3$, respectively:

$$\begin{aligned} m_1(\mathbf{X}_1 \times \{b\}) &= 0.4, \\ m_1(\mathbf{X}_1 \times \mathbf{X}_2) &= 0.6, \\ m_2(\mathbf{X}_2 \times \{c\}) &= 0.5, \\ m_2(\mathbf{X}_2 \times \mathbf{X}_3) &= 0.5. \end{aligned}$$

From Definition 2 one can immediately see that the formula in case **[a]** can assign a positive value to $(m_1 \triangleright m_2)(A)$ and/or $(m_2 \triangleright m_1)(A)$ only for those $A \subseteq \mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$ for which

$$A^{\downarrow\{1,2\}} = \mathbf{X}_1 \times \{b\} \quad \text{or} \quad A^{\downarrow\{1,2\}} = \mathbf{X}_1 \times \mathbf{X}_2,$$

and

$$A^{\downarrow\{2.3\}} = \mathbf{X}_2 \times \{c\} \quad \text{or} \quad A^{\downarrow\{2,3\}} = \mathbf{X}_2 \times \mathbf{X}_3.$$

There are only two such sets, namely:

$$\mathbf{X}_1 \times \mathbf{X}_2 \times \{c\} \quad \text{and} \quad \mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3.$$

For these sets we get

$$\begin{aligned} (m_1 \triangleright m_2)(\mathbf{X}_1 \times \mathbf{X}_2 \times \{c\}) &= \frac{m_1(\mathbf{X}_1 \times \mathbf{X}_2) \cdot m_2(\mathbf{X}_2 \times \{c\})}{m_2^{\downarrow\{2\}}(\mathbf{X}_2)} = \frac{0.6 \cdot 0.5}{1} = 0.3, \\ (m_1 \triangleright m_2)(\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3) &= \frac{m_1(\mathbf{X}_1 \times \mathbf{X}_2) \cdot m_2(\mathbf{X}_2 \times \mathbf{X}_3)}{m_2^{\downarrow\{2\}}(\mathbf{X}_2)} = \frac{0.6 \cdot 0.5}{1} = 0.3 \end{aligned}$$

and similarly

$$\begin{aligned} (m_2 \triangleright m_1)(\mathbf{X}_1 \times \mathbf{X}_2 \times \{c\}) &= \frac{m_2(\mathbf{X}_2 \times \{c\}) \cdot m_1(\mathbf{X}_1 \times \mathbf{X}_2)}{m_1^{\downarrow\{2\}}(\mathbf{X}_2)} = \frac{0.5 \cdot 0.6}{0.6} = 0.5, \\ (m_2 \triangleright m_1)(\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3) &= \frac{m_2(\mathbf{X}_2 \times \mathbf{X}_3) \cdot m_1(\mathbf{X}_1 \times \mathbf{X}_2)}{m_1^{\downarrow\{2\}}(\mathbf{X}_2)} = \frac{0.5 \cdot 0.6}{0.6} = 0.5. \end{aligned}$$

Since $m_2(\{b\}) = 0$, from case **[b]** of Definition 2 we will get yet another focal element for $m_1 \triangleright m_2$, namely

$$A = \mathbf{X}_1 \times \{b\} \times \mathbf{X}_3,$$

for which

$$A^{\downarrow\{1,2\}} = \mathbf{X}_1 \times \{b\} \quad \text{and} \quad A^{\downarrow\{3\}} = \mathbf{X}_3.$$

Table 1: Composed basic assignments.

A	$(m_1 \triangleright m_2)(A)$	$(m_2 \triangleright m_1)(A)$
$\mathbf{X}_1 \times \mathbf{X}_2 \times \{c\}$	0.3	0.5
$\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$	0.3	0.5
$\mathbf{X}_1 \times \{b\} \times \mathbf{X}_3$	0.4	0

For this set we get

$$(m_1 \triangleright m_2)(\mathbf{X}_1 \times \{b\} \times \mathbf{X}_3) = m_1(\mathbf{X}_1 \times \{b\}) = 0.4.$$

Notice that when computing a composition $m_2 \triangleright m_1$, case **[b]** of Definition 2 does not assign a positive value to any subset A of $\mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$, since if $m_2^{\downarrow\{2\}}(A^{\downarrow\{2\}}) > 0$ then also $m_1^{\downarrow\{2\}}(A^{\downarrow\{2\}}) > 0$.

Both the composed basic assignments $m_1 \triangleright m_2$ and $m_2 \triangleright m_1$ are outlined in Table 1 (recall once more that for all other $A \subseteq \mathbf{X}_1 \times \mathbf{X}_2 \times \mathbf{X}_3$ different from those included in Table 1, both assignments equal 0). It is also evident from the table that the operator $\triangleright$ is not commutative.

Let us present the most important properties of the operator of composition for basic assignments, which were proved in [2].

Lemma 2. *Let $K, L \subseteq N$. For arbitrary basic assignments m_1, m_2 defined on $\mathbf{X}_K$ and $\mathbf{X}_L$, respectively:*

(i) *$m_1 \triangleright m_2$ is a basic assignment on $\mathbf{X}_{K\cup L}$;*

(ii) *$m_1 \triangleright m_2 = m_2 \triangleright m_1 \iff m_1^{\downarrow K_1\cap K_2} = m_2^{\downarrow K_1\cap K_2}$;*

(iii) *$(m_1 \triangleright m_2)^{\downarrow K_1} = m_1$.*

3 Relation of combinations and compositions

3.1 Disjoint domains

Theorem 1. *Let $K, L \subseteq N$ and m_1, m_2 be basic assignments defined on $\mathbf{X}_K$ and $\mathbf{X}_L$, respectively. If $K \cap L = \emptyset$ then*

$$m_1 \triangleright m_2 = m_2 \triangleright m_1 = m_1 \oplus m_2.$$

Proof. For disjoint K, L and $A \subseteq \mathbf{X}_K, B \subseteq \mathbf{X}_L$ one gets $A \otimes B = A \times B$ and $m_2^{\downarrow K\cap L} \equiv 1$. Therefore, for computation of $m_1 \triangleright m_2$ (for any focal element $C \subseteq \mathbf{X}_{K\cup L}$ of $m_1 \triangleright m_2$) only case [a] of Definition 2 is employed, and therefore

$$(m_1 \triangleright m_2)(C) = m_1(C^{\downarrow K}) \cdot m_2(C^{\downarrow L}) = \sum_{A=C^{\downarrow K}} \sum_{B=C^{\downarrow L}} m_1(A)m_2(B)$$

$$= \sum_{A\subseteq \mathbf{X}_K} \sum_{B\subseteq \mathbf{X}_L : A\otimes B=C} m_1(A)m_2(B) = (m_1 \oplus m_2)(C),$$

because, in this case,

$$\sum_{A\subseteq \mathbf{X}_K} \sum_{B\subseteq \mathbf{X}_L: A\cap B=\emptyset} m_1(A)m_2(B) = 0.$$

The fact that $m_1 \triangleright m_2 = m_2 \triangleright m_1$ follows immediately from property (ii) of Lemma 2. □

3.2 Identical domains

Theorem 2. *If for arbitrary two basic assignments m_1, m_2 on $\mathbf{X}_K$ each focal element of m_2 contains all the focal elements of m_1, i.e.*

$$m_1(A) > 0, m_2(B) > 0 \quad \Longrightarrow \quad A \subseteq B,$$

then

$$m_1 \triangleright m_2 = m_1 \oplus m_2.$$

Proof. First, compute

$$\begin{aligned} \sum_{A,B\subseteq \mathbf{X}_K: A\otimes B=\emptyset} m_1(A)m_2(B) &= \sum_{A,B\subseteq \mathbf{X}_K: A\cap B=\emptyset} m_1(A)m_2(B) \\ &= \sum_{A\subseteq \mathbf{X}_K} m_1(A) \sum_{B\subseteq \mathbf{X}_K: A\cap B=\emptyset} m_2(B) = 0, \end{aligned}$$

because, under the given assumptions, for each focal element A of m_1

$$\sum_{B\subseteq \mathbf{X}_K: A\cap B=\emptyset} m_2(B) = 0.$$

Now, we can easily compute $(m_1 \oplus m_2)(C)$ for any focal element C of m_1.

$$\begin{aligned} (m_1 \oplus m_2)(C) &= \sum_{A\subseteq \mathbf{X}_K} m_1(A) \sum_{B\subseteq \mathbf{X}_K: A\otimes B=C} m_2(B) = \sum_{A\subseteq \mathbf{X}_K: A=C} m_1(A) \\ &= m_1(C). \end{aligned}$$

In this way we obtained that $(m_1 \oplus m_2)(C) = m_1(C)$ for all focal elements C of m_1. Therefore, since

$$\sum_{C\subseteq \mathbf{X}_K} (m_1 \oplus m_2)(C) = \sum_{C\subseteq \mathbf{X}_K} m_1(C) = 1,$$

it is clear that $(m_1 \oplus m_2)(C) = m_1(C)$ for all $C \subseteq \mathbf{X}_K$, and therefore also

$$m_1 \oplus m_2 = m_1 = m_1 \triangleright m_2.$$ □

As a special case of Theorem 2 one gets the following assertion.

Corollary 1. *Let m_1 be an arbitrary basic assignment on $\mathbf{X}_K$ and let $\mathcal{F}$ denote the set of its focal elements. If m_2 is a simple basic assignment on $\mathbf{X}_K$ focused on B such that $B \supseteq \cup_{A\in\mathcal{F}} A$, then*

$$m_1 \triangleright m_2 = m_1 \oplus m_2.$$

3.3 General situation

Let us start studying general overlapping (but not identical) frames of discernment by an example illustrating the fact that a sufficient condition describing situations when combination and composition results in the same basic assignments cannot be obtained as a generalization of results from the previous two subsections.

Example 2. Consider two basic assignments m_1 on $\mathbf{X}_{\{1,2,3\}}$ and m_2 on $\mathbf{X}_{\{2,3,4\}}$ (with $\mathbf{X}_1 = \{a, \bar{a}\}, \mathbf{X}_2 = \{b, \bar{b}\}, \mathbf{X}_3 = \{c, \bar{c}\}, \mathbf{X}_4 = \{d, \bar{d}\}$), each having only two focal elements:

$$\begin{array}{lll} m_1: & A_1 = \{abc\}, A_2 = \{abc, \bar{a}\bar{b}\bar{c}\} & m_1(A_1) = 1/4, m_1(A_2) = 3/4. \\ m_2: & B_1 = \{bcd, \bar{b}\bar{c}d\}, B_2 = \{bcd, b\bar{c}d, \bar{b}\bar{c}\bar{d}\} & m_2(B_1) = 1/3, m_2(B_2) = 2/3. \end{array}$$

The reader can immediately see that each focal element of $m_2^{\downarrow\{2,3\}}$ contains all the focal elements of $m_1^{\downarrow\{2,3\}}$; i.e. $A_1^{\downarrow\{2,3\}} = \{bc\}$ and $A_2^{\downarrow\{2,3\}} = \{bc, \bar{b}\bar{c}\}$ are subsets of both $B_1^{\downarrow\{2,3\}} = \{bc, \bar{b}\bar{c}\}$ and $B_2^{\downarrow\{2,3\}} = \{bc, b\bar{c}, \bar{b}\bar{c}\}$.

Realizing that

$$\begin{aligned} A_1 \otimes B_1 &= \{abcd\}, \\ A_1 \otimes B_2 &= \{abcd\}, \\ A_2 \otimes B_1 &= \{abcd, \bar{a}\bar{b}\bar{c}d\}, \\ A_2 \otimes B_2 &= \{abcd, \bar{a}\bar{b}\bar{c}\bar{d}\}, \end{aligned}$$

it is clear that

$$\sum_{A \subseteq \mathbf{X}_{\{1,2,3\}}} \sum_{B \subseteq \mathbf{X}_{\{2,3,4\}} : A \otimes B = \emptyset} m_1(A) m_2(B) = 0,$$

and therefore

$$\begin{aligned} (m_1 \oplus m_2)(\{abcd\}) &= \sum_{A \subseteq \mathbf{X}_{\{1,2,3\}}} \sum_{B \subseteq \mathbf{X}_{\{2,3,4\}} : A \otimes B = \{abcd\}} m_1(A) m_2(B) \\ &= m_1(A_1) m_2(B_1) + m_1(A_1) m_2(B_2) = 1/4. \end{aligned}$$

When computing $m_1 \triangleright m_2$ one has to realize that even though

$$\{abcd\} = \{abcd\}^{\downarrow\{1,2,3\}} \otimes \{abcd\}^{\downarrow\{2,3,4\}},$$

$m_2^{\downarrow\{2,3\}}(\{bc\}) = 0$ and therefore neither case [a] nor [b] of Definition 2 is applicable for computing $(m_1 \triangleright m_2)(\{abcd\})$, and therefore it equals 0 according to case [c]. So we obtained that in this example $m_1 \oplus m_2 \neq m_1 \triangleright m_2$.

Theorem 3. *Let m_1 on $\mathbf{X}_K$ and m_2 on $\mathbf{X}_L$ be such basic assignments that each focal element A of m_1 and each focal element B of m_2 projects to a unique set in $\mathbf{X}_{K \cap L}$. Then*

$$m_1 \triangleright m_2 = m_1 \oplus m_2.$$

Proof. First, let us note that the assumption that all focal elements of both m_1 and m_2 project to a unique set implies, that $m_2^{\downarrow K \cap L}(A^{\downarrow K \cap L}) = 1$ for any focal element A of m_1.

Now, consider any $C \subseteq \mathbf{X}_{K\cup L}$ for which $C = C^{\downarrow K} \otimes C^{\downarrow L}$. For this C

$$(m_1 \oplus m_2)(C) = \frac{\sum\limits_{A\subseteq \mathbf{X}_K} \sum\limits_{B\subseteq \mathbf{X}_L : A\otimes B = C} m_1(A)m_2(B)}{1 - \sum\limits_{A\subseteq \mathbf{X}_K} \sum\limits_{B\subseteq \mathbf{X}_L : A\otimes B = \emptyset} m_1(A)m_2(B)}$$
$$\geq \sum_{A\subseteq \mathbf{X}_K} \sum_{B\subseteq \mathbf{X}_L : A\otimes B = C} m_1(A)m_2(B) \geq m_1(C^{\downarrow K}) \cdot m_2(C^{\downarrow L}).$$

Simultaneously, if $m_1(C^{\downarrow K}) > 0$,

$$(m_1 \triangleright m_2)(C) = \frac{m_1(C^{\downarrow K}) \cdot m_2(C^{\downarrow L})}{m_2^{\downarrow K\cap L}(C^{\downarrow K\cap L})} = m_1(C^{\downarrow K}) \cdot m_2(C^{\downarrow L}).$$

Since if $m_1(C^{\downarrow K}) = 0$ then also

$$(m_1 \triangleright m_2)(C) = 0 = m_1(C^{\downarrow K}) \cdot m_2(C^{\downarrow L}),$$

one can see that for all $C \subseteq \mathbf{X}_{K\cup L}$ for which $C = C^{\downarrow K} \otimes C^{\downarrow L}$

$$(m_1 \oplus m_2)(C) \geq m_1(C^{\downarrow K}) \cdot m_2(C^{\downarrow L}) = (m_1 \triangleright m_2)(C).$$

Regarding Definition 2, according to which $(m_1 \triangleright m_2)(C) = 0$ for $C \neq C^{\downarrow K} \otimes C^{\downarrow L}$, we see that

$$(m_1 \oplus m_2)(C) \geq (m_1 \triangleright m_2)(C)$$

holds true for all $C \subseteq \mathbf{X}_{K\cup L}$, from which, because both $m_1 \oplus m_2$ and $m_1 \triangleright m_2$ are normalized basic assignments, we get that $m_1 \oplus m_2 = m_1 \triangleright m_2$. □

Example 3. Let X_1, X_2 and X_3 be three binary variables with values in $\mathbf{X}_1 = \{a, \bar{a}\}, \mathbf{X}_2 = \{b, \bar{b}\}$, $\mathbf{X}_3 = \{c, \bar{c}\}$ and m_1 and m_2 be two basic assignments on $\mathbf{X}_1 \times \mathbf{X}_3$ and $\mathbf{X}_2 \times \mathbf{X}_3$ respectively, both of them having only two focal elements:

$$\begin{array}{lll} m_1: & A_1 = \{a\bar{c}, \bar{a}\bar{c}\}, A_2 = \{a\bar{c}, \bar{a}c\} & m_1(A_1) = 1/2, m_1(A_2) = 1/2. \\ m_2: & B_1 = \{b\bar{c}, \bar{b}\bar{c}\}, B_2 = \{b\bar{c}, \bar{b}c\} & m_2(B_1) = 1/2, m_2(B_2) = 1/2. \end{array} \quad (5)$$

One can immediately see that both $A_1 \otimes B_2$ and $A_2 \otimes B_1$ are empty and therefore $m_1 \oplus m_2$ has only two focal elements, namely $A_1 \otimes B_1 = \mathbf{X}_1 \times \mathbf{X}_2 \times \{\bar{c}\}$ and $A_2 \otimes B_2 = \{ab\bar{c}, \bar{a}\bar{b}c\}$. For these focal elements we have

$$(m_1 \oplus m_2)(\mathbf{X}_1 \times \mathbf{X}_2 \times \{\bar{c}\}) = \frac{m_1(A_1)m_2(B_1)}{1 - (m_1(A_1)m_2(B_2) + m_1(A_2)m_2(B_1))} = 1/2,$$
$$(m_1 \oplus m_2)(\{ab\bar{c}, \bar{a}\bar{b}c\}) = \frac{m_1(A_2)m_2(B_2)}{1 - (m_1(A_1)m_2(B_2) + m_1(A_2)m_2(B_1))} = 1/2$$

and simultaneously

$$(m_1 \triangleright m_2)(\mathbf{X}_1 \times \mathbf{X}_2 \times \{\bar{c}\}) = 1/2,$$
$$(m_1 \triangleright m_2)(\{ab\bar{c}, \bar{a}\bar{b}c\}) = 1/2.$$

Thus we got that for the basic assignments defined in expressions (5) $m_1 \oplus m_2 = m_1 \triangleright m_2$. Nevertheless, it does not mean that for any couple of basic assignments m_1, m_2 defined on $\mathbf{X}_1 \times \mathbf{X}_2$, $\mathbf{X}_2 \times \mathbf{X}_3$, respectively, with the respective focal elements A_1, A_2 and B_1, B_2, the coincidence must hold. This happened because we chose special values of the considered basic assignments. If we change the values of m_1 and m_2 e.g. in the following way:

$$m'_1(A_1) = 1/3 \quad m'_1(A_2) = 2/3,$$
$$m'_2(B_1) = 1/3 \quad m'_2(B_2) = 2/3,$$

we will get, analogously to (6),

$$(m'_1 \oplus m'_2)(\mathbf{X}_1 \times \mathbf{X}_2 \times \{\bar{c}\}) = 1/5,$$
$$(m_1 \oplus m_2)(\{ab\bar{c}, \bar{a}\bar{b}c\}) = 4/5,$$

and

$$(m_1 \triangleright m_2)(\mathbf{X}_1 \times \mathbf{X}_2 \times \{\bar{c}\}) = 1/3,$$
$$(m_1 \triangleright m_2)(\{ab\bar{c}, \bar{a}\bar{b}c\}) = 2/3.$$

Special property holds for Bayesian basic assignments.

Theorem 4. *Let $K, L \subseteq N$ and m_1, m_2 be Bayesian basic assignments defined on $\mathbf{X}_K$ and $\mathbf{X}_L$, respectively. Then*

$$m_1 \triangleright m_2 = m_1 \oplus m_2$$

if $m_2^{\downarrow K\cap L}$ corresponds to uniform probability distribution.

Proof. The assumption that $m_2^{\downarrow K\cap L}$, being Bayesian basic assignment, corresponds to the uniform probability distribution implies that $m_2^{\downarrow K\cap L}$ is positive for any singleton from $\mathbf{X}_{K\cap L}$. This shows that case [b] of Definition 2 is not applicable to any $C \subseteq \mathbf{X}_{K\cup L}$ such that $C^{\downarrow K\cap L}$ is singleton.

Now consider an arbitrary singleton $C \subset \mathbf{X}_{K\cup L}$. It is obvious that $C = C^{\downarrow K} \otimes C^{\downarrow L}$ and therefore, according to case [a] of Definition 2,

$$(m_1 \triangleright m_2)(C) = \frac{m_1(C^{\downarrow K}) \cdot m_2(C^{\downarrow L})}{\beta}, \tag{6}$$

where $\beta = m_2^{\downarrow K\cap L}(C^{\downarrow K\cap L})$ is, due to the assumption posed on $m_2^{\downarrow K\cap L}$, the same for all singletons $C \subset \mathbf{X}_{K\cup L}$. On the other hand, if $C \subset \mathbf{X}_{K\cup L}$ is not singleton then either $C^{\downarrow K}$ or $C^{\downarrow L}$ cannot be singleton and therefore, if $(m_1 \triangleright m_2)(C)$ is assigned by case [a] of Definition 2, the value of $(m_1 \triangleright m_2)(C)$ is 0. In case that $(m_1 \triangleright m_2)(C)$ is assigned by case [b] of Definition 2, the resulting value is also 0, because this case is applicable only when $m_2^{\downarrow K\cap L}(C^{\downarrow K\cap L}) = 0$, which may appear only when $C^{\downarrow K\cap L}$ is not singleton and therefore neither $C^{\downarrow K}$ is a singleton, which means that $m_1(C^{\downarrow K}) = 0$. So, we showed that $m_1 \triangleright m_2$ is defined by (6) for singletons and for non-singletons it equals 0.

Let us denote

$$\alpha = \sum_{A\subseteq \mathbf{X}_K} \sum_{B\subseteq \mathbf{X}_L : A\otimes B=\emptyset} m_1(A) \cdot m_2(B).$$

For the considered Bayesian assignments

$$m_1(A) \cdot m_2(B)$$

can be positive only when both $A \subseteq \mathbf{X}_K$ and $B \subseteq \mathbf{X}_L$ are singletons. Therefore for any singleton $C \subseteq \mathbf{X}_{K\cup L}$

$$\begin{aligned}(m_1 \oplus m_2)(C) &= \frac{\sum_{A\subseteq\mathbf{X}_K} \sum_{B\subseteq\mathbf{X}_L : A\otimes B=C} m_1(A)\cdot m_2(B)}{1-\alpha} \\ &= \frac{m_1(C^{\downarrow K})\cdot m_2(C^{\downarrow L})}{1-\alpha}, \end{aligned} \tag{7}$$

and for non-singletons C

$$(m_1 \oplus m_2)(C) = 0 = (m_1 \triangleright m_2)(C).$$

To prove the required equality

$$(m_1 \oplus m_2)(C) = (m_1 \triangleright m_2)(C)$$

also for singletons it is enough to compare equalities (7) and (6) and again realize that both $m_1 \oplus m_2$ and $m_1 \triangleright m_2$ are normalized basic assignments and therefore $1 - \alpha = \beta$. □

4 Conclusions

In the paper we introduced the operator of composition for basic assignments and compared it with the famous Dempster's rule of combination. We showed that though Dempster's rule of combination and operator of composition were designed for different purposes they coincide in special situations; $m_1 \oplus m_2 = m_1 \triangleright m_2$

- when the combined basic assignments m_1 and m_2 are defined on disjoint frames of discernment;
- when all the focal elements of m_1 are contained in each focal element of m_2 and the basic assignments in question are defined on the same frame of discernment;
- when all the focal elements of both m_1 and m_2 project to the same subset of the overlapping frame of discernment.

Naturally, as shown in Example 3, the above described situations do not form a complete list of conditions under which the studied two operators coincide.

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