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A complete-damage problem at small strains

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Abstract. Damage of a linearly-responding material that can completely disintegrate is addressed at small strains. Using time-varying Dirichlet boundary conditions we set up a rate-independent evolution problem in multidimensional situations. The stored energy involves the gradient of the damage variable. This variable as well as the stress and energies are shown to be well defined even under complete damage, in contrast to displacement and strain. Existence of an energetic solution is proved, in particular, by detailed investigating the Γ-limit of the stored energy and its dependence on boundary conditions. Eventually, the theory is illustrated on a one-dimensional example.

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1. Introduction

Damage, as a special sort of *inelastic response* of solid materials, originates from microstructural changes under mechanical load. In applications routine computational simulations using various models are performed, although mostly without being supported by rigorous mathematical and numerical analysis. This convincingly indicates the mathematical non-triviality of the damage problem.

We will consider damage as a *rate-independent* process by neglecting all rate dependent processes like viscosity and inertia. This is often, although not always, an appropriate concept and has applications in a variety of industrially important materials, especially concrete [14, 17, 34], filled polymers [11], or filled rubbers [19, 25, 26]. Being rate-independent, it is necessarily an *activated* process, i.e. to trigger a damage the mechanical stress must achieve a certain activation threshold. The mathematical difficulty is reflected by the fact that only local-in-time existence for a simplified scalar model or for a rate-dependent 0- or 1-dimensional model has recently been performed in [2, 10, 15, 16]. The 3-dimensional situation was investigated in [12, 28, 29] for the case of incomplete damage. The main focus of this paper is on *complete damage*, i.e. the material can completely *disintegrate* and its displacement completely loses any sense in such regions. The related mathematical troubles are immediately expected and specific mathematical techniques urgently needed.

We consider a nonhomogeneous anisotropic material but confine ourselves to materials with *linear elastic response* under *small strains* and an *isotropic damage* using only one scalar damage parameter under *small strains* (as in [1, 2, 14, 18]) and the *gradient-of-damage* theory [9, 14, 17, 18, 23, 24, 35] expressing a certain nonlocality in the sense that damage of a particular spot is to some extent influenced by its surrounding, leading to possible hardening or softening-like effects, and introducing a certain internal length scale eventually preventing damage microstructure development. From the mathematical viewpoint, the damage gradient has a compactifying character and opens possibilities for the successful analysis of the model. Anyhow, some investigations are still possible without gradient of damage, as shown in [12] for incomplete damage, leading to the possibility of microstructure in the damage variable.

To present a relevant formulation of the rate-independent evolution of the damage, in Section 2 we first scrutinize the static problem with a prescribed damage profile under a prescribed boundary condition. Then, in Section 3, the energetic solution to the evolution problem is formulated in terms of the damage profile and stress (or, equivalently, of the shape of completely damaged part and the strain in the rest) and its existence is proved with help of results from [27, 28, 29]. Eventually, an illustrative one-dimensional example is presented in some detail in Section 4.

2. Static problem and its perturbation analysis

We consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, an open nonempty part $\Gamma \subset \partial\Omega$ of its boundary $\partial\Omega$ on which we prescribe the *Dirichlet boundary condition* $w \in W^{1/2,2}(\Gamma;\mathbb{R}^d)$. We use the standard notation $W^{k,p}$ for Sobolev or Sobolev-Slobodetskiĭ spaces whose p-power of the k-order derivatives is integrable, allowing for $k > 0$ non-integer. Further, we will consider $\zeta \in W^{1,r}(\Omega)$ valued in $[0,1]$ as a scalar *damage variable* assumed to be prescribed in this section; but later, in Sections 3 and 4, it will evolve in time. The meaning of ζ is the portion of the undamaged material, i.e. $\zeta(x) = 1$ means that the material is completely undamaged at the current point $x \in \Omega$ while $\zeta(x) = 0$ means just the opposite, i.e. complete damage at x. Let us abbreviate the set of admissible damage profiles

$$Z := \big\{\zeta \in W^{1,r}(\Omega);\ \ \zeta(\cdot) \in [0,1] \text{ a.e. on } \Omega\big\} \tag{2.1}$$

and denote the set of the complete damage by

$$N_\zeta := \big\{x \in \Omega;\ \ \zeta(x) = 0\big\}, \tag{2.2}$$

then $u : \Omega\backslash N_\zeta \to \mathbb{R}^d$ will denote a displacement. Naturally, we do not consider u defined on the damaged part N_ζ where the material is completely disintegrated.

Our aim is to investigate a minimization problem that can be *formally* written as

$$\left.\begin{array}{ll}\text{minimize} & V_0(u,\zeta) := \displaystyle\int_\Omega \zeta(x)\varphi\big(x,[e(u)](x)\big) + \frac{\kappa(x)}{r}|\nabla\zeta(x)|^r \,\mathrm{d}x \\ \text{subject to} & u \text{ is a displacement such that } \; u|_\Gamma = w,\end{array}\right\} \tag{2.3}$$

where $\kappa : \Omega \to \mathbb{R}$ is a so-called factor of influence of damage and $\varphi : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ is a Carathéodory function such that $\varphi(x,\cdot) : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$ is a quadratic coercive form on the set of the symmetric $(d\times d)$-matrices $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ describing the elastic stored energy, say

$$\varphi(e) = \frac{1}{2}\sum_{i,j,k,l=1}^{d} \mathbb{C}_{ijkl}(x)e_{ij}e_{kl}, \tag{2.4}$$

and where, as usual in linear elasticity (where small strains are assumed), $e(u)$ denotes the linearized strain tensor, called the *small-strain tensor*:

$$e(u) = \frac{1}{2}(\nabla u)^\top + \frac{1}{2}\nabla u.$$

The 4-th order tensor $\mathbb{C}(x)$ of *elastic moduli* satisfies the usual symmetries, uniform positive-definiteness and boundedness:

$$\begin{array}{l}\forall(\text{a.a.})\, x\in\Omega : \qquad \mathbb{C}_{ijkl}(x) = \mathbb{C}_{jikl}(x) = \mathbb{C}_{klij}(x), \\ \exists\eta>0 \quad \forall(\text{a.a.})\, x\in\Omega \quad \forall e\in\mathbb{R}^{d\times d}_{\mathrm{sym}} : \quad \displaystyle\sum_{i,j,k,l=1}^{d} \mathbb{C}_{ijkl}(x)e_{ij}e_{kl} \ge \eta|e|^2, \\ \mathbb{C}_{ijkl} \in L^\infty(\Omega).\end{array} \tag{2.5}$$

The term $\frac{1}{r}\kappa(x)|\nabla\zeta(x)|^r$ models a certain nonlocality as mentioned in Sect. 1 and is quite often used in literature [9, 14, 17, 18, 23, 24]. The scalar coefficient κ determines a certain length-scale of the possible fine structure that might develop in a damage profile and, in accord with the adopted nonhomogeneous-material concept, is assumed possibly x-dependent and to satisfy

$$\kappa \in L^\infty(\Omega), \qquad \operatorname*{ess\,inf}_{x\in\Omega} \kappa(x) > 0. \tag{2.6}$$

In particular, for the usage in Sect. 3, we are interested in a certain stability of this problem with respect to *perturbations of the damage profile* ζ in the weak $W^{1,r}(\Omega)$-topology. Here, in accord with [28], we consider $r > d$. Then N_ζ from (2.2) is closed in Ω since $\zeta \in W^{1,r}(\Omega) \subset C(\bar\Omega)$ with $r > d$. Let us remark that the theory of incomplete damage was alternatively developed also for $\zeta \in W^{\alpha,2}(\Omega)$ with $\alpha > 0$ in [29]. But it is not obvious how it would be transferred to complete damage because, in the following consideration, we will heavily rely on the compact embedding $\zeta \in W^{1,r}(\Omega) \subset C(\bar\Omega)$.

Let us agree that occasionally we will omit the explicit x-dependence of φ for brevity.

2.1. Regularized problem

The mentioned essential trouble with (2.3) is that the displacement u has no obvious meaning on the completely damaged part N_ζ, which is why (2.3) must be considered only formally, as said above. For the purpose of further analysis based on the results from [28, Sect.4] and, perhaps even more importantly, for a conceptual numerical strategy (see Remark 3.10 below), it is relevant to investigate limit behaviour (for $\varepsilon \to 0+$) of a *regularized problem*

$$\left.\begin{array}{ll}\text{minimize} & V_\varepsilon(u,\zeta) := \displaystyle\int_\Omega \big(\zeta(x)+\varepsilon\big)\varphi\big(x,[e(u)](x)\big) + \frac{\kappa(x)}{r}|\nabla\zeta(x)|^r \,\mathrm{d}x \\ \text{subject to} & u \in W^{1,2}(\Omega;\mathbb{R}^d), \quad u|_\Gamma = w. \end{array}\right\} \tag{2.7}$$

Obviously, V_0 from (2.3) is just V_ε for $\varepsilon = 0$. For $\varepsilon \geq 0$, let us define

$$G_\varepsilon(u,\zeta) := \begin{cases} V_\varepsilon(u,\zeta) & \text{if } u|_\Gamma = w \text{ and } \zeta \in Z, \\ +\infty & \text{elsewhere,} \end{cases} \tag{2.8}$$

where Z is from (2.1). The theory for complete damage developed in [28, Sect.4] relies on a *substantial stored energy* defined, for a given damage profile ζ and a hard-device loading w, as the Γ-limit of the sequence $\{g_\varepsilon\}_{\varepsilon>0}$ (considering only a countable number of ε converging to 0) where

$$g_\varepsilon(\zeta) := \min_{u\in W^{1,2}(\Omega;\mathbb{R}^d)} G_\varepsilon(u,\zeta). \tag{2.9}$$

Let us note that the minimum in (2.9) is attained by the standard coercivity arguments.

Thanks to the regularization term $\int_\Omega \frac{\kappa}{r}|\nabla\zeta|^r\,\mathrm{d}x$, the relevant topology used for the damage variable ζ will be the weak topology of $W^{1,r}(\Omega)$. It is important for the subsequent analysis that we assumed $r > d$ so that the weak convergence of a sequence $\{\zeta_\varepsilon\}$ (denoted as usual by $\zeta_\varepsilon \rightharpoonup \zeta$) implies the uniform convergence as continuous functions on $\overline{\Omega}$.

Recall now that the sequence $\{g_\varepsilon\}_{\varepsilon>0}$ is said to be *sequentially* Γ*-convergent to* $\mathfrak{g}$ *in the weak topology* of $W^{1,r}(\Omega)$ if the following properties hold:

(i) *lower bound*: for every sequence $\{\zeta_\varepsilon\}_{\varepsilon>0}$ converging weakly to $\zeta \in Z$, we have:

$$\liminf_{\varepsilon\to 0} g_\varepsilon(\zeta_\varepsilon) \geq \mathfrak{g}(\zeta), \tag{2.10}$$

(ii) *recovering sequence*: for every $\zeta \in Z$ there exists a sequence $\{\zeta_\varepsilon\}_{\varepsilon>0} \subset Z$ converging weakly to ζ such that

$$\limsup_{\varepsilon\to 0} g_\varepsilon(\zeta_\varepsilon) \leq \mathfrak{g}(\zeta). \tag{2.11}$$

When properties (i) and (ii) are satisfied, we write $\mathfrak{g} = \Gamma\text{-}\lim_{\varepsilon\to 0} g_\varepsilon$. In our case the sequence $\{g_\varepsilon\}_{\varepsilon>0}$ is monotone and the existence of a Γ-limit is guaranteed by the following lemma:

Lemma 2.1. (See [7].) *Assume that g_ε is nonincreasing with respect to ε and let $g_0(\zeta) := \inf_{\varepsilon>0} g_\varepsilon(\zeta)$. Then $\{g_\varepsilon\}_{\varepsilon>0}$ does Γ-converge to the lower semicontinuous envelope of g_0 with respect to the weak topology on $W^{1,r}(\Omega)$. Moroever, we have*

$$\mathfrak{g}(\zeta) = \liminf_{\substack{\varepsilon\to 0,\ \tilde\zeta\in Z\\ \tilde\zeta\rightharpoonup\zeta\ \mathit{in}\ W^{1,r}(\Omega)}} g_\varepsilon(\tilde\zeta). \tag{2.12}$$

In our case, the computation of g_0 is quite easy: by using (2.9) and by switching the infimum in ε with the infimum in u, one has

$$g_0(\zeta) = \inf_{\varepsilon>0} \inf_{u\in W^{1,2}(\Omega)} G_\varepsilon(u,\zeta) = \inf_{u\in W^{1,2}(\Omega)} \inf_{\varepsilon>0} G_\varepsilon(u,\zeta) = \inf_{u\in W^{1,2}(\Omega)} G_0(u,\zeta)\ .$$

As a consequence of Lemma 2.1, g_0 will be the Γ-limit we are looking for provided g_0 given above enjoys the lower semicontinuity property. Unfortunately, as shown in Section 2.2, this property fails and the determination of $\mathfrak{g}$ is a more involved problem which we are going to solve later, see Proposition 2.10.

Also note that $\mathfrak{g}$ is always bounded from below because we do not consider any external dead loading like gravity force; obviously, we always have $\mathfrak{g} \geq 0$. In fact, due to the regularization term $\int_\Omega \frac{\kappa}{r}|\nabla\zeta|^r\,\mathrm{d}x$ and (2.6), we have even the coercivity $\mathfrak{g}(\zeta) \geq (\operatorname{ess\,inf}\frac{\kappa}{r})\|\nabla\zeta\|^r_{L^r(\Omega;\mathbb{R}^d)}$ and therefore the sequential Γ-limit $\mathfrak{g}$ is weakly lower semicontinuous.

Remark 2.2. (*Mosco convergence.*) In fact, later in the proof of (3.20) we will show even strong convergence of recovery sequences. This allows for replacing the weak topology in (ii) by the strong one, which means that the convergence of g_ε to $\mathfrak{g}$ in the sense of U. Mosco [33].

2.2. A 1-dimensional counterexample

Let us show a 1-dimensional example of a failure of weak lower-semicontinuity of g_0. Here and in the following Sections 2.3 and 2.4, the damage profile ζ will be considered essentially given, and we therefore omit the term $\frac{\kappa}{r}|\nabla\zeta|^r$ for a moment to simplify the notation. Thus, we introduce the notation

$$G_\varepsilon^{\mathrm{red}}(u,\zeta) := G_\varepsilon(u,\zeta) - \int_\Omega \frac{\kappa}{r}|\nabla\zeta|^r\,\mathrm{d}x, \quad g_\varepsilon^{\mathrm{red}}(\zeta) := g_\varepsilon(\zeta) - \int_\Omega \frac{\kappa}{r}|\nabla\zeta|^r\,\mathrm{d}x.$$

Also, we define $\mathfrak{g}^{\mathrm{red}} := \Gamma\text{-}\lim_{\varepsilon\to 0} g_\varepsilon^{\mathrm{red}}$.

Being inspired by [4, Example 3] and by [3], let us consider $d = 1$, the interval $\Omega := (-1,1)$, the Dirichlet condition w prescribed on $\Gamma := \{-1,1\}$ as $w(x) := x$, $\varphi(e) = \frac{1}{2}|e|^2$, and the damage profile

$$\zeta(x) := \big|x\big|^\alpha \qquad \text{with} \qquad 1 - \frac{1}{r} < \alpha < 1. \tag{2.13}$$

Direct calculations easily show that $\zeta \in W^{1,r}(\Omega)$. Then we consider the sequence

$\{\zeta_n\}_{n\in\mathbb{N}}$ of

$$\zeta_n(x) := \Big(\max\big(0, |x| - \frac{1}{n}\big)\Big)^{\alpha}. \tag{2.14}$$

Obviously $\zeta_n \to \zeta$ for $n \to \infty$ even in the norm topology of $W^{1,r}(\Omega)$. Moreover, $g_0^{\mathrm{red}}(\zeta_n) = 0$ because obviously $g_0^{\mathrm{red}}(\zeta_n) = G_0^{\mathrm{red}}(u_n, \zeta_n) = 0$ for the piecewise affine displacement profile

$$u_n(x) := \begin{cases} -1 & \text{for} \quad -1 \le x \le -\frac{1}{n}, \\ nx & \text{for} \quad -\frac{1}{n} < x < \frac{1}{n}, \\ 1 & \text{for} \quad \frac{1}{n} \le x \le 1. \end{cases} \tag{2.15}$$

Therefore $\mathfrak{g}^{\mathrm{red}}(\zeta) = 0$ because

$$0 \le \mathfrak{g}^{\mathrm{red}}(\zeta) \le \liminf_{n\to\infty} G_0^{\mathrm{red}}(u_n, \zeta_n) = \lim_{n\to\infty} 0 = 0.$$

On the other hand, we will show that $\inf_{u\in W^{1,2}(\Omega;\mathbb{R}^d)} G_0^{\mathrm{red}}(u, \zeta) = 2(1-\alpha) > 0$. To this end, choose any $p \in (1, 2/(1+\alpha))$ and set $\beta = \alpha p/2$, $q = 2/p$, and $q' = q/(q-1)$, then we have

$$\|u'\|_{L^p(\Omega)} = \Big(\int_\Omega |x|^{-\beta}(|x|^{\beta}|u'|^p)\,dx\Big)^{1/p} \le \|\,|x|^{-\beta}\|_{L^{q'}(\Omega)} \|\,|x|^{\beta}|u'|^p\|_{L^q(\Omega)}.$$

Using $\beta q = \alpha$ and $pq = 2$ the last term equals $G_0^{\mathrm{red}}(u, \zeta)$ and we have the lower estimate

$$G_0^{\mathrm{red}}(u, \zeta) = \int_\Omega |x|^{\alpha}|u'|^2\,dx \ge \frac{1}{C}\|u'\|^q_{L^p(\Omega)} \quad \text{with} \quad C = \|\,|x|^{-\beta}\|^p_{L^{q'}(\Omega)} < \infty$$

Thus, the functional is coercive and strictly convex on the reflexive Banach space $W^{1,p}(\Omega)$, when including the boundary conditions. Hence, there is a unique minimizer, which is easily identified to be $u_*(x) = \mathrm{sign}(x)|x|^{1-\alpha}$. Since $W^{1,2}(\Omega)$ is densely embedded into $W^{1,p}(\Omega)$ we conclude that

$$g_0^{\mathrm{red}}(\zeta) = G_0^{\mathrm{red}}(u_*, \zeta) = 2(1-\alpha) > 0.$$

We summarize the result in the following statement.

Corollary 2.3. *For the scalar situation and Ω, φ, and ζ from the above example, it holds $\mathfrak{g}^{\mathrm{red}}(\zeta) = 0 < 2(1-\alpha) = \inf_{u\in W^{1,2}(\Omega;\mathbb{R}^d)} G_0^{\mathrm{red}}(u, \zeta)$.*

In fact, the above Corollary 2.3 gives a counterexample for the (thus wrong) conjecture in [28, Remark 4.1].

2.3. Realizable strain, stress and energy

The important question is the behaviour of the stress

$$\sigma_\varepsilon = (\zeta_\varepsilon + \varepsilon)\varphi'_e(e(u_\varepsilon)) = (\zeta_\varepsilon + \varepsilon)\mathbb{C}e(u_\varepsilon), \tag{2.16}$$

where u_ε is the minimizer of $G_\varepsilon^{\rm red}(\cdot,\zeta_\varepsilon)$ as well as the corresponding strain $e(u_\varepsilon)$ and the energy $G_\varepsilon^{\rm red}(\cdot,\zeta_\varepsilon)$ itself, when ζ_ε approaches ζ weakly in $W^{1,r}(\Omega)$ and $\varepsilon\to 0+$. We will denote such sort of limit objects by the adjective "realizable". For this, let us first define (possibly nonuniquely) a *realizable strain* $\mathfrak{e}$. Let us define standardly

$$L^2_{\rm loc}(\Omega\backslash N_\zeta;\mathbb{R}^d) \;:=\; \big\{u:\Omega\backslash N_\zeta\to\mathbb{R}^d;\ \forall A\subset\Omega\backslash N_\zeta \text{ open},\\ \mathrm{cl}(A)\cap N_\zeta=\emptyset:\quad u|_A\in L^2(A;\mathbb{R}^d)\big\}. \tag{2.17}$$

Lemma 2.4. ***(Realizable strains.)*** *The sequence $\{e(u_\varepsilon)\}_{\varepsilon>0}$ is bounded in $L^2_{\rm loc}(\Omega\backslash N_\zeta;\mathbb{R}^{d\times d}_{\rm sym})$ and there are $\mathfrak{e}\in L^2_{\rm loc}(\Omega\backslash N_\zeta;\mathbb{R}^{d\times d}_{\rm sym})$ and a subsequence such that $e(u_\varepsilon)\rightharpoonup\mathfrak{e}$ weakly in $L^2_{\rm loc}(\Omega\backslash N_\zeta;\mathbb{R}^{d\times d}_{\rm sym})$, i.e. $e(u_\varepsilon)|_A\rightharpoonup\mathfrak{e}|_A$ weakly in $L^2(A;\mathbb{R}^{d\times d}_{\rm sym})$ for any $A\subset\Omega\backslash N_\zeta$ as in (2.17).*

Proof. Let $N_\zeta\neq\Omega$, otherwise the statement is trivial. Without loss of generality, we can assume A's in (2.17) to be organized into an increasing sequence whose union is just $\Omega\backslash N_\zeta$. As $\zeta_\varepsilon\to\zeta$ in $C(\bar\Omega)$, for any A_j from this sequence there are $\delta_{A_j}>0$ and $\varepsilon_0>0$ such that $\zeta_\varepsilon+\varepsilon\geq\delta_{A_j}$ provided $\varepsilon\leq\varepsilon_0$. Then, for $\varepsilon\leq\varepsilon_0$,

$$\int_{A_j}\varphi(e(u_\varepsilon))\,\mathrm{d}x \;\leq\; \frac{1}{\delta_{A_j}}\int_{A_j}(\zeta_\varepsilon+\varepsilon)\varphi(e(u_\varepsilon))\,\mathrm{d}x\\ \leq\; \frac{1}{\delta_{A_j}}\int_{\Omega}(\zeta_\varepsilon+\varepsilon)\varphi(e(u_\varepsilon))\,\mathrm{d}x=\frac{G_\varepsilon^{\rm red}(u_\varepsilon,\zeta_\varepsilon)}{\delta_{A_j}},$$

which is bounded uniformly with respect to $\varepsilon>0$. By the assumed coercivity of φ, we have $e(u_\varepsilon)$ bounded in $L^2(A_j;\mathbb{R}^{d\times d}_{\rm sym})$. Then we can select a subsequence of ε's such that $\{e(u_\varepsilon)|_{A_j}\}$ converges weakly in $L^2(A_j;\mathbb{R}^{d\times d}_{\rm sym})$ if $\varepsilon\to0$ to some limit, let us denote it by $\mathfrak{e}_{A_j}$. Then we can take A_{j+1} and select further subsequence from this already selected one. This will keep the convergence of $\{e(u_\varepsilon)|_{A_j}\}$ and gives some $\mathfrak{e}_{A_{j+1}}$ as a weak limit of the sub-subsequence $\{e(u_\varepsilon)|_{A_{j+1}}\}$. Of course, $\mathfrak{e}_{A_{j+1}}|_{A_j}=\mathfrak{e}_{A_j}$. Inflating A_j's by passing $j\to\infty$ gives by the diagonalization procedure a subsequence of $\{e(u_\varepsilon)\}_{\varepsilon>0}$ and $\mathfrak{e}$ defined a.e. on $\Omega\backslash N_\zeta$ by $\mathfrak{e}|_{A_j}:=\mathfrak{e}_{A_j}$ with the claimed properties. ☐

The following assertion introduces and characterizes *realizable stresses* $\mathfrak{s}$ using the strains $\mathfrak{e}$ constructed in Lemma 2.4.

Proposition 2.5. ***(Realizable stresses.)*** *The sequence $\{\sigma_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$, and each subsequence selected in Lemma 2.4 converges weakly to a realizable stress $\mathfrak{s}$ that satisfies*

$$\mathfrak{s}=\begin{cases}\zeta\varphi'_e(\mathfrak{e}) & \text{on } \Omega\backslash N_\zeta,\\ 0 & \text{on } N_\zeta.\end{cases} \tag{2.18}$$

Moreover, this convergence is even strong on N_ζ.

Proof. It has already been observed in [28, Formula (4.11)] that $\{\sigma_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$. Indeed, using the property of the quadratic form φ

$$\exists C_\varphi<+\infty \quad \forall e\in\mathbb{R}^{d\times d}_{\rm sym}: \qquad |\varphi'_e(e)|^2=\varphi'_e(e):\varphi'_e(e)\le C_\varphi\varphi(e),$$

we obtain

$$\begin{aligned}
\limsup_{\varepsilon\to0}\big\|\sigma_\varepsilon\big\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})} &= \limsup_{\varepsilon\to0}\int_\Omega(\zeta_\varepsilon+\varepsilon)^2|\varphi'_e(e(u_\varepsilon))|^2\,\mathrm{d}x\\
&\le \limsup_{\varepsilon\to0}\Big(\|\zeta_\varepsilon\|_{L^\infty(\Omega)}+\varepsilon\Big)\int_\Omega(\zeta_\varepsilon+\varepsilon)|\varphi'_e(e(u_\varepsilon))|^2\,\mathrm{d}x\\
&\le \limsup_{\varepsilon\to0}\Big(\|\zeta_\varepsilon\|_{L^\infty(\Omega)}+\varepsilon\Big)C_\varphi\int_\Omega(\zeta_\varepsilon+\varepsilon)\varphi(e(u_\varepsilon))\,\mathrm{d}x\\
&= \|\zeta\|_{L^\infty(\Omega)}C_\varphi\limsup_{\varepsilon\to0}\int_\Omega(\zeta_\varepsilon+\varepsilon)\varphi(e(u_\varepsilon))\,\mathrm{d}x<+\infty. \qquad (2.19)
\end{aligned}$$

Hence we can consider a subsequence and a limit *realizable stress* $\mathfrak{s}$ such that $\sigma_\varepsilon\rightharpoonup\mathfrak{s}$ in $L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$.

Having $\zeta_\varepsilon\to\zeta$ weakly in $W^{1,r}(\Omega)$, hence strongly in $L^\infty(\Omega)$, and $e(u_\varepsilon)|_A\rightharpoonup\mathfrak{e}|_A$ (a subsequence) in $L^2(A;\mathbb{R}^{d\times d}_{\rm sym})$ for each A as in (2.17), we can just pass to the limit in (2.16) to get the equality $\mathfrak{s}=\zeta\varphi'_e(\mathfrak{e})$ on A. For this, we used that φ'_e in (2.16) is linear. Inflating A yields this equality on the whole $\Omega\backslash N_\zeta$ in the sense of $L^2_{\rm loc}(\Omega\backslash N_\zeta;\mathbb{R}^{d\times d}_{\rm sym})$ and thus also $L^2(\Omega\backslash N_\zeta;\mathbb{R}^{d\times d}_{\rm sym})$ because $\mathfrak{s}\in L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$. On the other hand, $\mathfrak{s}=0$ on N_ζ because $\zeta_\varepsilon\to0$ in $L^\infty(N_\zeta)$ and, similarly as in (2.19), we can estimate

$$\big\|\sigma_\varepsilon\big\|^2_{L^2(N_\zeta;\mathbb{R}^{d\times d}_{\rm sym})}\le \underbrace{\Big(\sup_{N_\zeta}\zeta_\varepsilon+\varepsilon\Big)}_{\text{converges to }0}\;C_\varphi\underbrace{\int_{N_\zeta}(\zeta_\varepsilon+\varepsilon)\varphi(e(u_\varepsilon))\,\mathrm{d}x}_{\text{remains bounded}}\;\overset{\text{for }\varepsilon\to0}{\longrightarrow}\;0.$$

Hence we have the complete formula (2.18) for the realizable stress. As we identified the limit by means of $\mathfrak{e}$ constructed by a subsequence selected for Lemma 2.4, we do not need to select a further subsequence here. $\square$

In view of (2.4), we obtained

$$\mathfrak{s}_{ij}=\begin{cases}\zeta\sum_{k,l=1}^d\mathbb{C}_{ijkl}\mathfrak{e}_{kl} & \text{on }\Omega\backslash N_\zeta,\\ 0 & \text{on }N_\zeta.\end{cases} \qquad (2.20)$$

The further important quantity is the *realizable energy density* $\mathfrak{E}$ describing the limit behaviour of the specific stored energy $E_\varepsilon:=(\zeta_\varepsilon+\varepsilon)\varphi(e(u_\varepsilon))$ related to the unique minimizer u_ε of the regularized problem $G^{\rm red}_\varepsilon(\cdot,\zeta_\varepsilon)$. Since u_ε is the minimizer of $G^{\rm red}_\varepsilon(\cdot,\zeta_\varepsilon)$, it satisfies the *Euler-Lagrange equation*, i.e. in the weak form,

$$\forall v\in W^{1,2}(\Omega;\mathbb{R}^d),\ v|_\Gamma=0: \qquad \int_\Omega(\zeta_\varepsilon+\varepsilon)\varphi'_e(e(u_\varepsilon)):e(v)\,\mathrm{d}x=0. \qquad (2.21)$$

Considering $u_{\rm D}$ is a *continuation of the Dirichlet boundary data* w onto Ω, using $v = u_\varepsilon - u_{\rm D}$ in (2.21) and realizing also (2.4) and (2.16) then yield the formula for the total energy

$$\begin{aligned}\int_\Omega E_\varepsilon(x)\,\mathrm{d}x &= \int_\Omega (\zeta_\varepsilon+\varepsilon)\varphi(e(u_\varepsilon))\,\mathrm{d}x\\ &= \frac{1}{2}\int_\Omega (\zeta_\varepsilon+\varepsilon)\varphi'_e(e(u_\varepsilon)) : e(u_\varepsilon)\,\mathrm{d}x\\ &= \frac{1}{2}\int_\Omega (\zeta_\varepsilon+\varepsilon)\varphi'_e(e(u_\varepsilon)) : e(u_{\rm D})\,\mathrm{d}x = \frac{1}{2}\int_\Omega \sigma_\varepsilon : e(u_{\rm D})\,\mathrm{d}x.\end{aligned} \tag{2.22}$$

Proposition 2.6. ***(Realizable energy.)*** *The sequence $\{E_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^1(\Omega)$, and thus, as a subsequence, converges weakly* to a realizable energy density, let us denote it by $\mathfrak{E}$. This density is a measure on $\bar\Omega$ such that $\lim_{\varepsilon\to0} G^{\rm red}_\varepsilon(u_\varepsilon,\zeta_\varepsilon) = \lim_{\varepsilon\to0}\int_\Omega E_\varepsilon(x)\,\mathrm{d}x = \int_{\bar\Omega}\mathfrak{E}(\mathrm{d}x)$. In particular, it holds for the subsequence selected already in Lemma 2.4 and then, for $\mathfrak{e}$ from Lemma 2.4 and $\mathfrak{s}$ from (2.20), it holds*

$$\int_{\bar\Omega}\mathfrak{E}(\mathrm{d}x) = \frac{1}{2}\int_\Omega \mathfrak{s} : e(u_{\rm D})\,\mathrm{d}x = \int_{\Omega\setminus N_\zeta}\zeta\sum_{k,l=1}^{d}\mathbb{C}_{ijkl}\mathfrak{e}_{kl} : e(u_{\rm D})\,\mathrm{d}x, \tag{2.23}$$

where $u_{\rm D}\in W^{1,2}(\Omega;\mathbb{R}^d)$ is an (arbitrary) continuation of w onto Ω.

Proof. It just suffices to apply Proposition 2.5 to (2.22) and apply (2.20). □

Example 2.7. (*Nonuniqueness of $\mathfrak{e}$, $\mathfrak{s}$, and $\mathfrak{E}$.*) Referring to Section 2.2, we consider $\zeta_\varepsilon := \zeta_n$ from (2.14) with $n = n(\varepsilon)$ such that $n\to\infty$ but $\varepsilon n(\varepsilon)^{1/\alpha}\to 0$ for $\varepsilon\to0$. Then, for ε small, $\zeta_\varepsilon+\varepsilon$ and the corresponding u_ε essentially approach the profiles $\zeta_{n(\varepsilon)}$ and $u_{n(\varepsilon)}$ from (2.14) and (2.15), respectively. This is because the overall stiffness of the slot of the length $2n(\varepsilon)^{-1/\alpha}$ filled of "material" with the elastic modulus ε is $\frac{1}{2}\varepsilon n(\varepsilon)^{1/\alpha}$ and asymptotically goes to zero so that asymptotically we approach the situation in Section 2.2. For this $u_{n(\varepsilon)}$, we have got $e(u_{n(\varepsilon)}) = 0$ on $\Omega\setminus[-\frac{1}{n(\varepsilon)},\frac{1}{n(\varepsilon)}]$. For $\zeta_\varepsilon+\varepsilon$, this holds only asymptotically but, nevertheless, the limit is the same, namely $\mathfrak{e}=0$ on $\Omega\setminus\{0\}$. Also the corresponding stress and the energy are (asymptotically) zero, and thus in the limit both, $\mathfrak{s}$ and $\mathfrak{E}$, are zero. On the other hand, for $\zeta_\varepsilon := \zeta$ from (2.13), the displacement profile $u_\varepsilon\in W^{1,2}(\Omega)$ corresponding to $\zeta_\varepsilon+\varepsilon$ essentially imitates the example constructed in Section 2.2, i.e. $G^{\rm red}_0(u_\varepsilon,\zeta_\varepsilon+\varepsilon)$ converges to $G^{\rm red}_0(u,\zeta)>0$. In particular, $\mathfrak{e}=e(u)\neq0$, $\mathfrak{s}=\zeta e(u)\neq0$, and also $\int_{[-1,1]}\mathfrak{E}(\mathrm{d}x)>0$. Of course, in both cases $\zeta_\varepsilon+\varepsilon$ converges to the same limit profile ζ.

In view of the above Example 2.7, it makes sense to consider the set of all realizable stresses $\mathfrak{s}$ for a given damage profile:

$$\begin{aligned}\mathfrak{S}(\zeta) := \big\{\mathfrak{s}\in L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym});\ \ &\exists\zeta_\varepsilon\rightharpoonup\zeta \text{ weakly in } W^{1,r}(\Omega):\\ &\sigma_\varepsilon\rightharpoonup\mathfrak{s} \text{ weakly in } L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}) \text{ with } \sigma_\varepsilon \text{ from (2.16)}\big\}.\end{aligned} \tag{2.24}$$

Proposition 2.8. *The set* $\mathfrak{S}(\zeta)$ *is weakly compact in* $L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$.

Proof. By arguments like in the proof of Proposition 2.5 we can see that the set $\mathfrak{S}(\zeta)$ is bounded in $L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$; in fact, all its elements must share the bound in (2.19). Due to metrizability of the weak topology on bounded sets of $L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$, we can equally focus on sequential compactness. Take a sequence $\{\mathfrak{s}_j\}_{j\in\mathbb{N}}\subset\mathfrak{S}(\zeta)$. As it is bounded in $L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$, it contains a subsequence (for simplicity denoted by the same indexes) converging weakly in $L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$; let $\mathfrak{s}$ denote its limit. As $\mathfrak{s}_j\in\mathfrak{S}(\zeta)$ for each j, there are sequences $\{\zeta_{\varepsilon_{jk}}\}_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}\varepsilon_{jk}=0$, w-$\lim_{k\to\infty}\zeta_{\varepsilon_{jk}}=\zeta_j$ (meant weakly in $W^{1,r}(\Omega)$) and w-$\lim_{k\to\infty}\sigma_{\varepsilon_{jk}}=\mathfrak{s}_j$ with $\sigma_{\varepsilon_{jk}}=(\zeta_{\varepsilon_{jk}}+\varepsilon_{jk})\varphi'_e(e(u_{\varepsilon_{jk}}))$. By the diagonalization procedure we obtain a sequence $\{\sigma_{\varepsilon_{j_nk_n}}\}_{n\in\mathbb{N}}$ converging to $\mathfrak{s}$, which shows that $\mathfrak{s}\in\mathfrak{S}(\zeta)$. □

Proposition 2.9. *It holds*

$$\mathfrak{g}^{\rm red}(\zeta)=\min_{\mathfrak{s}\in\mathfrak{S}(\zeta)}\frac{1}{2}\int_\Omega\mathfrak{s}:e(u_{\rm D})\,{\rm d}x \tag{2.25}$$

where $u_{\rm D}\in W^{1,2}(\Omega;\mathbb{R}^d)$ *is as in Proposition 2.6.*

Proof. As $u_{\rm D}\in W^{1,2}(\Omega;\mathbb{R}^d)$, also $e(u_{\rm D})\in L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$, and $\mathfrak{s}\mapsto\frac{1}{2}\int_\Omega\mathfrak{s}:e(u_{\rm D})\,{\rm d}x$ is a weakly continuous functional which obviously attains its minimum on the set $\mathfrak{S}(\zeta)$ which is, due to Proposition 2.8, weakly compact.

By the definition (2.9) of $\mathfrak{g}^{\rm red}$, the sequence $(\varepsilon,\tilde\zeta)\to(0,\zeta)$ infimizing the expression in (2.9) gives a cluster point $\mathfrak{s}$ of the corresponding sequence $\{\sigma_{\varepsilon,\tilde\zeta}\}$ with $\sigma_{\varepsilon,\tilde\zeta}=(\tilde\zeta+\varepsilon)\varphi'_e(e(u_{\varepsilon,\tilde\zeta}))$ where $\sigma_{\varepsilon,\tilde\zeta}$ minimizes $G^{\rm red}_\varepsilon(\cdot,\tilde\zeta)$, cf. (2.16). This yields $\mathfrak{s}\in\mathfrak{S}(\zeta)$ and, using also (2.22),

$$\begin{aligned}\mathfrak{g}^{\rm red}(\zeta)\;&=\;\lim_{(\varepsilon,\tilde\zeta)\to(0,\zeta)}\int_\Omega(\tilde\zeta+\varepsilon)\varphi(e(u_{\varepsilon,\tilde\zeta}))\,{\rm d}x=\lim_{(\varepsilon,\tilde\zeta)\to(0,\zeta)}\frac{1}{2}\int_\Omega\sigma_{\varepsilon,\tilde\zeta}:e(u_{\rm D})\,{\rm d}x\\&=\;\frac{1}{2}\int_\Omega\mathfrak{s}:e(u_{\rm D})\,{\rm d}x\geq\min_{\tilde{\mathfrak{s}}\in\mathfrak{S}(\zeta)}\frac{1}{2}\int_\Omega\tilde{\mathfrak{s}}:e(u_{\rm D})\,{\rm d}x.\end{aligned}$$

Conversely, taking $\mathfrak{s}\in\mathfrak{S}(\zeta)$ at which the minimum in (2.25) is attained and, by (2.24), the sequence $\{\zeta_\varepsilon\}_{\varepsilon>0}$ such that the corresponding $\{\sigma_\varepsilon\}_{\varepsilon>0}$ attains $\mathfrak{s}$, using again also (2.22), we obtain

$$\begin{aligned}\mathfrak{g}^{\rm red}(\zeta)\;&\leq\;\liminf_{\varepsilon\to0}\int_\Omega(\zeta_\varepsilon+\varepsilon)\varphi(e(u_\varepsilon))\,{\rm d}x=\lim_{\varepsilon\to0}\frac{1}{2}\int_\Omega\sigma_\varepsilon:e(u_{\rm D})\,{\rm d}x\\&=\;\frac{1}{2}\int_\Omega\mathfrak{s}:e(u_{\rm D})\,{\rm d}x=\min_{\tilde{\mathfrak{s}}\in\mathfrak{S}(\zeta)}\frac{1}{2}\int_\Omega\tilde{\mathfrak{s}}:e(u_{\rm D})\,{\rm d}x.\end{aligned}$$ □

Let us note that the formula (2.25) determines (still nonuniquely) a stress $\mathfrak{s}$ that realizes the minimum in (2.25). Let us call it a *minimizing realizable stress*. Naturally, we can think also about the corresponding *minimizing realizable strain*

$\mathfrak{e} \in L^2_{\rm loc}(\Omega\backslash N_\zeta;\mathbb{R}^{d\times d}_{\rm sym})$ related with $\mathfrak{s}$ by

$$\mathfrak{e}(x) = \left[\varphi_e'\right]^{-1}\Big(\frac{\mathfrak{s}(x)}{\zeta(x)}\Big) \qquad \text{for a.a. } x \in \Omega\backslash N_\zeta. \tag{2.26}$$

Let us agree to call the realizable stress $\mathfrak{s} \in \mathfrak{S}(\zeta)$ which realizes the minimum in (2.25) an *effective stress* and $\mathfrak{e}$ corresponding to it via (2.26) the *effective strain*.

2.4. Effective stress and strain, and sensitivity to the boundary data

Now, we construct a particular effective stress, i.e. a minimizer for (2.25), that provides a characterization of the Γ-limit (2.10)–(2.11) as a pointwise limit and it leads to a selection of a particular effective stress and that this effective stress can be recovered by using a particular approximating sequence ζ_ε. Thus we will be able to prove a specific differentiable behaviour (sometimes, in optimization theory, called a *sensitivity*) of this Γ-limit with respect to varying boundary conditions.

For this, we apply the standard shift of the Dirichlet condition. Let us abbreviate the linear space $W^{1,2}_\Gamma(\Omega;\mathbb{R}^d) := \{v \in W^{1,2}(\Omega;\mathbb{R}^d);\ v|_\Gamma = 0\}$. Considering $e_{\rm D} \in L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$, we define

$$F_\varepsilon(e_{\rm D}, v, \zeta) := \int_\Omega (\zeta{+}\varepsilon)\varphi\big(x, e_{\rm D} + e(v)\big)\,{\rm d}x. \tag{2.27}$$

Note that, considering again the continuation $u_{\rm D}$ of the Dirichlet condition w as in Proposition 2.6 and $G^{\rm red}_\varepsilon$ from (2.8), we have

$$G^{\rm red}_\varepsilon(u,\zeta) = F_\varepsilon(e_{\rm D}, v, \zeta) \qquad \text{with } e_{\rm D} := e(u_{\rm D}) \quad \text{and} \quad v := u - u_{\rm D}, \tag{2.28}$$

for any $v \in W^{1,2}_\Gamma(\Omega;\mathbb{R}^d)$ or, equally, for any $u \in W^{1,2}(\Omega;\mathbb{R}^d)$ such that $u|_\Gamma = w$. For $e_{\rm D} \in L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$ let

$$\mathfrak{f}_\varepsilon(e_{\rm D}, \zeta) := \min_{v\in W^{1,2}_\Gamma(\Omega;\mathbb{R}^d)} F_\varepsilon(e_{\rm D}, v, \zeta). \tag{2.29}$$

For $\varepsilon > 0$, the strictly convex quadratic functional $F_\varepsilon(e_{\rm D},\cdot,\zeta)$ on $W^{1,2}_\Gamma(\Omega;\mathbb{R}^d)$ has a unique minimizer, say v, and the mapping $L_{\zeta+\varepsilon}$ defined as

$$e_{\rm D} \mapsto L_{\zeta+\varepsilon} v : L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}) \to W^{1,2}_\Gamma(\Omega;\mathbb{R}^d), \qquad v \text{ minimizes } F_\varepsilon(e_{\rm D},\cdot,\zeta), \tag{2.30}$$

is linear and bounded. Hence, we conclude that, for each ζ, the functional

$$e_{\rm D} \mapsto \mathfrak{f}_\varepsilon(e_{\rm D}, \zeta) = F_\varepsilon(e_{\rm D}, L_{\zeta+\varepsilon}e_{\rm D}, \zeta) \tag{2.31}$$

is a quadratic form on $L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$ which, moreover, is bounded uniformly, namely $0 \le \mathfrak{f}_\varepsilon(e_{\rm D},\zeta) \le C\|e_{\rm D}\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})}$ with $C := (\|\zeta\|_{C(\bar\Omega)} + \varepsilon)\|\mathbb{C}\|_{L^\infty(\Omega;\mathbb{R}^{d\times d\times d\times d})}$.

Now, like in (2.9), we consider the Γ-limit of the collection $\{\mathfrak{f}_\varepsilon(\cdot,\zeta)\}_{\varepsilon>0,\zeta\in Z}$ as

$$\mathfrak{f}(e_{\rm D},\zeta) := \liminf_{\substack{\varepsilon\to0+\\ \tilde\zeta\rightharpoonup\zeta,\ \tilde\zeta\in Z}} \mathfrak{f}_\varepsilon(e_{\rm D},\tilde\zeta) \tag{2.32}$$

with Z defined in (2.1). The following assertion is based on an explicit construction to a universal recovery sequence for the Γ-limit (2.32).

Proposition 2.10. ***(A formula for the*** Γ***-limit*** $\mathfrak{f}$***.)*** *For all* $\zeta \in Z$ *the functional* $\mathfrak{f}(\cdot,\zeta) : L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}) \to \mathbb{R}$ *is convex and quadratic, and can be obtained as follows:*

$$\mathfrak{f}(e_{\rm D},\zeta) = \lim_{\delta\to 0+}\Big(\lim_{\varepsilon\to 0+} \mathcal{F}(\varepsilon,\delta,e_{\rm D},\zeta)\Big), \tag{2.33}$$

where

$$\mathcal{F}(\varepsilon,\delta,e_{\rm D},\zeta) = f_\varepsilon\big(e_{\rm D},(\zeta-\delta)^+\big) \qquad \textit{with} \ \ (\zeta-\delta)^+ := \max\{\zeta-\delta,0\}. \tag{2.34}$$

Proof. Each $\mathcal{F}(\varepsilon,\delta,\cdot,\zeta)$ is a bounded convex quadratic form on $L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$. If the limit exists, then it will be a convex quadratic form again.

For the existence of the limits, we use the following monotonicities of $\mathcal{F}$:

$$\begin{aligned} 0<\varepsilon_1<\varepsilon_2 \quad &\Longrightarrow \quad \mathcal{F}(\varepsilon_1,\delta,e_{\rm D},\zeta) < \mathcal{F}(\varepsilon_2,\delta,e_{\rm D},\zeta);\\ 0<\delta_1<\delta_2 \quad &\Longrightarrow \quad \mathcal{F}(\varepsilon,\delta_1,e_{\rm D},\zeta) \geq \mathcal{F}(\varepsilon,\delta_2,e_{\rm D},\zeta). \end{aligned} \tag{2.35}$$

This follows easily from the monotonicity $F_{\varepsilon_1}(e_{\rm D},v,\zeta_1) \leq F_{\varepsilon_2}(e_{\rm D},v,\zeta_2)$, and hence also $f_{\varepsilon_1}(e_{\rm D},\zeta_1) \leq f_{\varepsilon_2}(e_{\rm D},\zeta_2)$, whenever $0<\varepsilon_1{+}\zeta_1 \leq \varepsilon_2{+}\zeta_2$.

Thus, the existence of the inner limit $\varepsilon \to 0+$ follows because the function is nonincreasing in ε, let us denote it as $\mathcal{F}_0(\delta,e_{\rm D},\zeta) := \lim_{\varepsilon\to 0+}\mathcal{F}(\varepsilon,\delta,e_{\rm D},\zeta)$. Hence, $\mathcal{F}_0(\delta,\cdot,\zeta)$ exists and is a bounded quadratic form on $L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$. Moreover, $\mathcal{F}_0(\cdot,u_{\rm D},\zeta)$ is still non-decreasing on $[0,1]$. Hence, $\mathcal{F}_{00}(e_{\rm D},\zeta) := \lim_{\delta\to 0+}\mathcal{F}_0(\delta,u_{\rm D},\zeta)$ exists and for each $\zeta \in Z$, the functional $\mathcal{F}_{00}(\cdot,\zeta) : L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}) \to \mathbb{R}$ is a bounded quadratic form.

As $\mathcal{F}_{00}(u_{\rm D},\zeta)$ is just the right-hand side of (2.33), it remains to show that $\mathfrak{f} = \mathcal{F}_{00}$.

To show $\mathfrak{f} \geq \mathcal{F}_{00}$ we take a recovery sequence ζ_ε for (2.32), i.e. such that $\zeta_\varepsilon \rightharpoonup \zeta$, $\zeta_\varepsilon \geq 0$, and $f_\varepsilon(e_{\rm D},\zeta_\varepsilon) \to \mathfrak{f}(e_{\rm D},\zeta)$. For each $\delta > 0$ there exists $\varepsilon_\delta > 0$ such that $\zeta_\varepsilon \geq (\zeta-\delta)^+$ for $\varepsilon \in (0,\varepsilon_\delta)$; note that here $r > d$ was essential. Hence, we find $f_\varepsilon(e_{\rm D},\zeta_\varepsilon) \geq \mathcal{F}(\varepsilon,\delta,e_{\rm D},\zeta)$. Keeping $\delta > 0$ fixed and letting $\varepsilon \to 0+$ we find $\mathfrak{g}^{\rm red}(e_{\rm D},\zeta) \geq \mathcal{F}_0(\delta,e_{\rm D},\zeta)$. Now taking the limit $\delta \to 0+$ we obtain $\mathfrak{f}(e_{\rm D},\zeta) \geq \mathcal{F}_{00}(e_{\rm D},\zeta)$.

To show $\mathfrak{f} \leq \mathcal{F}_{00}$, we use a diagonalization argument to find a sequence $0 < \delta_\varepsilon \to 0$ for $\varepsilon \to 0+$ such that $\mathcal{F}(\varepsilon,\delta_\varepsilon,e_{\rm D},\zeta) \to \mathcal{F}_{00}(e_{\rm D},\zeta)$. Now consider the functions $\zeta_\varepsilon = (\zeta-\delta_\varepsilon)^+$, so that $\mathcal{F}(\varepsilon,\delta_\varepsilon,e_{\rm D},\zeta) = f_\varepsilon(e_{\rm D},\zeta_\varepsilon)$. Because of $\delta_\varepsilon \to 0$ we easily find that $\zeta_\varepsilon \rightharpoonup \zeta$ in $W^{1,r}(\Omega)$ because obviously $\zeta_\varepsilon \to \zeta$ in $C(\bar{\Omega})$ and because always $|\nabla\zeta_\varepsilon| \leq |\nabla\zeta|$ a.e. on Ω. Also, $\zeta_\varepsilon \in Z$ because $\zeta \in Z$ and $\delta_\varepsilon \geq 0$. Hence we conclude by the definition of the Γ-limit that

$$\mathfrak{f}(e_{\rm D},\zeta) \leq \liminf_{\varepsilon\to 0+} f_\varepsilon(e_{\rm D},\zeta_\varepsilon) = \lim_{\varepsilon\to 0+} \mathcal{F}(\varepsilon,\delta_\varepsilon,e_{\rm D},\zeta) = \mathcal{F}_{00}(e_{\rm D},\zeta). \tag{2.36}$$

$\square$

Let us now focus on sensitivity with respect to the boundary condition w or, more conveniently, to its extension $u_{\rm D}$. In the "language" of this subsection, it means rather sensitivity with respect to $e_{\rm D}$. As $\mathfrak{f}(\cdot,\zeta)$ was proved to be a bounded quadratic form, its derivative is a bounded linear operator, let us denote it by $\mathfrak{T}_\zeta : L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}) \to L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$. Thus we define a stress

$$\tau = \tau(e_{\rm D},\zeta) := \mathfrak{T}_\zeta e_{\rm D} := \mathfrak{f}'_{e_{\rm D}}(e_{\rm D},\zeta). \tag{2.37}$$

Let us now relate this to the original quantities as defined before. The following lemma uses an argument developed in [27, Proposition 5.6], which in turn is an abstract version of a result in [8].

Lemma 2.11. *Let $\{\zeta_\varepsilon\}_{\varepsilon>0}$ be a recovery sequence for $\mathfrak{f}(e_{\rm D},\zeta)$ as defined by (2.32), let $e_{\rm D} = e(u_{\rm D})$, and let σ_ε be the stress corresponding to ζ_ε and $u_{\rm D}$ due to the formula (2.16). Then, referring to (2.37), it holds $\sigma_\varepsilon \rightharpoonup \tau$ in $L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$.*

Proof. In view of (2.32), having assumed $\{\zeta_\varepsilon\}$ a recovery sequence, we just assume $f_\varepsilon(e_{\rm D},\zeta_\varepsilon) \to \mathfrak{f}(e_{\rm D},\zeta)$, $\varepsilon \to 0+$, and $\zeta_\varepsilon \rightharpoonup \zeta$. For any other $e \in L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$, we have only

$$\liminf_{\varepsilon\to 0} f_\varepsilon(e,\zeta_\varepsilon) \geq \mathfrak{f}(e,\zeta) \tag{2.38}$$

just by the definition of the Γ-limit (2.32). Let us put $\tau_\varepsilon := [f_\varepsilon]'_{e_{\rm D}}(e_{\rm D},\zeta_\varepsilon)$. We want to show that $\tau_\varepsilon \rightharpoonup \tau$ with τ from (2.37). As $\{\tau_\varepsilon\}_{\varepsilon>0}$ is bounded in $L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$, there is at least a subsequence converging to some $\tilde\tau$ weakly. By the definition of τ_ε and by the convexity of $f_\varepsilon(\cdot,\zeta_\varepsilon)$, for any $h > 0$ and any $e \in L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$, we have

$$\int_\Omega \tau_\varepsilon : \tilde e \,{\rm d}x \leq \frac{f_\varepsilon(e_{\rm D},\zeta_\varepsilon) - f_\varepsilon(e_{\rm D} - h\tilde e,\zeta_\varepsilon)}{h}\ . \tag{2.39}$$

Passing $\varepsilon \to 0+$ in (2.39) and using (2.38) for $e := e_{\rm D} - h\tilde e$, we obtain

$$\begin{aligned}\int_\Omega \tilde\tau : \tilde e \,{\rm d}x &= \lim_{\varepsilon\to 0+} \int_\Omega \tau_\varepsilon : \tilde e \,{\rm d}x \leq \limsup_{\varepsilon\to 0+} \frac{f_\varepsilon(e_{\rm D},\zeta_\varepsilon) - f_\varepsilon(e_{\rm D} - h\tilde e,\zeta_\varepsilon)}{h} \\ &= \frac{1}{h} \lim_{\varepsilon\to 0+} f_\varepsilon(e_{\rm D},\zeta_\varepsilon) - \frac{1}{h}\liminf_{\varepsilon\to 0+} f_\varepsilon(e_{\rm D} - h\tilde e,\zeta_\varepsilon) \leq \frac{\mathfrak{f}(e_{\rm D},\zeta) - \mathfrak{f}(e_{\rm D} - h\tilde e,\zeta)}{h}\ . \end{aligned} \tag{2.40}$$

Passing $h \to 0+$ in (2.40), by (2.37) we obtain $\int_\Omega \tilde\tau : \tilde e \,{\rm d}x \leq \int_\Omega \mathfrak{f}'_{e_{\rm D}}(e_{\rm D},\zeta) : \tilde e \,{\rm d}x = \int_\Omega \tau : \tilde e \,{\rm d}x$. Making the same procedure with $-\tilde e$ instead of $\tilde e$, we get also the opposite inequality. Taking $\tilde e$ arbitrary, we can see that $\tilde\tau = \tau$. In particular, the whole sequence $\{\tau_\varepsilon\}_{\varepsilon>0}$ converges to τ.

Now it remains to show that $\sigma_\varepsilon = \tau_\varepsilon$. Referring to $L_{\zeta+\varepsilon}$ from (2.30) and the definition of u_ε from (2.16) as a minimizer of $G^{\rm red}_\varepsilon(\cdot,\zeta_\varepsilon)$, by using the shift $v_\varepsilon = u_\varepsilon - u_{\rm D}$ (cf. 2.28)) and $v_\varepsilon := L_{\zeta_\varepsilon+\varepsilon} e(u_{\rm D})$, we have $u_\varepsilon = u_{\rm D} + L_{\zeta_\varepsilon+\varepsilon} e(u_{\rm D})$. By (2.31) with (2.27), we have

$$\begin{aligned} f_\varepsilon(e_{\rm D},\zeta_\varepsilon) = F_\varepsilon(e_{\rm D}, L_{\zeta_\varepsilon+\varepsilon} e(u_{\rm D}),\zeta_\varepsilon) &= \int_\Omega (\zeta_\varepsilon+\varepsilon)\varphi(x, e_{\rm D} + e(L_{\zeta_\varepsilon+\varepsilon} e(u_{\rm D}))\,{\rm d}x \\ &= \int_\Omega (\zeta_\varepsilon+\varepsilon)\varphi(x, e_{\rm D} + e(u_\varepsilon - u_{\rm D}))\,{\rm d}x\ . \end{aligned}$$

Differentiating both sides with respect to $e_{\rm D}$, we obtain

$$\tau_\varepsilon := [f_\varepsilon]'_{e_{\rm D}}(e_{\rm D},\zeta_\varepsilon) = (\zeta_\varepsilon{+}\varepsilon)\varphi'_e(x, e_{\rm D} + e(u_\varepsilon{-}u_{\rm D})).$$

In particular, for $e_{\rm D} = e(u_{\rm D})$, we can still continue as

$$(\zeta_\varepsilon{+}\varepsilon)\varphi'_e(x, e_{\rm D} + e(u_\varepsilon{-}u_{\rm D})) = (\zeta_\varepsilon{+}\varepsilon)\varphi'_e(x, e(u_\varepsilon)) =: \sigma_\varepsilon.$$

This concludes the proof. □

Corollary 2.12. *Setting*

$$\mathfrak{s} \equiv \mathfrak{s}(\zeta) := \tau(e_{\rm D},\zeta) \qquad \textit{for} \quad e_{\rm D} = e(u_{\rm D}) \quad \textit{with} \quad u_{\rm D}|_\Gamma = w, \tag{2.41}$$

we obtain an effective stress and, moreover, it holds

$$\mathfrak{g}^{\rm red}(\zeta) = \frac{1}{2}\int_\Omega \mathfrak{s}(\zeta) : e_{\rm D}\,{\rm d}x. \tag{2.42}$$

Proof. As $\mathfrak{f}(\cdot,\zeta)$ is quadratic, in view of (2.37), we have the formula

$$\mathfrak{f}(e_{\rm D},\zeta) = \frac{1}{2}\int_\Omega \tau(e_{\rm D},\zeta) : e_{\rm D}\,{\rm d}x. \tag{2.43}$$

As a consequence of (2.28) with (2.9) and (2.29), we have $g^{\rm red}_\varepsilon(\zeta) = f_\varepsilon(u_{\rm D},\zeta)$, and this equality is inherited by the respective Γ-limits defined in (2.9) and (2.32), i.e. we have

$$\mathfrak{g}^{\rm red}(\zeta) = \mathfrak{f}(e_{\rm D},\zeta) \qquad \text{for } \ e_{\rm D} = e(u_{\rm D}) \ \text{ with } \ u_{\rm D}|_\Gamma = w. \tag{2.44}$$

Substituting $\mathfrak{s}$ defined by (2.41) into (2.43) and using (2.44), we obtain (2.42).

For the specific recovery sequence $\{\zeta_\varepsilon\}$ from the proof of Proposition 2.10, by Lemma 2.11, the corresponding stresses σ_ε converge and we have $\sigma_\varepsilon \rightharpoonup \mathfrak{s}(\zeta)$ so that, by the definition (2.24), we have $\mathfrak{s}(\zeta) \in \mathfrak{S}(\zeta)$. In view of (2.25), we can see that we have constructed a particular realizable stress $\mathfrak{s}(\zeta)$ that attains the minimum in (2.25), i.e. an effective stress. □

For further use it is important that (2.42) yields an explicit information about sensitivity of $\mathfrak{g}^{\rm red}(\zeta)$ with respect to $u_{\rm D}$.

3. Rate-independent damage evolution

Now, we will let the "hard-device" *loading vary in time* t ranging $[0,T]$ with $T > 0$ a fixed time horizon, i.e. $w = w(t,x)$. Then the damage parameter will depend on both x and t, i.e. $\zeta = \zeta(t,x)$. Instead of $G_\varepsilon(u,\zeta)$ from (2.8) with (2.7), we will consider

$$\mathcal{G}_\varepsilon(t,u,\zeta) := \begin{cases} V_\varepsilon(u,\zeta) & \text{if } u|_\Gamma = w(t,\cdot) \text{ and } \zeta \in Z, \\ +\infty & \text{elsewhere,} \end{cases} \tag{3.1}$$

where Z is again from (2.1). A further important concept consists in specific *dissipation of energy* during the damage process, which is given by a phenomenological *activation threshold*, denoted by $a(x) > 0$ (of a physical dimension $\mathrm{J/m}^d$) at a given spot $x \in \Omega$. Roughly speaking, the damage starts evolving when the elastic energy $\varphi(e(u))$ reaches the activation threshold a, cf. (3.4b) and Sect. 3.1 for more details. At the same time, $a(x)$ says how much energy (per d-dimensional "volume") is dissipated by accomplishing the damage process, i.e. by decreasing $\zeta(x)$ from 1 to 0.

The rate of energy dissipated in the whole body is then

$$R(\dot\zeta) := \int_\Omega \varrho\big(x, \dot\zeta(x)\big)\,\mathrm{d}x, \qquad \text{where} \qquad \varrho(x,\dot z) = \begin{cases} -a(x)\dot z & \text{if } \dot z \le 0, \\ +\infty & \text{elsewhere.} \end{cases} \tag{3.2}$$

The value $+\infty$ reflects that we consider damage as a *unidirectional process*, i.e. damage can only develop, but the material can never heal. We qualify the activation-threshold profile as:

$$a \in L^\infty(\Omega), \qquad \operatorname{ess\,inf}_{x\in\Omega} a(x) > 0. \tag{3.3}$$

3.1. Classical formulation of the regularized evolution problem

Let us first consider the regularized case with $\varepsilon > 0$ where the displacement $u_\varepsilon = u_\varepsilon(t,x)$ is well defined a.e. on the whole $Q := (0,T)\times\Omega$. The evolving damage profile will now also depend on ε hence we denote it by ζ_ε. Taking into account our Gibbs energy (3.1) and the dissipation potential (3.2), the classical considerations in rational thermodynamics lead to the generalized force $f \in -\partial_{(u,\zeta)}G_\varepsilon(t,u_\varepsilon(t),\zeta_\varepsilon(t))$ to belong to $(0, \partial R(\frac{\mathrm{d}\zeta_\varepsilon}{\mathrm{d}t}))$, where the notation ∂ stands for subdifferential of the involved convex functionals. This, at least formally, leads to the classical formulation (cf. [13]) consisting in the balance of the stress and the evolution of the damage parameter:

$$\operatorname{div}\big(\sigma_\varepsilon\big) = 0 \qquad\qquad \text{with} \qquad\qquad \sigma_\varepsilon = (\zeta_\varepsilon{+}\varepsilon)\varphi_e'\big(e(u_\varepsilon)\big), \tag{3.4a}$$

$$\left.\begin{aligned} \frac{\partial\zeta_\varepsilon}{\partial t} &\le 0, \\ \varphi(e(u_\varepsilon)) - r_{\zeta_\varepsilon} - a - \operatorname{div}\big(\kappa|\nabla\zeta_\varepsilon|^{r-2}\nabla\zeta_\varepsilon\big) &\le 0, \\ \frac{\partial\zeta_\varepsilon}{\partial t}\Big(a - \varphi(e(u_\varepsilon)) + \operatorname{div}\big(\kappa|\nabla\zeta_\varepsilon|^{r-2}\nabla\zeta_\varepsilon\big) + r_{\zeta_\varepsilon}\Big) &= 0 \end{aligned}\right\} \tag{3.4b}$$

on Q, where $r_{\zeta_\varepsilon} \in \partial\chi_{[0,1]}(\zeta_\varepsilon)$. The notation $\chi_{[0,1]}$ stands for the indicator function of the interval $[0,1]$ where the damage parameter ranges; in fact, $[0,+\infty)$ can be used equally. The complementarity problem (3.4b) represents the evolution inclusion

$$\partial_{\dot\zeta}\varrho\Big(x, \frac{\partial\zeta_\varepsilon}{\partial t}\Big) - \kappa\operatorname{div}\big(|\nabla\zeta_\varepsilon|^{r-2}\nabla\zeta_\varepsilon\big) + \varphi(x,e(u_\varepsilon)) + \partial\chi_{[0,1]}(\zeta_\varepsilon) \ni 0\,. \tag{3.5}$$

The second inequality in (3.4b) can bear the interpretation that the *driving force* for the damage process can be identified as the specific energy $\varphi(x,e(u_\varepsilon))$. Moreover, damage evolves if it reaches the *activation threshold* $a(x)$ modified by the term $\mathrm{div}(\kappa(x)|\nabla\zeta_\varepsilon(x)|^{r-2}\nabla\zeta_\varepsilon(x))$ which reflects in some way a *hardening-like* effect (if the spot x is surrounded by a less damaged material) or softening (in an opposite case); we refer to [1].

We must complete the system by some boundary conditions not only for u_ε but now also for the damage ζ_ε. In accord with previous sections, we assume the mentioned Dirichlet conditions for u_ε combined with zero normal stress implicitly imposed already in (2.3) while for ζ_ε we assumed, for simplicity, zero Neumann condition as any condition for it is a bit artificial anyhow. Hence,

$$u_\varepsilon = w \qquad \text{on } \Gamma, \tag{3.6a}$$

$$\sigma_\varepsilon \nu = 0 \qquad \text{on } \partial\Omega\setminus\Gamma, \tag{3.6b}$$

$$\frac{\partial\zeta_\varepsilon}{\partial\nu} = 0 \qquad \text{on } \partial\Omega. \tag{3.6c}$$

An initial condition should be prescribed for the damage parameter, considering some prescribed initial profile ζ_0 and, rather formally, also the initial displacement u_0 (qualified later):

$$\zeta_\varepsilon(0,\cdot) = \zeta_0, \quad u_\varepsilon(0,\cdot) = u_0 \quad \text{ on } \ \Omega. \tag{3.7}$$

3.2. Energetic solution of the regularized problem

The relevant and mathematically amenable concept of a “weak solution” to the doubly-nonlinear problem (3.5) with degree-1 homogeneous $\varrho(x,\cdot)$ is a so-called *energetic solution*, formulated in [31, 32], see also [27] for a survey. Recently, this concept was also exposed in the context of Γ-limits in [30].

Let us first derive it formally from (3.4). For this, let us consider $u_{\rm D}(t,\cdot)$ as a suitable (qualified later) extension of $w(t,\cdot)$. The weak formulation of the Euler-Lagrange equation (3.4a) tested by $\frac{\partial}{\partial t}(u_\varepsilon - u_{\rm D})$, which has zero traces and is thus a legal test function, yields $\int_\Omega \sigma_\varepsilon : e(\frac{\partial}{\partial t}u_\varepsilon)\,\mathrm{d}x = \int_\Omega \sigma_\varepsilon : e(\frac{\partial}{\partial t}u_{\rm D})\,\mathrm{d}x$. Then, as there is no explicit dependence of $\mathcal{G}_\varepsilon$ on t in (3.1), $\frac{\partial}{\partial t}\mathcal{G}_\varepsilon = 0$ and we can formally apply the chain rule in the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}_\varepsilon\big(t,u_\varepsilon(t),\zeta_\varepsilon(t)\big) &= \int_\Omega \sigma_\varepsilon{:}e\Big(\frac{\partial u_\varepsilon}{\partial t}\Big) + \varphi\big(e(u_\varepsilon)\big)\frac{\partial\zeta_\varepsilon}{\partial t} + \kappa|\nabla\zeta_\varepsilon|^{r-2}\nabla\zeta_\varepsilon{\cdot}\nabla\frac{\partial\zeta_\varepsilon}{\partial t}\,\mathrm{d}x\\ &= \int_\Omega \sigma_\varepsilon{:}e\Big(\frac{\partial u_{\rm D}}{\partial t}\Big) + \varphi\big(e(u_\varepsilon)\big)\frac{\partial\zeta_\varepsilon}{\partial t} + \kappa|\nabla\zeta_\varepsilon|^{r-2}\nabla\zeta_\varepsilon{\cdot}\nabla\frac{\partial\zeta_\varepsilon}{\partial t}\,\mathrm{d}x.\end{aligned} \tag{3.8}$$

Using (3.5) in the weak formulation tested formally by $\frac{\partial}{\partial t}\zeta_\varepsilon$ together with (3.6c), one gets

$$\int_\Omega \varphi\big(x, e(u_\varepsilon)\big)\frac{\partial \zeta_\varepsilon}{\partial t} + \kappa|\nabla\zeta_\varepsilon|^{r-2}\nabla\zeta_\varepsilon\cdot\nabla\frac{\partial \zeta_\varepsilon}{\partial t}\,\mathrm{d}x = -\int_\Omega \partial_{\dot\zeta}\varrho\Big(x, \frac{\partial \zeta_\varepsilon}{\partial t}\Big)\frac{\partial \zeta_\varepsilon}{\partial t}\,\mathrm{d}x$$
$$= -\int_\Omega \varrho\Big(x, \frac{\partial \zeta_\varepsilon}{\partial t}\Big)\,\mathrm{d}x = -R\Big(\frac{\partial \zeta_\varepsilon}{\partial t}\Big) \tag{3.9}$$

due to the degree-1 homogeneity of $\varrho(x,\cdot)$, see definition (3.2). Putting (3.9) into (3.8), integrating it over a time interval $[t_1, t_2]$, and expressing the dissipated energy $\int_{t_1}^{t_2} R(\frac{\partial}{\partial t}\zeta(t))\,\mathrm{d}t$ as the total variation without referring explicitly to the time derivative $\frac{\partial}{\partial t}\zeta$, i.e.

$$\mathrm{Var}_R(\zeta; t_1, t_2) := \sup\sum_{i=1}^{j} R\big(\zeta(s_i) - \zeta(s_{i-1})\big) \tag{3.10}$$

with the supremum taken over all $j \in \mathbb{N}$ and over all partitions of $[t_1, t_2]$ in the form $t_1 = s_0 < s_1 < ... < s_{j-1} < s_j = t_2$, we eventually obtain

$$\mathcal{G}_\varepsilon\big(t_2, u_\varepsilon(t_2), \zeta_\varepsilon(t_2)\big) + \mathrm{Var}_R(\zeta_\varepsilon; t_1, t_2)$$
$$= \mathcal{G}_\varepsilon\big(t_1, u_\varepsilon(t_1), \zeta_\varepsilon(t_1)\big) + \int_{t_1}^{t_2}\int_\Omega \sigma_\varepsilon : e\Big(\frac{\partial u_{\mathrm{D}}}{\partial t}\Big)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.11}$$

In our special situation with R defined via (3.2), we have simply

$$\mathrm{Var}_R(\zeta; t_1, t_2) = R\big(\zeta(t_1) - \zeta(t_2)\big) = \begin{cases} \displaystyle\int_\Omega a(x)\big(\zeta(t_1,x) - \zeta(t_2,x)\big)\,\mathrm{d}x & \text{if } \zeta(\cdot,x) \text{ is nondecreasing on } [t_1,t_2] \text{ for a.a. } x\in\Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

The particular terms in (3.11) represent respectively:

- the stored energy at the final time t_2,
- the energy dissipated by damage during the time interval $[t_1, t_2]$,
- the stored energy at the initial time t_1, and
- the work done by external loadings during the time interval $[t_1, t_2]$.

The global-minimization hypothesis related to (3.4a) is a consequence of the *stability condition*

$$\forall(\tilde u, \tilde\zeta) \in W^{1,2}(\Omega; \mathbb{R}^d)\times Z \text{ with } \tilde u|_\Gamma = w(t):$$
$$\mathcal{G}_\varepsilon\big(t, u_\varepsilon(t), \zeta_\varepsilon(t)\big) \le \mathcal{G}_\varepsilon(t, \tilde u, \tilde\zeta) + R\big(\tilde\zeta - \zeta_\varepsilon(t)\big). \tag{3.12}$$

The philosophy of (3.12) is that the gain of Gibbs' energy $\mathcal{G}_\varepsilon(t, u_\varepsilon(t), \zeta_\varepsilon(t)) - \mathcal{G}_\varepsilon(t, \tilde u, \tilde\zeta)$ at any other state $(\tilde u, \tilde\zeta)$ is not larger than the dissipation $R(\tilde\zeta - \zeta_\varepsilon(t))$; cf. [32] for discussion.

Now, following [31], see also [27, 32], we introduce a definition of an energetic solution to the considered problem. By $\mathrm{B}([0,T]; X)$ or $\mathrm{BV}([0,T]; X)$ we denote the Banach space of bounded Bochner-measurable or bounded-variation X-valued mappings defined everywhere on $[0,T]$, respectively.

Definition 3.1. ***(Energetic solution to the regularized problem.)*** A process $(u_\varepsilon,\zeta_\varepsilon):[0,T]\to W^{1,2}(\Omega;\mathbb{R}^d)\times Z$ is called an *energetic solution* to the problem (3.4) and (3.6)–(3.7), i.e. given by the data φ, κ, ϱ, r, w, u_0, ζ_0, and $\varepsilon>0$, if, beside (3.7), also

(i) $(u_\varepsilon,\zeta_\varepsilon)\in \mathrm{B}([0,T];W^{1,2}(\Omega;\mathbb{R}^d))\times\big(\mathrm{BV}([0,T];L^1(\Omega))\cap\mathrm{B}([0,T];W^{1,r}(\Omega))\big)$,

(ii) it is stable in the sense that (3.12) holds for all $t\in[0,T]$, and

(iii) the energy balance (3.11) holds for any $0\le t_1<t_2\le T$ and, in particular, the function $t\mapsto\int_\Omega\sigma_\varepsilon:e(\frac{\partial}{\partial t}u_{\rm D})\,\mathrm{d}x$ belongs to $L^1(0,T)$.

Remark 3.2. In fact, Definition 3.1 is based on a *global-minimization* hypothesis competing with the *maximum-dissipation* principle (or rather *Levitas' realizability principle* [22]).

Remark 3.3. (*Normal stress: reaction to the Dirichlet loading.*) Due to (2.16) and Definition 3.1(i), $\sigma_\varepsilon\in B([0,T];L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}))$ and, in order to ensure that $t\mapsto\int_\Omega\sigma_\varepsilon:e(\frac{\partial}{\partial t}u_{\rm D})\,\mathrm{d}x$ belongs to $L^1(0,T)$, one needs just $u_{\rm D}\in W^{1,1}([0,T];W^{1,2}(\Omega;\mathbb{R}^d))$. In fact, one needs only to qualify $w\in W^{1,1}([0,T];W^{1/2,2}(\Gamma;\mathbb{R}^d))$ because then such extension $u_{\rm D}$ of it will always exists. Even more, (3.11) and thus the whole Definition 3.1 depends only on w and not on any particular choice of its extension $u_{\rm D}$. Actually, we could define the normal stress $\vec{\sigma}_\varepsilon$ as the linear bounded functional on $W^{1/2,2}(\Gamma;\mathbb{R}^d)$ by the formula

$$\big\langle\vec{\sigma}_\varepsilon,v|_\Gamma\big\rangle=\int_\Omega\sigma_\varepsilon:e(v(x))\;\mathrm{d}x.\tag{3.13}$$

It is a consequence of the stability (3.12) with $\tilde{\zeta}:=\zeta_\varepsilon(t)$ that $u_\varepsilon(t)$ minimizes $G_\varepsilon(t,\cdot,\zeta_\varepsilon(t))$ so that the corresponding Euler-Lagrange equation, cf. (2.21) for the static case, says in particular that

$$\mathrm{div}(\sigma_\varepsilon)=0\qquad\text{in the sense of distributions on } Q.\tag{3.14}$$

Then the right-hand side of (3.13) is independent of the particular extension v of $v|_\Gamma$ into Ω and thus the normal stress $\vec{\sigma}_\varepsilon$ is well defined by (3.13). This can easily be seen by an extension of Green's formula using Neumann boundary conditions (3.6b) and by the symmetry of the stress tensor

$$0=\int_\Omega\mathrm{div}(\sigma_\varepsilon)\cdot v\;\mathrm{d}x=\int_{\partial\Omega}(\sigma_\varepsilon\nu)\cdot v\,\mathrm{d}S-\int_\Omega\sigma_\varepsilon:\nabla v\;\mathrm{d}x=\int_\Gamma(\sigma_\varepsilon\nu)\cdot v\,\mathrm{d}S-\int_\Omega\sigma_\varepsilon:e(v)\;\mathrm{d}x.$$

In a regular case thus $\vec{\sigma}_\varepsilon=\sigma_\varepsilon\nu$. The last term in (3.11) can equivalently be expressed as $\int_{t_1}^{t_2}\langle\vec{\sigma}_\varepsilon,\frac{\partial w}{\partial t}\rangle\,\mathrm{d}t$, which is just the more explicit form of the work of the external "hard-device" load $\int_{t_1}^{t_2}\int_\Gamma\vec{\sigma}_\varepsilon\cdot\frac{\partial w}{\partial t}\;\mathrm{d}S\,\mathrm{d}t$. In what follows, we will confine ourselves to

$$w\in C^1(I;W^{1/2,2}(\Gamma;\mathbb{R}^d)),\tag{3.15}$$

which has nearly the same generality in the context of rate-independent processes and makes the proofs easier, cf. in particular [30, Assumption (2.8)] pointed also out later in Remark 3.9. Then, assumption (3.15) allows for considering $u_{\rm D} \in C^1([0,T];W^{1,2}\ (\Omega;\mathbb{R}^d))$.

Proposition 3.4. ***(Existence of energetic solutions to ε-problems.)*** (See [28].) *Let (2.5), (3.3), (3.15), $(u_0,\zeta_0)\in W^{1,2}(\Omega;\mathbb{R}^d)\times Z$ be stable in the sense*

$$\forall\,(\tilde u,\tilde\zeta)\in W^{1,2}(\Omega;\mathbb{R}^d)\times Z \ \textit{ with } \tilde u|_\Gamma = w(0,\cdot):$$
$$\mathcal{G}_\varepsilon(0,u_0,\zeta_0)\le\mathcal{G}_\varepsilon(0,\tilde u,\tilde\zeta)+R(\zeta_0-\tilde\zeta), \tag{3.16}$$

and let $\varepsilon>0$. Then a solution $(u_\varepsilon,\zeta_\varepsilon)$ in the sense of Definition 3.1 does exist.

Comments to the proof. The above assertion has been proved, except the Bochner measurability of u_ε, in [28] for the case φ and ϱ independent of x; but our x-dependent generalization is trivial. Also, a special loading and initial stable initial condition was chosen in [28], namely $w(0,\cdot)=0$, $u_0=0$, $\zeta_0=1$, i.e. unloaded undamaged body at the original time. Our, only slightly more general initial condition makes just a trivial and standard modification, cf. [13, 27, 29, 30]. Also, $w\in W^{1,1}(I;W^{1,\infty}(\Gamma;\mathbb{R}^d))$ has been used in [28]; but the generalization to $w\in W^{1,1}(I;W^{1/2,2}(\Gamma;\mathbb{R}^d))$ is routine since, unlike [28], we do not treat any contact problem at large strains and then (3.15) works, too.

Due to our formula $u_\varepsilon(t)=u_{\rm D}(t)+L_{\zeta_\varepsilon(t)+\varepsilon}e(u_{\rm D}(t))$, the claimed Bochner measurability of u_ε in time, not proved in [28], is here a simple consequence of the measurability of $\zeta_\varepsilon:[0,T]\to W^{1,r}(\Omega)$ and of the continuity of the mapping $(e_{\rm D},\zeta)\mapsto v:=L_{\zeta+\varepsilon}e_{\rm D}$ as a mapping $L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})\times W^{1,r}(\Omega)\to W^{1,2}_\Gamma(\Omega;\mathbb{R}^d)$. The mentioned measurability of ζ_ε follows from measurability of the BV-function $\zeta_\varepsilon:[0,T]\to L^1(\Omega)$ and from the a-priori estimate of $\{\zeta_\varepsilon(t)\}_{t\in[0,T]}$ in the separable space $W^{1,r}(\Omega)$ by Pettis' theorem. The mentioned continuity of $(e_{\rm D},\zeta)\mapsto v:=L_{\zeta+\varepsilon}e_{\rm D}$ (even locally Lipschitz continuity in $(L^2\times L^\infty,W^{1,2})$) can be proved quite standardly: We take the Euler-Lagrange equation for $v:=L_{\zeta+\varepsilon}e_{\rm D}$ defined in (2.30), i.e. in the weak formulation $\int_\Omega\zeta\mathbb{C}(e_{\rm D}+e(v)):e(z)\,{\rm d}x=0$ for all $z\in W^{1,2}_\Gamma(\Omega;\mathbb{R}^d)$. Considering other $\tilde e_{\rm D}$, $\tilde\zeta$, and $\tilde v:=L_{\tilde\zeta+\varepsilon}\tilde e_{\rm D}$, we have $\int_\Omega\tilde\zeta\mathbb{C}(\tilde e_{\rm D}+e(\tilde v)):e(z)\,{\rm d}x=0$. Subtracting these equations and testing the difference by $z:=v-\tilde v$ give, after some algebra and Hölder's and Young's inequalities,

$$\begin{aligned}\varepsilon\eta\big\|e(v-\tilde v)\big\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})}&\le\int_\Omega(\zeta+\varepsilon)\mathbb{C}(e(v-\tilde v)):e(v-\tilde v)\,{\rm d}x\\&=\int_\Omega(\zeta-\tilde\zeta)\mathbb{C}(e_{\rm D}+e(\tilde v)):e(v-\tilde v)+(\tilde\zeta+\varepsilon)\mathbb{C}(e_{\rm D}-\tilde e_{\rm D}):e(v-\tilde v)\,{\rm d}x\\&\le C\|\zeta-\tilde\zeta\|^2_{L^\infty(\Omega)}+C\|e_{\rm D}-\tilde e_{\rm D}\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})}+\frac{\varepsilon\eta}{2}\big\|e(v-\tilde v)\big\|^2_{L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})}\end{aligned}$$

with $\eta>0$ from (2.5b) and with $C=\max(\|e_{\rm D}+e(\tilde v)\|_{L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})},\|\tilde\zeta\|_{L^\infty(\Omega)}+\varepsilon)^2/(\varepsilon\eta)$. Absorbing the last term in the left-hand side and involving still Korn's inequality $\|v-\tilde{v}\|_{W^{1,2}(\Omega;\mathbb{R}^d)} \le K_{\Omega,\Gamma}\|e(v-\tilde{v})\|_{L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})}$, we clearly get the claim continuity. $\square$

3.3. Energetic solution of the complete-damage problem

Let us observe that, due to the definition (3.1) with (2.22),

$$\mathcal{G}_\varepsilon\big(t,u_\varepsilon(t),\zeta_\varepsilon(t)\big) = \int_\Omega \frac{1}{2}\sigma_\varepsilon(t,x){:}e(u_{\rm D}(t,x)) + \frac{\kappa(x)}{r}\big|\nabla\zeta_\varepsilon(t,x)\big|^r\,{\rm d}x, \tag{3.17}$$

hence both (3.11) and (3.12) can be expressed in terms of σ_ε and ζ_ε. Moreover, as explained above, (3.14) implies that σ_ε itself is essentially determined by $\zeta_\varepsilon(t,\cdot)$ and $w(t,\cdot)$.

Like (2.9) and (2.12) let us now define

$$\boldsymbol{g}(t,\zeta) := \liminf_{\substack{\varepsilon\to0+,\ \ \tilde\zeta\in Z,\\ \tilde\zeta\rightharpoonup\zeta \text{ in } W^{1,r}(\Omega)}} \ \min_{u\in W^{1,2}(\Omega;\mathbb{R}^d)} \mathcal{G}_\varepsilon(t,u,\tilde\zeta) \tag{3.18}$$

with $\mathcal{G}_\varepsilon$ defined in (3.1). Since $\min_{u\in W^{1,2}(\Omega;\mathbb{R}^d)}\mathcal{G}_\varepsilon(t,u,\tilde\zeta) = f_\varepsilon(e(u_{\rm D}(t)),\tilde\zeta) + \int_\Omega \frac{\kappa}{r}|\nabla\tilde\zeta|^r\,{\rm d}x$ with f_ε from (2.29), we have equivalently

$$\boldsymbol{g}(t,\zeta) = \liminf_{\substack{\varepsilon\to0+,\ \ \tilde\zeta\in Z,\\ \tilde\zeta\rightharpoonup\zeta \text{ in } W^{1,r}(\Omega)}} f_\varepsilon(e(u_{\rm D}(t)),\tilde\zeta) + \int_\Omega \frac{\kappa}{r}\big|\nabla\tilde\zeta\big|^r\,{\rm d}x. \tag{3.19}$$

Lemma 3.5. *Any recovery sequence $\{\zeta_\varepsilon\}_{\varepsilon>0}\subset Z$ for (3.19), i.e. $\zeta_\varepsilon\rightharpoonup\zeta$ and $f_\varepsilon(e(u_{\rm D}(t)),\zeta_\varepsilon)+\int_\Omega\frac{\kappa}{r}|\nabla\zeta_\varepsilon|^r\,{\rm d}x \to \boldsymbol{g}(t,\zeta)$, in fact converges strongly. Moreover, referring to $\mathfrak{f}(u_{\rm D},\zeta)$ defined by (2.32), we have now*

$$\boldsymbol{g}(t,\zeta) = \mathfrak{f}(e(u_{\rm D}(t),\zeta) + \int_\Omega \frac{\kappa}{r}\big|\nabla\zeta\big|^r\,{\rm d}x. \tag{3.20}$$

Proof. First, we prove (3.20). The inequality "$\geq$" is by the weak lower semicontinuity of $\zeta\mapsto\int_\Omega\kappa|\nabla\zeta|^r\,{\rm d}x$ and by the definition of the Γ-limits $\boldsymbol{g}$ and $\mathfrak{f}$ in (3.18) and (2.32), respectively. It suffices to take any recovery sequence $\{\zeta_\varepsilon\}_{\varepsilon>0}$ for $\boldsymbol{g}$ and make a limit passage in

$$\begin{aligned}\boldsymbol{g}(t,\zeta) &= \lim_{\varepsilon\to0+}\min_{u\in W^{1,2}(\Omega)}\mathcal{G}_\varepsilon(t,u,\zeta_\varepsilon) = \lim_{\varepsilon\to0+}\Big(f_\varepsilon(e(u_{\rm D}(t)),\zeta_\varepsilon) + \int_\Omega\frac{\kappa}{r}\big|\nabla\zeta_\varepsilon\big|^r\,{\rm d}x\Big)\\ &\ge \liminf_{\varepsilon\to0+} f_\varepsilon(e(u_{\rm D}(t)),\zeta_\varepsilon) + \liminf_{\varepsilon\to0+}\int_\Omega\frac{\kappa}{r}\big|\nabla\zeta_\varepsilon\big|^r\,{\rm d}x\\ &\ge \mathfrak{f}(e(u_{\rm D}(t)),\zeta) + \int_\Omega\frac{\kappa}{r}\big|\nabla\zeta\big|^r\,{\rm d}x.\end{aligned}$$

The opposite inequality "$\leq$" is by the same limit passage but now using the special recovery sequence $\zeta_\varepsilon = (\zeta-\delta_\varepsilon)^+$ for $\mathfrak{f}$ from the proof of Proposition 2.10. It converges to ζ not only weakly but also strongly. Indeed, $\nabla\zeta_\varepsilon(x)\to\nabla\zeta(x)$ for a.a. $x \in \Omega$ because $\nabla\zeta = 0 = \nabla\zeta_\varepsilon$ a.e. on N_ζ and because, for a.a. $x \in \Omega\backslash N_\zeta$, there is $\varepsilon_x > 0$ such that $0 < \zeta_\varepsilon(x) = \zeta(x) - \delta_\varepsilon$ and thus $\nabla\zeta_\varepsilon(x) = \nabla\zeta(x)$ for all $0 < \varepsilon < \varepsilon_x$, and then, by Lebesgue dominated-convergence theorem, $\int_\Omega |\nabla\zeta_\varepsilon(x)|^r \,\mathrm{d}x \to \int_\Omega |\nabla\zeta(x)|^r \,\mathrm{d}x$ and, having convergence of the norms as well as weak convergence, we can conclude strong convergence by uniform convexity of $W^{1,r}(\Omega)$ and a Fan-Glicksberg type theorem.

Let us now consider an arbitrary recovery sequence $\{\zeta_\varepsilon\}_{\varepsilon>0} \subset Z$ for (3.18). Denote $\widehat{\alpha} = \int_\Omega \frac{\kappa}{r}|\nabla\zeta|^r \,\mathrm{d}x$. For a subsequence and some α and β, $\int_\Omega \frac{\kappa}{r}|\nabla\zeta_\varepsilon|^r \,\mathrm{d}x \to \alpha$ and $f_\varepsilon(e(u_{\rm D}(t)), \zeta_\varepsilon) \to \beta$. Simultaneously, $f_\varepsilon(e(u_{\rm D}(t)), \zeta_\varepsilon) + \int_\Omega \frac{\kappa}{r}|\nabla\zeta_\varepsilon|^r \,\mathrm{d}x \to \boldsymbol{g}(t, \zeta) = \alpha + \beta$. By the weak lower semicontinuity, always $\widehat{\alpha} \le \alpha$. Assume $\widehat{\alpha} < \alpha$. Using (3.20), we would have

$$\begin{aligned}\beta = \lim_{\varepsilon\to 0+} f_\varepsilon(e(u_{\rm D}(t)), \zeta_\varepsilon) &= \lim_{\varepsilon\to 0+} \Big(\boldsymbol{g}(t,\zeta) - \int_\Omega \frac{\kappa}{r}\big|\nabla\zeta_\varepsilon\big|^r \,\mathrm{d}x\Big)\\ &= \boldsymbol{g}(t,\zeta) - \alpha < \boldsymbol{g}(t,\zeta) - \widehat{\alpha} = \mathfrak{f}\big(e(u_{\rm D}(t)), \zeta\big),\end{aligned}$$

a contradiction with (2.32). Hence $\widehat{\alpha} = \alpha$ and we have $\int_\Omega \frac{\kappa}{r}|\nabla\zeta_\varepsilon|^r \,\mathrm{d}x \to \alpha = \widehat{\alpha} = \int_\Omega \frac{\kappa}{r}\big|\nabla\zeta\big|^r \,\mathrm{d}x$. Due to the strict convexity of the integrand $\kappa(x)|\cdot|^r$ and due to the weak convergence $\zeta_\varepsilon \rightharpoonup \zeta$, we can conclude strong convergence, cf. e.g. [36]. $\square$

Considering an effective stress, as in (2.42), we can write

$$\boldsymbol{g}(t,\zeta) = \int_\Omega \frac{1}{2}\mathfrak{s}(t,\zeta) : e(u_{\rm D}(t)) + \frac{\kappa}{r}\big|\nabla\zeta\big|^r \,\mathrm{d}x. \tag{3.21}$$

Motivated by this and by the investigations for $\varepsilon \to 0$ in the static case in Sect. 2, we introduce the following "energetic" definition without referring to the problem (3.4) for $\varepsilon = 0$ because the displacement need not have a well defined sense any longer. For simplicity and without much restriction for possible applications, we consider the initial damage profile from Z away from zero

$$\min_{x\in\Omega} \zeta_0(x) > 0. \tag{3.22}$$

Then, prescribing the initial displacement u_0 makes sense and we thus automatically prescribe also the initial stress $\sigma(0) = \zeta_0\varphi_e'(e(u_0))$. As for the stability (3.16) of the initial conditions, for example, $w(0) = 0$, $u_0 = 0$ and $0 < \zeta_0 \le 1$ constant will satisfy (3.16) even for any $\varepsilon > 0$, which is what we will assume later in Theorem 3.7. This can be however satisfied for some non-constant damage profiles ζ_0 too, depending on $a(\cdot)$ and $\kappa(\cdot)$.

Definition 3.6. ***(Energetic solution to the complete-damage problem.)*** The process $(\mathfrak{s}, \zeta) : [0,T] \to L^2(\Omega; \mathbb{R}^{d\times d}_{\rm sym}) \times Z$ is called an *energetic solution* to the problem given by the data φ, ϱ, w, and ζ_0, if, beside (3.7), also

(i) $(\mathfrak{s}, \zeta) \in \mathrm{B}([0,T]; L^2(\Omega; \mathbb{R}^{d\times d})) \times \big(\mathrm{BV}([0,T]; L^1(\Omega)) \cap \mathrm{B}([0,T]; W^{1,r}(\Omega)\big)$,

(ii) it is stable in the sense that

$$\boldsymbol{g}(t, \zeta(t)) \le \boldsymbol{g}(t, \tilde{\zeta}) + \int_\Omega \varrho(x, \tilde{\zeta} - \zeta(t)) \,\mathrm{d}x \quad \text{for any } \tilde{\zeta} \in Z, \tag{3.23}$$

(iii) and, for any $0 \le t_1 < t_2 \le T$, the energy equality holds:

$$\boldsymbol{g}(t_2,\zeta(t_2)) + \mathrm{Var}_R(\zeta;t_1,t_2) = \boldsymbol{g}(t_1,\zeta(t_1)) + \int_{t_1}^{t_2}\int_\Omega \mathfrak{s}:e\Big(\frac{\partial u_{\mathrm{D}}}{\partial t}\Big)\,\mathrm{d}x\,\mathrm{d}t, \quad (3.24)$$

in particular, the function $t \mapsto \int_\Omega \mathfrak{s}(t,x) : e(\frac{\partial u_{\mathrm{D}}}{\partial t}(t,x))\,\mathrm{d}x$ belongs to $L^1(0,T)$,

(iv) $\mathrm{div}(\mathfrak{s}) = 0$ in the sense of distributions and $\mathfrak{s}(t)$ is an effective stress with respect to $\zeta(t)$ and $w(t)$ for any $t \in [0,T]$; in particular (3.21) holds.

Theorem 3.7. ***(Existence of energetic solutions, convergence of*** $(u_\varepsilon,\zeta_\varepsilon)$***.)*** *Let (2.5), (3.3), $w \in C^1([0,T];W^{1/2,2}(\Gamma;\mathbb{R}^d))$, $(u_0,\zeta_0) \in W^{1,2}(\Omega;\mathbb{R}^d)\times Z$ satisfy (3.16) for all $\varepsilon > 0$ and (3.22). Then, there exist a subsequence $\{\varepsilon_n\}_{n\in\mathbb{N}}$ converging to 0 and a process $(\mathfrak{s},\zeta) : [0,T] \to L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}) \times Z$ being an energetic solution according to Definition 3.6, in particular $u_{\mathrm{D}} \in C^1([0,T];W^{1,2}(\Omega;\mathbb{R}^d))$ is considered for (3.24) in accord with Remark 3.3, such that the following holds for all $t \in [0,T]$:*

(i) $\mathcal{E}_{\varepsilon_n}(t,u_{\varepsilon_n}(t),\zeta_{\varepsilon_n}(t)) \to \boldsymbol{g}(t,\zeta(t))$,

(ii) $\mathrm{Var}_R(\zeta_{\varepsilon_n};0,t) \to \mathrm{Var}_R(\zeta;0,t)$,

(iii) $\zeta_{\varepsilon_n}(t) \to \zeta(t)$ *strongly in* $W^{1,r}(\Omega)$,

(iv) $\sigma_{\varepsilon_n}(t) = (\zeta_{\varepsilon_n}(t)+\varepsilon)\varphi'_e(e(u_{\varepsilon_n}(t))) \rightharpoonup \mathfrak{s}(t)$ *weakly in* $L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$.

Proof. Most of the assertions have been proved in [28, Sect.4] but the most essential properties remained open in the context of non-quadratic quasiconvex φ considered there. Namely, only an energy inequality in (3.24) has been proved in [28], only the weak convergence of $\zeta_{\varepsilon_n}(t) \rightharpoonup \zeta(t)$ instead of (iii), and, instead of the properties claimed in Definition 3.6(iv), $\mathfrak{s}(t)$ was shown to be a realizable stress only. Moreover, instead of (iv), only $\sigma_{\varepsilon_n} \rightharpoonup \mathfrak{s}$ weakly* in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$ was proved in [28]. Let us remark that, in fact, instead of $(\zeta+\varepsilon)\varphi(e)$, the regularization $\zeta\varphi(e)+\varepsilon|e|^2$ has been used in [28], homogeneous material (i.e. φ, ϱ, a, and κ independent of x), and only special initial conditions $u_0 = 0$, $\zeta_0 = 1$, $w(0) = 0$ were considered, but these modifications are easy under our data qualification. Let us now prove the remaining properties.

The property $\mathrm{div}(\mathfrak{s}) = 0$ claimed in Definition 3.6(iv) is inherited by a trivial limit passage from (3.14).

Due to (i), $\{\zeta_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is a recovery sequence for (3.19), by Lemma 3.5 we have strong convergence in (iii). Moreover, by Lemma 2.11, we have $\sigma_{\varepsilon_n}(t) \rightharpoonup \tau(e(u_{\mathrm{D}}(t)),\zeta(t))$. Hence, modifying $\mathfrak{s}$ obtained in [28], if necessary on a zero-measure set on $[0,T]$, we have $\mathfrak{s}(t) = \tau(e(u_{\mathrm{D}}(t)),\zeta(t))$ and $\mathfrak{s}(t)$ being thus proved an essential stress.

Energy equality in (3.24) is then a consequence of [27, Proposition 5.7] provided one shows the power of external loading to be in $L^\infty(0,T)$ and the last term in (3.24) to be equal to $\int_{t_1}^{t_2} \frac{\partial \boldsymbol{g}}{\partial t}(t,\zeta(t))\,\mathrm{d}t$. Here, by using successively (3.20), (2.43), and (2.37), for any $\zeta \in Z$ fixed, we have

$$\begin{aligned}\boldsymbol{g}(t,\zeta) &= \mathfrak{f}(e(u_{\rm D}(t),\zeta) + \int_\Omega \frac{\kappa}{r}\big|\nabla\zeta\big|^r\,{\rm d}x \\ &= \int_\Omega \frac{1}{2}\tau(e(u_{\rm D}(t)),\zeta) : e(u_{\rm D}(t)) + \frac{\kappa}{r}\big|\nabla\zeta\big|^r\,{\rm d}x \\ &= \int_\Omega \frac{1}{2}\mathfrak{T}_\zeta e(u_{\rm D}(t)) : e(u_{\rm D}(t)) + \frac{\kappa}{r}\big|\nabla\zeta\big|^r\,{\rm d}x. \end{aligned} \tag{3.25}$$

In particular, $u_{\rm D} \in C^1([0,T];{\rm W}^{1,2}(\Omega;\mathbb{R}^{\rm d}))$ implies $\mathfrak{g}(\cdot,\zeta) \in C^1([0,T])$ for each $\zeta \in Z$. Also, by using (3.25) and (2.41), we have the desired formula for the power of external loading:

$$\begin{aligned}\frac{\partial \boldsymbol{g}}{\partial t}(t,\zeta) &= \int_\Omega \mathfrak{T}_\zeta e(u_{\rm D}(t)) : e\Big(\frac{\partial u_{\rm D}}{\partial t}\Big)\,{\rm d}x \\ &= \int_\Omega \tau(e(u_{\rm D}(t)),\zeta) : e\Big(\frac{\partial u_{\rm D}}{\partial t}\Big)\,{\rm d}x = \int_\Omega \mathfrak{s}(t) : e\Big(\frac{\partial u_{\rm D}}{\partial t}\Big)\,{\rm d}x. \end{aligned} \tag{3.26}$$

The Bochner measurability of $\mathfrak{s}$ follows from the measurability of $u_\varepsilon : [0,T] \to W^{1,2}(\Omega;\mathbb{R}^d)$ proved in Proposition 3.4 implying measurability of $\sigma_\varepsilon : [0,T] \to L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$ and from the point (iv) together with Pettis' theorem. □

Remark 3.8. (*Alternative formulation in terms of strains.*) Based on formula (2.26), we could define the energetic solution to the complete-damage problem not as a couple $(\mathfrak{s},\zeta)$ but as a couple $(\mathfrak{e},\zeta)$ with $\mathfrak{e}(t)$ defined on $\Omega\backslash N_{\zeta(t)}$ and belonging to the time-dependent locally-convex space $L^2_{\rm loc}(\Omega\backslash N_{\zeta(t)};\mathbb{R}^{d\times d}_{\rm sym})$. Taking into account (2.18), the energy equality (3.24) would then take the form

$$\boldsymbol{g}(t_2,\zeta(t_2)) + {\rm Var}_R(\zeta;t_1,t_2) = \boldsymbol{g}(t_1,\zeta(t_1)) + \int_{t_1}^{t_2}\!\int_{\Omega\backslash N_{\zeta(t)}} \zeta\varphi_e'(\mathfrak{e}) : e\Big(\frac{\partial u_{\rm D}}{\partial t}\Big)\,{\rm d}x\,{\rm d}t. \tag{3.27}$$

Remark 3.9. (*Direct Γ-limit convergence.*) In terms of ζ only, we could obtain existence of the energetic solutions and convergence of solutions of our ε-regularized problem by using abstract results about Γ-limits, see [30, Theorem 3.1]. In fact, [30, Assumptions (2.9)–(2.10)] had been proved here in Section 2, [30, Assumption (2.8)] can be easily verified if $w \in C^1(I;W^{1/2,2}(\Gamma))$, and [30, Assumptions (2.11)] had been proved in [28], while the other assumptions in [30] are satisfied quite obviously. However, by this way, we would lose tack on the mechanical interpretation involving stress; in particular, the key information in (3.26) would be completely out.

Remark 3.10. (*Numerical strategies.*) The regularized problem introduced in Section 3.1 suggests a direct numerical treatment: applying implicit discretization in time with a time step $\tau > 0$ and, considering a polyhedral domain Ω triangulated by simplicial finite elements with a mesh-parameter $h > 0$, applying P1-finite elements for spatial discretization of both u and ζ (let us denote the corresponding discrete spaces U_h and Z_h, respectively), we get a recursive coercive mathematical-programming problem with a nonlinear objective and box-constraints for $(u^k_{\tau h\varepsilon}, \zeta^k_{\tau h\varepsilon})$:

$$\left.\begin{array}{ll}\text{Minimize} & \displaystyle\int_\Omega \frac{\zeta^k_{\tau h\varepsilon}+\varepsilon}{2}\mathbb{C}e(\nabla u^k_{\tau h\varepsilon}) : e(\nabla u^k_{\tau h\varepsilon}) - a\zeta^k_{\tau h\varepsilon} + \frac{\kappa}{r}\big|\nabla\zeta^k_{\tau h\varepsilon}\big|^r \,\mathrm{d}x \\ \text{subject to} & 0 \le \zeta^k_{\tau h\varepsilon} \le \zeta^{k-1}_{\tau h\varepsilon}, \\ & u^k_{\tau h\varepsilon}|_\Gamma = w(k\tau), \\ & u^k_{\tau h\varepsilon} \in U_h\ , \quad \zeta^k_{\tau h\varepsilon} \in Z_h \end{array}\right\} \tag{3.28}$$

for $k = 1, ..., K := T/\tau$ with $(u^0_{\tau h\varepsilon}, \zeta^0_{\tau h\varepsilon}) := (u_0, \zeta_0)$. This is an implementable conceptual algorithm. Unfortunately, it does not have a quadratic cost functional, which makes it not entirely simple for numerical treatment; for a similar problem with tri-linear objectives we refer to numerical simulations in [21]. On the other hand, the approximate solution $(u_{\tau h\varepsilon}, \zeta_{\tau h\varepsilon})$ considered as a piece-wise constant interpolant $(u_{\tau h\varepsilon}(t), \zeta_{\tau h\varepsilon}(t)) := (u^k_{\tau h\varepsilon}, \zeta^k_{\tau h\varepsilon})$ for $t \in ((k-1)\tau, k\tau]$ has a guaranteed convergence (in terms of suitable subsequences), based on the abstract results from [30, Theorem 3.3], cf. also [29, Sect.5.5].

Remark 3.11. (*Bourdin's approach to cracks.*) A functional that is of a similar type as (3.28), namely $\int_\Omega (\zeta + \varepsilon^\alpha)\varphi(\nabla u) + \varepsilon|\nabla\zeta|^2 + \varepsilon^{-\beta}(1-\zeta)\,\mathrm{d}x$, was used in the context of approximation of Francfort-Marigo's crack model [5, 6]. At least for fixed $\varepsilon > 0$ the mathematical properties of that functional are exactly as those of ours. However, suitable scalings in ε yields in the limit $\varepsilon \to 0$ the mentioned crack problem.

4. A one-dimensional example

Let us illustrate the above introduced objects on a one-dimensional situation, having an interpretation of a bar undergoing a tension/compression experiment by a "hard-device" loading, where all mathematical objects can be described explicitly. We consider a bar of the length L fixed at the end-points with a (possibly spatially varying) elastic modulus $\mathbb{C}$ (that may reflect a possibly varying thickness of the bar). Let us thus put $d := 1$, $\Omega := (0, L)$, $\Gamma := \partial\Omega = \{0, 1\}$, $w(0) := w_0$, $w(L) := w_L$, and now $\mathbb{C} : (0, L) \to \mathbb{R}^+$. In accord with (2.5b), $\mathbb{C}(x) \ge \eta > 0$ for a.a. $x \in (0, L)$.

4.1. Static case

Minimization of

$$V_\varepsilon(u, \zeta) = \int_0^L (\zeta(x) + \varepsilon)\frac{\mathbb{C}(x)}{2}\Big(\frac{\mathrm{d}u}{\mathrm{d}x}\Big)^2\,\mathrm{d}x$$

on $\{u \in W^{1,2}(0,L);\ u(0) = w_0,\ u(L) = w_L\}$ gives the *Euler-Lagrange equation*

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big((\zeta(x) + \varepsilon)\mathbb{C}(x)\frac{\mathrm{d}u}{\mathrm{d}x}\Big) = 0 \quad \text{on} \ \ (0,L).$$

The *stress* $\sigma_\varepsilon = (\zeta+\varepsilon)\mathbb{C}\frac{\mathrm{d}}{\mathrm{d}x}u$ is thus necessarily *constant* along the whole bar, and its value can be calculated by using $\zeta+\varepsilon \geq \varepsilon > 0$ and

$$w_L - w_0 = u(L) - u(0) = \int_0^L \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int_0^L \frac{\sigma_\varepsilon}{(\zeta(x){+}\varepsilon)\mathbb{C}(x)}\,\mathrm{d}x.$$

Thus we find the formulas for the (constant) stress and for the strain:

$$\sigma_\varepsilon = \mathcal{H}\big((\zeta{+}\varepsilon)\mathbb{C}\big)\frac{w_L - w_0}{L} \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{w_L - w_0}{L}\,\frac{\mathcal{H}\big((\zeta{+}\varepsilon)\mathbb{C}\big)}{(\zeta(x){+}\varepsilon)\mathbb{C}(x)}\ , \tag{4.1}$$

where $\mathcal{H}$ denotes the *harmonic mean* of an indicated profile over the interval $[0,L]$, i.e.

$$\mathcal{H}(z) := \frac{1}{\frac{1}{L}\int_0^L \frac{\mathrm{d}x}{z(x)}}\ . \tag{4.2}$$

In particular, we find the explicit formula for g_ε from (2.9):

$$g_\varepsilon(\zeta) = \mathcal{H}\big((\zeta{+}\varepsilon)\mathbb{C}\big)\frac{(w_L - w_0)^2}{2L}\ .$$

Similarly, the functional f_ε from (2.29) as a quadratic function of $e_{\mathrm{D}} \in L^2(0,L)$ can explicitly be written down as:

$$f_\varepsilon(e_{\mathrm{D}},\zeta) = \frac{\mathcal{H}\big((\zeta{+}\varepsilon)\mathbb{C}\big)}{2L}\Big(\int_0^L e_{\mathrm{D}}(x)\,\mathrm{d}x\Big)^2\ .$$

The counterexample from Section 2.2 (where $L = 2$ and $\mathbb{C} = 1$ were considered) is easily obtained by letting $\zeta(x) := |x - L/2|^\alpha$. Clearly,

$$\lim_{\varepsilon\to 0+} g_\varepsilon(\zeta) = g_0(\zeta) = \mathcal{H}\big(\zeta\mathbb{C}\big)\frac{(w_L - w_0)^2}{2L}. \tag{4.3}$$

However the Γ-limit $\mathfrak{f}(e_{\mathrm{D}},\zeta)$ vanishes for this particular damage profile ζ. Indeed, for all $\delta > 0$, we have $(\zeta - \delta)^+ = 0$ on the interval $[L/2 - \delta^{1/\alpha}, L/2 + \delta^{1/\alpha}]$ and therefore by (4.3) and (2.34):

$$\begin{aligned}\mathcal{F}(\varepsilon,\delta,e_{\mathrm{D}},\zeta) \ &= \ \frac{(w_L - w_0)^2}{2\int_0^L \frac{\mathrm{d}x}{((\zeta(x)-\delta)^+ + \varepsilon)\mathbb{C}(x)}} \\ &\leq \ \frac{(w_L - w_0)^2}{2\int_{L/2-\delta^{1/\alpha}}^{L/2+\delta^{1/\alpha}} \frac{\mathrm{d}x}{\varepsilon\mathbb{C}(x)}} \ \leq \ \frac{(w_L{-}w_0)^2}{4}\big\|\mathbb{C}\big\|_{L^\infty(0,L)}\frac{\varepsilon}{\delta^{1/\alpha}}\end{aligned}$$

so that the limit in ε already vanishes. By using the same reasoning for a general $\zeta \in Z$, one checks easily that $\mathfrak{f}(e_{\mathrm{D}},\zeta)$ is given as follows:

$$\mathfrak{f}(e_{\mathrm{D}},\zeta) = \frac{(w_L - w_0)^2}{2}\begin{cases} 1/\int_0^L \frac{\mathrm{d}x}{\zeta(x)\mathbb{C}(x)} & \text{if } \min_{[0,L]}\zeta(\cdot) > 0, \\ 0 & \text{if } \min_{[0,L]}\zeta(\cdot) = 0.\end{cases}$$

Note that $\mathfrak{f}(e_{\mathrm{D}}, \cdot) : Z \to \mathbb{R}^+$ is *not continuous* in the strong topology of $W^{1,r}(0, L)$, $r > 1$.

This example can also be used to show that the set $\mathfrak{S}(t, \zeta)$ of realizable stresses may contain more than one stress distribution. For this, take any $\zeta \in Z$ such that $\int_0^L \frac{\mathrm{d}x}{\zeta(x)\mathbb{C}(x)}$ is finite. Now, choosing $\zeta_\varepsilon \equiv \zeta$, we find the stress σ_ε from (4.1) and the limit reads $\sigma_0 = (w_L - w_0)/\int_0^L \frac{\mathrm{d}x}{\zeta(x)\mathbb{C}(x)}$. On the other hand, for a suitable sequence $\delta_\varepsilon \to 0+$, the sequence $\hat{\zeta}_\varepsilon = (\zeta - \delta_\varepsilon)^+$ satisfies $\int_0^L \frac{\mathrm{d}x}{(\zeta_\varepsilon(x)+\varepsilon)\mathbb{C}(x)} \to 0$ and the corresponding stresses $\hat{\sigma}_\varepsilon$ converge to zero. Thus $\mathfrak{S}(t, \zeta)$ contains at least two constant stress profiles. In fact, it is not difficult to see that all intermediate constant stresses are realizable, namely

$$\mathfrak{S}(t, \zeta) = \begin{cases} \big\{\sigma \ \text{ constant}; \ \ 0 \le \sigma(\cdot) \le \sigma_0\big\} & \text{under tension, i.e. if } w_L \le w_0, \\ \big\{\sigma \ \text{ constant}; \ \ 0 \ge \sigma(\cdot) \ge \sigma_0\big\} & \text{under compression, i.e. } w_L \ge w_0. \end{cases}$$

The *effective stress* is obviously *zero*. This is well intuitive for tension experiment but a bit paradoxical for a pressure experiment, but this is a usual consequence of (infinitesimally) small strain concept.

This is a general observation that, as the stress distributions are constant in this 1-dimensional case, the set of $\mathfrak{S}(t, \zeta)$ realizable stresses is composed from constants and is therefore linearly ordered. Thus, the minimizer in (2.25), i.e. the *effective stress*, is always unique.

4.2. Stability

Further, we investigate the global stability of the undamaged state $\zeta = 1$. For simplicity, we consider $r = 2$ and homogeneous material, i.e. constant coefficients $\mathbb{C}$, a, and κ. Let us abbreviate

$$\zeta_{\min} := \min_{0 \le x \le L} \zeta(x) \qquad \text{and} \qquad \zeta_{\max} := \max_{0 \le x \le L} \zeta(x).$$

Lemma 4.1. *Let* $E(\zeta) := \int_0^L \frac{\kappa}{2} |\frac{\mathrm{d}}{\mathrm{d}x}\zeta|^2 + a(1-\zeta)\,\mathrm{d}x$ *and* $z \in [0, 1)$*, then we have*

$$\min\Big\{E(\zeta);\ \zeta \in Z,\ \zeta_{\min} = z\Big\} \ = \ aL\,\lambda\Big(z, \frac{\sqrt{a}L}{\sqrt{2\kappa}}\Big) \tag{4.4}$$

with

$$\lambda(z, \varrho) = \begin{cases} 1 - z - \varrho^2/3 & \textit{for } 0 < \varrho \le \sqrt{1-z}, \\ 2(1-z)^{3/2}/(3\varrho) & \textit{for } \varrho \ge \sqrt{1-z}. \end{cases} \tag{4.5}$$

Proof. Since E is coercive on $Z \subset W^{1,2}((0, L))$, and convex, there is a minimizer ζ_* on the weakly closed (but non-convex!) set $\{\zeta \in Z;\ \zeta_{\min} = z\}$.

As the integrand of E is decreasing in ζ because of $a > 0$, it is easy to see that the graph of ζ_* on any interval $[x_1, x_2]$ has to lie above the segment connecting

$(x_1, \zeta_*(x_1))$ and $(x_2, \zeta_*(x_2))$ if $\zeta_*(\cdot) > z$ on $[x_1, x_2]$, i.e. the value $\zeta_*(\cdot) = z$ is attained somewhere outside $[x_1, x_2]$. Hence, ζ_* has at most one point $x_* \in [0, L]$ such that $\zeta_*(x_*) = z$ if $z < 1$, and it is strictly concave on both $[0, x_*]$ and $[x_*, L]$.

After some rather lengthy algebra, the formula (4.5) is obtained by assuming $x_* = 0$ (or, equally, $x_* = L$). For small L, we obtain a solution satisfying $\frac{\mathrm{d}}{\mathrm{d}t}\zeta_*(L) = 0$ and $\zeta_*(L) < 1$. For larger L, we have $\zeta_*(x) = 1$ for $x \geq \sqrt{2\kappa/a}$.

The condition $\zeta_*(x_*) = z$ with $x_* \in (0, L)$ then leads to $aL\,\lambda(z, \sqrt{a}L/\sqrt{2\kappa}) + a(L{-}x_*)\lambda(z, \sqrt{a}(L{-}x_*)/\sqrt{2\kappa})$ as the minimal value of $E(\zeta)$ under the (convex) condition $\zeta(x_*) = z$, $\zeta \in Z$. The concavity of $\xi \mapsto \xi\lambda(z, \xi/\sqrt{2a\kappa})$ now implies that only $x_* = 0$ or $x_* = L$ can be optimal. □

To study the stability of the undamaged state $\zeta = 1$ at a specific (and now considered fixed) time t, we define

$$m(\gamma) := \min_{\zeta\in Z} J_\gamma(\zeta) \qquad \text{with}$$

$$J_\gamma(\zeta) := \gamma\mathcal{H}_0(\zeta) + E(\zeta) \quad \text{and} \quad \mathcal{H}_0(\zeta) := \begin{cases} \mathcal{H}(\zeta) & \text{if } \zeta_{\min} > 0, \\ 0 & \text{if } \zeta_{\min} = 0, \end{cases} \tag{4.6}$$

where E from Lemma 4.1, $\mathcal{H}$ from (4.2) and

$$\gamma = \gamma(t) := \mathbb{C}\,\frac{\ell(t)^2}{2L} \geq 0 \qquad \text{with} \qquad \ell(t) := w(t, L) - w(t, 0) \tag{4.7}$$

is the energy stored in the body if no damage would occur, i.e. if $\zeta \equiv 1$; of course, we then have $J_\gamma(1) = \gamma$. Note that E, γ, J_γ, and m have a physical dimension as energy (i.e. J=kg m^{2}s^{-1}), while λ, ζ, z, and $\varrho = \sqrt{a}L/\sqrt{2\kappa}$ have a dimension 1. Also, $\gamma = g_0(1)$ with g_0 from (2.9) with $\varepsilon = 0$ or in the evolution context, equivalently, $\gamma = \min_{u\in W^{1,2}([0,L])} \mathcal{G}_0(t, \cdot, 1)$ with $\mathcal{G}_0$ from (3.1).

Also, we can see that stability of $\zeta = 1$ at time t is equivalent to $m(\gamma) = \gamma$ whereas $m(\gamma) < \gamma$ means that the (global!) stability of ζ is lost.

Proposition 4.2. ***(Some conditions for stability of the undamaged state.)*** *Let us define functions $\Lambda_1, \Lambda_2 : \mathbb{R}^+ \to [0, 1]$ (of physical dimension 1) by*

$$\Lambda_1(\varrho) := \frac{2}{4 + 3\varrho} \qquad \text{and} \qquad \Lambda_2(\varrho) := \begin{cases} 1 - \varrho^2/3 & \text{if } 0 < \varrho \leq 1, \\ 2/(3\varrho) & \text{if } \varrho \geq 1. \end{cases} \tag{4.8}$$

Then we have $\Lambda_1(\varrho) < \Lambda_2(\varrho)$ and

(i) $\gamma > aL\Lambda_2(\sqrt{a}L/\sqrt{2\kappa})$ *implies $m(\gamma) < \gamma$, i.e. $\zeta = 1$ is not globally stable,*

(ii) $\gamma \leq aL\Lambda_1(\sqrt{a}L/\sqrt{2\kappa})$ *implies $m(\gamma) = \gamma$, i.e. $\zeta = 1$ is globally stable.*

Proof. Part (i) follows easily by using the minimizers of Lemma 1 for $z \in (0, L)$ and then taking the limit $z \to 0$.

For Part (ii), the argument is more involved. First, note that a global minimizer ζ_γ of J_γ in Z must exist. As we only consider $0 < \gamma \leq aL\Lambda_1(\sqrt{a}L/\sqrt{2\kappa})$ and $\Lambda_1 \leq \Lambda_2$, we use the arguments of Part (i) to conclude that $[\zeta_\gamma]_{\min} > 0$, and hence ζ_γ solves the Neumann boundary-value problem for the following differential inclusion:

$$-\kappa\frac{\mathrm{d}^2\zeta}{\mathrm{d}x^2} - a + \frac{\gamma\mathcal{H}(\zeta)^2}{L}\frac{1}{\zeta^2} + \partial\chi_{(-\infty,1]}(\zeta) \ni 0, \qquad \frac{\mathrm{d}\zeta}{\mathrm{d}x}(0) = 0 = \frac{\mathrm{d}\zeta}{\mathrm{d}x}(L). \tag{4.9}$$

By [20, Chap.3, Theorem 2.3], each solution lies in $W^{2,p}((0,L))$, $p < +\infty$ arbitrary; possibly it has a flat part with $\zeta(\cdot) = 1$.

Testing (4.9) by $\frac{\mathrm{d}}{\mathrm{d}x}\zeta$ gives

$$\frac{\kappa}{2}\Big|\frac{\mathrm{d}\zeta}{\mathrm{d}x}\Big|^2 + a\zeta + \frac{\gamma\mathcal{H}(\zeta)^2}{L}\frac{1}{\zeta^2} = ac \tag{4.10}$$

for a suitable constant c. Note that this also holds if the "reaction force" from $\partial\chi_{(-\infty,1]}(\zeta)$ does not vanish. It holds either $\zeta = 1$ (and then (4.10) is trivial) or $0 < \zeta_{\min} < 1$. In the latter case, $\frac{\mathrm{d}}{\mathrm{d}x}\zeta(x) = 0$ whenever $\zeta(x) = 1$ and (4.10) again holds on $[0, L]$.

Now, assume $0 < \zeta_{\min} \le \zeta_{\max} \le 1$. Then inserting these values into (4.10) (using that $\frac{\mathrm{d}}{\mathrm{d}x}\zeta(\cdot) = 0$ when these values are attained) gives

$$a\zeta_{\max} + \frac{\gamma\mathcal{H}(\zeta)^2}{L}\frac{1}{\zeta_{\max}} = ac = a\zeta_{\min} + \frac{\gamma\mathcal{H}(\zeta)^2}{L}\frac{1}{\zeta_{\min}}. \tag{4.11}$$

First, consider $\zeta_{\min} = \zeta_{\max}$, then $\zeta \equiv \zeta_{\min}$ and $J_\gamma(\zeta_{\min}) = \gamma + (aL-\gamma)(1-\zeta_{\min})$. Because of $\gamma < aL$, we have $J_\gamma(\zeta_{\min}) > J_\gamma(1)$ for $\zeta_{\min} < 1$. Hence we have a contradiction. Second, assuming that we have a minimizer with $\zeta_{\min} < \zeta_{\max} \le 1$, we conclude from (4.11) that

$$c = \zeta_{\min} + \zeta_{\max} \quad \text{and} \quad \mathcal{H}(\zeta)^2 = \frac{aL}{\gamma}\zeta_{\min}\zeta_{\max}.$$

Using $\mathcal{H}(\zeta) \le \zeta_{\max}$ and $\zeta_{\max} \le 1$, we find $\zeta_{\min} \le \gamma/(aL)$. Now, using $J_\gamma(\zeta) \ge \mathcal{E}(\zeta)$, we employ Lemma 4.1 and find $J_\gamma(\zeta_\gamma) \ge aL\,\lambda(\gamma/(aL), \sqrt{a}L/\sqrt{2\kappa})$. Elementary calculations show that $\gamma \le aL\Lambda_1(\sqrt{a}L/\sqrt{2\kappa})$ implies $aL\,\lambda(\gamma/(aL), \sqrt{a}L/\sqrt{2\kappa}) > \gamma$. In fact, since $\gamma \mapsto \lambda(\gamma/(aL), \varrho)$ strictly decreases on $[0, aL]$ and attains the value 0 at $\gamma = aL$, there is a unique solution γ_* of $\gamma = \lambda(\gamma/(aL), \varrho)$, and $J_\gamma(\zeta_\gamma) \ge \gamma$ holds for any $\gamma \in [0, \gamma_*]$. An explicit calculation gives $\gamma_* = aL\Lambda(\sqrt{a}L/\sqrt{2\kappa})$, where $\Lambda(\varrho)$ is the unique solution of $z = \lambda(z, \varrho)$. We find $\Lambda(\varrho) = 1/2 + \varrho^2/6$ for $\varrho^2 \le 3/7$ and the estimate $\Lambda(\varrho) \ge 2/(3(1+\varrho))$ for $\varrho^2 \ge 3/7$. Hence we obtain a contradiction to the assumption that a nontrivial (i.e. not identically 1) global minimizer exists, and Part (ii) is proved. □

4.3. Evolution conjectured

We conjecture that the bound Λ_2 in Proposition 4.2 is sharp, i.e. the upper bound Λ_1 can be replaced by Λ_2. In such case, we could give an exact solution for the

1-dimensional damage evolution problem as follows. We now consider $\gamma = \gamma(t)$ evolving in time, cf. (4.7).

Consider $\boldsymbol{g}(t,\zeta) = \gamma(t)\mathcal{H}_0(\zeta) + \int_0^L \frac{\kappa}{2}|\frac{\mathrm{d}}{\mathrm{d}t}\zeta|^2\,\mathrm{d}x$ and R as before, cf. (4.6)–(4.7) and (3.2). The prescribed elongation/shrinkage $\ell(t)$ is continuous, cf. (3.15) where even C^1-smoothness was assumed. Let ℓ be strictly monotone, say decreasing, in time, starting from $\ell(0) = 0$, and the body is initially undamaged and undeformed, i.e. $\zeta_0 \equiv 1$ and $u_0 \equiv 0$, which is compatible with (3.16). Then

$$\zeta(t,x) = \begin{cases} 1 & \text{for } 0 \le t < t_*, \ \ x \in [0,L], \\ \zeta_{\mathrm{dam}}(x) & \text{for } t \ge t_*, \qquad x \in [0,L], \end{cases} \tag{4.12}$$

where t_* is the unique value such that

$$\ell(t_*)^2 = \frac{2a}{\mathbb{C}} L^2 \Lambda_2\Big(\frac{\sqrt{a}L}{\sqrt{2\kappa}}\Big) \tag{4.13}$$

and where ζ_{dam} is one of the two minimizers of E under the constraint $\zeta_{\min} = 0$, see (4.4) with $z = 0$. We have immediate total damage at one point since the instability criterion in Proposition 4.2(i) is obtained by complete damage. From (4.13), we can identify a *critical strain* $e_{\mathrm{crit}} := |\ell(t_*)|/L$ above which the (even total) damage starts evolving, namely

$$e_{\mathrm{crit}} := \frac{|\ell(t_*)|}{L} = \sqrt{\frac{2a}{\mathbb{C}} \Lambda_2\Big(\frac{\sqrt{a}L}{\sqrt{2\kappa}}\Big)}\ . \tag{4.14}$$

For very *short bars*, i.e. small L, we have asymptotically $\varrho = \sqrt{a}L/\sqrt{2\kappa} \to 0$ and then $\Lambda_2(\varrho) \to 1$, cf. (4.8), so that, from (4.14), we can see that

$$e_{\mathrm{crit}} \approx \sqrt{2a/\mathbb{C}}. \tag{4.15}$$

In particular, we can see that the resistivity to damage is determined by the ratio (physically of dimension 1) of the activation stress and the elastic modulus, while $\kappa > 0$ plays (asymptotically) no role as well as the length L itself.

Conversely, for *long bars*, in particular for $L \ge \sqrt{2\kappa/a}$, we have $\varrho = \sqrt{a}L/\sqrt{2\kappa} \ge 1$ and thus $\Lambda_2(\varrho) = 2/(3\varrho)$, cf. (4.8), so that, substituting it into (4.14), we can see that

$$e_{\mathrm{crit}} = 2\frac{\sqrt[4]{2a\kappa}}{\sqrt{3L\mathbb{C}}}\ . \tag{4.16}$$

In particular, we can see that e_{crit} decays with increasing length L as $\mathcal{O}(1/\sqrt{L})$. A paradoxical effect can thus be expected (at least asymptotically if $L \to \infty$) that the bar tends to break already even when a very small strain is achieved by the loading (although the boundary displacement, i.e. the loading $\ell(t_*) = e_{\mathrm{crit}}L \approx \sqrt{L}$ itself, must be sufficiently large). This effect is caused by the adopted concept of global stability (3.12) which is ultimately favorite for damage at small spots if there is enough energy stored in the whole body. Fortunately, large engineering workpieces (as e.g. long bridges or tall towers) rely rather on local stability principles for which, however, a rigorous mathematical theory is not developed yet. This reveals certain limits of applications for the presented model.

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