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SUFFICIENT CONDITIONS FOR BOUNDEDNESS OF THE RIESZ POTENTIAL IN LOCAL MORREY-TYPE SPACES

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Abstract. The problem of the boundedness of the Riesz potential I_α, $0 < \alpha < n$ in local Morrey-type spaces is reduced to the problem of the boundedness of the Hardy operator in weighted L_p-spaces on the cone of non-negative non-increasing functions. This allows obtaining sharp sufficient conditions for the boundedness for all admissible values of the parameters.

1. Introduction

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at x of radius r and ${}^{\complement}B(x,r)$ denote the set $\mathbb{R}^n \backslash B(x,r)$.

Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. The fractional maximal operator M_α and the Riesz potential I_α is defined by

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |f(y)| dy, \ 0 \le \alpha < n,$$

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \ 0 < \alpha < n,$$

where $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$.

The operators $M \equiv M_0$, M_α and I_α play an important role in real and harmonic analysis. (see, for example [9] and [10])

In the theory of partial differential equations, together with weighted $L_{p,w}$ spaces, Morrey spaces $\mathcal{M}_{p,\lambda}$ play an important role. They were introduced by C. Morrey in 1938 [12] and defined as follows: *For* $\lambda \ge 0$, $1 \le p \le \infty$, $f \in \mathcal{M}_{p,\lambda}$ *if*

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$f \in L_p^{\rm loc}(\mathbb{R}^n)$ *and*

$$\|f\|_{\mathcal{M}_{p,\lambda}} \equiv \|f\|_{\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\lambda/p} \|f\|_{L_p(B(x,r))} < \infty$$

holds .

These spaces appeared to be quite useful in the study of local behavior of the solutions of elliptic partial differential equations.

Also by $W\mathcal{M}_{p,\lambda}$ we denote the weak Morrey space of all functions $f \in {\rm WL}_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{W\mathcal{M}_{p,\lambda}} \equiv \|f\|_{W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\lambda/p} \|f\|_{WL_p(B(x,r))} < \infty,$$

where ${\rm WL}_p$ denotes the weak L_p-space.

The classical result by Hardy-Littlewood-Sobolev states that if $1 < p_1 < p_2 < \infty$, then I_α is bounded from $L_{p_1}(\mathbb{R}^n)$ to $L_{p_2}(\mathbb{R}^n)$ if and only if $\alpha = n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$ and for $p_1 = 1 < p_2 < \infty$, I_α is bounded from $L_1(\mathbb{R}^n)$ to $WL_{p_2}(\mathbb{R}^n)$ if and only if $\alpha = n\left(1 - \frac{1}{p_2}\right)$. D.R. Adams [1] studied the boundedness of the Riesz potential in Morrey spaces and proved the following statement.

Theorem 1.1. *Let $1 < p_1 < p_2 < \infty$. Then I_α is bounded from $\mathcal{M}_{p_1,\lambda}$ to $\mathcal{M}_{p_2,\lambda}$ if and only if*

$$0 < \alpha \le n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \text{ and } \lambda = \left(n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) - \alpha\right)\left(\frac{1}{p_1} - \frac{1}{p_2}\right)^{-1} \qquad (1.1)$$

If $\alpha = n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$, then $\lambda = 0$ and the statement of Theorem 1.1 reduces to the above mentioned result by Hardy-Littlewood-Sobolev.

Recall that, for $0 < \alpha < n$,

$$M_\alpha f(x) \le \upsilon_n^{\frac{\alpha}{n}-1} I_\alpha(|f|)(x), \qquad (1.2)$$

hence Theorem 1.1 also implies the boundedness of the fractional maximal operator M_α. F. Chiarenza and M. Frasca [8] proved that the maximal operator M is also bounded from $\mathcal{M}_{p,\lambda}$ to $\mathcal{M}_{p,\lambda}$ for all $1 < p < \infty$ and $0 < \lambda < n$.

If in the place of the power function $r^{-\lambda/p}$ in the definition of $\mathcal{M}_{p,\lambda}$ we consider any positive weight function w defined on $(0,\infty)$, then it becomes the Morrey-type space $\mathcal{M}_{p,w}$. T. Mizuhara [11] and E. Nakai [13] generalized Theorem 1.1 and obtained sufficient conditions on a weights w_1 and w_2 ensuring the boundedness of the Riesz potential I_α where $\alpha = n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$ from $\mathcal{M}_{p_1,w_1}$ to $\mathcal{M}_{p_2,w_2}$. In [13] the following statement, containing the result from [11], was proved.

Theorem 1.2. *Let $1 \le p_1 < p_2 < \infty$ and $\alpha = n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$. Moreover, let w be a positive function satisfying the following conditions: there exists $c_1 > 0$ such that*

$$0 < r \le t \le 2r \Rightarrow c_1^{-1} w(t) \le w(r) \le c_1 w(t) \qquad (1.3)$$

and there exists $c_2 > 0$ such that for all $r > 0$.

$$\left\| w^{-1}(t) t^{\alpha - \frac{n+1}{p_1}} \right\|_{L_{p_1}(r,\infty)} \le c_2 w^{-1}(r) r^{\alpha - \frac{n}{p-1}}. \tag{1.4}$$

Then for $p_1 > 1$ I_α is bounded from $\mathcal{M}_{p_1,w}$ to $\mathcal{M}_{p_2,w}$ and for $p = 1$ I_α is bounded from $\mathcal{M}_{1,w}$ to $W\mathcal{M}_{p_2,w}$.

In [5] V.I.Burenkov, V.S.Guliyev considered general local and global Morrey-type spaces $LM_{p_1,\theta_1,\omega_1}$ and studied the boundedness of the Riesz potential operator I_α from $LM_{p_1\theta_1,w_1}$ to $LM_{p_2\theta_2,w_2}$ for all admissible values of α. Moreover, for some values of the parameters necessary and sufficient conditions for the operator I_α to be bounded from $LM_{p_1\theta_1,w_1}$ to $LM_{p_2\theta_2,w_2}$ were obtained.

2. Definitions and basic properties of Morrey-type spaces

Definition 2.1. Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0, \infty)$. We denote by $LM_{p_1,\theta_1,\omega_1}$, $GM_{p,\theta,\omega}$, the local Morrey-type spaces, the global Morrey-type spaces respectively, the spaces of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorms

$$\|f\|_{LM_{p_1,\theta_1,\omega_1}} \equiv \|f\|_{LM_{p_1,\theta_1,\omega_1}(\mathbb{R}^n)} = \left\| w(r) \|f\|_{L_p(B(0,r))} \right\|_{L_\theta(0,\infty)},$$

$$\|f\|_{GM_{p,\theta,\omega}} = \sup_{x \in \mathbb{R}^n} \|f(x + \cdot)\|_{LM_{p_1,\theta_1,\omega_1}}$$

respectively.

Note that

$$\|f\|_{LM_{p\infty,1}} = \|f\|_{GM_{p\infty,1}} = \|f\|_{L_p}.$$

Furthermore, $GM_{p\infty,r^{-\lambda/p}} \equiv \mathcal{M}_{p,\lambda}$, $0 < \lambda < n$. The interpolation properties of the spaces $GM_{p\infty,w}$ were studied by S. Spanne in [16]. The spaces $GM_{p\theta,r^{-\lambda}}$ were used by G. Lu [15] for studying the embedding theorems for vector fields of Hörmander type. The boundedness of various integral operators in the spaces $GM_{p\infty,w}$ was studied by T. Mizuhara [11] and E. Nakai [13]. In [6, 7] the boundedness of the maximal operator M from $LM_{p_1\theta_1,w_1}$ to $LM_{p_2\theta_2,w_2}$ and from $GM_{p_1\theta_1,w_1}$ to $GM_{p_2\theta_2,w_2}$ was investigated.

In [7] the following statement was proved.

Lemma 2.2. *Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0, \infty)$.*

1. If for all $t > 0$

$$\|w(r)\|_{L_\theta(t,\infty)} = \infty, \tag{2.1}$$

then $LM_{p_1,\theta_1,\omega_1} = GM_{p,\theta,\omega} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

2. If for all $t > 0$

$$\|w(r) r^{n/p}\|_{L_\theta(0,t)} = \infty, \tag{2.2}$$

then, for all functions $f \in LM_{p_1,\theta_1,\omega_1}$, continuous at 0, $f(0) = 0$, and for $0 < p < \infty$ $GM_{p,\theta,\omega} = \Theta$.

Definition 2.3. Let $0 < p, \theta \le \infty$. We denote by Ω_θ the set of all functions w which are non-negative, measurable on $(0,\infty)$, not equivalent to 0 and such that for some $t > 0$

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty. \tag{2.3}$$

Moreover, we denote by $\Omega_{p,\theta}$ the set of all functions w which are non-negative, measurable on $(0,\infty)$, not equivalent to 0 and such that for some $t_1, t_2 > 0$

$$\|w(r)\|_{L_\theta(t_1,\infty)} < \infty, \qquad \|w(r)r^{n/p}\|_{L_\theta(0,t_2)} < \infty. \tag{2.4}$$

In the sequel, keeping in mind Lemma 2.2, we always assume that either $w \in \Omega_\theta$ or $w \in \Omega_{p,\theta}$.

In [5] the following statements were proved.

Lemma 2.4. *Let $1 < p_1 \le \infty$, $0 < p_2 \le \infty$, $0 < \alpha < n$, $0 < \theta_1, \theta_2 \le \infty$, $\omega_1 \in \Omega_{\theta_1}$, and $\omega_2 \in \Omega_{\theta_2}$. Then the condition*

$$\alpha < \frac{n}{p_1}$$

is necessary for the boundedness of I_α from $LM_{p_1,\theta_1,\omega_1}$ to $LM_{p_2,\theta_2,\omega_2}$.

Lemma 2.5. *Let $1 \le p_1 \le \infty$, $0 < p_2 \le \infty$, $0 < \alpha < n$, $0 < \theta_1, \theta_2 \le \infty$, $\omega_1 \in \Omega_{\theta_1}$, and $\omega_2 \in \Omega_{\theta_2}$. Moreover, let $\omega_1 \in L_{\theta_1}(0,\infty)$. Then the condition* [1]

$$\alpha \ge n\left(\frac{n}{p_1} - \frac{n}{p_2}\right)_+ \tag{2.5}$$

is necessary for the boundedness of I_α from $LM_{p_1,\theta_1,\omega_1}$ to $LM_{p_2,\theta_2,\omega_2}$.

Remark 2.6. If $\omega_1 \in \Omega_{\theta_1}$ but $\omega_1 \notin L_{\theta_1}(0,\infty)$, then condition (2.5) is not necessary for the boundedness of I_α from $LM_{p_1,\theta_1,\omega_1}$ to $LM_{p_2,\theta_2,\omega_2}$.

Throughout this paper $a \lesssim b$, $(b \gtrsim a)$, means that $a \le \lambda b$, where $\lambda > 0$ depends on inessential parameters. If $b \lesssim a \lesssim b$, then we write $a \approx b$.

3. L_p-estimates over balls

Our aim is to obtain the following inequality

$$\|I_\alpha f\|_{LM_{p_2,\theta_2,\omega_2}} \lesssim \|f\|_{LM_{p_1,\theta_1,\omega_1}}.$$

In order to obtain conditions on ω_1 and ω_2 ensuring the boundedness of I_α we shall reduce the problem of the boundedness of I_α in the local Morrey-type spaces to the problem of the boundedness of the Hardy operator in weighted L_p-spaces on the cone of non-negative monotone functions.

Let $1 < p < \infty$, $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$. For any $r > 0$ we have

$$\|I_\alpha f\|_{L_{p_2}(B(0,r))} \le \|I_\alpha(f\chi_{B(0,2r)})\|_{L_{p_2}(B(0,r))} + \|I_\alpha(f\chi_{\complement_{B(0,2r)}})\|_{L_{p_2}(B(0,r))} \tag{3.1}$$

[1] Here and in the sequel $t_+ = t$ if $t \ge 0$ and $t_+ = 0$ if $t < 0$ and $t_- = -t$ if $t \le 0$ and $t_- = 0$ if $t > 0$.

$$\text{If } |x| \le r, |y| \ge 2r, \text{ then } |y|/2 \le |x-y| \le 3|y|/2. \tag{3.2}$$

Therefore

$$\|I_\alpha(f\chi_{\complement_{B(0,2r)}})\|_{L_{p_2}(B(0,r))} = \left(\int_{B(0,r)} \left|\int_{\complement_{B(0,2r)}} \frac{f(y)}{|x-y|^{n-\alpha}} dy\right|^{p_2} dx\right)^{\frac{1}{p_2}}$$
$$\le c r^{\frac{n}{p_2}} \int_{\mathbb{R}^n \setminus B(0,2r)} \frac{|f(y)|}{|y|^{n-\alpha}} dy \tag{3.3}$$

Let us estimate $\|I_\alpha(f\chi_{B(0,2r)})\|_{L_{p_2}(B(0,r))}$. The next lemma is true

Lemma 3.1. *Let* $0<\alpha<n$, $0<p_2<\infty$. *Moreover, let* $1<\frac{p_2 n}{n+\alpha p_2} \le p_1 < \infty$, *or* $\frac{p_2 n}{n+\alpha p_2} < 1 \le p_1 < \infty$, *or* $\frac{p_2 n}{n+\alpha p_2} = 1 < p_1 < \infty$. *Then*

$$\|I_\alpha(f\chi_{B(0,2r)})\|_{L_{p_2}(B(0,r))} \lesssim r^{\alpha - n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \|f\|_{L_{p_1}(B(0,2r))}. \tag{3.4}$$

Proof. Suppose that $1<\frac{p_2 n}{n+\alpha p_2} \le p_1 < \infty$. Then by Sobolev's theorem we have

$$\|I_\alpha(f\chi_{B(0,2r)})\|_{L_{p_2}(B(0,r))} \lesssim \|f\|_{L_{\frac{p_2 n}{n+\alpha p_2}}(B(0,2r))}.$$

If $\frac{p_2 n}{n+\alpha p_2} = p_1$, then we arrive at (3.4). If $p_1 > \frac{p_2 n}{n+\alpha p_2}$, then applying Hölder's inequality (with exponents $\frac{p_1(n+\alpha p_2)}{p_2 n}$ and $(\frac{p_1(n+\alpha p_2)}{p_2 n})'$) we get (3.4).

Assume that $\frac{p_2 n}{n+\alpha p_2} < 1 \le p_1 < \infty$. Since

$$\int_{B(0,r)} \left(I_\alpha(f\chi_{B(0,2r)})(x)\right)^{p_2} dx = \int_0^{|B(0,r)|} \left[\left(I_\alpha(f\chi_{B(0,2r)})\right)^*(t)\right]^{p_2} dt$$
$$\le \left[\sup_{0<t<|B(0,r)|} t^{\frac{n-\alpha}{n}} \left(I_\alpha(f\chi_{B(0,2r)})\right)^*(t)\right]^{p_2} \int_0^{|B(0,r)|} t^{\frac{\alpha-n}{n}p_2} dt \tag{3.5}$$

Using the boundedness of I_α from $L_1(\mathbb{R}^n)$ to $WL_{\frac{n}{n-\alpha}}(\mathbb{R}^n)$ we have

$$\int_{B(0,r)} \left(I_\alpha(f\chi_{B(0,2r)})(x)\right)^{p_2} dx \lesssim \|f\|^{p_2}_{L_1(B(0,2r))} |B(0,r)|^{\frac{\alpha-n}{n}p_2+1}. \tag{3.6}$$

Therefore

$$\|I_\alpha(f\chi_{B(0,2r)})\|_{L_{p_2}(B(0,r))} \lesssim r^{\alpha-n+\frac{1}{p_2}} \|f\|_{L_1(B(0,2r))}. \tag{3.7}$$

If $p_1 = 1$, then we arrive at (3.4). If $p_1 > 1$, then applying Hölder's inequality (with exponents p_1 and $p_1{}'$) we get (3.4).

Suppose that $\frac{p_2 n}{n+\alpha p_2} = 1 < p_1 < \infty$. Let $p_0 > p_1$ be defined by $n\left(\frac{1}{p_1} - \frac{1}{p_0}\right) = \alpha$. Then by Hölder's inequality (with exponents $\frac{p_0}{p_2}$ and $(\frac{p_0}{p_2})'$) we have

$$\|I_\alpha(f\chi_{B(0,2r)})\|_{L_{p_2}(B(0,r))} \lesssim r^{\frac{1}{p_2}-\frac{1}{p_0}} \|I_\alpha(f\chi_{B(0,2r)})\|_{L_{p_0}(B(0,r))}. \tag{3.8}$$

Then by Sobolev's theorem we arrive at (3.4). □

The statement of the next lemma follows from (3.1), (3.3) and Lemma 3.1.

Lemma 3.2. *Let* $0 < \alpha < n$, $0 < p_2 < \infty$. *Moreover, let* $1 < \frac{p_2 n}{n+\alpha p_2} \le p_1 < \infty$, *or* $\frac{p_2 n}{n+\alpha p_2} < 1 \le p_1 < \infty$, *or* $\frac{p_2 n}{n+\alpha p_2} = 1 < p_1 < \infty$. *Then*

$$\|I_\alpha f\|_{L_{p_2}(B(0,r))} \le c r^{\frac{n}{p_2}} \int_{\mathbb{R}^n \setminus B(0,2r)} \frac{|f(y)|}{|y|^{n-\alpha}} dy + c r^{\alpha - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)} \|f\|_{L_{p_1}(B(0,2r))}, \quad (3.9)$$

where constant c *does not depend on* r.

The next lemma is true.

Lemma 3.3. *Let* $0 < \alpha < n$, $0 < p_2 < \infty$. *Moreover, let* $1 < \frac{p_2 n}{n+\alpha p_2} \le p_1 < \infty$, *or* $\frac{p_2 n}{n+\alpha p_2} < 1 \le p_1 < \infty$, *or* $\frac{p_2 n}{n+\alpha p_2} = 1 < p_1 < \infty$. *Then*

$$\|I_\alpha f\|_{L_{p_2}(B(0,r))} \le c r^{\frac{n}{p_2}} \int_r^\infty \left(\int_{B(0,t)} |f(x)|^{p_1} dx \right)^{\frac{1}{p_1}} \frac{dt}{t^{\frac{n}{p_1} - \alpha + 1}}, \quad (3.10)$$

where constant c *does not depend on* r.

Proof. Denote by

$$I_1 := r^{\frac{n}{p_2}} \int_{\mathbb{R}^n \setminus B(0,2r)} \frac{|f(y)|}{|y|^{n-\alpha}} dy \text{ and } I_2 := r^{\alpha - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)} \|f\|_{L_{p_1}(B(0,2r))}.$$

Let estimate I_1. By Fubini's theorem we have

$$\begin{aligned} I_1 &= c r^{\frac{n}{p_2}} \int_{\mathbb{R}^n \setminus B(0,2r)} |f(y)| \int_{|y|}^\infty \frac{dt}{t^{n-\alpha+1}} dy \\ &= c r^{\frac{n}{p_2}} \int_{2r}^\infty \left(\int_{2r \le |x| \le t} |f(x)| dx \right) \frac{dt}{t^{n-\alpha+1}} \\ &\le c r^{\frac{n}{p_2}} \int_{2r}^\infty \int_{B(0,t)} |f(x)| dx \frac{dt}{t^{n-\alpha+1}}. \end{aligned}$$

Applying Hölder's inequality

$$I_1 \le c r^{\frac{n}{p_2}} \int_{2r}^\infty \left(\int_{B(0,t)} |f(x)|^{p_1} dx \right)^{\frac{1}{p_1}} \frac{dt}{t^{\frac{n}{p_1} - \alpha + 1}} \quad (3.11)$$

In the other hand

$$\begin{aligned} &\int_{2r}^\infty \left(\int_{B(0,t)} |f(x)|^{p_1} dx \right)^{\frac{1}{p_1}} \frac{dt}{t^{\frac{n}{p_1} - \alpha + 1}} \\ &\quad \ge \left(\int_{B(0,2r)} |f(x)|^{p_1} dx \right)^{\frac{1}{p_1}} \int_{2r}^\infty \frac{dt}{t^{\frac{n}{p_1} - \alpha + 1}} \\ &\quad = c r^{\alpha - \frac{n}{p_1}} \left(\int_{B(0,2r)} |f(x)|^{p_1} dx \right)^{\frac{1}{p_1}}. \end{aligned}$$

Then

$$I_2 \le c r^{\frac{n}{p_2}} \int_{2r}^{\infty} \left(\int_{B(0,t)} |f(x)|^{p_1} dx \right)^{\frac{1}{p_1}} \frac{dt}{t^{\frac{n}{p_1} - \alpha + 1}} \tag{3.12}$$

The statement of the lemma follows from (3.11) and (3.12). □

Remark 3.4. Note that inequality (36) in [5]

$$\|I_\alpha f\|_{L_{p_2}(B(0,r))} \le c r^{\frac{n}{p_2} - \delta} \left(\int_r^{\infty} \left(\int_{B(0,t)} |f(x)|^{p_1} dx \right) \frac{dt}{t^{n - (\alpha+\delta) p_1 + 1}} \right)^{\frac{1}{p!}}$$

follows from the inequality (3.10) by applying Hölder's inequality.

Proof. For any $\delta > 0$

$$\begin{aligned} \|I_\alpha f\|_{L_{p_2}(B(0,r))} &\le c r^{\frac{n}{p_2}} \int_r^{\infty} \left(\int_{B(0,t)} |f(x)|^{p_1} dx \right)^{\frac{1}{p_1}} \frac{dt}{t^{\frac{n}{p_1} - \alpha + 1}} \\ &= c r^{\frac{n}{p_2}} \int_r^{\infty} \left(\int_{B(0,t)} |f(x)|^{p_1} dx \right)^{\frac{1}{p_1}} \frac{dt}{t^{\frac{n}{p_1} - (\alpha+\delta) + \frac{1}{p_1} + \delta + \frac{1}{p_1{}'}}}. \end{aligned}$$

By applying Hölder's inequality

$$\begin{aligned} &\|I_\alpha f\|_{L_{p_2}(B(0,r))} \\ &\quad \le c r^{\frac{n}{p_2}} \left(\int_r^{\infty} \left(\int_{B(0,t)} |f(x)|^{p_1} dx \right) \frac{dt}{t^{n-(\alpha+\delta)p_1+1}} \right)^{\frac{1}{p_1}} \left(\int_r^{\infty} \frac{dt}{t^{p_1{}'\delta+1}} \right)^{\frac{1}{p_1{}'}} \\ &\quad \le c r^{\frac{n}{p_2} - \delta} \left(\int_r^{\infty} \left(\int_{B(0,t)} |f(x)|^{p_1} dx \right) \frac{dt}{t^{n-(\alpha+\delta)p_1+1}} \right)^{\frac{1}{p_1}}. \end{aligned}$$

□

Lemma 3.5. $0 < p_2 < \infty$, $0 < \alpha < n$ *and* $f \in L_1^{loc}(\mathbb{R}^n)$. *Then the next inequality holds*

$$\|I_\alpha |f|\|_{L_{p_2}(B(0,r))} \gtrsim r^{\frac{n}{p_2}} \int_r^{\infty} \int_{B(0,t)} |f(x)| dx \frac{dt}{t^{n-\alpha+1}}, \tag{3.13}$$

where the constant c does not depend on r.

Proof. It easy to see that

$$\|I_\alpha |f|\|_{L_{p_2}(B(0,r))} \approx \|I_\alpha(|f| \chi_{B(0,2r)})\|_{L_{p_2}(B(0,r))} + \|I_\alpha(|f| \chi_{\complement_{B(0,2r)}})\|_{L_{p_2}(B(0,r))} \tag{3.14}$$

Taking into account (3.2), and then, applying Fubini's theorem, we have

$$\begin{aligned} \|I_\alpha(|f| \chi_{\complement_{B(0,2r)}})\|_{L_{p_2}(B(0,r))} &\approx r^{\frac{n}{p_2}} \int_{\mathbb{R}^n \setminus B(0,2r)} \frac{|f(y)|}{|y|^{n-\alpha}} dy \\ &\approx r^{\frac{n}{p_2}} \int_{\mathbb{R}^n \setminus B(0,2r)} |f(y)| \int_{|y|}^{\infty} \frac{dt}{t^{n-\alpha+1}} dy \end{aligned}$$

$$\approx r^{\frac{n}{p_2}} \int_{2r}^{\infty} \int_{B(0,t)\setminus B(0,2r)} |f(x)|dx \frac{dt}{t^{n-\alpha+1}}. \tag{3.15}$$

In the other hand the next inequality is true for all $x \in B(0,r)$

$$(I_\alpha |f|\chi_{B(0,2r)})(x) = \int_{B(0,2r)} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \gtrsim r^{\alpha-n} \int_{B(0,2r)} |f(y)|dy.$$

Then

$$\|I_\alpha(|f|\chi_{B(0,2r)})\|_{L_{p_2}(B(0,r))} \gtrsim r^{\alpha-n+\frac{n}{p_2}} \int_{B(0,2r)} |f(y)|dy$$

$$\approx r^{\frac{n}{p_2}} \int_{2r}^{\infty} \int_{B(0,2r)} |f(y)|dy \frac{dt}{t^{n-\alpha+1}}. \tag{3.16}$$

From (3.14), (3.15) and (3.16) we get the next inequality

$$\|I_\alpha|f|\|_{L_{p_2}(B(0,r))} \gtrsim r^{\frac{n}{p_2}} \int_{2r}^{\infty} \int_{B(0,t)} |f(x)|dx \frac{dt}{t^{n-\alpha+1}}$$

$$\approx r^{\frac{n}{p_2}} \int_{r}^{\infty} \int_{B(0,t)} |f(x)|dx \frac{dt}{t^{n-\alpha+1}}. \tag{3.17}$$

□

Theorem 3.6. *Let $0 < \alpha < n$, $0 < p_2 < \infty$ and $\frac{p_2 n}{n+\alpha p_2} < 1$. Then*

$$\|I_\alpha|f|\|_{L_{p_2}(B(0,r))} \approx r^{\frac{n}{p_2}} \int_{\mathbb{R}^n\setminus B(0,r)} \frac{|f(y)|}{|y|^{n-\alpha}} dy + r^{\alpha-n\left(1-\frac{1}{p_2}\right)} \int_{B(0,r)} |f(y)|dy. \tag{3.18}$$

Proof. The statement of the Theorem follows from Lemma 3.1 and Lemma 3.5. □

4. Riesz potential and Hardy operator

Let H be the Hardy operator

$$(Hg)(t) := \frac{1}{t}\int_0^t g(r)dr,\ 0 < t < \infty,$$

Lemma 4.1. *Let $0 < \alpha < n$, $0 < p_2 < \infty$, $0 < \theta_2 \le \infty$ and $\omega_2 \in \Omega_{\theta_2}$. Moreover, let $1 < \frac{p_2 n}{n+\alpha p_2} \le p_1 < \infty$, or $\frac{p_2 n}{n+\alpha p_2} < 1 \le p_1 < \infty$, or $\frac{p_2 n}{n+\alpha p_2} = 1 < p_1 < \infty$.*
Then

$$\|I_\alpha f\|_{LM_{p_2,\theta_2,\omega_2}} \lesssim \|Hg\|_{L_{\theta_2,\upsilon_2}(0,\infty)} \tag{4.1}$$

for all $f \in L_{p_l}^{loc}$, where

$$g(t) = \left(\int_{B(0,t^{-\frac{1}{\sigma}})} |f(y)|^{p_1} dy\right)^{\frac{1}{p_1}},\ \sigma = \frac{n}{p_1} - \alpha \tag{4.2}$$

and

$$\upsilon_2(r) = \omega_2^{\theta_2}(r^{-\frac{1}{\sigma}}) r^{\theta_2\left(1-\frac{n}{\sigma p_2}\right)-\frac{1}{\sigma}-1}. \tag{4.3}$$

Proof. By Lemma 3.3 we have

$$\begin{aligned}
\|I_\alpha f\|_{LM_{p_2,\theta_2,\omega_2}} &\lesssim \left\| \omega_2(r) r^{\frac{n}{p_2}} \int_r^\infty \left(\int_{B(0,t)} |f(x)|^{p_1} dx \right)^{\frac{1}{p_1}} \frac{dt}{t^{\sigma+1}} \right\|_{L_{\theta_2(0,\infty)}} \\
&\approx \left\| \omega_2(r) r^{\frac{n}{p_2}} \int_0^{r^{-\sigma}} \left(\int_{B(0,\tau^{-\frac{1}{\sigma}})} |f(x)|^{p_1} dx \right)^{\frac{1}{p_1}} d\tau \right\|_{L_{\theta_2(0,\infty)}} \\
&= \left\| \omega_2(r) r^{\frac{n}{p_2}} \int_0^{r^{-\sigma}} g(\tau) d\tau \right\|_{L_{\theta_2(0,\infty)}} \\
&= \left(\int_0^\infty \left(\omega_2(r) r^{\frac{n}{p_2}} \right)^{\theta_2} \left(\int_0^{r^{-\sigma}} g(\tau) d\tau \right)^{\theta_2} dr \right)^{\frac{1}{\theta_2}} \\
&\approx \left(\int_0^\infty \left(\omega_2(\rho^{-\frac{1}{\sigma}}) \rho^{-\frac{1}{\sigma}\cdot\frac{n}{p_2}+1} \right)^{\theta_2} \rho^{-\frac{1}{\sigma}-1} \left(\frac{1}{\rho} \int_0^\rho g(\tau) d\tau \right)^{\theta_2} dr \right)^{\frac{1}{\theta_2}} \\
&= \|Hg\|_{L_{\theta_2,\upsilon_2}(0,\infty)}. \qquad (4.4)
\end{aligned}$$

□

Theorem 4.2. *Let* $0 < \alpha < n$, $0 < p_2 < \infty$, $0 < \theta_1, \theta_2 \leq \infty$, $\omega_1 \in \Omega_{\theta_1}$ *and* $\omega_2 \in \Omega_{\theta_2}$. *Moreover, let* $1 < \frac{p_2 n}{n+\alpha p_2} \leq p_1 < \infty$, *or* $\frac{p_2 n}{n+\alpha p_2} < 1 \leq p_1 < \infty$, *or* $\frac{p_2 n}{n+\alpha p_2} = 1 < p_1 < \infty$.

Assume that the operator H *is bounded from* $L_{\theta_1,\upsilon_1}(0,\infty)$ *to* $L_{\theta_2,\upsilon_2}(0,\infty)$ *on the cone of all non-negative and non-increasing functions on* $(0,\infty)$, *that is,*

$$\|Hg\|_{L_{\theta_2,\upsilon_2}(0,\infty)} \lesssim \|g\|_{L_{\theta_1,\upsilon_1}(0,\infty)}, \qquad (4.5)$$

where

$$\upsilon_1(r) = \omega_1^{\theta_1}(r^{-\frac{1}{\sigma}}) r^{-\frac{1}{\sigma}-1}, \qquad (4.6)$$

$$\upsilon_2(r) = \omega_2^{\theta_2}(r^{-\frac{1}{\sigma}}) r^{\theta_2\left(1-\frac{n}{\sigma p_2}\right)-\frac{1}{\sigma}-1}. \qquad (4.7)$$

Then I_α *is bounded from* $LM_{p_1,\theta_1,\omega_1}$ *to* $LM_{p_2,\theta_2,\omega_2}$.

Proof. Since g is non-negative and non-increasing on $(0,\infty)$ and H is bounded from $L_{\theta_1,\upsilon_1}(0,\infty)$ to $L_{\theta_2,\upsilon_2}(0,\infty)$ on the cone of functions containing g, by Lemma 4.1 we have

$$\|I_\alpha f\|_{LM_{p_2,\theta_2,\omega_2}} \lesssim \|g\|_{L_{\theta_1,\upsilon_1}(0,\infty)}.$$

Hence

$$\begin{aligned}
\|I_\alpha f\|_{LM_{p_2,\theta_2,\omega_2}} &\lesssim \left(\int_0^\infty \upsilon_1(r) \|f\|^{\theta_1}_{L_p(B(0,r^{-\frac{1}{\sigma}}))} dr \right)^{\frac{1}{\theta_1}} \\
&\approx \left(\int_0^\infty \omega_1^{\theta_1}(r^{-\frac{1}{\sigma}}) r^{-\frac{1}{\sigma}-1} \|f\|^{\theta_1}_{L_p(B(0,^{-\frac{1}{\sigma}}))} dr \right)^{\frac{1}{\theta_1}}
\end{aligned}$$

$$\approx \left(\int_0^\infty \omega_1^{\theta_1}(r)\|f\|_{L_p(B(0,r))}^{\theta_1} dr\right)^{\frac{1}{\theta_1}}$$

$$= \|f\|_{LM_{p_1,\theta_1,\omega_1}}.$$

□

5. Two-weighted Hardy inequalities for non-increasing functions

In order to obtain sufficient conditions on the weight functions ensuring the boundedness of I_α, we shall apply the following Theorem ensuring the boundedness of the Hardy operator H from one weighted Lebesgue space to another one (see [3] and [4]).

Theorem 5.1. *Let p, $q \in (0,\infty]$ and let v, w be weights. Denote by*

$$V(t) := \int_0^t v(s)ds, \;\; W(t) := \int_0^t w(s)ds, \;\; \frac{1}{r} = \frac{1}{q} - \frac{1}{p}.$$

(i) Let $1 < p \le q < \infty$. Then the inequality

$$\|Hg\|_{L_{q,w}(0,\infty)} \le c\|g\|_{L_{p,v}(0,\infty)} \tag{5.1}$$

holds for all non-negative and non-increasing g on $(0,\infty)$ if and only if

$$A_1^1 := \sup_{t>0} W^{\frac{1}{q}}(t) V^{-\frac{1}{p}}(t) < \infty \tag{5.2}$$

and

$$A_2^1 := \sup_{t>0} \left(\int_t^\infty \frac{w(s)}{s^q}\right)^{\frac{1}{q}} \left(\int_0^t \frac{v(s)s^{p'}}{V^{p'}(s)} ds\right)^{\frac{1}{p'}} < \infty, \tag{5.3}$$

and the best constant c in (5.1) satisfies $c \approx A_1^1 + A_2^1$.

(ii) Let $0 < p \le 1$, $0 < p \le q < \infty$. Then (5.1) holds if and only if $A_1^1 < \infty$ and

$$A_1^2 := \sup_{t>0} t \left(\int_t^\infty \frac{w(s)}{s^q} ds\right)^{\frac{1}{q}} V^{-\frac{1}{p}}(t) < \infty, \tag{5.4}$$

$c \approx A_1^1 + A_1^2$.

(iii) Let $1 < p < \infty$, $0 < q < p < \infty$, $q \ne 1$. Then the inequality (5.1) holds if and only if

$$A_1^3 := \left(\int_0^\infty \left(\frac{W(t)}{V(t)}\right)^{\frac{r}{p}} w(t)dt\right)^{\frac{1}{r}}$$

$$= \left(\frac{q}{r}\frac{W^{\frac{r}{q}}(\infty)}{V^{\frac{r}{q}}(\infty)} + \frac{q}{p}\int_0^\infty \left(\frac{W(t)}{V(t)}\right)^{\frac{r}{p}} v(t)dt\right)^{\frac{1}{r}} < \infty \tag{5.5}$$

and

$$A_2^3 := \left(\int_0^\infty \left[\left(\int_t^\infty \frac{w(s)}{s^q}ds\right)^{\frac{1}{q}} \left(\int_0^t \frac{v(s)s^{p'}}{V^{p'}(s)}ds\right)^{\frac{q-1}{q}}\right]^r \frac{v(t)t^{p'}}{V^{p'}(t)}dt\right)^{\frac{1}{r}}$$

$$\approx \left(\int_0^\infty \left[\left(\int_t^\infty \frac{w(s)}{s^q}ds\right)^{\frac{1}{p}} \left(\int_0^t \frac{v(s)s^{p'}}{V^{p'}(s)}ds\right)^{\frac{1}{p'}}\right]^r \frac{w(t)}{t^q}dt\right)^{\frac{1}{r}} < \infty, \tag{5.6}$$

and $c \approx A_1^3 + A_2^3$.

(iv) Let $1 = q < p < \infty$*. Then* (5.1) *holds if and only if* $A_1^3 < \infty$ *and*

$$A_2^4 := \left(\int_0^\infty \left(\frac{W(t) + t\int_t^\infty \frac{w(s)}{s}ds}{V(t)}\right)^{p'-1} \int_t^\infty \frac{w(s)}{s}dsdt\right)^{\frac{1}{p'}}$$

$$\approx \frac{W(\infty)}{V^{\frac{1}{p}}(\infty)} + \left(\int_0^\infty \left(\frac{W(t) + t\int_t^\infty \frac{w(s)}{s}ds}{V(t)}\right)^{p'} v(t)dt\right)^{\frac{1}{p'}} < \infty, \tag{5.7}$$

and $c \approx A_1^3 + A_2^4$.

(v) Let $0 < q < p = 1$*. Then* (5.1) *holds if and only if* $A_1^3 < \infty$ *and*

$$A_2^5 := \left(\int_0^\infty \left(\int_t^\infty \frac{w(s)}{s^q}ds\right)^{\frac{q}{1-q}} \left(\operatorname*{ess\,inf}_{0<s<t} \frac{V(s)}{s}\right)^{\frac{q}{q-1}} \frac{w(t)}{t^q}dt\right)^{\frac{1-q}{q}} < \infty, \tag{5.8}$$

and $c \approx A_1^3 + A_2^5$.

(vi) Let $0 < q < p < 1$*. Then* (5.1) *holds if and only if* $A_1^3 < \infty$ *and*

$$A_2^6 := \left(\int_0^\infty \sup_{0<s\le t} \frac{s^r}{V(s)^{\frac{r}{p}}} \left(\int_t^\infty \frac{w(s)}{s^q}ds\right)^{\frac{r}{p}} \frac{w(t)}{t^q}dt\right)^{\frac{1}{r}} < \infty, \tag{5.9}$$

and $c \approx A_1^6 + A_2^6$.

6. Sufficient conditions

From Theorem 5.1 follows the next statement

Corollary 6.1. *Let* $0 < \theta_1, \theta_2 < \infty$ *and weight functions* υ_1, υ_2 *are determined by* (4.6) *and* (4.7).

(a) Let $1 < \theta_1 \le \theta_2 < \infty$*. Then the inequality* (4.5) *holds if and only if*

$$B_1^1 := \sup_{t>0} \left(\int_t^\infty \omega_2^{\theta_2}(r) r^{\theta_2\left(\alpha + \frac{n}{p_2} - \frac{n}{p_1}\right)} dr\right)^{\frac{1}{\theta_2}} \left(\int_t^\infty \omega_1^{\theta_1}(r)dr\right)^{-\frac{1}{\theta_1}} < \infty, \tag{6.1}$$

and

$$B_2^1 := \sup_{t>0} \left(\int_0^t \omega_2^{\theta_2}(r) r^{\theta_2 \frac{n}{p_2}} dr \right)^{\frac{1}{\theta_2}} \left(\int_t^\infty \frac{\omega_1^{\theta_1}(r) r^{\theta_1'\left(\alpha - \frac{n}{p_1}\right)}}{\left(\int_r^\infty \omega_1^{\theta_1}(\rho) d\rho\right)^{\theta_1'}} dr \right)^{-\frac{1}{\theta_1'}} < \infty. \tag{6.2}$$

(b) Let $0 < \theta_1 \le 1$, $0 < \theta_1 \le \theta_2 < \infty$. *Then* (4.5) *holds if and only if* $B_1^1 < \infty$ *and*

$$B_2^2 := \sup_{t>0} t^{\alpha - \frac{n}{p_1}} \left(\int_0^t \omega_2^{\theta_2}(r) r^{\theta_2 \frac{n}{p_2}} dr \right)^{\frac{1}{\theta_2}} \left(\int_t^\infty \omega_1^{\theta_1}(r) dr \right)^{-\frac{1}{\theta_1}} < \infty. \tag{6.3}$$

(c) Let $1 < \theta_1 < \infty$, $0 < \theta_2 < \theta_1 < \infty$, $\theta_2 \ne 1$. *Then the inequality* (4.5) *holds if and only if*

$$B_1^3 := \left(\int_0^\infty \left(\frac{\int_t^\infty \omega_2^{\theta_2}(r) r^{\theta_2\left(\alpha + \frac{n}{p_2} - \frac{n}{p_1}\right)} dr}{\int_t^\infty \omega_1^{\theta_1}(r) dr} \right)^{\frac{r}{p}} \omega_2^{\theta_2}(t) t^{-\theta_2\left(\alpha + \frac{n}{p_2} - \frac{n}{p_1}\right)} dt \right)^{\frac{1}{r}} < \infty, \tag{6.4}$$

and

$$B_2^3 := \left(\int_0^\infty \left[\left(\int_0^t \omega_2^{\theta_2}(r) r^{\theta_2 \frac{n}{p_2}} dr \right)^{\frac{1}{\theta_2}} \left(\int_t^\infty \frac{\omega_1^{\theta_1}(r) r^{\theta_1'\left(\alpha - \frac{n}{p_1}\right)}}{\left(\int_r^\infty \omega_1^{\theta_1}(\rho) d\rho\right)^{\theta_1'}} dr \right)^{\frac{\theta_2 - 1}{\theta_2}} \right]^{\frac{\theta_1 \theta_2}{\theta_1 - \theta_2}} \times \right.$$
$$\left. \times \frac{\omega_1^{\theta_1}(t) t^{\theta_1'\left(\alpha - \frac{n}{p_1}\right)}}{\left(\int_t^\infty \omega_1^{\theta_1}(\rho) d\rho\right)^{\theta_1'}} dt \right)^{\frac{\theta_1 - \theta_2}{\theta_1 \theta_2}} < \infty. \tag{6.5}$$

(d) Let $1 = \theta_2 < \theta_1 < \infty$. *Then* (4.5) *holds if and only if* $B_1^3 < \infty$ *and*

$$B_2^4 := \left(\int_0^\infty \left(\frac{\int_t^\infty \omega_2^{\theta_2}(r) r^{\theta_2\left(\alpha + \frac{n}{p_2} - \frac{n}{p_1}\right)} dr + t^{\alpha - \frac{n}{p_1}} \int_0^t \omega_2^{\theta_2}(r) r^{\alpha + \frac{n}{p_2} - \frac{n}{p_1} - 1} dr}{\int_t^\infty \omega_1^{\theta_1}(r) dr} \right)^{\theta_1' - 1} \times \right.$$
$$\left. \times \int_0^t \omega_2^{\theta_2}(r) r^{\alpha + \frac{n}{p_2} - \frac{n}{p_1} - 1} dr\, t^{\alpha - \frac{n}{p_1} - 1} dt \right)^{\theta_1'} < \infty. \tag{6.6}$$

(e) Let $0 < \theta_2 < \theta_1 = 1$*. Then* (4.5) *holds if and only if* $B_1^3 < \infty$ *and*

$$B_2^5 := \left(\int_0^\infty \left(\int_0^t \omega_2^{\theta_2}(r) r^{\theta_2 \frac{n}{p_2}} dr \right)^{\frac{\theta_2}{1-\theta_2}} \left(\operatorname*{ess\,inf}_{t<s<\infty} s^{\frac{n}{p_1}-\alpha} \int_s^\infty \omega_1^{\theta_1}(\rho) d\rho \right)^{\frac{\theta_2}{\theta_2-1}} \times \right. \\ \left. \times \omega_2^{\theta_2}(t) t^{\theta_2 \frac{n}{p_2}} dt \right)^{\frac{1-\theta_2}{\theta_2}} < \infty. \tag{6.7}$$

(f) Let $0 < \theta_2 < \theta_1 < 1$*. Then* (4.5) *holds if and only if* $B_1^3 < \infty$ *and*

$$B_2^6 := \left(\int_0^\infty \sup_{t \le s < \infty} \frac{s^{\left(\alpha - \frac{n}{p_1}\right)\frac{\theta_1\theta_2}{\theta_1-\theta_2}}}{\left(\int_s^\infty \omega_1^{\theta_1}(\rho) d\rho \right)^{\frac{\theta_2}{\theta_1-\theta_2}}} \left(\int_0^t \omega_2^{\theta_2}(r) r^{\theta_2 \frac{n}{p_2}} dr \right)^{\frac{\theta_2}{\theta_1-\theta_2}} \times \right. \\ \left. \times \omega_2^{\theta_2}(t) t^{\theta_2 \frac{n}{p_2}} dt \right)^{\frac{\theta_1-\theta_2}{\theta_1\theta_2}} < \infty. \tag{6.8}$$

From Theorem 4.2 and Corollary 6.1 follows the next theorem.

Theorem 6.2. *Let* $0 < \alpha < n$, $0 < p_2 < \infty$, $0 < \theta_1, \theta_2 \le \infty$, $\omega_1 \in \Omega_{\theta_1}$ *and* $\omega_2 \in \Omega_{\theta_2}$*. Moreover, let* $1 < \frac{p_2 n}{n+\alpha p_2} \le p_1 < \infty$*, or* $\frac{p_2 n}{n+\alpha p_2} < 1 \le p_1 < \infty$*, or* $\frac{p_2 n}{n+\alpha p_2} = 1 < p_1 < \infty$*.*

Assume that any of conditions (a)-(f) be satisfied. Then I_α *is bounded from* $LM_{p_1,\theta_1,\omega_1}$ *to* $LM_{p_2,\theta_2,\omega_2}$*.*

Remark 6.3. We can combine two conditions (6.1) and (6.3) into one condition

$$\sup_{t>0} \left(\int_0^\infty \omega_2^{\theta_2}(r) \frac{r^{\theta_2 \frac{n}{p_2}}}{(t+r)^{\theta_2(\frac{n}{p_1}-\alpha)}} dr \right)^{\frac{1}{\theta_2}} \left(\int_t^\infty \omega_1^{\theta_1}(r) dr \right)^{-\frac{1}{\theta_1}} < \infty, \tag{6.9}$$

which coincide with the necessary condition for boundedness of the Riesz potential from $LM_{p_1,\theta_1,\omega_1}$ to $LM_{p_2,\theta_2,\omega_2}$ in the case $0 < \theta_1 \le 1$, $0 < \theta_1 \le \theta_2 < \infty$, $1 < p_1 < p_2 < \infty$, $\alpha = n(1/p_1 - 1/p_2)$ (see [5]).

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