Covering an uncountable square by countably many continuous functions

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Abstract

We prove that there exists a countable family of continuous real functions whose graphs together with their inverses cover an uncountable square, i.e. a set of the form $X \times X$, where $X \subseteq \mathbb{R}$ is uncountable. This extends Sierpiński's theorem from 1919, saying that $S \times S$ can be covered by countably many graphs of functions and inverses of functions if and only if $|S| \leqslant \aleph_1$. Our result is also motivated by Shelah's study of planar Borel sets without perfect rectangles.

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1 Introduction

A classical result of Sierpiński from 1919 (see [11, 12] or [13, Chapter I]) says that, given a set S of cardinality $\aleph_1$, there exists a countable family of functions $f_n \colon S \to S$ such that

$$S \times S = \bigcup_{n \in \omega} (f_n \cup f_n^{-1}), \tag{1}$$

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where f_n^{-1} is the inverse of f_n, i.e. $f_n^{-1} = \{\langle f_n(x), x\rangle : x \in S\}$. A typical proof proceeds as follows. Assume $S = \omega_1$ and for each positive $\beta < \omega_1$ choose a surjection $g_\beta \colon \omega \to \beta$. Define $f_n \colon \omega_1 \to \omega_1$ by the equation $f_n(\beta) = g_\beta(n)$. For every $\langle \alpha, \beta\rangle \in S \times S$ with $\alpha < \beta$ there exists n such that $g_\beta(n) = \alpha$; thus $\langle \alpha, \beta\rangle \in f_n$ and $\langle \beta, \alpha\rangle \in f_n^{-1}$. Finally, it suffices to add the identity function to the family $\{f_n\}_{n\in\omega}$ in order to get (1). It is worth noting that the sets $f_n^{-1}(\alpha)$ form an *Ulam matrix* on ω_1. See e.g. [4, Chapter 10] or [8, Chapter II, §6] for applications of Ulam matrices.

An easy argument (also noted by Sierpiński) shows that the above statement fails for a set S of cardinality $\aleph_2$. In particular, the continuum hypothesis is equivalent to the statement "there exists a countable family of functions which, together with their inverses, cover the plane".

Let us say that a set M is *covered by* a family of functions $\mathcal{F}$, if for every $\langle x, y\rangle \in M$ there is $f \in \mathcal{F}$ such that either $y = f(x)$ or $x = f(y)$.

Question 1. Does there exist a sequence $\{f_n \colon \mathbb{R} \to \mathbb{R}\}_{n\in\omega}$ of continuous functions that covers an uncountable square?

One can hope for a positive answer only when the side S of the square has some smallness properties, besides having cardinality $\aleph_1$. In fact, by a result of Zakrzewski [15, Theorem 2.1], if $S \times S$ is covered by countably many functions (and their inverses) whose graphs are Borel sets, then S is *universally small*, i.e. S belongs to every Borel σ-ideal $I \subseteq \mathcal{P}(\mathbb{R})$ such that $\mathrm{Borel}(\mathbb{R})/I$ satisfies the countable chain condition.

Actually, consistent affirmative answers to Question 1 already exist in the literature. Namely, Abraham and Geschke [1] showed that for every set $X \subseteq \mathbb{R}$ of cardinality $\aleph_1$ there is a ccc forcing notion adding countably many continuous functions that cover $X \times X$. Consequently, under Martin's Axiom every $\aleph_1$-square in the plane is covered by a countable family of continuous functions. The Open Coloring Axiom of Abraham, Rubin and Shelah [2] implies that for every set $X \subseteq 2^\omega$ of size $\aleph_1$ there is a countable family of 1-Lipschitz functions that covers $X \times X$ (see [3] for more details).

Yet another motivation for addressing Question 1 comes from the work of Shelah [10], continued in [7], where planar Borel sets without perfect squares were studied. It is not hard to prove (see e.g. [6, Thm. 2.2]) that a G_δ subset of the plane containing countable squares of arbitrarily large countable Cantor-Bendixson ranks, contains also a perfect square. On the other hand, using Keisler's completeness, it has been proved in [10] that there exists in ZFC a planar F_σ set C such that $S \times S \subseteq C$ for some uncountable set S, while $P_0 \times P_1 \not\subseteq C$ whenever P_0, P_1 are perfect sets. A significant part of [7] is devoted to a ZFC construction of certain F_σ sets in the plane which do not contain perfect squares, while consistently they contain squares of a prescribed cardinality below $\aleph_{\omega_1}$. These sets moreover have certain universality property, among sets of the same type (see [7] for details). Based on the results of [10] and [7], it is natural to ask for the existence of a more special planar F_σ set which covers an uncountable square: namely, a set consisting of countably many continuous real functions and their inverses. There are natural restrictions here. Namely, such a set cannot contain rectangles of the form $S_0 \times S_1$, where $|S_i| = \aleph_1$ and $|S_{1-i}| \geqslant \aleph_2$. Easy absoluteness arguments show that it cannot contain perfect rectangles, therefore the best property we can expect is covering a square of cardinality $\aleph_1$.

In the present note we find a family of continuous functions $\mathcal{F} = \{f_n \colon 2^\omega \to 2^\omega\}_{n\in\omega}$ such that every maximal square covered by $\mathcal{F}$ is uncountable. The functions f_n are not Lipschitz with respect to any natural metric on 2^ω, however we describe a natural ccc forcing notion which introduces a countable family of 1-Lipschitz functions on the Cantor set that covers an uncountable square. Using Keisler's completeness theorem [5], we deduce that such a family exists in ZFC, although we do not know any direct construction.

Finally, we observe that it is impossible to cover the square of any uncountable compact Hausdorff space by countably many continuous functions and their inverses.

It is worth noting that there exists no uncountable set $S \subseteq \mathbb{R}$ whose square can be covered by countably many non-decreasing functions and their inverses. This is because the graphs of such functions (and of their inverses) are chains with respect to the coordinatewise ordering and consequently the order of S would be a Countryman type (see [9] or [14, p. 258]), not embeddable into the real line.

2 Main result

Theorem 2.1. *Let X be a space containing at least two points and suppose there exists a continuous onto mapping $\varphi \colon X \to X^\omega$. Then there exists a countable family of continuous mappings of X to itself such that every maximal square covered by this family is uncountable.*

Proof. Consider a mapping $f \colon X \to X^\omega$ which is defined as a composition $\pi_0 \circ \varphi^\omega \circ \varphi$, where $\varphi^\omega \colon X^\omega \to (X^\omega)^\omega$ is a product of countable many copies of the mapping φ and $\pi_0 \colon (X^\omega)^\omega \to X^\omega$ is the projection to the first coordinate. We get that f is a continuous onto mapping with the property, that preimages of points are uncountable.

Define $f_n \colon X \to X$ to be a composition $\pi_n \circ f$ where $\pi_n \colon X^\omega \to X$ is the n-th projection. It remains to show that the system $\{f_n \colon n \in \omega\}$ together with the identity function is the desired family. Thus suppose for a contradiction that there is a maximal countable set $S = \{s_n \colon n \in \omega\}$ whose square is covered by our family of functions. Take arbitrary point x from the uncountable set $f^{-1}\big(\langle s_0, s_1, \ldots\rangle\big) \setminus S$. Clearly the set $\big(S \cup \{x\}\big)^2$ is covered by our family of functions since $f_n(s_n) = x$ for every $n \in \omega$ and the point $\langle x, x\rangle$ lies on the graph of the identity. This contradicts the maximality of S. $\square$

Notice that the Axiom of Choice was not used in the above proof but it is heavily used in the simple proof of the following corollary.

Corollary 2.2. *There exists a family of continuous functions of the Cantor space to itself which covers an uncountable square.*

This answers Question 1 since any continuous function of the Cantor space to itself can be extended to a continuous real function.

3 Forcing countably many Lipschitz functions

It is easy to see that the functions constructed in Theorem 2.1 are not Lipschitz with respect to the natural metric on the Cantor set. We do not know a direct construction of a countable family of 1-Lipschitz functions covering an uncountable square. This section is devoted to showing that such a family can be introduced by a natural forcing notion. Using Keisler's completeness, we later conclude that this family exists in ZFC, namely:

Theorem 3.1. *There exists a countable family of* 1*-Lipschitz functions on the Cantor set which covers an uncountable square.*

We first show the consistency of the above statement with the axioms of ZFC. The metric on 2^ω which we have in mind is given by the formula $d(x,y) = 2^{-k}$, where k is the smallest natural number such that $x \restriction k \neq y \restriction k$.

Given a natural number n, we shall denote by 2^n the complete binary tree consisting of all zero-one sequences of length n. Trees of the form 2^n serve as finite approximations of the Cantor set 2^ω. We consider 2^n with the lexicographic ordering and with the metric defined above, like in the case of 2^ω. Denote by $\mathbb{L}_1(n)$ the set of all 1-Lipschitz functions of the form $g\colon 2^n \to 2^n$.

We are going to define a forcing notion $\mathbb{P}$ which will introduce a countable family of Lipschitz functions covering an uncountable square.

A condition $p \in \mathbb{P}$ is, by definition, of the form $p = \langle n^p, s^p, v^p, \mathcal{F}^p, \gamma^p, \varrho^p \rangle$, where

(1) $n^p \in \omega$, $s^p \in [\omega]^{<\omega}$ and $v^p \in [\omega_1]^{<\omega}$;

(2) $\mathcal{F}^p = \{f_i^p\}_{i \in s^p} \subseteq \mathbb{L}_1(n^p)$ and $\varrho^p\colon [v^p]^2 \to s^p$;

(2') $\varrho^p(\alpha,\beta) \neq \varrho^p(\alpha',\beta)$ whenever $\alpha < \alpha' < \beta$;

(3) $\gamma^p\colon v^p \to 2^{n^p}$ is one-to-one;

(4) $\gamma^p(\alpha) = f^p_{\varrho^p(\alpha,\beta)}(\gamma^p(\beta))$ whenever $\alpha < \beta$ and $\alpha, \beta \in v^p$.

Note that condition (2') is actually implied by the conjunction of (3) and (4). The order of $\mathbb{P}$ is defined naturally. Namely, $p \leqslant q$ (q is stronger than p) iff

(5) $n^p \leqslant n^q$, $s^p \subseteq s^q$, $v^p \subseteq v^q$;

(6) $f_i^q(\eta) \restriction n^p = f_i^p(\eta \restriction n^p)$ for each $i \in s^p$ and for every $\eta \in 2^{n^q}$;

(7) $\gamma^q(\alpha) \restriction n^p = \gamma^p(\alpha)$ for every $\alpha \in s^p$;

(8) $\varrho^q \restriction [v^p]^2 = \varrho^p$.

It is easy to see (the details are given below) that the forcing $\mathbb{P}$ introduces a countable family $\{f_n\}_{n\in\omega}$ of continuous functions on the Cantor set together with a function $\varrho\colon [\omega_1]^2 \to \omega$ and a one-to-one function $\gamma\colon \omega_1 \to 2^\omega$ such that $\gamma(\alpha) = f_{\varrho(\alpha,\beta)}(\gamma(\beta))$ for every $\alpha < \beta < \omega_1$. We need to prove that $\mathbb{P}$ does not collapse $\aleph_1$.

Lemma 3.2. *$\mathbb{P}$ satisfies the countable chain condition.*

Proof. Fix a family $\mathcal{G} \subseteq \mathbb{P}$ with $|\mathcal{G}| = \aleph_1$. Replacing $\mathcal{G}$ by an uncountable subfamily, we may assume that there exist $n \in \omega$, $s \in [\omega]^{<\omega}$ and $\mathcal{F} = \{f_i\}_{i\in s} \subseteq \mathbb{L}_1(n)$ such that $n^p = n$, $s^p = s$ and $\mathcal{F}^p = \mathcal{F}$ for every $p \in \mathcal{G}$. Further refining $\mathcal{G}$, we may assume that

(9) $\{v^p : p \in \mathcal{G}\}$ forms a Δ-system with root $a \subseteq \omega_1$.

(10) For every $p, q \in \mathcal{G}$ the structures $\langle v^p, \gamma^p, \varrho^p, <\rangle$ and $\langle v^q, \gamma^q, \varrho^q, <\rangle$ are isomorphic, where $<$ is the linear order inherited from ω_1. In other words, there exists an order preserving bijection $\varphi\colon v^p \to v^q$ such that $\gamma^p(\alpha) = \gamma^q(\varphi(\alpha))$ and $\varrho^p(\alpha,\beta) = \varrho^q(\varphi(\alpha),\varphi(\beta))$ for every $\alpha, \beta \in v^p$.

For the remaining part of the proof we fix $p, q \in \mathcal{G}$ such that $\max(a) < \min(v^p \setminus a)$ and $\max(v^p) < \min(v^q \setminus a)$. Our aim is to construct $r \in \mathbb{P}$ with $p \leqslant r$ and $q \leqslant r$.
Define $n^r = n + 1$ and $v^r = v^p \cup v^q$. Note that by (10), ϱ^p, ϱ^q coincide on $a = v^p \cap v^q$. Extend $\varrho^p \cup \varrho^q$ to a function $\varrho^r\colon [v^r]^2 \to \omega$ in such a way that ϱ^r restricted to the set

$$\sigma = [v^r]^2 \setminus ([v^p]^2 \cup [v^q]^2) = \{\{\alpha,\beta\} : \alpha \in v^p \setminus a,\ \ \beta \in v^q \setminus a\}$$

is a bijection onto $t \subseteq \omega \setminus s$. Clearly, ϱ^r satisfies (2'), i.e. $\varrho^(\alpha,\beta) \neq \varrho^r(\alpha',\beta)$ whenever $\alpha < \alpha' < \beta$. Define $s^r = s \cup t$. Then $\varrho^r\colon [v^r]^2 \to s^r$. Further, define

$$\gamma^r(\alpha) = \begin{cases} \gamma^p(\alpha)^\frown 0 & \text{if } \alpha \in v^p, \\ \gamma^q(\alpha)^\frown 1 & \text{if } \alpha \in v^q \setminus a. \end{cases}$$

Observe that $\gamma^r\colon v^r \to 2^{n^r}$ is one-to-one. It remains to define $\mathcal{F}^r = \{f_i^r\}_{i\in s^r}$.
If $i \in t$ then we define f_i^r to be the constant function with value $\gamma^r(\alpha)$, where $\alpha \in v^p \setminus a$, $\beta \in v^q \setminus a$ are such that $i = \varrho^r(\alpha,\beta)$. Note that α, β are uniquely determined, so there is no ambiguity here and f_i^r satisfies (4). Finally, fix $i \in s$, $\eta \in 2^n$, $\varepsilon \in 2$ and define

$$f_i^r(\eta^\frown\varepsilon) = \begin{cases} f_i(\eta)^\frown\varepsilon & (\exists\, \alpha,\beta \in v^p \setminus a)\ \alpha < \beta \wedge i = \varrho^p(\alpha,\beta) \wedge \eta = \gamma^p(\beta), \\ f_i(\eta)^\frown 0 & \text{otherwise.} \end{cases}$$

By this way we have finished the definition of $r = \langle n^r, s^r, v^r, \mathcal{F}^r, \gamma^r, \varrho^r \rangle$. In order to show that $r \in \mathbb{P}$, we need to verify condition (4) only, since conditions (1)–(3) are rather clear.

For fix $\alpha < \beta$ in v^r and let $\ell = \varrho^r(\alpha, \beta)$. If $\ell \in t$ then f^r_ℓ is constantly equal to $\gamma^r(\alpha)$, therefore (4) holds in this case. So assume $\ell \in s$ and let $\eta = \gamma^r(\beta) \restriction n$. We consider the following two cases.

Case 1. $\alpha \in v^q \setminus a$.

Notice that also $\beta \in v^q \setminus a$, because $\alpha < \beta$. By (10), there exist $\alpha', \beta' \in v^p$ such that $\varrho^p(\alpha', \beta') = \varrho^q(\alpha, \beta) = \ell$ and $\gamma^p(\beta') = \gamma^q(\beta) = \eta$. Thus the first possibility in the definition of f^r_ℓ occurs and we have

$$f^r_\ell(\gamma^r(\beta)) = f^r_\ell(\eta^\frown 1) = f_\ell(\eta)^\frown 1 = \gamma^q(\alpha)^\frown 1 = \gamma^r(\alpha),$$

therefore (4) holds.

Case 2. $\alpha \in v^p$.

Now $\gamma^r(\alpha) = \gamma^p(\alpha)^\frown 0$ and either $\beta \in v^p$ or else $\alpha \in a$ and $\beta \in v^q$ (because $\ell \in s$ implies that either $\{\alpha, \beta\} \subseteq v^p$ or $\{\alpha, \beta\} \subseteq v^q$). Observe that $f^r_\ell(\gamma^r(\beta)) \restriction n = \gamma^r(\alpha) \restriction n$, by the definition of f^r_ℓ and by the fact that $p, q \in \mathbb{P}$. Thus, the only possibility for the failure of (4) is that $f^r_\ell(\gamma^r(\beta)) = \gamma^p(\alpha)^\frown 1$. Suppose this is the case. By the definition of f^r_ℓ, we conclude that $\gamma^r(\beta) = \eta^\frown 1$ and in particular $\beta \in v^q \setminus a$ and $\alpha \in a$. Moreover, the first case in the definition of f^r_ℓ occurs, so there exist $\alpha' < \beta'$ in $v^p \setminus a$ such that $\ell = \varrho^p(\alpha', \beta')$ and $\eta = \gamma^p(\beta')$. Let $\varphi \colon v^p \to v^q$ be the bijection appearing in condition (10). In particular $\gamma^p(\beta) = \eta = \gamma^q(\varphi(\beta'))$, therefore $\varphi(\beta') = \beta$, because γ^q is one-to-one. Further,

$$\varrho^r(\varphi(\alpha'), \beta) = \varrho^q(\varphi(\alpha'), \varphi(\beta')) = \varrho^p(\alpha', \beta') = \ell = \varrho^r(\alpha, \beta).$$

Thus $\varphi(\alpha') = \alpha$, because ϱ^r satisfies (2'). This leads to a contradiction, because $\alpha \in a$, $\alpha' \in v^p \setminus a$ and $\varphi[v^p \setminus a] = v^q \setminus a$. Thus (4) holds.

We have proved that $r \in \mathbb{P}$. Clearly $p \leqslant r$ and $q \leqslant r$. $\square$

Lemma 3.3. *Let $k \in \omega$ and $\xi \in \omega_1$. The sets*

$$\mathcal{D}(k) = \{p \in \mathbb{P} \colon n^p \geqslant k \text{ and } k \in s^p\}, \qquad \mathcal{E}(\xi) = \{p \in \mathbb{P} \colon \xi \in v^p\}.$$

are dense in $\mathbb{P}$.

Proof. Fix $p \in \mathbb{P}$. Define $n^q = n^p + 1$, $s^q = s^p \cup \{k\}$, $v^q = v^p$, $\varrho^q = \varrho^p$, $\gamma^q(\eta) = \gamma^p(\eta)^\frown 0$ and $f^q_i(\eta^\frown \varepsilon) = f^p_i(\eta)^\frown \varepsilon$ for $i \in s^p$, $\eta \in 2^{n^p}$, $\varepsilon \in 2$. Finally, if $k \notin s^p$, let f^q_k be any function from $\mathbb{L}_1(n^p + 1)$. By this way we have extended p to a condition $q = \langle n^q, s^q, v^q, \mathcal{F}^q, \gamma^q, \varrho^q \rangle \in \mathbb{P}$ so that $n^q > n^p$, $k \in s^q$. Repeating this procedure finitely many times we obtain $r \geqslant p$ such that $n^r \geqslant k$ and $k \in s^r$. This shows that $\mathcal{D}(k)$ is dense in $\mathbb{P}$.

In order to show the density of $\mathcal{E}(\xi)$ again fix $p \in \mathbb{P}$ and assume $\xi \notin v^p$. Define $n^q = n^p+1$ and $v^q = v^p \cup \{\xi\}$. Let $\sigma = \{\{\xi, \alpha\} \colon \alpha \in v^p\}$. Extend ϱ^p to a function $\varrho^q \colon [v^q]^2 \to \omega$ so that $\varrho^q \restriction \sigma$ is one-to-one onto $t \subseteq \omega \setminus s^p$. Let $s^q = s^p \cup t$. Further, define $\gamma^q(\alpha) = \gamma^p(\alpha)^\frown 0$ for $\alpha \in v^p$ and let $\gamma^q(\xi)$ be the constant one function in 2^{n^q}. It remains to define $\mathcal{F}^q$. Given $i \in s^p$, define $f_i^q(\eta^\frown \varepsilon) = f_i^p(\eta)^\frown \varepsilon$ for every $\eta \in 2^{n^p}$, $\varepsilon \in 2$. Fix $i \in t$ and let $\alpha \in v^p$ be such that $i = \varrho^q(\alpha, \xi)$. If $\xi < \alpha$, define f_i^q to be the constant function with value $\gamma^q(\xi)$. If $\alpha < \xi$, define f_i^q to be the constant function with value $\gamma^q(\alpha)$. Observe that conditions (1) – (4) are satisfied, therefore $q = \langle n^q, s^q, v^q, \mathcal{F}^q, \gamma^q, \varrho^q \rangle \in \mathbb{P}$. It is clear that $p \leqslant q$ and $q \in \mathcal{E}(\xi)$. □

Lemma 3.4. *The poset $\mathbb{P}$ forces a family $\mathcal{F} = \{f_n \colon n \in \omega\}$ of 1-Lipschitz functions on the Cantor set 2^ω and an uncountable set $X \subseteq 2^\omega$ whose square is covered by $\mathcal{F}$.*

Proof. Let G be a $\mathbb{P}$-generic filter over a fixed ground model $\mathbb{V}$. Define functions $f_k \colon 2^\omega \to 2^\omega$ $(k \in \omega)$, $\gamma \colon \omega_1 \to 2^\omega$ and $\varrho \colon [\omega_1]^2 \to \omega$ by the following equations:

$$\begin{aligned} f_k(x) \restriction n^p &= f_k^p(x \restriction n^p), \\ \gamma(\alpha) \restriction n^p &= \gamma^p(\alpha), \\ \varrho(\alpha, \beta) &= \varrho^p(\alpha, \beta), \end{aligned}$$

where $x \in 2^\omega$ and p is any element of G such that $\alpha, \beta \in v^p$ and $k \in s^p$. The fact that G is a filter and the density of sets $\mathcal{D}(k)$ and $\mathcal{E}(\xi)$ (Lemma 3.3) imply that the above definitions are correct. Let $X = \{\gamma(\xi) \colon \xi < \omega_1\}$. By the definition of $\mathbb{P}$, the set $X \subseteq 2^\omega$ is uncountable, the functions f_k are 1-Lipschitz and for every $\alpha < \beta < \omega_1$ we have that $\gamma(\alpha) = f_{\varrho(\alpha,\beta)}(\gamma(\beta))$. It follows that $X^2 \subseteq \{\mathrm{id}_{2^\omega}\} \cup \bigcup_{n\in\omega}(f_n \cup f_n^{-1})$. □

Proof of Theorem 3.1. The above forcing shows that the existence of a countable family of 1-Lipschitz functions on the Cantor set covering an uncountable square is consistent with ZFC. We are going to argue, using Keisler's completeness theorem [5], that such a family actually exists in ZFC.

To be more precise, we shall use completeness theorem for the logic $L^\omega(Q)$, see Corollary 3.10 in [5]. Here, L^ω stands for the *ω-logic*, which is an extension of the classical predicate logic with identity by adding a unary predicate symbol N and constant symbols $0, 1, 2, \dots$ which are supposed to denote natural numbers. The meaning of $N(x)$ is "x is a natural number". The letter Q stands for a new quantifier that means "there exist uncountably many". For the completeness theorem, the language is assumed to be countable, i.e. only countably many function, relation and constant symbols are allowed. A *standard model* for $L^\omega(Q)$ is a model M of $L^\omega(Q)$ in which constant symbols $0, 1, 2, \dots$ are interpreted as the "real" natural numbers (in particular $\omega \subseteq M$), further $M \models N(x)$ if and only if $x \in \omega$ and M satisfies $(Q\, x)\varphi$ if and only if the set

$$\{t \in M \colon M \models \varphi[t]\}$$

is uncountable, where $\varphi[t]$ is obtained from φ by replacing each free occurence of x by t.

Completeness theorem for $L^\omega(Q)$ says that a set of sentences of $L^\omega(Q)$ is consistent if and only if it has a standard model. We remark here that the logic $L^\omega(Q)$ has some natural axioms—we refer the reader to [5] for details. For instance, the following is an axiom

$$\neg(Q\ x)N(x)$$

which simply says that the set of natural numbers is countable. Finally, $L^\omega(Q)$ has three rules of inference: modus ponens, generalization and the ω-*rule* that allows to conclude $(\forall\ x)(N(x) \implies \varphi(x))$ from

$$\varphi(0), \varphi(1), \varphi(2), \ldots.$$

Thus, it is rather clear that the property of being consistent in $L^\omega(Q)$ is absolute between transitive models of ZFC (In fact, we need downward absoluteness only).

We can use finitely many predicates and the quantifier Q to describe a countable family of 1-Lipschitz functions on the Cantor set that covers an uncountable square.

More precisely, let C and L be unary predicates which will denote elements of the Cantor set and 1-Lipschitz self-maps of the Cantor set respectively. For elements of the Cantor set we need to compute their coordinates and for maps of the Cantor set we need to compute their values, therefore we need another two function symbols P and V, where $P(x,k)$ will mean "the k-th coordinate of x" and $V(f,x)$ will mean "$f(x)$". Finally, we need a binary function symbol D such that, assuming $x, y \in 2^\omega$, $D(x,y)$ will denote the minimal k with $x(k) \neq y(k)$. It will be convenient to add the relation symbol $\leqslant$ describing the usual linear ordering of natural numbers.

We now describe (omitting some details) the required set of sentences θ. First of all, let θ_0 be a finite set of sentences saying that the sets described by C and L are disjoint and do not consist of natural numbers. Let θ_1 be a finite set of sentences describing that $P(x,k)$ is $x(k)$ for $x \in 2^\omega$ and $k \in \omega$. Namely, the following sentences should be in θ_1:

$$(\forall\ x)(\forall\ k)\ C(x) \wedge N(k) \implies P(x,k) = 0 \vee P(x,k) = 1,$$

$$(\forall\ x,y)\ (C(x) \wedge C(y) \wedge x \neq y \implies (\exists\ k)N(k) \wedge P(x,k) \neq P(y,k)).$$

Further, let θ_2 be a finite set of sentences describing the meaning of $V(f,x)$ and the fact that f is a 1-Lipschitz map of the Cantor set. The following sentence says that f is 1-Lipschitz:

$$(\forall\ x,y,f)\ (C(x) \wedge C(y) \wedge L(f) \wedge x \neq y)) \implies D(V(f,x), V(f,y)) \leqslant D(x,y).$$

Let θ_3 be an infinite set of sentences describing the order $\leqslant$. Namely, θ_3 should consist of sentences of the form "$i \leqslant j$", where $i, j \in \omega$ are such that $i \leqslant j$ in ω.

Using the predicate C we cannot describe the full Cantor set, however we can add an infinite set of sentences θ_4 which says that C is dense in the Cantor set. This will ensure that the 1-Lipschitz functions are indeed defined on the full Cantor set, not only on its proper closed subset. So, θ_4 should consist of sentences of the form

$$(\exists\ x)\ C(x) \wedge P(x,0) = s(0) \wedge P(x,1) = s(1) \wedge \cdots \wedge P(x,n-1) = s(n-1)$$

where $s \in 2^n$ and $n \in \omega$.

Finally, let θ_5 consist of the following three sentences:

$$(\forall\, x, y)\ C(x) \wedge C(y) \implies (\exists\, f)\ L(f) \wedge (V(f, x) = y \vee V(f, y) = x),$$
$$(Q\, x)\ C(x),$$
$$\neg(Q\, f)\ L(f).$$

These sentences say that the square of C is covered by functions from the set L and, what is most important, C is uncountable while L is countable.

Let $\theta = \theta_0 \cup \cdots \cup \theta_5$. A standard model M of θ formally consists of ω and two other disjoint sets C^M and L^M, however it is obviously isomorphic to a model of the form $\omega \cup C \cup L$, where C is an uncountable dense subset of 2^ω and L is a countable family of 1-Lipschitz functions of the Cantor set into itself. Finally, $C \times C$ is covered by L.

The forcing arguments described above show that θ is consistent in some extension of the universe of set theory. Since this property is absolute, θ is consistent and hence, by Keisler's theorem, it has a standard model. This model, by the above remarks, gives the desired countable family of 1-Lipschitz functions. $\square$

4 Final remarks

It is natural to ask whether there exists an uncountable (necessarily scattered) compact space K such that K^2 is covered by countably many graphs of continuous functions and their inverses. Below we show that the answer is negative.

Theorem 4.1. *Let K be a compact Hausdorff space and let $\{f_n\}_{n\in\omega}$ be a family of continuous functions such that for each $n \in \omega$ the set $\operatorname{dom}(f_n)$ is closed in K and $K \times K = \bigcup_{n\in\omega}(f_n \cup f_n^{-1})$. Then $|K| \leqslant \aleph_0$.*

Proof. By the Baire Category Theorem, a compact K satisfying the above assertion must be scattered. Suppose the theorem is false and fix a counterexample K of minimal Cantor-Bendixson rank λ. Denote by $K^{(\alpha)}$ the α-th derivative of K. Passing to a subspace, we may further assume that $K^{(\lambda)}$ is a singleton, which we shall denote by ∞. Note that every closed set not containing ∞ is countable. Indeed, if $A \subseteq K$ is closed and $\infty \notin A$, then by compactness, $A \cap K^{(\gamma)} = \emptyset$ for some $\gamma < \lambda$. Thus the Cantor-Bendixson rank of A is $\leqslant \gamma$, therefore by the minimality of λ, A must be countable because it satisfies the above assertion.

Let $M = \{f_n(\infty) \colon n \in \omega \text{ and } \infty \in \operatorname{dom}(f_n)\}$ and choose $y \in K \setminus M$. Let

$$A = K \setminus (M \cup \{f_n(y) \colon n \in \omega\}).$$

Then A is uncountable and for each $x \in A$ there exists $k \in \omega$ such that $y = f_k(x)$. Find $k \in \omega$ such that the set $B = \{x \in A \colon y = f_k(x)\}$ is uncountable. Note that $\infty \in \operatorname{cl} B$, because every closed set not containing ∞ is countable. Thus $\infty \in \mathrm{dom}(f_k)$ and, by continuity, $y = f_k(\infty) \in M$; a contradiction. $\square$

By the above result, it is impossible to cover $\omega_1 \times \omega_1$ by countably many functions which are continuous with respect to the order topology. Indeed, all these functions would be extendable onto the Čech-Stone compactification of ω_1 which equals ω_1+1 and therefore, adding one more function, we would obtain a countable family of continuous functions covering the square of $\omega_1 + 1$.

It is easy to see, using Sierpiński's theorem, that the one point compactification of the discrete space of cardinality $\aleph_1$ can be covered by countably many *partial* continuous functions and their inverses. Thus, Theorem 4.1 fails when we drop the assumption that $\mathrm{dom}(f_n)$ be closed.

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