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ON THE UNIVERSALITY OF A CLASS OF NON CONVOLUTION OPERATORS ON $\mathcal{H}(\mathbb{C})$

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Abstract. In this note we study the universal behaviour of some non-convolution operators on the space $\mathcal{H}(\mathbb{C})$ of entire functions endowed with the compact open topology.

1. Introduction

Let $\mathcal{T} = \{T_n\} \subset L(X)$ be a sequence of continuous linear operators acting on a separable infinite-dimensional $\mathcal{F}$-space X. This sequence is said to be *universal* if there exists $x \in X$ such that its orbit under $\mathcal{T}$, that is $\mathrm{Orb}(\mathcal{T}, x) := \{T_n x : n \in \mathbb{N}\}$, is dense in X. If X is a function space, such a vector x is said to be a *universal function* for $\mathcal{T}$. If $T_n := T^n$, $n \in \mathbb{N}$, for some $T \in L(X)$, the universal elements are called *hyperyclic.*

Some of the first examples of universal behavior of linear operators on linear spaces were provided on the space of entire functions $\mathcal{H}(\mathbb{C})$, endowed with the compact open topology. Birkhoff [Bir29] proved the universality of the sequence of translations $T_n f(z) := f(z + n)$, $n \in \mathbb{N}$, on $\mathcal{H}(\mathbb{C})$ using Runge's theorem. MacLane [Mac52] did it for the sequence of powers of the derivative operator $Df(z) := f'(z)$. For a recent account on the proof of these results see [AM04]. The derivative operator has also been treated by Bonet on weighted inductive limits of spaces of holomorphic functions [Bon00], and on weighted spaces of entire functions [Bon08]. Details concerning the influence of Birkhoff and MacLane's operators on the development of universality (and hypercyclicity) can be found in [GE99, Sec. 4.a and 4.c], see also [GE99, BMGP03, GE03, Mül07].

The above mentioned seminal examples led to a more comprehensive study of these phenomena by Godefroy and Shapiro in [GS91]. They showed that every convolution operator on $\mathcal{H}(\mathbb{C}^N)$ (a continuous linear operator that commutes with all translations, or equivalently commutes with each partial differential operator) that is not a multiple of the identity is hypercyclic [GS91, Th. 5.1].

On the other hand, examples of non-convolution operators on $\mathcal{H}(\mathbb{C})$ can be found in [FH05], see also [AM04], where the hypercyclicity of the operators $(T_{\lambda,b} f)(z) := f'(\lambda z + b)$ for $|\lambda| \geq 1$ and $b \neq 0$ is shown. Other examples in $\mathcal{H}(\mathbb{C}^N)$ can be found in [Pet06].

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For a nonempty compact set $K \subset \mathbb{C}$, let $\mathcal{A}(K)$ be the Banach space of all functions which are continuous on K and holomorphic in its interior. In [Luh96], Luh studied the existence of holomorphic functions which have some universal properties simultaneously, for several sequences of operators. The existence of such functions is easy to see in the case of hypercyclicity, since the set of all hypercyclic vectors of an operator is a residual set. For instance, there are entire functions which are simultaneously hypercyclic for a translation and the derivative operator on $\mathcal{H}(\mathbb{C})$. In fact, these operators share a dense subspace, whose non null functions are hypercyclic for both operators [GE03, Prop. 1], and there is a residual set of entire functions which are hypercyclic for the translation operator, and all of their powers are hypercyclic for the derivative operator [ACPSS07]. The study of existence of common universal or hypercyclic vectors for families of operators is an area of intensive work (see, e.g., [AG03, ABLP05, Bay04, BM07, CS04, LSM04, CMP07]).

Theorem 1.1. [Luh96] *Let $\{z_n\}_n$ be an unbounded sequence in $\mathbb{C}$. For an entire function φ we denote by $\varphi^{(j)}$ its derivative of order j, if $j \in \mathbb{N}_0$; and the "normalized" antiderivative of order $-j$, if $-j \in \mathbb{N}$. There exists an entire function φ such that for each nonempty simply connected compact subset $K \subset \mathbb{C}$ we have:*

(1) *for any fixed $j \in \mathbb{Z}$ the sequence $\{\varphi^{(j)}(z+z_n)\}_n$ is dense in $\mathcal{A}(K)$;*
(2) *for any fixed $j \in \mathbb{Z}$ the sequence $\{\varphi^{(j)}(zz_n)\}_n$ is dense in $\mathcal{A}(K)$, if $0 \notin K$;*
(3) *the sequence of derivatives $\{\varphi^{[|z_n|]}(z)\}_n$ is dense in $\mathcal{A}(K)$.*

In this result, 1.1(1) is an extension of Birkhoff's example for the derivatives of φ. On the other hand, 1.1(2) is an extension of a result of Zappa on the existence of a holomorphic function on $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ such that $\{\varphi(nz)\}_n$ is dense in $\mathcal{A}(K)$ for every nonempty compact set $K \subset \mathbb{C}^*$ whose complement is connected in $\mathbb{C}^*$ [Zap90]. Approximation theorems of Runge and Mergelyan can be refined considering lacunary polynomials [DK77, Mül91, GLM03, GLM04]. So it is not surprising that functions which simultaneously verify 1.1.(1) and 1.1.(2) can have lacunary power series [LMM98].

A non-convolution operator T on $\mathcal{H}(\mathbb{C})$ defined as $Tf(z) = (zf)'(z)$ was introduced in [MM04]. It generalizes MacLane's operator, but preserves the initial lacunas in the power series of a function along all the orbit under $\{T^n\}_n$. This operator cannot be hypercyclic in $X = \mathcal{H}(\mathbb{C})$ nor in $X = \{f \in \mathcal{H}(\mathbb{C}) : f(0) = 0\}$: Given $f(z) := \sum_{k=0}^{\infty} a_k z^k$, we have $T(\sum_{k=0}^{\infty} a_k z^k) = \sum_{k=0}^{\infty}(k+1)a_k z^k$, and hence $T^n(\sum_{k=0}^{\infty} a_k z^k) = \sum_{k=0}^{\infty}(k+1)^n a_k z^k$. Take an arbitrary $k_0 \in \mathbb{N}$. By continuity of the projection $\pi_{k_0} : \sum_{k=0}^{\infty} a_k z^k \to a_{k_0}$, we have that $\sum_{k=1}^{\infty} a_k z^k$ can only be hypercyclic for T on X if $((k_0+1)^n a_{k_0})_n$ is dense in $\mathbb{C}$, which leads to a contradiction.

However, in [MM04] there can be found results in the sense of Theorem 1.1 for the operator T instead of the derivative operator. But in [MM04, Th. 2] the computation of T^{n_k} results in a serious gap (just take $\{z_n\}_n = \{n\}_n$

and $Q = \mathbb{N}_0$). This was pointed out by Grosse-Erdmann [GE08]. In this note we provide a general result that provides a correct proof of this statement.

2. Main Section

Let $K \subset \mathbb{C}$ be a nonempty compact set. We define the radius of K as $\mathrm{rad}(K) := \sup\{|z| : z \in K\}$, and given $\lambda \in \mathbb{C}$ let $d(\lambda, K) := \inf\{|\lambda - z| : z \in K\}$ be the distance of λ to K. In addition, for every $f \in \mathcal{H}(\mathbb{C})$ we define its supremum norm on K as $||f||_K := \sup\{|f(z)| : z \in K\}$.

Let $K \subset \mathbb{C}$ be a nonempty simply connected compact set and $\lambda \notin K$. By the Runge theorem, the function $\frac{1}{\lambda - z}$ can be approximated by polynomials uniformly on K. The next lemma is a quantitative version of the Runge theorem — it gives an estimate of the degree of the approximating polynomial.

Lemma 2.1. *Let $K \subset \mathbb{C}$ be a nonempty simply connected compact subset. Let $\lambda \in \mathbb{C} \setminus K$ and $s > 0$. Then, the following property holds, namely (*):*

There exists $k \in \mathbb{N}$ such that for every $n, j \in \mathbb{N}$, with $j \leq n$, there is a polynomial p satisfying $\deg(p) \leq kn$ and

$$\left|\left|p(z) - \frac{1}{(\lambda - z)^j}\right|\right|_K \leq \frac{1}{s^n}.$$

Proof. Fix K as in the hypothesis. The proof will be carried out in several steps.

Step 1: Property (*) holds for every $\lambda \in \mathbb{C} \backslash K$ with $|\lambda| > \max\{4\,\mathrm{rad}(K), 2\}$.

Fix λ as indicated, and $s > 0$. Let $k \in \mathbb{N}$ satisfy $2^k > s$. For every $z \in K$ we have

$$\frac{1}{\lambda - z} = \frac{1}{\lambda} \sum_{l=0}^{\infty} \left(\frac{z}{\lambda}\right)^l.$$

A direct computation shows that

$$\frac{1}{(\lambda - z)^j} = \frac{1}{\lambda^j} \sum_{l=0}^{\infty} \left(\frac{z}{\lambda}\right)^l \binom{l+j-1}{j-1} \quad \text{for every } j \in \mathbb{N}.$$

Let $n, j \in \mathbb{N}$ and $j \leq n$. Consider the polynomial

$$p_j(z) := \frac{1}{\lambda^j} \sum_{l=0}^{kn} \left(\frac{z}{\lambda}\right)^l \binom{l+j-1}{j-1}$$

with $\deg(p_j) \leq kn$. Since $|z/\lambda| < 1/4$ for all $z \in K$, and $\binom{n}{m} \leq 2^n$ for all $m \leq n$, we get

$$\left|\left|p_j(z) - \frac{1}{(\lambda - z)^j}\right|\right|_K \leq \frac{1}{|\lambda|^j} \sum_{l=kn+1}^{\infty} \frac{2^{l+j-1}}{4^l} \leq \frac{2^j}{|\lambda|^j} \sum_{l=kn+1}^{\infty} \frac{1}{2^l} \leq \frac{1}{2^{kn}} \leq \frac{1}{s^n}.$$

Step 2: If property (*) holds for some $\lambda \in \mathbb{C} \setminus K$, then it also holds for each $\lambda' \in \mathbb{C} \setminus K$ such that $|\lambda' - \lambda| < \min\{1, \mathrm{d}(\lambda, K)/4\}$.

Fix λ, λ' as in the statement. Let $s' > 0$, without loss of generality we can assume that $s' \geq \max\{1, 4|\lambda|\}$ and $\mathrm{d}(\lambda, K) > 1/s'$.

Choose $s > 4s'$. By the assumption, there exists $k \in \mathbb{N}$ such that (*) holds for λ, s and k. Let us take $k'' \in \mathbb{N}$ satisfying $2^{k''} > 4s'^2$ and define $k' := k(k''+1)$. We show that property (*) holds for λ', s' and k'.

Let $n, j \in \mathbb{N}$ with $j \leq n$. For every $z \in K$ we have

$$\frac{1}{\lambda' - z} = \frac{1}{\lambda - z} \sum_{l=0}^{\infty} \left(\frac{\lambda - \lambda'}{\lambda - z} \right)^l,$$

and therefore

$$\frac{1}{(\lambda' - z)^j} = \frac{1}{(\lambda - z)^j} \sum_{l=0}^{\infty} \left(\frac{\lambda - \lambda'}{\lambda - z} \right)^l \binom{l+j-1}{j-1}.$$

We have

$$\begin{aligned} & \left\| \frac{1}{(\lambda - z)^j} \sum_{l=k''n+1}^{\infty} \left(\frac{\lambda - \lambda'}{\lambda - z} \right)^l \binom{l+j-1}{j-1} \right\|_K \\ \leq \quad & s'^j \sum_{l=k''n+1}^{\infty} \frac{2^{l+j-1}}{4^l} \leq \frac{(2s')^j}{2^{k''n}} \leq \frac{(2s')^n}{(4s'^2)^n} \leq \frac{1}{2s'^n}, \end{aligned}$$

which let us construct the required polynomial. By hypothesis, for every $0 \leq r \leq k''n+n$ there exists a polynomial q_r such that $\deg(q_r) \leq (k''n+n)k = k'n$ and

$$\left\| q_r(z) - \frac{1}{(\lambda - z)^r} \right\|_K \leq \frac{1}{s^{k''n+n}}.$$

We show that the polynomial

$$p_j(z) := \sum_{l=0}^{k''n} \binom{l+j-1}{j-1} (\lambda - \lambda')^l q_{l+j}(z)$$

satisfies the required conditions. Clearly, $\deg(p_j) \leq k'n$. Furthermore,

$$\begin{aligned}
&\left|\left|p_j(z) - \frac{1}{(\lambda' - z)^j}\right|\right|_K \\
\leq\ & \sum_{l=0}^{k''n} \binom{l+j-1}{j-1} |\lambda - \lambda'|^l \left|\left|q_{l+j} - \frac{1}{(\lambda - z)^{l+j}}\right|\right|_K \\
&\qquad + \left|\left|\frac{1}{(\lambda - z)^j} \sum_{l=k''n+1}^{\infty} \left(\frac{\lambda - \lambda'}{\lambda - z}\right)^l \binom{l+j-1}{j-1}\right|\right|_K \\
\leq\ & \frac{1}{2s'^n} + \frac{1}{s^{k''n+n}} \sum_{l=0}^{k''n} 2^{l+j-1} |\lambda - \lambda'|^l \leq \frac{1}{2s'^n} + \frac{2^{k''n+n}}{s^{k''n+n}} \\
\leq\ & \frac{1}{2s'^n} + \left(\frac{1}{2s'}\right)^{k''n+n} \leq \frac{1}{2s'^n} + \frac{1}{2s'^n} = \frac{1}{s'^n}.
\end{aligned}$$

Step 3: Property (*) holds for all $\lambda \in \mathbb{C} \setminus K$.

Denote by A the set of all $\lambda \in \mathbb{C} \setminus K$ for which (*) holds. By Step 1, A is nonvoid. Since, by Step 2, A is open in $\mathbb{C} \setminus K$, we only have to show that it is also relatively closed, and then $A = \mathbb{C} \setminus \mathbb{K}$. Take an arbitrary sequence $(\lambda_n)_n \subset A$ tending to $\lambda \in \mathbb{C} \setminus \mathbb{K}$. We take $n' \in \mathbb{N}$ such that $|\lambda - \lambda_{n'}| < 1$ and $|\lambda - \lambda_{n'}| < \mathrm{d}(\lambda, K)/5$. So $\mathrm{d}(\lambda_{n'}, K) \geq \mathrm{d}(\lambda, K) - |\lambda_{n'} - \lambda| \geq 4|\lambda_{n'} - \lambda|$, and by Step 2 we get that $\lambda \in A$. Hence $A \subset \mathbb{C} \setminus K$ is a nonempty, relatively open and closed subset. Since K is simply connected, we have $A = \mathbb{C} \setminus K$. $\square$

Given a sequence $\{\gamma_k\} \subset \mathbb{C}$, we consider the multiplier operator $T : \mathcal{H}(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ defined for every $f(z) := \sum_{k=0}^{\infty} a_k z^k \in \mathcal{H}(\mathbb{C})$ as

$$(Tf)(z) := \sum_{k=0}^{\infty} \gamma_k a_k z^k.$$

This operator is well defined whenever $\limsup_{k\to\infty} \sqrt[k]{|\gamma_k|} < \infty$.

Theorem 2.2. *Let T be the multiplier operator associated to a sequence $\{\gamma_k\} \subset \mathbb{C}^*$ which verifies $\lim_{k\to} |\gamma_k| = \infty$ and $\limsup_{k\to\infty} \sqrt[k]{|\gamma_k|} < \infty$. Let $K \subset \mathbb{C}$ be a nonempty simply connected compact set with $0 \notin K$. Let p be a polynomial and let $\varepsilon, R > 0$. Then:*

(1) *for each n large enough there exists a polynomial h such that*

$$||T^n h - p||_K < \varepsilon \textit{ and } ||h||_{B(0,R)} < \varepsilon;$$

(2) *for each n large enough there exists a polynomial g such that*

$$||T^n g - T^n p||_K < \varepsilon \textit{ and } ||g||_{B(0,R)} < \varepsilon.$$

Proof. Choose a simply connected compact set K' such that $K \subset \mathrm{int}(K')$ and $0 \notin K'$. Fix $a \in K$, an arbitrary polynomial $p(z) := \sum_{j=0}^{\deg(p)} \beta_j z^j$, and choose s such that $B(a, 1/s) \subset K'$, and

$$s > \max \left\{ \frac{1}{\varepsilon}, 2, R, \mathrm{rad}(K'), \|z^{-1}\|_{K'}, \|p\|_{K'}, \sum_{j=0}^{\deg(p)} |\beta_j|, \deg(p), |\gamma_0|, \ldots, |\gamma_{\deg(p)}| \right\}.$$

Let k be the number obtained in Lemma 2.1 such that for every $n \in \mathbb{N}$ there exists a polynomial q_n with $\deg(q_n) \leq kn$, and $||q_n - 1/z^n||_{K'} < 1/s^{4n}$. Clearly

$$||q_n||_{K'} \leq \left|\left| q_n - \frac{1}{z^n} \right|\right|_{K'} + \left|\left| \frac{1}{z^n} \right|\right|_{K'} \leq \frac{1}{s^{4n}} + \left|\left| \frac{1}{z^n} \right|\right|_{K'} \leq 1 + s^n \leq s^{n+1}.$$

Firstly, let us prove (1): Take $n \in \mathbb{N}$ with $n > \deg(p)$, $n \geq 3$ and $|\gamma_m| \geq s^{3k+7} \quad (m \geq n)$. The following estimation yields

$$||p - z^n p q_n||_K \leq ||p||_K \, ||z^n||_K \left|\left| q_n - \frac{1}{z^n} \right|\right|_K \leq \frac{s^{n+1}}{s^{4n}} < \frac{1}{s} < \varepsilon$$

and $\deg(pq_n) \leq n + kn$. Therefore we can write $pq_n(z) = \sum_{j=0}^{kn+n} \alpha_j (z-a)^j$. By the Cauchy formula, $|\alpha_j| \leq ||pq_n||_{K'} \, s^j \leq s^{n+j+2}$ for every $0 \leq j \leq kn+n$. Now, the polynomial

$$h(z) := \sum_{j=0}^{kn+n} \alpha_j \sum_{l=0}^{j} \binom{j}{l} \frac{z^{n+l}}{\gamma_{n+l}^n} (-a)^{j-l}$$

verifies

$$T^n h(z) = z^n \sum_{j=0}^{kn+n} \alpha_j \sum_{j=0}^{l} \binom{j}{l} z^l (-a)^{j-l} = z^n p(z) q_n(z),$$

hence $||T^n h - p||_K < \varepsilon$.

Furthermore,

$$\begin{aligned} ||h||_{B(0,R)} &\leq \sum_{j=0}^{kn+n} |\alpha_j| \sum_{l=0}^{j} \binom{j}{l} \frac{s^{n+l} s^{j-l}}{|\gamma_{n+l}|^n} \\ &\leq \frac{1}{\min\{|\gamma_m|^n : m \geq n\}} \sum_{j=0}^{kn+n} s^{2n+3j+2} \\ &\leq \frac{s^{2n+3kn+3n+3}}{s^{(3k+7)n}} \leq \left(\frac{s^{3k+6}}{s^{3k+7}} \right)^n = \frac{1}{s^n} < \varepsilon. \end{aligned}$$

Finally, let us prove (2): For every $n \in \mathbb{N}$, $n > \deg(p)$ and $|\gamma_m| > 3k+9 \quad (m \geq n)$, we have $T^n p(z) = \sum_{j=0}^{\deg(p)} \beta_j \gamma_j^n z^j$ and

$$||T^n p||_{K'} \leq \sum_{j=0}^{\deg(p)} |\beta_j| \cdot |\gamma_j|^n s^j \leq s^n s^{\deg(p)} \sum_{j=0}^{\deg(p)} |\beta_j| \leq s^{2n}.$$

Hence

$$\begin{aligned} ||T^n p - z^n (T^n p) q_n||_{K'} &\leq ||T^n p||_{K'} \, ||z^n||_{K'} \left|\left| \frac{1}{z^n} - q_n \right|\right|_{K'} \\ &\leq \frac{s^{2n} s^n}{s^{4n}} \leq \frac{1}{s^n} < \varepsilon. \end{aligned}$$

Now, let us write, $(T^n p)(z) q_n(z) = \sum_{j=0}^{kn+n} \alpha'_j (z-a)^j$. As above, for each $0 \leq j \leq kn+n$ we have $|\alpha'_j| \leq ||(T^n p)(z) q_n(z)||_{K'} \, s^j \leq s^{3n+j+1}$.

Let $g(z) := \sum_{j=0}^{kn+n} \alpha'_j \sum_{l=0}^{j} \binom{j}{l} \frac{z^{n+l}}{\gamma_{n+l}^n} (-a)^{j-l}$. Then g is a polynomial satisfying

$$T^n g(z) = z^n \sum_{j=0}^{kn+n} \alpha'_j \sum_{l=0}^{j} \binom{j}{l} z^l (-a)^{j-l} = z^n \sum_{j=0}^{kn+n} \alpha'_j (z-a)^j = z^n T^n p(z) q_n(z).$$

Hence $||T^n g - T^n p||_K < \varepsilon$. Furthermore,

$$\begin{aligned} ||g||_{B(0,R)} &\leq \sum_{j=0}^{kn+n} |\alpha'_j| \sum_{l=0}^{j} \binom{j}{l} \frac{s^{n+l} s^{j-l}}{|\gamma_{n+l}|^n} \\ &\leq \sum_{j=0}^{kn+n} s^{3n+j+1} \frac{2^j s^{n+j}}{\min\{|\gamma_m|^n : m \geq n\}} \\ &\leq \frac{1}{s^{(3k+9)n}} \sum_{j=0}^{kn+n} s^{4n+3j+1} \\ &\leq \frac{s^{3kn+7n+2}}{s^{(3k+9)n}} \leq \frac{1}{s^n} < \varepsilon. \end{aligned}$$

□

The following result should be compared with the universal criteria given in [BGGE03, BP99, CP05, BBP04].

Theorem 2.3. *Let X be a Baire locally convex space, and $T : X \to X$ a continuous linear mapping. Let Λ be a countable system of continuous seminorms on X, and let X_0 be a dense subset of X. Suppose that for all $y \in X_0, q \in \Lambda, \varepsilon > 0$, and each continuous seminorm p on X there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ there are $u, u' \in X$ with*

1. *$p(u) < \varepsilon$,*
2. *$p(u') < \varepsilon$,*
3. *$q(T^n u - y) < \varepsilon$, and*

(4) $q(T^n(u'+y)) < \varepsilon$.

Then there exists a residual set $A \subset X$ such that for every $a \in A, y \in X, q \in \Lambda$ and $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $q(T^n a - y) < \varepsilon$. That is, the sequence $\{T^n : X \to (X, \Lambda)\}_n$ is generically universal, see [GE99, Part 1, Def. 2].

Proof. Fix $q \in \Lambda$ and consider the normed space $X_q := X/\ker(q)$, with the norm $\tilde{q}$ induced by q. Consider the sequence $\hat{T}_n : X \to X_q$ induced by T^n.

Let $x, y \in X, \varepsilon > 0$ and let p be a continuous seminorm on X. Let $x_0, y_0 \in X_0$ satisfy $p(x - x_0) < \varepsilon/3$ and $q(y - y_0) < \varepsilon/3$. By assumptions, there exist $u, u' \in X$ and $n \in \mathbb{N}$ such that $p(u) < \varepsilon/3$, $p(u') < \varepsilon/3$, $q(T^n u - y_0) < \varepsilon/3$ and $q(T^n u' + T^n x_0) < \varepsilon/3$.

Set $x' = x_0 + u + u'$. Then

$$p(x' - x) \leq p(x_0 - x) + p(u) + p(u') < \varepsilon$$

and

$$q(T^n x' - y) \leq q(T^n x_0 + T^n u') + q(T^n u - y_0) + q(y_0 - y) < \varepsilon.$$

By [GE99, Part 1, Th. 1], the set A_q of universal elements for the sequence $\hat{T}_n$ is residual in X. Since the family Λ is countable, there is a residual set $A = \bigcap_{q \in \Lambda} A_q$ of elements verifying the conclusion. □

We apply the previous theorem to the operator $T(f) \mapsto (zf)'$.

Lemma 2.4. [Luh96] *There exists a sequence of nonempty simply connected sets $K_n \subset \mathbb{C}$ such that $0 \notin K_n$, $n \in \mathbb{N}$, and for any simply connected compact set $K \subset \mathbb{C}$ with $0 \notin K$ there exists some $n \in \mathbb{N}$ for which $K \subset K_n$.*

To see a simple argument, note that for each simply connected compact set $K \subset \mathbb{C}$ with $0 \notin K$ there exist $n, r \in \mathbb{N}$ and a finite sequence $a_1, \ldots, a_k$ of complex numbers with rational real and imaginary parts such that $K \subset \{z \in \mathbb{C} : n^{-1} < |z| < n\}$ and $\mathrm{d}\{K, \Gamma\} > r^{-1}$, where Γ is the piecewise linear curve connecting n^{-1} and n, with vertices $a_1, \ldots, a_k$. Hence $K \subset \{z : n^{-1} \leq |z| \leq n, \mathrm{d}\{z, \Gamma\} \geq r^{-1}\}$. Note that the sets on the right-hand side of the last formula form a countable system of simply connected compact sets not containing 0.

Finally, we obtain as a corollary the statement of Theorem 2 in [MM04].

Corollary 2.5. *There exists $f \in \mathcal{H}(\mathbb{C})$ with the following property: for every nonempty simply connected compact set $K \subset \mathbb{C}$ with $0 \notin K$, $g \in \mathcal{H}(\mathbb{C})$ and $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $||T^n f(z) - g(z)||_K < \varepsilon$.*

Proof. We just have to consider Theorem 2.3 with Z as the set of polynomials with rational coefficients, and $\{q_m\}_m = \{\| \cdot \|_{K_m}\}$. The conditions of Theorem 2.3 follow directly from Theorem 2.2. □

The former statement of Martirosian and Martirosyan was concerning the existence of such universal function with a prescribed lacunary power series. The following question remains open:

Question 2.6. For which subsets $A \subset \mathbb{N}$ is it possible to find $\varphi \in \mathcal{H}(\mathbb{C})$ of the form $\varphi(z) := \sum_{n=0; n\in A}^{\infty} a_n z^n$ verifying the statement of Corollary 2.5?

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