# Belief Functions on Formulas in Łukasiewicz Logic

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**Abstract**

Belief functions are generalized to formulas in Łukasiewicz logic. It is shown that they generalize probabilities on formulas (so-called states) and that they are completely monotone mappings with respect to the lattice operations.

## 1 Introduction

Belief measures are certain non-additive real-valued set functions introduced by Dempster and Shafer [10, 12]. Roughly speaking, models based on belief measures are used in situations in which the precise probabilistic model consisting of one probability measure is not available due to the lack of information about the conditions or results of some random experiment. From the mathematical point of view, belief measures are completely monotone set functions in the sense of Choquet [11], who studied complete monotonicity of capacities in the systematic way.

The aim of this paper is to introduce belief functions in the framework of Łukasiewicz logic. This is accomplished by an extension procedure that assigns a functional to some belief measure via Choquet integral. In this general setting the key issue is to clarify the meaning of total monotonicity, which can be expressed on an arbitrary Abelian semigroup according to Choquet. The concept of belief function proposed in this paper includes many-valued analogues of probabilities on formulas, the so-called states. States were introduced by Mundici [9] in order to model the notion of "average truth-value" of formulas. It was proved in [5] and [6] that the mathematical properties of states indeed fits this idea, namely, every state is the Lebesgue integral of (an equivalence class of) a formula w.r.t. a Borel probability measure on possible worlds. Since this result is of an independent interest and motivates the forthcoming definition of a belief function, a new proof is given in Section 3.

The paper is structured as follows. Section 2 contains necessary definitions and results concerning Łukasiewicz infinite-valued propositional logic and its associated Lindenbaum algebra $L_k$ of (equivalence classes of) formulas over $k$ propositional variables. Section 3 is devoted to states. In particular, it will be shown that the geometrical structure of formulas in $L_k$ makes possible to derive the integral representation of states (Theorem 1). In Section 4 we investigate belief functions on formulas in $L_k$ and show a number of generalizations of results known for classical belief measures on events (Theorem 3 and 4).

## 2 Preliminary Notions

The aim of this section is to provide a survey of Łukasiewicz infinite-valued propositional logic [1, Chapter 4] and its associated Lindenbaum algebra. Formulas $\varphi, \psi, \ldots$ are constructed from propositional variables $A_1, \ldots, A_k$ by applying the standard rules known in Boolean logic. The connectives are negation, disjunction and conjunction, which are denoted by $\neg$, $\oplus$ and $\odot$, respectively. This is already a complete set of connectives so that, for instance, the implication $\varphi \to \psi$ can be defined as $\neg\varphi \oplus \psi$. The set of all formulas in propositional variables $A_1, \ldots, A_k$ is denoted by $\mathrm{Form}(A_1, \ldots, A_k)$.

Semantics for connectives of Łukasiewicz logic is defined by operations in algebras called MV-algebras [1]. The algebra of truth degrees of Łukasiewicz logic is the *standard MV-algebra*, which is the unit interval $[0,1]$ endowed with the operations $\neg, \oplus, \odot$ defined as follows:

$$\begin{aligned} \neg a &= 1 - a \\ a \oplus b &= \min\{a + b, 1\} \\ a \odot b &= \max\{a + b - 1, 0\} \end{aligned}$$

A *valuation* is a mapping $V : \mathrm{Form}(A_1, \ldots A_k) \to [0,1]$ such that $V(\neg\varphi) = 1 - V(\varphi)$, $V(\varphi \oplus \psi) = V(\varphi) \oplus V(\psi)$ and $V(\varphi \odot \psi) = V(\varphi) \odot V(\psi)$. Formulas $\varphi, \psi \in \mathrm{Form}(A_1, \ldots, A_k)$ are called *equivalent* when $V(\varphi) = V(\psi)$, for every valuation $V$. The *equivalence class* of $\varphi$ is denoted $[\varphi]$. The set of all such equivalence classes is an MV-algebra $L_k$ with the operations $\neg[\varphi] = [\neg\varphi]$, $[\varphi] \oplus [\psi] = [\varphi \oplus \psi]$ and $[\varphi] \odot [\psi] = [\varphi \odot \psi]$, for every $\varphi, \psi \in \mathrm{Form}(A_1, \ldots, A_k)$.

Since every valuation $V$ is uniquely determined by its restriction to the propositional variables $V \mapsto V(A_1, \ldots, A_k) \in [0,1]^k$, every "possible world" $V$ is matched with a unique point $x_V$ from the $k$-dimensional unit cube $[0,1]^k$ and vice versa. Let $V_x$ be the valuation corresponding to $x \in [0,1]^k$. Put $[\varphi](x) = V_x(\varphi)$, for every $x \in [0,1]^k$. Hence the equivalence class $[\varphi]$ of every $\varphi \in \mathrm{Form}(A_1, \ldots A_k)$ can be viewed as a function $[0,1]^k \to [0,1]$ and $L_k$ is the algebra of all such functions endowed with the pointwise operations $\neg, \oplus, \odot$.

**McNaughton theorem** ([2]). $(L_k, \oplus, \odot, \neg)$ *is precisely the algebra of all functions* $[0,1]^k \to [0,1]$ *that are continuous and piecewise linear, where each linear piece has integer coefficients.*

Let $f \vee g = \neg(\neg f \oplus g) \oplus g$, $f \wedge g = \neg(\neg f \vee \neg g)$. These operations are in fact the pointwise supremum and infimum of functions in $L_k$, respectively, and they make $L_k$ into a distributive lattice.

A *filter* in $L_k$ is a subset $\mathcal{F}$ of $L_k$ such that (i) $1 \in \mathcal{F}$; (ii) if $f \in \mathcal{F}$ and $f \leq g$ with $g \in L_k$, then $g \in \mathcal{F}$; (iii) if $f, g \in \mathcal{F}$, then $f \odot g \in \mathcal{F}$. In this article we consider only filters $\mathcal{F}$ with $\mathcal{F} \neq L_k$. A *maximal filter* is a filter $\mathcal{F}$ such that no filter in $L_k$ strictly contains $\mathcal{F}$.

Theory of Schauder hats and bases in $L_k$, which was developed for the purely geometrical proof of McNaughton theorem [1, Section 9.1], is briefly repeated in this paragraph. The basic familiarity with polyhedral geometry and topology is assumed, see [3, 4], for instance. A *polyhedral complex (in* $[0,1]^k$*)* is a finite set of polyhedra $\mathcal{R}$ such that: (i) each polyhedron of $\mathcal{R}$ is included in $[0,1]^k$, all its vertices have rational coordinates; (ii) if $P \in \mathcal{R}$ and $Q$ is a face of $P$, then $Q \in \mathcal{R}$; (iii) if $P, Q \in \mathcal{R}$, then $P \cap Q$ is a face of both $P$ and $Q$. The set $\bigcup_{P \in \mathcal{R}} P$ is called a *support* of $\mathcal{R}$. When all the polyhedra of a polyhedral complex $\mathcal{S}$ are simplices, then $\mathcal{S}$ is said to be a *simplicial complex.* Alternatively, a simplicial complex $\mathcal{S}$ with the support $S$ is called a *triangulation* of $S$. The *denominator* $\operatorname{den}(q)$ of a point $q \in [0,1]^k$ with rational coordinates $(\frac{r_1}{s_1}, \ldots, \frac{r_k}{s_k})$, where $r_i \geq 0, s_i > 0$ are the uniquely determined relatively prime integers, is the least common multiple of $s_1, \ldots, s_k$. Passing to homogeneous coordinates in $\mathbb{R}^k$, put

$$\tilde{q} = \left( \tfrac{\operatorname{den}(q)}{s_1} r_1, \ldots, \tfrac{\operatorname{den}(q)}{s_k} r_k, \operatorname{den}(q) \right)$$

and note that $\tilde{q} \in \mathbb{Z}^{k+1}$. A $k$-simplex with vertices $v^0, \ldots, v^k$ is *unimodular* if $\{\tilde{v}^0, \ldots, \tilde{v}^k\}$ is a basis of the free Abelian group $\mathbb{Z}^{k+1}$. An $n$-simplex with $n < k$ is *unimodular* when it is a face of some unimodular $k$-simplex. We say that a triangulation $\Sigma$ is *unimodular* if each simplex of $\Sigma$ is unimodular. When $\mathcal{R}$ is a polyhedral complex, $\mathrm{V}(\mathcal{R})$ denotes the set of all the vertices of $\mathcal{R}$. Let $\Sigma$ be a unimodular triangulation with a support $S \subseteq [0,1]^k$. For each $x \in \mathrm{V}(\Sigma)$, the *Schauder hat (at* $x$ *over* $\Sigma$*)* is the uniquely determined continuous piecewise linear function $h_x : S \to [0,1]$ which attains the value $\frac{1}{\operatorname{den}(x)}$ at $x$, vanishes at each vertex from $\mathrm{V}(\Sigma) \setminus \{x\}$, and is a linear function on each simplex of $\Sigma$. The *basis* $H_\Sigma$ (over $\Sigma$) is the set $\{h_x \mid x \in \mathrm{V}(\Sigma)\}$.

# 3 States

States on MV-algebras are many-valued analogues of probabilities on Boolean algebras. The disjointness of functions in $L_k$ is captured by the relation $f \odot g = 0$, for $f, g \in L_k$. This condition also implies $f \oplus g = f + g$.

**Definition 1.** *A* state $s$ *on* $L_k$ *is a mapping* $s : L_k \to [0,1]$ *such that* $s(1) = 1$ *and* $s(f \oplus g) = s(f) + s(g)$, *for every* $f, g \in L_k$ *with* $f \odot g = 0$.

States on any (semisimple) MV-algebra were completely characterized in [5] and independently in [6] as integrals.

**Theorem 1.** *If $s$ is a state on $L_k$, then there exists a uniquely determined Borel probability measure $\mu$ on $[0,1]^k$ such that $s(f) = \int f \, \mathrm{d}\mu$, for each $f \in L_k$.*

In the rest of this section we give an alternative, a purely geometrical proof of Theorem 1. By $\mathcal{M}^1$ we denote the convex set of all Borel probability measures on $[0,1]^k$, which is a compact metric space in $w^*$-topology. For every sequence $(\mu_n)$ in $\mathcal{M}^1$,

$$\mu_n \xrightarrow{w^*} \mu \text{ iff } \int f d\mu_n \longrightarrow \int f d\mu,$$

for every continuous function $f : [0,1]^k \to \mathbb{R}$. Let $s$ be a state on $L_k$. In the sequel $\mathfrak{T}$ denotes the collection of all unimodular triangulations of $[0,1]^n$. Theorem 1 will be established in three steps.

**Claim 1.** *For every $\Sigma \in \mathfrak{T}$, the set of Borel probability measures*

$$\mathcal{M}_\Sigma = \{\mu \mid s(h_x) = \int h_x d\mu, \text{ for each } h_x \in H_\Sigma\}$$

*is nonempty and $w^*$-closed.*

*Proof.* Let $\delta_x$ denotes the Dirac measure concentrated at a point $x \in [0,1]^n$. Put

$$\delta = \sum_{x \in \mathrm{V}(\Sigma)} \mathrm{den}(x) s(h_x) \delta_x,$$

and observe that $\mathrm{den}(x)s(h_x) = s(\mathrm{den}(x)h_x) \in [0,1]$ for each $x \in \mathrm{V}(\Sigma)$. The sum $\sum_{x \in \mathrm{V}(\Sigma)} \mathrm{den}(x) h_x$ is constantly equal to 1 since it is equal to 1 at every vertex of $\mathrm{V}(\Sigma)$ and every Schauder hat is linear over each simplex of $\Sigma$. This gives

$$\sum_{x \in \mathrm{V}(\Sigma)} \mathrm{den}(x) s(h_x) = \sum_{x \in \mathrm{V}(\Sigma)} s(\mathrm{den}(x) h_x) = s\left(\sum_{x \in \mathrm{V}(\Sigma)} \mathrm{den}(x) h_x\right) = s(1) = 1.$$

Hence $\delta$ is a convex combination of Borel probability measures and therefore itself a Borel probability measure. We will show that $\delta \in \mathcal{M}_\Sigma$. For each vertex $x' \in \mathrm{V}(\Sigma)$, we get

$$\begin{aligned} \int h_{x'} \, \mathrm{d}\delta &= \sum_{x \in \mathrm{V}(\Sigma)} \int \mathrm{den}(x) s(h_x) h_{x'} \, \mathrm{d}\delta_x = \sum_{x \in \mathrm{V}(\Sigma)} \mathrm{den}(x) s(h_x) h_{x'}(x) \\ &= \mathrm{den}(x') s(h_{x'}) h_{x'}(x') = \mathrm{den}(x') s(h_{x'}) \frac{1}{\mathrm{den}(x')} = s(h_{x'}). \end{aligned} \tag{1}$$

In order to show that $\mathcal{M}_\Sigma$ is $w^*$-closed, consider a sequence $(\mu_n)$ in $\mathcal{M}_\Sigma$ with $\mu_n \xrightarrow{w^*} \mu$, for some $\mu \in \mathcal{M}^1$. It follows that for each $h_x \in H_\Sigma$ we obtain $s(h_x) = \int h_x \, \mathrm{d}\mu_n \longrightarrow \int h_x \, \mathrm{d}\mu$. Hence $s(h_x) = \int h_x \, \mathrm{d}\mu$ and $\mu \in \mathcal{M}_\Sigma$. □

**Claim 2.** *The collection of subsets $(\mathcal{M}_\Sigma)_{\Sigma \in \mathfrak{T}}$ of $\mathcal{M}^1$ has the finite intersection property.*

*Proof.* Let $\mathfrak{T}' \subseteq \mathfrak{T}$ be nonempty and finite. We will show that $\bigcap_{\Sigma\in\mathfrak{T}'} \mathcal{M}_\Sigma \neq \emptyset$. First, we will show that every pair of bases $H_{\Sigma_1}, H_{\Sigma_2}$, where $\Sigma_1, \Sigma_2 \in \mathfrak{T}'$, has a joint refinement (that is, there exists a basis $H$ such that both $H_{\Sigma_1}$ and $H_{\Sigma_2}$ are included in the MV-algebra generated by $H$). This is proved directly as follows. The triangulations $\Sigma_1, \Sigma_2$ have a joint subdivison (that is, there exists a triangulation of $[0,1]^k$ with the property that each of its simplices is included in some simplex of $H_{\Sigma_1}$ or $H_{\Sigma_2}$) by taking all the intersections of simplices of $H_{\Sigma_1}$ and $H_{\Sigma_2}$, and eventually triangulating the resulting polyhedral complex. This triangulation can be in turn subdivided to a unimodular triangulation $\Sigma^* \in \mathfrak{T}$ [7, Claim 2]. The joint refinement of the bases $H_{\Sigma_1}, H_{\Sigma_2}$ is then the basis $H_{\Sigma^*}$. The same argument straightforwardly applies to the finite set of bases $\{H_\Sigma \mid \Sigma \in \mathfrak{T}'\}$. Let $H_{\Sigma'}$ be the basis refining each basis $H_\Sigma$, $\Sigma \in \mathfrak{T}'$.

Precisely, if $\Sigma \in \mathfrak{T}'$, then for each $h_y \in H_\Sigma$ there exist uniquely determined nonnegative integers $\alpha_x$, where $x \in \mathrm{V}(\Sigma')$, such that $h_y = \sum_{x\in\mathrm{V}(\Sigma')} \alpha_x h_x$. Put $\delta = \sum_{x\in\mathrm{V}(\Sigma')} \mathrm{den}(x) s(h_x)\delta_x$. It follows that

$$\int h_y \,\mathrm{d}\delta = \sum_{x\in\mathrm{V}(\Sigma')} \alpha_x \int h_x \,\mathrm{d}\delta = \sum_{x\in\mathrm{V}(\Sigma')} \alpha_x s(h_x),$$

where the last equality results from the calculation completely analogous to (1). Since $\sum_{x\in\mathrm{V}(\Sigma')} \alpha_x h_x \leq 1$, we obtain $\sum_{x\in\mathrm{V}(\Sigma')} \alpha_x s(h_x) = s\left(\sum_{x\in\mathrm{V}(\Sigma')} \alpha_x h_x\right) = s(h_y)$, and thus $\delta \in \bigcap_{\Sigma\in\mathfrak{T}'} \mathcal{M}_\Sigma$. □

**Claim 3.** *The intersection* $\bigcap_{\Sigma\in\mathfrak{T}} \mathcal{M}_\Sigma$ *contains a single element* $\mu$ *which satisfies* $s(f) = \int f \,\mathrm{d}\mu$, *for every* $f \in L_k$.

*Proof.* As $\mathcal{M}^1$ is $w^*$-compact and $(\mathcal{M}_\Sigma)_{\Sigma\in\mathfrak{T}}$ is a collection of $w^*$-closed subsets having the finite intersection property, the intersection $\bigcap_{\Sigma\in\mathfrak{T}} \mathcal{M}_\Sigma$ is nonempty. Every probability measure $\mu \in \bigcap_{\Sigma\in\mathfrak{T}} \mathcal{M}_\Sigma$ represents the state $s$. Indeed, given a McNaughton function $f \in L_k$, find $\Sigma^* \in \mathfrak{T}$ and the basis $H_{\Sigma^*}$ such that $f = \sum_{x\in\mathrm{V}(\Sigma^*)} \alpha_x h_x$, for uniquely determined nonnegative integers $\alpha_x$ [1, Theorem 9.1.5]. It results that

$$\begin{aligned} s(f) = s\left(\sum_{x\in\mathrm{V}(\Sigma^*)} \alpha_x h_x\right) &= \sum_{x\in\mathrm{V}(\Sigma^*)} \alpha_x s(h_x) = \sum_{x\in\mathrm{V}(\Sigma^*)} \alpha_x \int h_x \,\mathrm{d}\mu \\ &= \int \sum_{x\in\mathrm{V}(\Sigma^*)} \alpha_x h_x \,\mathrm{d}\mu = \int f \,\mathrm{d}\mu. \end{aligned}$$

It remains to show that $\bigcap_{\Sigma\in\mathfrak{T}} \mathcal{M}_\Sigma$ is a singleton. By the way of contradiction, assume that there are Borel probability measures $\mu, \nu \in \bigcap_{\Sigma\in\mathfrak{T}} \mathcal{M}_\Sigma$ such that $\mu \neq \nu$. The Borel subsets of $[0,1]^n$ are generated by the collection of all open (in the subspace Euclidean topology of $[0,1]^n$) (hyper)rectangles with rational vertices: indeed, every open subset of $[0,1]^n$ can be written as a countable union of such rectangles. As a consequence, [8, Theorem 3.3] yields that there exists an open rectangle $R \subseteq [0,1]^n$ with rational vertices and $\mu(R) \neq \nu(R)$.

Let $\mathcal{R}$ be the polyhedral complex consisting of all the faces of the closure $\overline{R}$ of $R$. Taking an arbitrary point $r \in R$ with rational coordinates, consider the stellar subdivision $\mathcal{R}'$ of $\mathcal{R}$ (see [4, p.15]). The polyhedral complex $\mathcal{R}'$ can be triangulated without introducing any new vertices [4, Proposition 2.9]. In turn, the resulting simplicial complex can be subdivided into a unimodular triangulation $\Sigma$ of $\overline{R}$ with a possible introduction of new vertices (see [7, Claim 2], for example).

For each $v \in \mathrm{V}(\Sigma) \cap R$, let $h_v$ be the Schauder hat at $v$ over $\Sigma$, and define a function $f_v : [0,1]^n \to [0,1]$ by

$$f_v(x) = \begin{cases} h_v(x), & x \in \overline{R}, \\ 0, & \text{otherwise.} \end{cases}$$

When $f = \bigoplus_{v \in \mathrm{V}(\Sigma) \cap R} f_v$, then it follows directly from unimodularity of $\Sigma$ and the definition of $f_v$ that $f \in L_k$. In particular, note that $f(x)$ vanishes iff $x \in [0,1]^n \setminus R$ and thus

$$\sup_{m \in \mathbb{N}} \bigoplus_{i=1}^{m} f = \chi_R, \tag{2}$$

where $\chi_R$ is the characteristic function of $R$. For every $m \in \mathbb{N}$, the function $\bigoplus_{i=1}^{m} f$ is an $n$-variable McNaughton function, and (2) together with Lebesgue's dominated convergence theorem leads to the equality

$$\mu(R) = \sup_{m \in \mathbb{N}} \int \bigoplus_{i=1}^{m} f \, \mathrm{d}\mu = \sup_{m \in \mathbb{N}} \int \bigoplus_{i=1}^{m} f d\nu = \nu(R),$$

which is the contradiction. □

The state space of $L_k$ is a compact convex set. It can be completely described by its extreme boundary (Krein-Milman theorem), which is formed by the states $s_x : f \in L_k \mapsto f(x)$, for every $x \in [0,1]^k$. In addition, the set of all such states can be bijectively mapped onto the set of all maximal filters in $L_k$ [9, Theorem 2.5] by the mapping $s_x \mapsto \mathcal{F}_x = \{f \in L_k \mid s_x(f) = 1\}$.

**Theorem 2** ([9]). *The set $S(L_k)$ of all states on $L_k$ is a compact convex subset of the product space $[0,1]^{L_k}$. The set of all extreme points of $S(L_k)$ equals $\{s_x \mid x \in [0,1]^k\}$, which is a closed subset of $S(L_k)$ whose elements are in one-to-one correspondence with maximal filters in $L_k$.*

# 4 Belief Functions

Belief measures introduced in Dempster-Shafer theory [10, 12] are particular completely (totally) monotone mappings in the sense of Choquet [11]. The complete monotonicity of a real function can be defined on an arbitrary Abelian semigroup. Let $(G, *)$ be an Abelian semigroup and $\beta$ be a mapping $G \to \mathbb{R}$. Put $\Delta_a^* \beta(x) = \beta(x) - \beta(x * a)$, for every $x, a \in G$.

**Definition 2.** *A mapping* $\beta : G \to \mathbb{R}$ *is* completely monotone *if*

$$\Delta^*_{a_n} \cdots \Delta^*_{a_1} \beta(x) \geq 0 \tag{3}$$

*for every* $n \geq 1$ *and every* $x, a_1, \ldots, a_n \in G$.

A completely monotone, normalized and nonnegative real function on a family of sets equipped with $\cap$ is known as a belief measure (function) [12].

**Definition 3** (Belief measure)**.** *Let* $(G, *) = (\mathcal{A}, \cap)$*, where* $\mathcal{A}$ *is a family of subsets of some nonempty set* $X$ *closed w.r.t. finite intersections such that* $\emptyset, X \in \mathcal{A}$*. A completely monotone function* $\beta : \mathcal{A} \to [0, 1]$ *with* $\beta(X) = 1, \beta(\emptyset) = 0$ *is called a* belief measure.

In case that $\mathcal{A}$ is even an algebra of sets, the condition (3) can be equivalently expressed for belief measures as follows:

$$\beta\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{\substack{I \subseteq \{1,\ldots n\} \\ I \neq \emptyset}} (-1)^{|I|+1} \beta\left(\bigcap_{i \in I} A_i\right),$$

for every $A_1, \ldots, A_n \in \mathcal{A}$. In this case the nonnegativity of the first two successive differences in (3) implies that $\beta$ is a *monotone* and a *supermodular* set function, respectively, where the latter property means that

$$\beta(A_1 \cup A_2) + \beta(A_1 \cap A_2) \geq \beta(A_1) + \beta(A_2),$$

for every $A_1, A_2 \in \mathcal{A}$. In particular, note that every finitely additive probability measure on $\mathcal{A}$ is a belief measure due to the inclusion-exclusion principle.

A plain generalization of the classical notion of a belief measure from Definition 3 towards the MV-algebra of McNaughton functions $L_k$ leads to considering the Abelian semigroup $(L_k, \odot)$ together with the differences defined by the operator $\Delta^{\odot}$. This approach, however, does not seem to give the "right" concept of a belief function on $L_k$ since not every state is completely monotone w.r.t. $\Delta^{\odot}$. In fact it is possible to find a state $s$ and McNaughton functions $f, g_1, g_2 \in L_k$ such that $\Delta^{\odot}_{g_2} \Delta^{\odot}_{g_1} s(f) < 0$. The lack of complete monotonicity is caused by the absence of distributivity of $\odot$ over $\oplus$ (and vice versa), which is in a clear contrast to the properties of the lattice operations $\vee$ and $\wedge$ on $L_k$. Yet the requirement of complete monotonicity for states is rather natural due to the linearity of every state (cf. Theorem 1) and consistency with the classical definition of belief measure, which covers finitely additive probabilities. An alternative definition of belief function on $L_k$ is proposed in the next paragraph and it is shown how this concept relates to complete monotonicity w.r.t. the Abelian semigroup $(L_k, \wedge)$ together with the operator $\Delta^{\wedge}$.

In the sequel we consider belief measures on the family $\mathcal{C}$ of all closed subsets of $[0, 1]^k$. In particular, a belief measure $\beta$ on $\mathcal{C}$ is *outer regular (w.r.t.* $\mathcal{C}$*)* if $\beta(A) = \inf \{\beta(B) \mid B \in \mathcal{C} \text{ and } \operatorname{Int} B \supseteq A\}$, for every $A \in \mathcal{C}$.

**Definition 4** (Belief function)**.** *Let $\beta$ be an outer regular belief measure on $\mathcal{C}$. A* belief function *$\hat{\beta}$ on $L_k$ is given by*

$$\hat{\beta}(f) = \int_0^1 \beta(f^{-1}([t,1]))\,\mathrm{d}t, \quad f \in L_k. \tag{4}$$

Thus saying that "$\hat{\beta}$ is a belief function on $L_k$" is equivalent to the existence of an outer regular belief measure $\beta$ on $\mathcal{C}$ so that $\hat{\beta}$ and $\beta$ are related by the formula (4). The functional $f \mapsto \int_0^1 \beta(f^{-1}([t,1]))\,\mathrm{d}t$ is also called the *Choquet integral* of $f$ w.r.t. $\beta$ [13]. Every pre-image $f^{-1}([t,1])$ is a closed set in $[0,1]^k$ and $\beta(f^{-1}([t,1]))$ is thus well-defined. Since the function $t \mapsto \beta(f^{-1}([t,1]))$ is bounded and non-increasing on $[0,1]$ for a fixed $\beta$ and $f \in L_k$, the integral on the right-hand side of (4) exists as the Riemann integral. Definition 4 bears a resemblance to the approach of Goubault-Larrecq in [14], where, on the other hand, belief measures are defined on the lattice of open subsets of a certain topological space. The preference of closed sets over opens is immaterial from the viewpoint of Choquet integration (4) and it will be justified only in the following. In a nutshell, closed subsets of $[0,1]^k$ correspond one-to-one to particular basic belief functions.

States are special belief functions according to Definition 4. Indeed, if an outer regular belief measure $\beta$ satisfies

$$\beta(A \cup B) + \beta(A \cap B) = \beta(A) + \beta(B), \quad \text{for every } A, B \in \mathcal{C},$$

then $\beta$ determines a unique regular Borel measure [11, V.26.6], and, consequently, the corresponding $\hat{\beta}$ is a state on $L_k$ by Theorem 1 since the Choquet integral w.r.t. a measure is just the Lebesgue integral. Moreover, Choquet proved in [11, VII.52] that the integral in (4) preserves complete monotonicity of $\beta$ when the lattice operations on the domain of $\hat{\beta}$ are employed. Precisely, the following statement holds true.

**Theorem 3** ([11])**.** *Every belief function $\hat{\beta}$ is completely monotone w.r.t. the Abelian semigroup $(L_k, \wedge)$.*

Any belief function $\hat{\beta}$ thus satisfies the following properties that are jointly equivalent to its complete monotonicity:

(i) $\hat{\beta}$ is monotone,
(ii) for every $f_1, \ldots, f_n \in L_k$ with $n \geq 2$:

$$\hat{\beta}\left(\bigvee_{i=1}^{n} f_i\right) \geq \sum_{\substack{I \subseteq \{1,\ldots n\} \\ I \neq \emptyset}} (-1)^{|I|+1} \hat{\beta}\left(\bigwedge_{i \in I} f_i\right). \tag{5}$$

Further properties of belief functions on $L_k$ are direct consequences of the well-known properties of Choquet integral (see [13]).

**Proposition 1.** *Let $\hat{\beta}$ be a belief function on $L_k$. Then for every $f, g \in L_k$:*

*(i)* $\hat{\beta}(0) = 0, \hat{\beta}(1) = 1$
*(ii)* *if* $f \leq g$*, then* $\hat{\beta}(f) \leq \hat{\beta}(g)$
*(iii)* *if* $f \odot g = 0$*, then* $\hat{\beta}(f \oplus g) \geq \hat{\beta}(f) + \hat{\beta}(g)$
*(iv)* $\hat{\beta}(f) + \hat{\beta}(\neg f) \leq 1$
*(v)* $\hat{\beta}$ *is a state iff* $\beta$ *satisfies* $\beta(A \cup B) + \beta(A \cap B) = \beta(A) + \beta(B)$*, for every* $A, B \in \mathcal{C}$
*(vi)* *if* $f \odot g = 0$ *and there is no pair* $x, y \in [0,1]^k$ *with* $f(x) < g(x), f(y) > g(y)$*, then* $\hat{\beta}(f \oplus g) = \hat{\beta}(f) + \hat{\beta}(g)$

Basic examples of belief functions are minima of McNaughton functions over closed subsets of $[0,1]^k$.

**Example 1.** *Let* $C \in \mathcal{C}$ *be nonempty and*

$$b_C(f) = \min \{f(x) \mid x \in C\}, \quad f \in L_k.$$

*Then* $b_C$ *is a belief function since one can write* $b_C = \widehat{\beta_C}$*, where*

$$\beta_C(A) = \begin{cases} 1, & C \subseteq A, \\ 0, & \textit{otherwise}, \end{cases} \qquad A \in \mathcal{C},$$

*is an outer regular belief measure on* $\mathcal{C}$*.*

**Theorem 4.** *The set* $B(L_k)$ *of all belief functions on* $L_k$ *is a compact convex subset of the product space* $[0,1]^{L_k}$*. The set of extreme points* $\operatorname{ext} B(L_k)$ *of* $B(L_k)$ *is closed, equals* $\{b_C \mid C \in \mathcal{C}, C \neq \emptyset\}$*, and it is in one-to-one correspondence with filters in* $L_k$*.*

*Proof.* It is known that the set $B(\mathcal{C})$ of all outer regular belief measures on $\mathcal{C}$ is a compact convex subset of the product space $[0,1]^{\mathcal{C}}$ and that the set of extreme points of $B(\mathcal{C})$ is closed and equals $\{\beta_C \mid C \in \mathcal{C}, C \neq \emptyset\}$ (see [11, VII.50]). The mapping $\beta \mapsto \hat{\beta}$ is an affine and a continuous mapping of $B(\mathcal{C})$ onto $B(L_k)$ since Choquet integration is continuous for a fixed integrand. Moreover, it is also injective, which can be deduced from another result of Choquet [11, p. 266]. The one-to-one correspondence between $\{b_C \mid C \in \mathcal{C}, C \neq \emptyset\}$ and the filters in $L_k$ follows from [1, Section 3.4]: given $b_C$, put

$$\mathcal{F}_C = \{f \in L_k \mid f(x) = 1, \text{for every } x \in C\}. \tag{6}$$

Vice versa, if $\mathcal{F}$ is a filter in $L_k$, let

$$K_{\mathcal{F}} = \bigcap_{f \in \mathcal{F}} f^{-1}(1). \tag{7}$$

Compactness of $[0,1]^k$ and closedness of each $f^{-1}(1)$ gives that the closed set $K_{\mathcal{F}}$ is nonempty. The two mappings from (6)-(7) are mutually inverse since [1, Theorem 3.4.3(ii)] shows that $C = K_{\mathcal{F}_C}$, for every $C \in \mathcal{C}$. □

By Krein-Milman theorem, every belief function on $L_k$ is thus in the closure of some convex hull formed by belief functions $b_C$. In particular, the usual integral reformulation of Krein-Milman theorem together with Theorem 4 admits to prove another integral representation of $\hat{\beta}$. The uniqueness part of the next theorem can be deduced from the similar result [11, VII.50.1] for $B(\mathcal{C})$ by using the fact that $\beta \mapsto \hat{\beta}$ is an affine homeomorphism.

**Theorem 5.** *If $\hat{\beta}$ is a belief function on $L_k$, then there exists a unique regular Borel probability measure $\mu$ on $\operatorname{ext} B(L_k)$ such that*

$$\beta(f) = \int_{\operatorname{ext} B(L_k)} b_C(f)\, \mathrm{d}\mu, \quad f \in L_k.$$

## 4.1 Remarks

Every belief function of the form $b_C$ for some $C \in \mathcal{C}$ preserves finite minima:

$$b_C(f \wedge g) = b_C(f) \wedge b_C(g), \quad f, g \in L_k.$$

In general, every minimum-preserving function $b : L_k \to [0,1]$ with $b(0) = 1, b(1) = 1$ is a belief function. These functions are termed *necessity measures (functions)* and they were recently investigated on formulas of $n$-valued Łukasiewicz logic in [15].

Belief measures can be interpreted as certain lower probabilities. The corresponding upper probabilities are called *plausibility measures* in Dempster - Shafer theory. If $\mathcal{A}$ is an algebra of sets and $\beta : \mathcal{A} \to [0,1]$ is a belief measure, then the plausibility measure $\pi$ is defined by $\pi(A) = 1 - \beta(A^C)$, for every $A \in \mathcal{A}$. Properties of plausibility measures are "dual" to those of belief measures so that the general theory can be developed for any of them. Plausibility functions on $L_k$ are defined analogously: if $b$ is a belief function on $L_k$, then the function $p(f) = 1 - b(\neg f), f \in L_k$, is called a *plausibility function.* Observe that it is the involutivity of Łukasiewicz negation that makes $b$ and $p$ dual to each other:

$$b(f) = b(\neg\neg f) = 1 - p(\neg f), \quad f \in L_k.$$

## 4.2 Open problems

The important open question is whether complete monotonicity of a real mapping on $L_k$ is sufficient for its representation by the Choquet integral w.r.t. some belief measure on $\mathcal{C}$. Precisely, if $b : L_k \to [0,1]$ is such that $b(0) = 0, b(1) = 1$ and $b$ is completely monotone w.r.t. $(L_k, \wedge)$, is it true that there exists an outer regular belief measure $\beta$ on $\mathcal{C}$ satisfying $\hat{\beta} = b$?

Another question of interest is whether a belief function $b$ on $L_k$ is a "lower probability", that is, whether the equality

$$b(f) = \inf \{s(f) \mid s \text{ state with } s \geq b\}, \quad f \in L_k,$$

holds true or not.

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