Preprint, Institute of Mathematics, AS CR, Prague. 2007-12-27

Sigma-locally uniformly rotund and sigma-weak* Kadets dual norms

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Dedicated to the memory of our colleague and friend Jan Pelant

Abstract

The dual X^* of a Banach space X admits a dual $\sigma-$LUR norm if (and only if) X^* admits a $\sigma-$weak* Kadets norm if and only if X^* admits a dual weak* LUR norm and moreover X is $\sigma-$Asplund generated.

1 Introduction

M. Raja proved, in two different ways, that a dual Banach space, with weak* Kadets norm, admits an equivalent dual LUR [13, 15]. Actually, he proved that several assertions are equivalent each to other:

Theorem 1. (M. Raja [13, 15]) *Let X be a Banach space, with the topological dual X^*. Then the following assertions are equivalent:*

(i) X^* admits an equivalent dual LUR norm.

(ii) X^* admits an equivalent (dual) weak Kadets norm.*

(iii) The closed dual unit ball (B_{X^*}, w^*) is a descriptive compact space and moreover X is an Asplund space.

(iv) X^* admits an equivalent dual weak LUR norm and moreover X is an Asplund space.*

(v) X^* admits and equivalent dual norm such that, on the corresponding dual unit sphere S_{X^*}, the weak and the weak topologies coincide.*

*Supported by grants AINV06/050 (Generalitat Valenciana), IAA 100 190 610, and Institutional Research Plan of the Academy of Sciences of Czech Republic No. AVOZ 101 905 03. This author thanks the Department of Mathematics of the Polytechnical University of Valencia for their support and hospitality.

†Supported in part by Project MTM 2005-08210, the Universidad Politécnica de Valencia (Spain) and the Generalitat Valenciana.

‡Supported by grant IAA 100 190 502 and Institutional Research Plan of the Academy of Sciences of Czech Republic No. AVOZ 101 905 03.

Key words: dual Banach space, sigma-LUR norm, sigma-weak* Kadets norm, weak* LUR norm, sigma-Asplund generated space, sigma-isolated family, network, descriptive compact space.

AMS Subject Classification: 46B20, 46B50.

In [9], we left open a question if a dual Banach space, with $\sigma-$weak* Kadets norm, admits an equivalent dual norm which would be $\sigma-$LUR. Here we answer this question positively. This allows us to provide a $\sigma-$analogue of Theorem 1. Definitions of $\sigma-$concepts and of necessary topological notions are given below.

Theorem 2. *Let X be a Banach space, with topological dual X^*. Then the following assertions are equivalent:*

(i) X^* admits an equivalent dual $\sigma-LUR$ norm.

(ii) X^* admits an equivalent dual $\sigma-$weak* Kadets norm.

(iii) The closed dual unit ball (B_{X^*}, w^*) is a descriptive compact space and moreover X is $\sigma-$Asplund generated.

(iv) X^* admits an equivalent dual weak* LUR norm and moreover X is $\sigma-$Asplund generated.

(v) The closed dual unit ball (B_{X^*}, w^*) is a descriptive compact space and morever a quasi-Radon-Nikodým compact space.

Banach spaces which meet the statements of Theorem 2 are those with dual LUR norm (trivially) and subspaces of weakly compactly generated spaces [6, page 438]. If a compact space K is both descriptive and quasi-Radon-Nikodým, then $X := C(K)$ also satisfies the statements of Theorem 2, see [15, 1], [5, Proposition 6].

Note that, if X is weakly Lindelöf determined, then the conditions of Theorem 2 are equivalent with X being a subspace of a weakly compactly generated space [9].

2 Definitions and notation

The letters $\mathbb{N}$, $\mathbb{R}$ are used for denoting the sets of positive integers and real numbers, respectively.

Let $(X, \|\cdot\|)$ be a real Banach space with topological dual X^* and with the dual norm denoted also by the symbol $\|\cdot\|$. The closed unit balls in X and X^* are denoted by B_X and B_{X^*}, respectively. S_X and S_{X^*} mean the unit sphere in X and X^*, respectively. The weak* topology on X^* is denoted by w^*. We use this symbol also for denoting the restriction of w^* to B_{X^*} and S_{X^*}. The weak* convergence is denoted by the symbol $\rightharpoondown$. Let $\varepsilon > 0$ and let $\emptyset \neq M \subset B_X$ be given. We say that the norm $\|\cdot\|$ on X^* is $\varepsilon - M-LUR$ if $\limsup_{n\to\infty} \left\|x^* - x_n^*\right\|_M < \varepsilon$ whenever $x^*, x_n^* \in B_{X^*},\ n \in \mathbb{N}$, and $\lim_{n\to\infty} \left\|x^* + x_n^*\right\| = 2$; here and below, the symbol $\|\cdot\|_M$ means $\sup|\langle \cdot, M\rangle| = \sup\{|\langle \cdot, x\rangle|;\ x \in M\}$. We say that the dual norm $\|\cdot\|$ on X^* is $\varepsilon - M-$*weak* Kadets* if $\limsup_\tau \left\|x_\tau^* - x^*\right\|_M < \varepsilon$ whenever x^* and a net $(x_\tau^*)_{\tau\in T}$ lie in S_{X^*} and $x_\tau^* \rightharpoondown x^*$. We note that if the dual norm is $\varepsilon - M-$LUR or is $\varepsilon - M-$weak* Kadets for every $\varepsilon > 0$, and $M = B_X$, then we get the usual concepts of LUR, and weak* Kadets property, respectively. The norm $\|\cdot\|$ on X^* is called *weak* LUR* if $x_n^* \rightharpoondown x^*$ whenever $x^*, x_n^* \in B_{X^*},\ n \in \mathbb{N}$, and $\lim_{n\to\infty} \left\|x^* + x_n^*\right\| = 2$.

Given $\varepsilon > 0$, a nonempty subset M of B_X is called *$\varepsilon-$Asplund* if for every at most countable subset $\emptyset \neq A \subset M$ there exists a countable set $C \subset B_{X^*}$ such that for every $x^* \in B_{X^*}$ there is $c \in C$ satisfying $\|x^* - c\|_A < \varepsilon$. We note that *the union of finitely many $\varepsilon-$Asplund sets is a $2\varepsilon-$Asplund set.* This follows from [9, Propositions 6 and 8]. Clearly, if a set is $\varepsilon-$Asplund for every $\varepsilon > 0$, then it is an Asplund set, see [4, Definition 1.4.1].

We say that a Banach space $(X, \|\cdot\|)$ is *$\sigma-$Asplund generated* if for every $\varepsilon > 0$ there is a decomposition $B_X = \bigcup_{n\in\mathbb{N}} M_n^\varepsilon$ where each M_n^ε is an $\varepsilon-$Asplund set. We say that the norm $\|\cdot\|$ on X^*, dual to $\|\cdot\|$, is *$\sigma-LUR$* if for every $\varepsilon > 0$ there is a decomposition $B_X = \bigcup_{n\in\mathbb{N}} M_n^\varepsilon$ such that $\|\cdot\|$ is $\varepsilon - M_n^\varepsilon-$LUR for every $n \in \mathbb{N}$. We say that the norm $\|\cdot\|$ on X^* is *$\sigma-$weak* Kadets* if for every $\varepsilon > 0$ there is a decomposition $B_X = \bigcup_{n\in\mathbb{N}} M_n^\varepsilon$ such that $\|\cdot\|$ is $\varepsilon - M_n^\varepsilon-$weak* Kadets for every $n \in \mathbb{N}$.

A simple argument shows that *a norm $\|\cdot\|$ on X^* is $\sigma-LUR$ ($\sigma-$weak* Kadets) if and only if there exist sets $M_n \subset B_X,\ n \in \mathbb{N}$, such that for every $\varepsilon > 0$, every $k \in \mathbb{N}$, and every finite set $F \subset B_X$ there is $n \in \mathbb{N}$ so that $n > k$, $M_n \supset F$, and the norm $\|\cdot\|$ is $\varepsilon - M_n-LUR$ ($\varepsilon - M_n-$weak* Kadets).* Likewise, *a Banach space X is $\sigma-$Asplund generated if and only if there exist sets $M_n \subset B_X,\ n \in \mathbb{N}$, such that for every $\varepsilon > 0$, every $k \in \mathbb{N}$, and every finite set $F \subset B_X$ there is $n \in \mathbb{N}$ so that $n > k$ and M_n is an $\varepsilon-$Asplund set containing F.* These conditions will be useful in proofs. The $\varepsilon-$concepts and $\sigma-$concepts introduced above have appeared naturally in studying and characterizing uniformly Gateaux smooth Banach spaces, and subspaces of weakly compactly generated spaces, see [6, 9]. A sample result from [9] sounds as: *A weakly Lindelöf determined Banach space X is a subspace of a weakly compactly generated space, if and only if X^* admits a $\sigma-$weak* Kadets norm, if and oly if X is $\sigma-$Asplund generated.*

Let X be a topological space with a topology τ. Consider a family $\mathcal{F}$ of subsets of X. We say that $\mathcal{F}$ is *discrete* if every $x \in X$ has a neighbourhood which intersects at most one element of $\mathcal{F}$. We say that $\mathcal{F}$ is *isolated* if every $x \in \bigcup \mathcal{F}$ has a neighbourhood which intersects exactly one element of $\mathcal{F}$; this is equivalent with the requirement that $N \cap \overline{\bigcup(\mathcal{F}\backslash\{N\})} = \emptyset$ for every $N \in \mathcal{F}$. The family $\mathcal{F}$ is called *$\sigma-$discrete* or *$\sigma-$isolated* if it can be written as $\mathcal{F} = \bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ where each $\mathcal{F}_n$ is discrete and isolated, respectively. If $\mathcal{U} \subset \tau$ is given, we say that $\mathcal{F}$ is *$\mathcal{U}-$isolated* if for every $x \in \bigcup \mathcal{F}$ there is $x \in U \in \mathcal{U}$ so that $U \cap N' = \emptyset$ for every $N' \in \mathcal{F}\backslash\{N\}$. A *$\sigma - \mathcal{U}-$isolated* family is the union of countably many $\mathcal{U}-$isolated families. $\mathcal{F}$ is called a *network* for the topology τ if for every $U \in \tau$ there is $\mathcal{F}' \subset \mathcal{F}$ so that $\bigcup \mathcal{F}' = U$. Note that any basis for τ is a network for τ. Also, one family $\mathcal{F}$ can serve as a network for several topologies on X. A topological space is called *descriptive* if its topology admits a $\sigma-$isolated network. We note that *every Eberlein, even every Gull'ko compact space is descriptive* [15] and that *descriptive compact spaces are Gruenhage* [16]. The above topological concepts recently proved to be very useful in renorming dual Banach spaces, see, in particular, M. Raja's works [13, 14, 15] and R. Smith' paper [16].

A compact space K is called *quasi-Radon-Nikodým* if it admits a function $\rho : K \times K \to [0, +\infty)$ such that it distinguishes the points of K, is lower semicontinuous, and *fragments* K, that is, whenever $\emptyset \neq M \subset K$ and $\varepsilon > 0$ are given, then there is an open set $\Omega \subset K$ so that $M \cap \Omega \neq \emptyset$ and $\sup\{\rho(k_1, k_2);\ k_1, k_2 \in \Omega \cap M\} < \varepsilon$. This concept is a formal generalization of the continuous image of Radon-Nikodým compact space. It was introduced by A. Arvanitakis. He provided a topological proof of the theorem saying that *a compact space is Eberlein if (and only if) it is simultaneously Corson and quasi-Radon-Nikodým*, see [5]; for an analytical proof of this, see [9].

For standard notations and results used and not explained in this paper we refer to [2, 4, 7].

3 Tools

Proposition 3. *Let (X, τ) be a topological space admitting a mapping $G : \mathbb{N} \times X \to \tau$ such that*

(a) $\forall x \in X \ \ \forall m \in \mathbb{N} \ \ G(m, x) \ni x$, *and*

(b) $\forall\, \Omega \in \tau \ \ \forall x \in \Omega \ \ \exists\, m \in \mathbb{N} \ \ \forall z \in X \ \ \big[G(m, z) \ni x \Rightarrow G(m, z) \subset \Omega\big]$

Then (X, τ) admits a $\sigma-$discrete network.

Proof. We follow the argument and the notation from Gruenhage [10, Theorem 5.11]. Fix for a while any $m \in \mathbb{N}$. Put $\mathcal{U}_m = \{G(m, x);\ x \in X\}$ and let us well order this family by "$\prec$", say. Fix for a while any $n \in \mathbb{N}$ and define

$$V_n^U = U \backslash \Big[\bigcup \{U' \in \mathcal{U}_m;\ U' \prec U\} \cup \bigcup \big\{G(n, y);\ y \in X \backslash U\big\} \Big], \quad U \in \mathcal{U}_m.$$

Put then $\mathfrak{N}_n^m = \big\{V_n^U;\ U \in \mathcal{U}_m\big\}$. We shall show that the family $\mathfrak{N}_n^m$ is discrete. So fix any $x \in X$. Since $\bigcup \mathcal{U}_m = X$ by (a), there is $U \in \mathcal{U}_m$ so that $U \ni x$ and $U' \not\ni x$ whenever $U' \in \mathcal{U}_m$ and $U' \prec U$. Now, take any $U' \in \mathcal{U}_m$ different from U. First assume that $U' \succ U$. Then $U \cap V_n^{U'} \subset U \cap (U' \backslash U) = \emptyset$. Second, assume that $U' \prec U$. Since $x \notin U'$, we have $G(n, x) \cap V_n^{U'} = \emptyset$. Therefore the open set $W := U \cap G(n, x)$ contains x and has the property that $W \cap V_n^{U'} = \emptyset$ whenever $U' \in \mathcal{U}_m$ and $U' \neq U$. (Note that $U = G(m, z)$ where z may be different from x.) Having the above done for every $m \in \mathbb{N}$ and every $n \in \mathbb{N}$, we get a family $\bigcup_{m,n \in \mathbb{N}} \mathfrak{N}_n^m$ which is $\sigma-$discrete.

It remains to verify that this family is a network for the topology τ. So fix any $\emptyset \neq \Omega \in \tau$ and any $x \in \Omega$. Let $m \in \mathbb{N}$ be found by (b) for these Ω and x. Find $U \in \mathcal{U}_m$ so that $U \ni x$ and $U' \not\ni x$ whenever $U' \in \mathcal{U}_m$ and $U' \prec U$. Now, for these U and x find, by (b), $n \in \mathbb{N}$ so that

$$\forall\, y \in X \ \ \big[G(n, y) \ni x \Rightarrow G(n, y) \subset U\big]. \tag{1}$$

Then $x \in V_n^U$. Indeed, if not, then, by the definition of V_n^U, we have $x \in G(n, y)$ for a suitable $y \in X \backslash U$. But (1) yields $G(n, y) \subset U$; so $y \in U$, a contradiction. It remains to show that $V_n^U \subset \Omega$. We know that $V_n^U \subset U$. Find $z \in X$ so that $G(m, z) = U$ (may be that z is different from x). Then $x \in G(m, z)$ and, by (b), $G(m, z) \subset \Omega$. Therefore $x \in V_n^U \subset U \subset \Omega$. $\square$

The next proposition follows from Hansell [11, Theorem 7.2]. Here, imitating his argument, we present a more direct (but not simpler) proof of it.

Proposition 4. *Let $(X, \|\cdot\|)$ be a Banach space. Let $\mathcal{U} \subset w^*$ be a family such that $tU \in \mathcal{U}$ for every $U \in \mathcal{U}$ and every $t > 0$. Assume that (S_{X^*}, w^*) admits a $\sigma - \mathcal{U}-$isolated network. Then (X^*, w^*) also admits a $\sigma - \mathcal{U}-$isolated network.*

Proof. Let a network $\mathfrak{N} = \bigcup_{m\in\mathbb{N}} \mathfrak{N}_m$ witness for the premise. Fix, for a longer while, any $m \in \mathbb{N}$. We shall need to split every element of $\mathfrak{N}_m$ into countably many pieces. For $i \in \mathbb{N}$ and $N \in \mathfrak{N}_m$ we put

$$D_i^N = \big\{x^* \in N;\ \exists U \in \mathcal{U} \text{ so that } U \ni x^* \text{ and } \big(U + \tfrac{3}{i}B_{X^*}\big) \cap \big(\bigcup \big(\mathfrak{N}_m \backslash \{N\}\big)\big) = \emptyset\big\}.$$

Since the family $\mathfrak{N}_m$ is $\mathcal{U}-$isolated, we easily get that $\bigcup_{i=3}^{\infty} D_i^N = N$ for every $N \in \mathfrak{N}_m$. Fix for a while any $i > 2$. We shall show that *the family* $\big\{\big(1 - \frac{1}{i}, 1 + \frac{1}{i}\big)D_i^N;\ N \in \mathfrak{N}_m\big\}$ *of subsets of X^* is $\mathcal{U}-$isolated.* So fix any $N \in \mathfrak{N}_m$, with $D_i^N \neq \emptyset$, and any $y^* \in \big(1 - \frac{1}{i}, 1 + \frac{1}{i}\big)D_i^N$. We have to find $V \in \mathcal{U}$ so that $V \ni y^*$ and $V \cap \big(1 - \frac{1}{i}, 1 + \frac{1}{i}\big)D_i^{N'} = \emptyset$ for every $N' \in \mathfrak{N}_m \backslash \{N\}$. Write $y^* = tx^*$ where $x^* \in D_i^N$ and $t \in \big(1 - \frac{1}{i}, 1 + \frac{1}{i}\big)$. Find $x^* \in U \in \mathcal{U}$ so that $\big(U + \frac{3}{i}B_{X^*}\big) \cap N' = \emptyset$ whenever $N' \in \mathfrak{N}_m$ and $N' \neq N$. Put $V = tU$. Note that $y^* \in V \in \mathcal{U}$. Fix any $N' \in \mathfrak{N}_m \backslash \{N\}$. Then

$$V \cap \big(1 - \tfrac{1}{i}, 1 + \tfrac{1}{i}\big)D_i^{N'} \subset tU \cap \big(1 - \tfrac{1}{i}, 1 + \tfrac{1}{i}\big)N' \subset tU \cap \big(N' + \tfrac{1}{i}B_{X^*}\big) = \emptyset.$$

This shows that our V works. The last equality here can be proved as follows. Assume there is $z^* \in U$ so that $tz^* \in N' + \frac{1}{i}B_{X^*}$. Then

$$z^* \in \tfrac{1}{t}N' + \tfrac{1}{ti}B_{X^*} \subset N' + \big(\big|\tfrac{1}{t} - 1\big| + \tfrac{1}{ti}\big)B_{X^*} \subset N' + \tfrac{3}{i}B_{X^*},$$

which is in a contradiction with $\big(U + \frac{3}{i}B_{X^*}\big) \cap N' = \emptyset$. Here we used the fact that $|1 - t| < \frac{1}{i}$.

Do all the above for every $i \in \mathbb{N}$. Then do all the above for every $m \in \mathbb{N}$.

Put now

$$\mathfrak{M}_{m,i,r} = \big\{(r - \tfrac{r}{2i}, r + \tfrac{r}{2i})D_i^N;\ N \in \mathfrak{N}_m\big\}, \quad i, m \in \mathbb{N}, \quad i > 2, \quad r > 0 \ \text{ rational.}$$

Note that there are countably many such families. And, of course, by the above, each $\mathfrak{M}_{m,i,r}$ is $\mathcal{U}-$isolated as well. Thus $\mathfrak{M} := \bigcup \big\{\mathfrak{M}_{m,i,r};\ i, m \in \mathbb{N},\ i > 2,\ r > 0$ rational$\big\}$ *is a $\sigma - \mathcal{U}-$isolated family of subsets of X^*.*

It remains to prove that $\mathfrak{M} \cup \{\{0\}\}$ *is a network for (X^*, w^*).* So take any $\Omega \in w^*$ and any $0 \neq x^* \in \Omega$. Find $\Omega' \in w^*$ and $\Delta > 0$ so that $x^* \in \Omega' \subset \Omega' + \Delta\|x^*\|B_{X^*} \subset \Omega$. Find then $m \in \mathbb{N}$ and $N \in \mathfrak{N}_m$ so that $\frac{x^*}{\|x^*\|} \in N \subset \frac{1}{\|x^*\|}\Omega'$. Find $i > 2$ so that $\frac{x^*}{\|x^*\|} \in D_i^N$. As $D_3^N \subset D_4^N \subset \cdots$, we may and do take $i > \frac{1}{\Delta}$. Further pick a rational number r such that $\|x^*\| \cdot \frac{2i}{2i+1} < r < \|x^*\| \cdot \frac{2i+2}{2i+1}$. Then

$$\begin{aligned}
x^* &= \|x^*\|\tfrac{x^*}{\|x^*\|} \in \|x^*\|D_i^N \subset \Big(r - \tfrac{r}{2i}, r + \tfrac{r}{2i}\Big)D_i^N \\
&\subset \|x^*\|\big(1 - \tfrac{1}{i}, 1 + \tfrac{1}{i}\big)D_i^N \subset \|x^*\|\big(D_i^N + \tfrac{1}{i}B_{X^*}\big) \\
&\subset \|x^*\|\big(N + \tfrac{1}{i}B_X{}^*\big) \subset \|x^*\|\big(\tfrac{1}{\|x^*\|}\Omega' + \tfrac{1}{i}B_{X^*}\big) = \Omega' + \tfrac{\|x^*\|}{i}B_{X^*} \subset \Omega.
\end{aligned}$$

We thus verified that $\mathfrak{M}$ is a network for (X^*, w^*). $\square$

The result below is known. We present a self-contained proof of it.

Proposition 5. ([15, 12]) *Let $(X, \|\cdot\|)$ be a Banach space such that its dual norm on X^* is weak* LUR. Then the dual ball (B_{X^*}, w^*) is descriptive.*

Proof. For every $x^* \in S_{X^*}$ and every $m \in \mathbb{N}$ find $v(m, x^*) \in S_X$ so that $\langle x^*, v(m, x^*)\rangle > 1 - \frac{1}{m}$ and define

$$G(m, x^*) = \big\{y^* \in S_{X^*};\ \langle y^*, v(m, x^*)\rangle > 1 - \tfrac{1}{m}\big\};$$

this is a relatively weak* open set. We shall verify the assumptions of Proposition 3 for the space (S_{X^*}, w^*). That (a) holds is obvious. As regards (b), fix any nonempty relatively weak* open set Ω in S_{X^*} and any $x^* \in \Omega$. Since the norm $\|\cdot\|$ on X^* is weak* LUR, there is $m \in \mathbb{N}$ so big that $y^* \in \Omega$ whenever $y^* \in S_{X^*}$ and $\|x^* + y^*\| > 2 - \frac{2}{m}$. We shall show that this m works. So take any $z^* \in S_{X^*}$ such that $G(m, z^*) \ni x^*$. Then for every $y^* \in G(m, z^*)$ we have $\|x^* + y^*\| \geq \langle x^*, v(m, z^*)\rangle + \langle y^*, v(m, z^*)\rangle > 2 - \frac{2}{m}$, and hence $y^* \in \Omega$. The condition (b) was thus verified.

Now, Proposition 3 and Proposition 4, with $\mathcal{U} := w^*$, yield that (X^*, w^*) has a $\sigma - w^*-$isolated network, and therefore (B_{X^*}, w^*) is descriptive. $\square$

For a Banach space X let $\mathcal{H}(X)$ denote the family of all halfspaces in X^* of the form $\{x^* \in X^*;\ \langle x^*, x\rangle > \lambda\}$ where $x \in S_X$ and $\lambda \in \mathbb{R}$.

Proposition 6. *Let $(X, \|\cdot\|)$ be a Banach space whose dual norm $\|\cdot\|$ is weak* LUR. Consider a family $\mathcal{U} \subset \mathcal{H}(X)$ such that $\bigcup\mathcal{U} \supset S_{X^*}$ and assume that $\mathcal{U}$ is well ordered by "$\prec$". Then the family $\big\{(S_{X^*} \cap H)\backslash \bigcup\{H' \in \mathcal{U};\ H' \prec H\};\ H \in \mathcal{U}\big\}$ has a $\sigma - \mathcal{H}(X)-$isolated refinement, that is, there exists a family $\mathfrak{N} = \bigcup_{m\in\mathbb{N}} \mathfrak{N}_m$ of subsets of S_{X^*} such that*

(i) $\bigcup \mathfrak{N} = S_{X^*}$,

(ii) $\forall\, N \in \mathfrak{N}\ \ \exists\, H \in \mathcal{U}$, *with* $H \backslash \bigcup\{H' \in \mathcal{U};\ H' \prec H\} \supset N$, *and*

(iii) $\forall\, m \in \mathbb{N}\ \ \forall\, N \in \mathfrak{N}_m\ \ \forall\, x^* \in N\ \ \exists\, R \in \mathcal{H}(X)$ *such that* $R \ni x^*$ *and* $R \cap \bigcup \big(\mathfrak{N}_m \backslash \{N\}\big) = \emptyset$.

Proof. Our argument profits from the proof of [12, Lemma 3.19]. Express each $H \in \mathcal{U}$ as $H = \big\{u^* \in X^*;\ \langle u^*, x_H\rangle > \lambda_H\big\}$, with suitable $x_H \in S_X$ and $\lambda_H \in \mathbb{R}$. For $H \in \mathcal{U}$ put

$$M_H = \big(S_{X^*} \cap H\big)\backslash \bigcup \big\{H' \in \mathcal{U};\ H' \prec H\big\}$$

and

$$M_H^n = \big\{u^* \in M_H;\ \langle u^*, x_H\rangle > \lambda_H + \tfrac{1}{n}\big\}, \quad n \in \mathbb{N};$$

clearly, $M_H = \bigcup_{n\in\mathbb{N}} M_H^n$. Also $\bigcup \big\{M_H;\ H \in \mathcal{U}\big\} = S_{X^*}$.

For the construction of the families $\mathfrak{N}_m$'s we shall need a further splitting of each M_H^n into countably many pieces. To do so, fix for a while any $n \in \mathbb{N}$. For

$x^* \in S_{X^*}$ find $H_{x^*} \in \mathcal{U}$ such that $x^* \in M_{H_{x^*}}$; note that this H_{x^*} is unique. Then for $p \in \mathbb{N}$ define

$$S_p^n = \big\{x^* \in S_{X^*};\ \big|\big\langle x^* - y^*, x_{H_{x^*}}\big\rangle\big| < \tfrac{1}{n} \text{ whenever } y^* \in S_{X^*} \text{ and } \|x^* + y^*\| > 2 - \tfrac{1}{p}\big\}.$$

Keeping n still fixed, fix for a while any $p \in \mathbb{N}$.

`Claim.` *The family* $\big\{M_H^n \cap S_p^n;\ H \in \mathcal{U}\big\}$ *is* $\mathcal{H}(X)-$*isolated*, which means that for any $x^* \in \bigcup\big\{M_H^n \cap S_p^n;\ H \in \mathcal{U}\big\}$ there is $R \in \mathcal{H}(X)$, with $R \ni x^*$, such that $M_H^n \cap S_n^p \cap R \neq \emptyset$ for exactly one $H \in \mathcal{U}$. So take any $H \in \mathcal{U}$, with $M_H^n \cap S_p^n \neq \emptyset$, and take any $x^* \in M_H^n \cap S_p^n$. Find $x \in S_X$ so that $\langle x^*, x\rangle > 1 - \frac{1}{2p}$ and put $R = \big\{u^* \in X^*;\ \langle x^*, x\rangle > 1 - \frac{1}{2p}\big\}$; thus $R \in \mathcal{H}(X)$ and $x^* \in R \cap M_H^n \cap S_p^n$. Take any $H' \in \mathcal{U}$ different from H. Assume that $R \cap M_{H'}^n \cap S_p^n$ is a nonempty set; take any y^* in this intersection. We have $\|x^* + y^*\| \geq \langle x^* + y^*, x\rangle > 2 - \frac{1}{p}$, and, as $x^* \in S_p^n$, we get $\big|\big\langle x^* - y^*, x_{H_{x^*}}\big\rangle\big| < \frac{1}{n}$. Similarly, as $y^* \in S_p^n$, we also get $\big|\big\langle y^* - x^*, x_{H_{y^*}}\big\rangle\big| < \frac{1}{n}$. Thus

$$\max\big\{\big|\big\langle x^* - y^*, x_{H_{x^*}}\big\rangle\big|, \big|\big\langle y^* - x^*, x_{H_{y^*}}\big\rangle\big|\big\} < \tfrac{1}{n}. \tag{2}$$

We know that $x^* \in M_H^n$ and $y^* \in M_{H'}^n$. Assume first that $H' \succ H$. Since $M_{H'}^n \subset M_{H'} \subset H' \backslash H$, we have $y^* \notin H$. Thus $\langle x^* - y^*, x_H\rangle > \lambda_H + \frac{1}{n} - \lambda_H = \frac{1}{n}$. Second, let $H' \prec H$. Then $M_H^n \subset M_H \subset H \backslash H'$, and so $x^* \notin H'$. Thus we get $\langle y^* - x^*, x_{H'}\rangle > \lambda_{H'} + \frac{1}{n} - \lambda_{H'} = \frac{1}{n}$. And, since we necessarily have that $H_{x^*} = H,\ H_{y^*} = H'$, we get a contradiction with (2). Therefore $R \cap M_{H'}^n \cap S_p^n = \emptyset$ and the claim is proved.

Doing the above for every $n \in \mathbb{N}$ and then for every $p \in \mathbb{N}$, let us enumerate the set $\mathbb{N} \times \mathbb{N}$ as $\big\{(n_m, p_m);\ m \in \mathbb{N}\big\}$ and put

$$\mathfrak{N}_m = \big\{M_H^{n_m} \cap S_{p_m}^{n_m};\ H \in \mathcal{U}\big\}, \quad m \in \mathbb{N},$$

and $\mathfrak{N} = \bigcup_{m \in \mathbb{N}} \mathfrak{N}_m$. These families satisfy the conclusion of our proposition. Indeed, we already checked (iii), while (ii) is clear. And since the norm $\|\cdot\|$ on X^* is weak* LUR (Here is the only use of this property.), we have $\bigcup_{p \in \mathbb{N}} S_p^n = S_{X^*}$ for every $n \in \mathbb{N}$, and hence (i) is satisfied as well. □

Lemma 7. (M. Raja [13, Lemma 5]) *In a Banach space X, consider a nonempty set $M \subset B_X$, a nonempty bounded set $A \subset X^*$ and $\varepsilon > 0$. Then there exist bounded convex sets $C_k \subset X^*$, $k \in \mathbb{N}$, such that for every $x^* \in A$ and every $H \in \mathcal{H}(X)$ satisfying $H \ni x^*$ and $M-\mathrm{diam}(A \cap H) < \varepsilon$ there are $k \in \mathbb{N}$ and $R \in \mathcal{H}(X)$ so that $C_k \cap R \ni x^*$ and $M-\mathrm{diam}(C_k \cap R) < 3\varepsilon$.*

The crucial theorem below is a $\sigma-$variant of the implication 5)⇒1) in M. Raja's [13, Theorem 2].

Theorem 8. *Let X be a Banach space admitting sets $M_m \subset B_X$ and bounded convex sets $D_l^m \subset X^*$, $m, l \in \mathbb{N}$, such that for every $\varepsilon > 0$, every $0 \neq x^* \in X^*$, and every finite set $F \subset B_X$ there exist $m, l \in \mathbb{N}$ and $R \in \mathcal{H}(X)$ such that $M_m \supset F$, $D_l^m \cap R \ni x^*$, and $M_m-\mathrm{diam}(D_l^m \cap R) < \varepsilon$.*

Then X^* admits an equivalent dual $\sigma-LUR$ norm.

Proof. Just follow the proof of the implication 5)⇒1) of [13, Theorem 2]. □

4 Proof of Theorem 2

Proof. (iii)⇔(v) follows from Avilés' result that *$\sigma-$Asplund generated Banach spaces are exactly those X for which (B_{X^*}, w^*) is a quasi-Radon-Nikodým compact space,* see [1], [5, Proposition 6].

(i)⇒(ii) is simple, see [9, Proposition 9].

(ii)⇒(iii). Assume (ii) holds, with sets $M_m \subset B_X,\ m \in \mathbb{N}$, witnessing for that. Thus for every $\varepsilon > 0$, for every $k \in \mathbb{N}$, and for every finite set $F \subset B_X$ there is $m \in \mathbb{N}$ so that $m > k,\ M_m \supset F$, and $\|\cdot\|$ is $\varepsilon - M_m-$weak* Kadets. For $x^* \in S_{X^*},\ M \subset B_X$, and $\varepsilon > 0$ denote $B_M(x^*, \varepsilon) = \big\{z^* \in S_{X^*};\ \|z^* - x^*\|_M < \varepsilon\big\}$. For $m \in \mathbb{N}$ define

$$\varepsilon_m = \inf\big\{\varepsilon > 0;\ \|\cdot\| \text{ is } \varepsilon - M_m - \text{weak}^* \text{ Kadets}\big\} + \tfrac{1}{m}.$$

Using Proposition 3, we shall first prove that (S_{X^*}, w^*) *has a $\sigma-$discrete network.* Hence we need to define a mapping $G : \mathbb{N}\times S_{X^*} \to w^*$ and to verify the conditions (a) and (b) therein. For any $x^* \in S_{X^*}$ and any $m \in \mathbb{N}$ find an open set $G(m, x^*)$ in (S_{X^*}, w^*) such that $x^* \in G(m, x^*) \subset B_{M_m}(x^*, \varepsilon_m)$. Such a set does exist. Indeed, if not, then for every open set V in (S_{X^*}, w^*), with $V \ni x^*$, there is $x^*_V \in V\backslash B_{M_m}(x^*, \varepsilon_m)$. But then $x^*_V \to x^*$ when V's "approach" x^*. Hence, as the norm $\|\cdot\|$ is $\varepsilon_m - M_m-$weak* Kadets, $\|x^*_V - x^*\|_{M_m} < \varepsilon_m$ for all $x^* \in V \in w^*$ "sufficiently small". Taking one such V, we get that $x^*_V \in B_{M_m}(x^*, \varepsilon_m)$, a contradiction. Thus we have verified the condition (a) in Proposition 3.

As regards the condition (b) in Proposition 3, fix any weak* open set Ω in X^*, with $\Omega\cap S_{X^*} \neq \emptyset$, and fix any $x^* \in \Omega\cap S_{X^*}$. Find a finite set $F \subset B_X$ and $\Delta > 0$ such that $B_F(x^*, \Delta) \subset \Omega$. Find $m \in \mathbb{N}$ so that $m > \frac{6}{\Delta},\ M_m \supset F$, and that $\|\cdot\|$ is $\frac{\Delta}{3} - M_m-$weak* Kadets; thus $\varepsilon_m < \frac{\Delta}{3} + \frac{1}{m} < \frac{\Delta}{3} + \frac{\Delta}{6} = \frac{\Delta}{2}$. It remains to show that $G(m, z^*) \subset \Omega$ whenever $z^* \in S_{X^*}$ and $x^* \in G(m, z^*)$. So fix any such z^* and x^*; then $\|x^* - z^*\|_{M_m} < \varepsilon_m$. Now, for $y^* \in G(m, z^*)$ we have $\|y^* - z^*\|_{M_m} < \varepsilon_m$, and so

$$\|y^* - x^*\|_F \leq \|y^* - z^*\|_{M_m} + \|z^* - x^*\|_{M_m} < 2\varepsilon_m < \Delta,$$

and thus $y^* \in \Omega$. We verified (b), and therefore, by Proposition 3, (S_{X^*}, w^*) has a $\sigma-$discrete network.

Now, according to Proposition 4, (B_{X^*}, w^*) is a descriptive compact space.

Finally, X is $\sigma-$Asplund generated according to [9, Proposition 9]. Thus we obtained (iii).

(iii)⇒(iv). Here we refer to a deep result due to M. Raja that X^* *admits an equivalent dual weak* LUR norm provided that (B_{X^*}, w^*) is descriptive* [15].

(iv)⇒(i) can be done by adjusting the proof of [12, Corollary 3.24], which says that X^* admits an equivalent dual LUR norm provided that X is Asplund and X^* has a dual weak* LUR norm. For a reader's convenience we include a detailed proof. Let $\|\cdot\|$ be an equivalent dual weak* LUR norm on X^*. Let $M_m \subset B_X,\ m \in \mathbb{N}$, witness that the space X is $\sigma-$Asplund generated. This means that for every $\varepsilon > 0$, for every $k \in \mathbb{N}$, and for every finite set $F \subset B_X$

there is $m \in \mathbb{N}$ so that $m > k$ and M_m is an $\varepsilon-$Asplund set containing F. We shall verify the assumptions of Theorem 8. For $m \in \mathbb{N}$ define

$$\varepsilon_m = \inf\big\{\varepsilon > 0;\ M_m \text{ is } \varepsilon\text{−Asplund}\big\} + \tfrac{1}{m}.$$

According to [9, Propositions 8 and 6], for every set $\emptyset \neq S \subset B_{X^*}$ there is $H \in \mathcal{H}(X)$ such that the set $S \cap H$ is nonempty and has M_m-diameter less than $2\varepsilon_m$.

Fix for a while any $m \in \mathbb{N}$. We (easily) find, by induction, a family $\mathcal{U}_m = \big\{H^m_\gamma;\ \gamma < \xi_m\big\}$ of elements of $\mathcal{H}(X)$, indexed by ordinals, such that $\bigcup_{\gamma<\xi_m} H^m_\gamma \supset S_{X^*}$, and $M_m - \operatorname{diam}\Big(\big(S_{X^*} \cap H^m_\gamma\big)\backslash \bigcup_{\gamma'<\gamma} H^m_{\gamma'}\Big) < 2\varepsilon_m$ for every $\gamma < \xi_m$. To this $\mathcal{U}_m$, considered with the well order induced by the order of the ordinal subscripts, by Proposition 6 (here the weak* LUR is used), find $\mathcal{H}(X)-$isolated families $\mathfrak{N}^m_n,\ n \in \mathbb{N}$, of subsets of S_{X^*} such that $\bigcup_{n\in\mathbb{N}} \mathfrak{N}^m_n = S_{X^*}$. We recall that for every $n \in \mathbb{N}$ and every $N \in \mathfrak{N}^m_n$ there is $\gamma < \xi_m$ such that $H^m_\gamma \backslash \bigcup_{\gamma'<\gamma} H^m_{\gamma'} \supset N$. Also, we know that, whenever $n \in \mathbb{N}$ and $x^* \in N \in \mathcal{N}^m_n$, then there is $R \in \mathcal{H}(X)$ satisfying $R \ni x^*$ and $R \cap \bigcup\big(\mathfrak{N}^m_n \backslash \{N\}\big) = \emptyset$.

Keeping still m fixed, fix further for a while $n \in \mathbb{N}$ and put $A^m_n = \overline{\bigcup \mathfrak{N}^m_n}^{w^*} \cap S_{X^*}$. Take $N \in \mathfrak{N}^m_n$. From the above, for every $x^* \in N$ find $R^m_{n,x^*} \in \mathcal{H}(X)$ satisfying $R^m_{n,x^*} \ni x^*$ and $R^m_{n,x^*} \cap \bigcup\big(\mathfrak{N}^m_n \backslash \{N\}\big) = \emptyset$. Put then $U^m_{n,N} = \bigcup_{x^*\in N} R^m_{n,x^*}$. Note that $U^m_{n,N} \supset N$ and $U^m_{n,N} \cap \big(\bigcup \mathfrak{N}^m_n \backslash \{N\}\big) = \emptyset$. Do so for every $n \in \mathbb{N}$.

`Claim.` *For every $x^* \in S_{X^*}$ there are $n \in \mathbb{N}$ and $H \in \mathcal{H}(X)$ such that $H \cap A^m_n \ni x^*$ and $M_m - \operatorname{diam}\big(H \cap A^m_n\big) < 2\varepsilon_m$.* Indeed, fix such an x^*. For sure there are $n \in \mathbb{N}$ and $N \in \mathfrak{N}^m_n$ so that $x^* \in N$. And, taking $H = R^m_{n,x^*}$, we have

$$x^* \in H \cap A^m_n = (H \cap \overline{N}^{w^*} \cap S_{X^*}) \cup \Big(H \cap \overline{\bigcup(\mathfrak{N}^m_n \backslash \{N\})}^{w^*} \cap S_{X^*}\Big) = H \cap \overline{N}^{w^*} \cap S_{X^*} \subset \overline{N}^{w^*}.$$

But there is $\gamma < \xi_m$ such that $N \subset \big(S_{X^*} \cap H^m_\gamma\big) \backslash \bigcup_{\gamma'<\gamma} H^m_{\gamma'}$, where the latter set has the M_m- diameter less than $2\varepsilon_m$. This proves the claim.

Keep still m fixed. For every $n \in \mathbb{N}$, from Lemma 7 applied for $M := M_m,\ A := A^m_n$, and $\varepsilon := 2\varepsilon_m$, we find the corresponding bounded convex sets $C_1, C_2, \ldots$, called now $C^{m,n}_1,\ C^{m,n}_2, \ldots$

Do all the above for every $m \in \mathbb{N}$.

Thus, using the Claim, for every $m \in \mathbb{N}$ *and every* $x^* \in S_{X^*}$ *there are* $n \in \mathbb{N}$ and $H \in \mathcal{H}(X)$ such that $A^m_n \cap H \ni x^*$ and $M_m-\operatorname{diam}(A^m_n \cap H) < 2\varepsilon_m$, and hence, by Lemma 7, *there are $k \in \mathbb{N}$ and $R \in \mathcal{H}(X)$ so that $C^{m,n}_k \cap R \ni x^*$ and M_m-diam$(C^{m,n}_k \cap R) < 6\varepsilon_m$.*

Now, we are ready to verify the assumptions of Theorem 8. Fix any $\varepsilon > 0$, any $0 \neq x^* \in X^*$, and any finite set $F \subset B_X$. From the $\sigma-$Asplund generating, find $m \in \mathbb{N}$ such that $m > 12\|x^*\|/\varepsilon$, that $M_m \supset F$, and that M_m is an $\varepsilon/(12\|x^*\|)-$Asplund set. We observe that $\varepsilon_m < 2\varepsilon/(12\|x^*\|) = \varepsilon/(6\|x^*\|)$. From the previous paragraph find $n, k \in \mathbb{N}$ and $R \in \mathcal{H}(X)$ so that $C^{m,n}_k \cap R \ni x^*/\|x^*\|$ and $M_m - \operatorname{diam}\big(C^{m,n}_k \cap R\big) < 6\varepsilon_m\ \ (< \varepsilon/\|x^*\|)$. Put $R' = \|x^*\| R$ and note that $R' \in \mathcal{H}(X)$.

`Claim.` *There are rational numbers* $0 < s < \|x^*\| < t$ *such that such that* $(s,t)\, C_k^{m,n} \cap R' \ni x^*$ *and* $M_m - \operatorname{diam}\big((s,t)\, C_k^{m,n} \cap R'\big) < \varepsilon$. Assume this not true. Then there are sequences $0 < s_1 < s_2 < \cdots < \|x^*\|$ and $t_1 > t_2 > \cdots > \|x^*\|$ of rational numbers such that $\lim_{j\to\infty} s_j = \lim_{j\to\infty} t_j = \|x^*\|$ and $M_m - \operatorname{diam}\big((s_j,t_j)\, C_k^{m,n} \cap R'\big) \geq \varepsilon$ for every $j \in \mathbb{N}$. For every $j \in \mathbb{N}$ find $s'_j, t'_j \in (s_j,t_j)$ and $a_j, b_j \in C_k^{m,n}$ so that $s'_j a_j, t'_j b_j \in (s_j,t_j) C_k^{m,n} \cap R'$ and $\|s'_j a_j - t'_j b_j\|_{M_n} > \varepsilon - \frac{1}{j}$. Then $\lim_{j\to\infty} s'_j = \lim_{j\to\infty} t'_j = \|x^*\|$, and hence $\liminf_{j\to\infty} \big\| \|x^*\| a_j - \|x^*\| b_j \big\|_{M_n} \geq \varepsilon$. Therefore $M_m - \operatorname{diam}\big(C_k^{m,n} \cap R\big) \geq \varepsilon / \|x^*\|$, which is a contradiction. This proves the claim.

At this moment, we have verified the assumptions of Theorem 8. Indeed, given a fixed $m \in \mathbb{N}$, for the sets D_l^m, $l \in \mathbb{N}$, we take the (countable) family $(s,t)\, C_k^{m,n}$, $n,k \in \mathbb{N}$, $0 < s < t$ rational. Therefore X^* admits and equivalent weak* LUR norm, that is, (i) holds. $\square$

Remarks. 1. (iii)$\Rightarrow$(i) in Theorem 2 can be proved directly by following M. Raja's method from [15]. It needs just an adaptation of Lemma 2.2, Lemma 3.2, Theorem 3.3, and their proofs from this paper.

2. Let $\mathcal{D}, \mathcal{RN}, \mathcal{QRN}, \mathcal{IRN}$ denote the class of compact spaces which are descriptive, Radon-Nikodým, quasi-Radon-Nikodým, or continuous images of Radon-Nikodým compact spaces, respectively. J. Orihuela asked if $\mathcal{QRN} \cap \mathcal{D}$ is a subclass of $\mathcal{RN}$. Note that a converse is false as the long interval $[0, \omega_1]$ shows. We do not know of any Banach space counterpart to this. Yet a (weaker) question "*whether* $\mathcal{QRN} \cap \mathcal{D} \subset \mathcal{IRN}$*" is equivalent with the question "whether a* $\sigma-$*Asplund generated Banach space* X*, with* $\big(B_{X^*}, w^*\big) \in \mathcal{D}$*, is already a subspace of an Asplund generated space*". This follows from [15, 1], and [4, Theorem 1.5.4]. If, in the second question, the word "subspace" is dropped, we get a false statement — take any subspace of a WCG space which is not WCG, see [4, Section 1.6].

3. The following facta complete our knowledge; proofs are simple conseqeuences of [4, Theorem 1.5.4], [1], [5, Proposition 6], and [9, Theorem 2 (ii)].

Fact 1. *Given a compact space* K*, then*
(i) $K \in \mathcal{RN}$ *if and only if* $C(K)$ *is Asplund generated.*
(ii) $K \in \mathcal{QRN}$ *if and only if* $C(K)$ *is* $\sigma-$*Asplund generated.*
(iii) $K \in \mathcal{IRN}$ *if and only if* $C(K)$ *is a subspace of an Asplund generated space.*

Fact 2. *Given a Banach space* X*, then*
(i) $\big(B_{X^*}, w^*\big) \in \mathcal{QRN}$ *if and only if* X *is* $\sigma-$*Asplund generated.*
(ii) $\big(B_{X^*}, w^*\big) \in \mathcal{IRN}$ *if and only if* X *is a subspace of an Asplund generated space.*
(iii) If $\big(B_{X^*}, w^*\big) \in \mathcal{RN}$*, then* X *is a subspace of an Asplund generated space.*
(iv) $\big(B_{X^*}, w^*\big) \in \mathcal{RN}$ *provided that* X *is Asplund generated.*

Acknowledgement. We thank the anononymous referee for his suggestions that unabled us to improve the overal shape of our paper.

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