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THE CLOSED RANGE PROPERTY FOR BANACH SPACE OPERATORS

THOMAS L. MILLER AND VLADIMIR MÜLLER

Abstract. Let T be a bounded operator on a complex Banach space X. If V is an open subset of the complex plane, we give a condition sufficient for the mapping $f(z) \mapsto (T-z)f(z)$ to have closed range in the Fréchet space $H(V,X)$ of analytic X-valued functions on V. Moreover, we show that there is a largest open set U for which the map $f(z) \mapsto (T-z)f(z)$ has closed range in $H(V,X)$ for all $V \subseteq U$. Finally, we establish analogous results in the setting of the weak–$*$ topology on $H(V,X^*)$.

Introduction. Let X be a complex Banach space and denote by $B(X)$ the algebra of bounded linear operators on X. For $T \in B(X)$, let $\sigma(T)$ denote the spectrum of T, and denote by $\mathrm{Lat}\,(T)$ the collection of closed T-invariant subspaces of X. If $M \in \mathrm{Lat}\,(T)$, we write the restriction of T to M as $T|_M$.

A basic notion in local spectral theory is that of decomposability. Given an open subset U of the complex plane $\mathbb{C}$, $T \in B(X)$ is said to be decomposable on U provided that for any open cover $\{V_1,\dots,V_n\}$ of $\mathbb{C}$ with $\mathbb{C}\setminus U \subset V_1$, there exists $\{X_1,\dots,X_n\} \subset \mathrm{Lat}\,(T)$ such that $X = X_1+\dots+X_n$ and $\sigma(T|_{X_k}) \subset V_k$ for each k, $1 \le k \le n$; see [2], [5], [8], [11], and [12]. The fact that there exists for each $T \in B(X)$ a largest open set U on which T is decomposable was first shown by Nagy, [11].

An alternative characterization of decomposability may be given in terms of a property introduced by E. Bishop, [3]. For an open subset V of $\mathbb{C}$, let $H(V,X)$ denote the space of all analytic X-valued functions on V. Then $H(V,X)$ is a Fréchet space with generating semi-norms given by $p_K(f) := \sup\{\|f(\lambda)\| : \lambda \in K\}$, where K runs through the compact subsets of V. Every operator $T \in B(X)$ induces a continuous linear mapping T_V on $H(V,X)$, defined by $T_V f(\lambda) := (T-\lambda)f(\lambda)$ for all $f \in H(V,X)$ and $\lambda \in V$. An operator T is said to possess Bishop's property (β) on an open set $U \subset \mathbb{C}$ if for each open subset V of U, the operator T_V is injective with range $\mathrm{ran}\, T_V$ closed in $H(V,X)$; see [6, Prop. 1.2.6]. Clearly there exists a largest open set $\rho_\beta(T)$ on which T has property (β).

Fundamental work by Albrecht and Eschmeier established that an operator $T \in B(X)$ has property (β) on U precisely when there exists an operator $S \in B(Y)$ such that S is decomposable on U, $X \in \mathrm{Lat}\,(S)$ and $T = S|_X$, [2, Theorem 10]. Moreover, [2, Theorems 8 and 21], T is decomposable on U if and only if T and its adjoint T^* share property (β) on U. Thus Nagy's largest open set on which T is decomposable is the set $\rho_\beta(T) \cap \rho_\beta(T^*)$.

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An operator $T \in B(X)$ is said to have the single-valued extension property (SVEP) at a point $\lambda \in \mathbb{C}$ provided that, for every open disc V centered at λ, the mapping T_V is injective on $H(V, X)$. If $U \subset \mathbb{C}$ is open, then T is said to have SVEP on U if T has SVEP at every $\lambda \in U$, equivalently, if T_V is injective for each open set $V \subseteq U$. Let $\rho_{SVEP}(T)$ denote the largest open set on which T has SVEP.

Recently, M. Neumann, V. Miller and the first author of the current paper showed, [9, Theorem 2.5], that T_V has closed range in $H(V, X)$ for every open subset V of the "Kato-type" resolvent set of T, an open set that contains the semi-Fredholm region of T, thus extending a result of Eschmeier, [5]. Following Neumann, we say that an operator has the closed range property (CR) on an open set $U \subset \mathbb{C}$ provided $\operatorname{ran}(T_V)$ is closed in $H(V, X)$ for every open subset V of U. Thus T has property (β) on U if and only if T has both SVEP and (CR) on U.

In this note, we give a more general condition that suffices for $T \in B(X)$ to have (CR) on an open set U and prove that there is in fact a largest open set $\rho_{CR}(T)$ on which T has the closed range property. Thus $\rho_\beta(T) = \rho_{SVEP}(T) \cap \rho_{CR}(T)$. In the last section we establish corresponding results in the setting of the weak–$*$ topology on $H(V, X^*)$.

Main results. We denote the kernel of $T \in B(X)$ by $\ker(T)$ and define $N^\infty(T) := \bigcup_{n \geq 0} \ker(T^n)$ and $R^\infty(T) := \bigcup_{n \geq 0} \operatorname{ran}(T^n)$. If $T \in B(X)$ is such that $\operatorname{ran}(T)$ is closed and $N^\infty(T) \subseteq R^\infty(T)$, then T is said to be a Kato operator. A systematic exposition of this class, also referred to as semi-regular operators, may be found in [10, Section II.12]; also see [1, Section 1.2] and [6, Section 3.1]. In particular, an equivalent condition may be given in terms of the reduced minimum modulus function: for $S \in B(X)$, define $\gamma(S) := \inf\{\|Sx\| : \operatorname{dist}(x, \ker(S)) = 1\}$. Then an operator T is Kato if and only if $\gamma(T) > 0$ and the mapping $z \to \gamma(T - z)$ is continuous at 0, [10, II.12 Theorem 2]. Denote by $\sigma_K(T)$ the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda$ is not Kato. Then $\sigma_K(T)$ is a nonempty compact set, $z \mapsto R^\infty(T - z)$ is constant on each component of $\rho_K(T) := \mathbb{C} \setminus \sigma_K(T)$, $R^\infty(T - \lambda)$ is closed and $(T - \lambda)R^\infty(T - \lambda) = R^\infty(T - \lambda)$ for each $\lambda \in \rho_K(T)$, [10, II.12, Theorem 15 and Cor. 19]. Moreover, if G is a component of $\rho_K(T)$ and $S \subset G$ has an accumulation point in G, then $\bigcap_{z \in S} \operatorname{ran}(T - z) = R^\infty(T - \lambda)$ for each $\lambda \in G$, [6, 3.1.11].

For each closed subset F of $\mathbb{C}$, define the "glocal" analytic spectral subspace $\mathfrak{X}_T(F) := \{x \in X : x \in \operatorname{ran} T_{\mathbb{C} \setminus F}\}$. These spaces are T-invariant, but generally not closed. If $M \in \operatorname{Lat}(T)$ and $V \subset \mathbb{C}$ is such that $(T - z)M = M$ for all $z \in V$, then $M \subset \mathfrak{X}_T(\mathbb{C} \setminus V)$ by a theorem of Leiterer, [6, Theorem 3.2.1]. It follows from above that if G is a component of $\rho_K(T)$ and $V \subset G$ is open, then $\mathfrak{X}_T(\mathbb{C} \setminus V) = R^\infty(T - \lambda)$ for all $\lambda \in G$; in particular, $\mathfrak{X}_T(\mathbb{C} \setminus V)$ is closed. Also, it is easily seen that if T has (CR) on an open set U, then $\mathfrak{X}_T(\mathbb{C} \setminus V)$ is closed for every open $V \subset U$.

The content of Theorem 4 below is that the converse holds under the additional assumption that $\operatorname{ran}(T - z)$ is closed for all but countably many $z \in V$. Some additional assumption beyond closeness of the glocal spectral subspaces is seen to be necessary for (CR) by the facts that, on one hand, T has property (β) on all of $\mathbb{C}$ precisely when T has (CR) on $\mathbb{C}$, [6, Prop. 3.3.5], while on the other hand, there is an operator $T \in B(X)$ without property (β) but for which $\mathfrak{X}_T(F)$ is closed for all closed $F \subset \mathbb{C}$, [7].

Lemma 1. Let $T \in B(X)$ and let V be an open subset of $\mathbb{C}$. Let $(D_i)_{i\in A}$ be an cover of V consisting of simply connected open sets D_i such that $\mathfrak{X}_T(\mathbb{C}\setminus D_i)$ is closed for each $i \in A$ and $D_i \setminus D_j \neq \emptyset$ if $\mathfrak{X}_T(\mathbb{C}\setminus D_i) \neq \mathfrak{X}_T(\mathbb{C}\setminus D_j)$.

Let $M = \bigcap_{i\in A} \mathfrak{X}_T(\mathbb{C}\setminus D_i)$. Then M is closed, $TM \subset M$ and

(i) if $x \in M$ and $g_j \in H(D_j, X)$ is such that $T_{D_j} g_j = x$, then $g_j(D_j) \subset M$;

(ii) $\ker T_{D_j} \subset H(D_j, M)$;

(iii) $(T - z)M = M$ for all $z \in V$;

(iv) if $\widetilde{T} : X/M \to X/M$ is the quotient map induced by T then $\widetilde{T}_{D_j}$ is injective on $H(D_j, X/M)$.

Proof.

Clearly M is a closed subspace of X and $TM \subset M$.

(i) Let $x \in M$ and $g_j \in H(D_j, X)$ such that $T_{D_j} g_j = x$.

We show first that $g_j(D_j) \subset \mathfrak{X}_T(\mathbb{C}\setminus D_j)$. Let $z \in D_j$, and define $h_j : D_j \to X$ by $h_j(\omega) = (g_j(\omega) - g_j(z))/(\omega - z)$ if $\omega \in D_j \setminus \{z\}$ and $h_j(z) = g_j'(z)$. Then $h_j \in H(D_j, X)$ and

$$(T-\omega)h_j(\omega) = \frac{1}{\omega - z}\Big(x - ((T-z) + (z-\omega))g_j(z)\Big) = g_j(z).$$

Hence $g_j(z) \in \mathfrak{X}_T(\mathbb{C}\setminus D_j)$ and so $g_j(D_j) \subset \mathfrak{X}_T(\mathbb{C}\setminus D_j)$.

If i is such that $\mathfrak{X}_T(\mathbb{C}\setminus D_i) \neq \mathfrak{X}_T(\mathbb{C}\setminus D_j)$, let $g_i \in H(D_i, X)$ be such that $T_{D_i} g_i = x$, let $z \in D_j \setminus D_i$ and define $h_i : D_i \to X$ by $h_i(\omega) = \frac{g_i(\omega) - g_j(z)}{\omega - z}$. Then $h_i \in H(D_i, X)$ and again

$$\begin{aligned}(T-\omega)h_i(\omega) &= \frac{1}{\omega - z}\Big((T-\omega)g_i(\omega) - ((T-z) + (z-\omega))g_j(z)\Big)\\ &= \frac{1}{\omega - z}(x - x + (\omega - z)g_j(z)) = g_j(z).\end{aligned}$$

Thus $g_j(z) \in \mathfrak{X}_T(\mathbb{C}\setminus D_i)$ and $g_j(D_j \setminus D_i) \subset \mathfrak{X}_T(\mathbb{C}\setminus D_i)$.

Since the sets D_i and D_j are open, simply connected and $D_j \setminus D_i \neq \emptyset$, it is easy to see that $D_j \setminus D_i$ contains an accumulation point. Indeed, let $z_0 \in D_j \setminus D_i$. If $z_0 \notin \overline{D_i}$ then there is an open neighborhood of z_0 is contained in $D_j \setminus \overline{D_i}$. If $z_0 \in \partial D_i$, then there is a sequence $(z_n) \subset D_j \setminus D_i$ such that $z_n \to z_0$.

Since $\mathfrak{X}_T(\mathbb{C}\setminus D_i)$ is closed and $g_j(D_j \setminus D_i) \subset \mathfrak{X}_T(\mathbb{C}\setminus D_i)$, it follows that $g_j : D_j \to \mathfrak{X}_T(\mathbb{C}\setminus D_i)$.

This proves (i).

(ii) is an immediate consequence of (i).

(iii) Let $z \in D_j$ and $x \in M \subset \mathfrak{X}_T(\mathbb{C}\setminus D_j)$. There is a function $g_j : D_j \to X$ such that $T_{D_j} g_j = x$. By (i), $g_j(z) \in M$ and so $x = (T-z)g_j(z) \in (T-z)M$.

(iv) If $\pi : X \to X/M$ is the canonical projection, then Gleason's theorem implies that the sequence $0 \to H(\Omega, M) \to H(\Omega, X) \xrightarrow{\pi} H(\Omega, X/M) \to 0$ is exact, [6, Prop. 2.1.5]. Thus, if $\widetilde{T}_{D_j} h = 0$ for some $h \in H(D_j, X/M)$, then there exists $f \in H(D_j, X)$ such that $h = \tilde{f}$, where $\tilde{f} = \pi \circ f$. Clearly $T_{D_j} f \in H(D_j, M)$ and (iii) together with Leiterer's theorem implies that there exists $g \in H(D_j, M)$ such that $T_{D_j}f = T_{D_j}g$. Thus $f - g \in \ker T_{D_j} \subset H(D_j, M)$ by (ii). Consequently, $f \in H(D_j, M)$ and therefore, $h = \tilde{f} = 0$. □

Proposition 2. Let V_1, V_2 be open subsets of $\mathbb{C}$. If $T \in B(X)$ has (CR) on each $V_j \quad (j = 1, 2)$, then T has (CR) on $V_1 \cup V_2$.

Proof. Let $\Omega \subset V_1 \cup V_2$ be an open set. We show that T_Ω has closed range. Without loss of generality, assume that $\Omega_j = \Omega \cap V_j$ is nonempty for each j, $j = 1, 2$. So $\Omega = \Omega_1 \cup \Omega_2$ and T has (CR) on each Ω_j.

Let $\mathcal{U}$ be an cover of Ω consisting of open discs such that $\mathcal{U}$ contains a disc in each component of $\Omega_1 \cap \Omega_2$ and for each $D \in \mathcal{U}$, either $D \subset \Omega_1$ or $D \subset \Omega_2$. We may also assume that $D \backslash D' \neq \emptyset$ whenever $D, D' \in \mathcal{U}$ are distinct. Let $M = \bigcap_{D \in \mathcal{U}} \mathfrak{X}_T(\mathbb{C} \backslash D)$. By the assumptions, M is closed.

Let $f \in \overline{\operatorname{ran} T_\Omega}$. Then there are $g_j \in H(\Omega_j, X)$ such that $f|_{\Omega_j} = T_{\Omega_j} g_j$ for $j = 1, 2$. We have $T_{\Omega_1 \cap \Omega_2}(g_1 - g_2) = 0$, and so $(g_1 - g_2)(\Omega_1 \cap \Omega_2) \subset M$ by Lemma 1 (ii). So $\tilde{g}_1|_{(\Omega_1 \cap \Omega_2)} = \tilde{g}_2|_{(\Omega_1 \cap \Omega_2)}$ and we can define $h \in H(\Omega, X/M)$ by $h(z) = \tilde{g}_j(z)$ for $z \in \Omega_j$. We have $\tilde{f} = \widetilde{T}_\Omega h$ and, again by Gleason's theorem, there exists $g \in H(\Omega, X)$ such that $h = \tilde{g}$. Then $f - T_\Omega g \in H(\Omega, M)$ and so $f - T_\Omega g = T_\Omega k$ for some $k \in H(\Omega, M)$. Hence $f = T_\Omega(g + k) \in \operatorname{ran} T_\Omega$. □

Theorem 3. Let $T \in B(X)$. Then there is a largest open set $\rho_{CR}(T)$ on which T has (CR).

Proof. Let $\mathcal{W}$ be the family of all open subsets $V \subset \mathbb{C}$ such that T has (CR) on V. We show that T has (CR) on the union $W = \bigcup \mathcal{W}$, which is obviously the largest open set on which T has (CR).

Clearly W is the union of countably many open set W_n with (CR). Write $V_n = W_1 \cup \cdots \cup W_n$. By the previous proposition, T has (CR) on each V_n, $V_1 \subset V_2 \subset \cdots$ and $W = \bigcup_n V_n$.

Let $\Omega \subset W$ be a nonempty open subset. We show that T_Ω has closed range. For each n, let $\Omega_n = \Omega \cap V_n$. Then T has (CR) on each Ω_n and $\Omega = \bigcup_n \Omega_n$. Without loss of generality, we assume that $\Omega_1 \neq \emptyset$.

Let $\mathcal{U}_1$ be an open cover of Ω_1 consisting of open discs $D \subset \Omega_1$ such that $D \setminus D' \neq \emptyset$ whenever $D, D' \in \mathcal{U}_1$ are distinct. Similarly, for each $n \geq 2$, let $\mathcal{U}_n$ be a cover of $\Omega_n \setminus \Omega_{n-1}$ consisting of open discs such that $D \subset \Omega_n$, $D \setminus \Omega_{n-1} \neq \emptyset$ for each $D \in \mathcal{U}_n$, and $D \setminus D' \neq \emptyset$ whenever $D, D' \in \mathcal{U}_n$ are distinct. Let $\mathcal{U} = \bigcup_{n \geq 1} \mathcal{U}_n$. Then for each $D \in \mathcal{U}$ there is an n such that $D \subset \Omega_n$ and $D \setminus D' \neq \emptyset$ whenever $D, D' \in \mathcal{U}$ are distinct.

Let $M = \bigcap_n \mathfrak{X}_T(\mathbb{C} \setminus D)$. By Lemma 1, M is a closed subspace of X, $TM \subset M$ and $(T - z)M = M$ for all $z \in \Omega$. Denote by $\widetilde{T} : X/M \to X/M$ the operator induced by T and by $\pi : X \to X/M$ the canonical projection.

Let $f \in \overline{\operatorname{ran} T_\Omega}$. Then for each n there exists $g_n \in H(\Omega_n, X)$ such that $f|_{\Omega_n} = T_{\Omega_n} g_n$. If $n \geq 2$, then $T_{\Omega_{n-1}}(g_n|_{\Omega_{n-1}} - g_{n-1}) = 0$ and so, by Lemma 1 (ii), $g_n|_{\Omega_{n-1}} - g_{n-1} : \Omega_{n-1} \to M$, i.e.,

$$\tilde{g}_n|_{\Omega_{n-1}} = \tilde{g}_{n-1} \qquad \text{in } H(\Omega_{n-1}, X/M).$$

Define $h : \Omega \to X/M$ by $h|_{\Omega_n} = \tilde{g}_n$. Then h is well-defined and analytic on Ω. Also, $\tilde{f} = \widetilde{T}_\Omega h$ in $H(\Omega, X/M)$. By Gleason's theorem, there exists $g \in H(\Omega, X)$ such that $\tilde{g} = h$ and therefore, $\pi(f - T_\Omega g) = 0$. Exactness implies that $f - T_\Omega g \in$

$H(\Omega, M)$, and so it follows from Lemma 1 (ii) that there is a $k \in H(\Omega, M)$ such that $f - T_\Omega g = T_\Omega k$, i.e., $f = T_\Omega(g+k) \in \operatorname{ran} T_\Omega$. □

Next, we give a condition which implies that $T \in B(X)$ has (CR) on an open set V. Note that if T has (CR) on V then the spaces $\mathfrak{X}_T(\mathbb{C} \setminus U)$ are closed for each open set $U \subset V$.

Theorem 4. Let $T \in B(X)$ and let $V \subset \mathbb{C}$ be an open set. Suppose that the set $\{z \in V : \operatorname{ran}(T-z) \text{ is not closed}\}$ is countable and that, for all $z \in V$ and $r_0 > 0$, there is an $r, 0 < r < r_0$ such that the space $\mathfrak{X}_T(\{z : |z| \geq r\})$ is closed. Then T has (CR) on V.

Proof. Since the conditions of the theorem are inherited by every open subset U of V, it suffices to show that T_V has closed range in $H(V, X)$. Moreover, because the set $\{z \in \mathbb{C} : \operatorname{ran}(T-z) \text{ is closed and } T-z \text{ is not Kato}\}$ is countable by [10, II.12 Theorem 13], it follows that $E := V \cap \sigma_K(T)$ is countable. Let $E = \{\lambda_n : n = 1, 2, \dots\}$ (the sequence (λ_n) can be possibly finite). Note that the set $V \setminus E$ is open.

We can construct a sequence (B_j) of mutually disjoint open discs such that $E \subset \bigcup_j B_j$, $\overline{B_j} \subset V$ and $\mathfrak{X}_T(\mathbb{C} \setminus B_j)$ is closed for each j. Indeed, choose $r_1 > 0$ such that $B(\lambda_1, r_1)$, the open disc with center λ_1 and radius r_1 satisfies $\overline{B(\lambda_1, r_1)} \subset V$, $\mathfrak{X}_T(\mathbb{C} \setminus B(\lambda_1, r_1))$ is closed and $|\lambda_j - \lambda_1| \neq r_1 \quad (j \geq 2)$. Set $B_1 = B(\lambda_1, r_1)$. Let k be the smallest index such that $\lambda_k \notin B_1$ and find $r_2 > 0$ such that $B_2 := B(\lambda_k, r_2)$ satisfies $\overline{B_2} \subset V \setminus B_1$, the space $\mathfrak{X}_T(\mathbb{C} \setminus B_2)$ is closed and $|\lambda_j - \lambda_k| \neq r_2 \quad (j > k)$. If we continue in this way, we obtain the required sequence of open discs $\mathcal{U}_E = (B_j)_j$ covering E.

For each $z_0 \in V \setminus E$ we can find a simply connected open set W_{z_0} such that $z_0 \in W_{z_0} \subset V \setminus E$ and $W_{z_0} \cap (V \setminus \bigcup_n B_n) \neq \emptyset$. This is clear if $z_0 \notin \bigcup_n B_n$ — in this case there is an $r > 0$ such that $\{z : |z - z_0| < r\} \subset V \setminus E$ and we can take $W_{z_0} = B(z_0, r)$.

Suppose then that $z_0 \in \bigcup_n B_n \setminus E$. Since the sets B_n are mutually disjoint, there is only one j with $z_0 \in B_j$, and since the set E is countable, there is a $\theta,\ 0 \leq \theta < 2\pi$ such that $\{z_0 + te^{i\theta} : t \geq 0\} \cap E = \emptyset$. Let $t_0 = \min\{t \geq 0 : z_0 + te^{i\theta} \notin B_j\}$. Since the set $S := \{z_0 + te^{i\theta} : 0 \leq t \leq t_0\}$ is compact and the set $E \cup \partial V$ is closed, there is an $\varepsilon > 0$ such that the set $W_{z_0} := \{z \in \mathbb{C} : \operatorname{dist}\{z, S\} < \varepsilon\}$ is disjoint with $E \cup \partial V$. Clearly W_{z_0} is an open simply connected set, $z_0 \in W_{z_0} \subset V \setminus E$. Moreover, $W_{z_0} \subset \rho_K(T)$; if G is the component of $\rho_K(T)$ containing W_{z_0}, then $\mathfrak{X}_T(\mathbb{C} \setminus W_{z_0}) = R^\infty(T - \lambda)$ for every $\lambda \in G$. Thus $\mathfrak{X}_T(\mathbb{C} \setminus W_{z_0})$ is closed and $W_{z_0} \cap W_{z_1} = \emptyset$ if $z_0, z_1 \in V \setminus E$ are such that $\mathfrak{X}_T(\mathbb{C} \setminus W_{z_0}) \neq \mathfrak{X}_T(\mathbb{C} \setminus W_{z_1})$. By construction, $W_z \setminus B_j \neq \emptyset$ and $B_j \setminus W_z \neq \emptyset$ whenever $z \in V \setminus E$ and $B_j \in \mathcal{U}_E$. Thus, if $\mathcal{U}_K = \{W_z : z \in V \setminus E\}$ and $\mathcal{U} = \mathcal{U}_K \cup \mathcal{U}_E$, then $\mathcal{U}$ is an open cover of V satisfying the hypotheses of Lemma 1.

As in Lemma 1, let $M = \bigcap_{D \in \mathcal{U}} \mathfrak{X}_T(\mathbb{C} \setminus D)$ and let $\widetilde{T} : X/M \to X/M$ be the operator induced by T. By (iii), we have $(T - z)M = M$ for all $z \in V$. We show that $\widetilde{T} - z$ is bounded below for each $z \in V \setminus E$, equivalently, if $z \in V \setminus E$ and $(x_n)_n \subset X$ is such that $(\widetilde{T} - z)\tilde{x}_n \to 0$ in X/M, then $\tilde{x}_n \to 0$ in X/M.

Fix $z \in V \setminus E$ and let $x \in \ker(T - z)$. Then $\ker(T-z) \subset R^\infty(T-z) = \mathfrak{X}_T(\mathbb{C} \setminus W_z)$, and so there exists $g \in H(W_z, X)$ so that $(T - \omega)g(\omega) = x$ for all $\omega \in W_z$. If

$h = (T - z)g$, then $h \in \ker T_{W_z}$ and, since $W_z \in \mathcal{U}$, it follows that $h : W_z \to M$. In particular, $x = h(z) \in M$. Thus $\ker(T - z) \subset M$.

A sequence $(x_n)_n \subset X$ satisfies $(\widetilde{T} - z)\tilde{x}_n \to 0$ only if there exists $(y_n)_n \subset M$ so that $(T - z)x_n - y_n \to 0$ in X. Since $(T - z)M = M$, there exists $(w_n)_n \subset M$ so that $(T - z)w_n = y_n$ and therefore, $(T - z)(x_n - w_n) \to 0$. Since $\operatorname{ran}(T - z)$ is closed, it follows that $\operatorname{dist}(x_n - w_n, \ker(T - z)) \to 0$. But $\ker(T - z) \subset M$, and so $\operatorname{dist}(x_n, M) \to 0$, i.e., $\tilde{x}_n \to 0$ in X/M as required. Hence $\widetilde{T} - z$ is bounded below for each $z \in V \setminus E$.

The conclusion now follows as in [9]. Suppose that $(f_n)_n$ is a sequence in $H(V, X/M)$ such that $\widetilde{T}_V f_n \to 0$. If F is a compact subset of V, then there is a contour $\gamma \subset V \setminus E$ surrounding F in the sense of Cauchy's theorem. By continuity of $z \mapsto \gamma(T - z)$ on $V \setminus E$, there is a constant $c > 0$ so that $\sup_{z\in\gamma} \|f_n(z)\| \leq c \sup_{z\in\gamma} \|(T - z)f_n(z)\|$ for all n. Thus for each $\lambda \in F$ Cauchy's theorem implies that

$$\|f_n(\lambda)\| \leq \frac{c \sup_{z\in\gamma} \|(T - z)f_n(z)\|}{2\pi \operatorname{dist}(\gamma, F)} |\gamma|,$$

where $|\gamma|$ denotes the length of γ. Thus the seminorms $p_F(f_n) = \sup_{z\in F} \|f_n(z)\| \to 0$ as $n \to \infty$, and since F is arbitrary, it follows that $\widetilde{T}_V$ is injective with closed range. Since $(T - z)M = M$ for all $z \in V$ by part (iii) of Lemma 1, Leiterer's theorem implies that $T_V H(V, M) = H(V, M)$. T_V therefore has closed range in $H(V, X)$ by [9, Prop. 2.1], and the theorem is established. □

For $T \in B(X)$ denote by $K(T)$ the analytic core of T, i.e., the set of all $x_0 \in X$ such that there exists a sequence $(x_n) \subset X$ such that $Tx_n = x_{n-1}$ $(n \geq 1)$ and $\sup \|x_n\|^{1/n} < \infty$. Clearly $K(T) = \bigcup_n \mathfrak{X}_T(\mathbb{C} \setminus D(0, 1/n))$. This set has been shown to play a significant role in the Fredholm theory of Banach space operators; see, for example [1].

Corollary 5. Let $T \in B(X)$ and let $V \subset \mathbb{C}$ be an open set. Suppose that $K(T - z)$ is closed for each $z \in V$ and that the set $\{z \in V : \operatorname{ran}(T - z) \text{ is not closed}\}$ is countable. Then T has (CR) on V.

Proof. Let $z \in V$ and $K(T - z)$ be closed. Clearly $(T - z)K(T - z) = K(T - z)$ and, by the Banach open mapping theorem, there is an $\varepsilon > 0$ such that $K(T - z) = \mathfrak{X}_T(\mathbb{C} \setminus B(z, \varepsilon))$. (In fact, $\varepsilon = \gamma((T - z)|_{K(T-z)})^{-1})$. Clearly $\mathfrak{X}_T(\mathbb{C} \setminus W) = K(T - z)$ for each open set W with $z \in W \subset B(z, \varepsilon)$. By Theorem 4, T has (CR) on V. □

A generalized Kato decomposition for $T \in B(X)$ is a pair of subspaces $X_1, X_2 \in \operatorname{Lat}(T)$ such that $X = X_1 \oplus X_2$, $T|_{X_1}$ is Kato and $T|_{X_2}$ is quasinilpotent. The operator T is said to be of Kato-type if $T|_{X_2}$ is nilpotent. It is well known that semi-Fredholm operators are of Kato-type, see e.g. [1], [10].

If $\rho_{gk}(T)$ denotes the set of $\lambda \in \mathbb{C}$ such that $T - \lambda$ has a generalized Kato decomposition, then $\rho_{gk}(T)$ is open and $\rho_{gk}(T) \cap \sigma_K(T)$ accumulates only on $\partial\rho_{gk}(T)$. Indeed, suppose that $0 \in \rho_{gk}(T)$ and that $X_1, X_2 \in \operatorname{Lat}(T)$ such that $X = X_1 \oplus X_2$, $T|_{X_1}$ is Kato and $T|_{X_2}$ is quasinilpotent. If $\varepsilon > 0$ is such that $B(0, \varepsilon) \subset \rho_K(T|_{X_1})$, then for $0 < |z| < \varepsilon$, $(T - z)X_2 = X_2$. Thus $\operatorname{ran}(T - z) = (T - z)X_1 \oplus X_2$ is closed and $N^\infty(T - z) = N^\infty(T|_{X_1} - z) \subset R^\infty(T|_{X_1} - z)$.

Moreover, if T has generalized Kato decomposition (X_1, X_2) as above, then by the remarks preceding Lemma 1, $R^\infty(T|_{X_1}) \subseteq K(T)$. On the other hand, if $x \in K(T)$, write $x = u_0 + v_0$ with $u_0 \in X_1$ and $v_0 \in X_2$. We show that $v_0 = 0$.

Suppose on the contrary that $v_0 \neq 0$. Then, by definition, there are sequences $(u_n) \subset X_1$ and $(v_n) \subset X_2$ such that $Tu_n = u_{n-1}$ and $Tv_n = v_{n-1}$ for all n and $C := \sup \|u_n + v_n\|^{1/n} < \infty$. Let $P \in B(X)$ be the projection with $\ker P = X_1$ and $\operatorname{ran} P = X_2$. We have $\|v_n\|^{1/n} = \|P(u_n + v_n)\|^{1/n} \leq \|P\|^{1/n} \cdot C$. Thus

$$\lim \|T^n|_{X_2}\|^{1/n} \geq \limsup \Big(\frac{\|v_0\|}{\|v_n\|}\Big)^{1/n} = \frac{1}{\liminf \|v_n\|^{1/n}} \geq 1/C > 0,$$

a contradiction to the assumption that $T|X_2$ is quasinilpotent. Hence $v_0 = 0$ and $K(T) \subseteq X_1$. Therefore

$$K(T) = K(T|_{X_1}) = R^\infty(T|_{X_1});$$

in particular, $K(T)$ is closed.

Thus we have established the following special case of Corollary 5, generalizing [9, Theorem 2.5].

Corollary 6. $T \in B(X)$ has (CR) on $\rho_{gk}(T)$.

Duality and weak–∗ closed ranges. Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and for U an open neighborhood of ∞, let $P(U, X)$ denote the Fréchet space of analytic functions $f : U \to X$ with $f(\infty) = 0$. If $T \in B(X)$, then T induces a continuous mapping T^U on $P(U, X)$ defined by $T^U f(z) = (T - z)f(z) + \lim_{|\omega| \to \infty} \omega f(\omega)$. If F is closed in $\mathbb{C}_\infty$ with $\infty \in F$, let $P(F, X)$ denote the inductive limit of the spaces $P(U, X)$, $U \supset F$ open; i.e., $P(F, X)$ is the (LF)-space consisting of germs of analytic X–valued functions defined in a neighborhood of F and vanishing at infinity. The mappings T^U induce a continuous mapping T^F on $P(F, X)$. Recall that if V is open in $\mathbb{C}$, then the Fréchet space $H(V, X^*)$ may be canonically identified with the strong dual of $P(\mathbb{C}_\infty \setminus V, X)$ via

$$\langle f, g \rangle = \int_\gamma \langle f(z), \tilde{g}(z) \rangle \, dz,$$

where $f \in H(V, X^*)$, $\tilde{g} \in P(U, X)$ is a representative of $g \in P(\mathbb{C}_\infty \setminus V, X)$ and γ is a contour surrounding $\mathbb{C} \setminus U$ in V; see [6, Chapter 2] for details. In particular, we have that $T_V^* = (T^F)^*$, where $F = \mathbb{C}_\infty \setminus V$, [6, Theorem 2.5.12 and Lemma 2.5.13]. Moreover, by the duality results of Albrecht and Eschmeier, specifically, Theorem 21 and the proof of Theorem 5 of [2], T^* has property (β) on U if and only if $T^F P(F, X) = P(F, X)$ for every closed set $F \subseteq \mathbb{C}_\infty$ with $\mathbb{C}_\infty \setminus U \subseteq F$. In this case, for every open $V \subseteq U$, T_V^* is injective with weak–∗ closed range in $H(V, X^*)$ by a theorem of Köthe, [6, Theorem 2.5.9].

Let us say that T^* has the property, $(\mathrm{CR})^{\text{weak}-*}$, on U provided that $\operatorname{ran} T_V^*$ is weak–∗ closed in $H(V, X^*)$ for every open $V \subseteq U$.

Proposition 7. Let $T \in B(X)$ and $U \subset \mathbb{C}$ open.

(i) If T has (CR) on U, then for every closed $F \supset \mathbb{C} \setminus U$, $\mathfrak{X}_T(F) = {}^\perp\mathfrak{X}^*_{T^*}(\mathbb{C} \setminus F)$, the preannihilator of $\mathfrak{X}^*_{T^*}(\mathbb{C} \setminus F) := \bigcup\{\mathfrak{X}^*_{T^*}(K) : K \text{ compact}, \ K \subset (\mathbb{C} \setminus F)\}$.

(ii) If T^* has $(\mathrm{CR})^{\text{weak}-*}$ on U and F is closed with $F \supset \mathbb{C} \setminus U$, then $\mathfrak{X}^*_{T^*}(F) = \mathfrak{X}_T(\mathbb{C} \setminus F)^\perp$, the annihilator of $\mathfrak{X}_T(\mathbb{C} \setminus F) = \bigcup\{\mathfrak{X}_T(K) : K \text{ compact}, \ K \subset (\mathbb{C} \setminus F)\}$. In particular, $\mathfrak{X}^*_{T^*}(\mathbb{C} \setminus V)$ is weak–$*$ closed whenever $V \subseteq U$ is open.

Proof. If F is closed and $\mathbb{C} \setminus U \subseteq F$, then $V := \mathbb{C} \setminus F$ is an open subset of U. Thus $\operatorname{ran} T_V$ is closed and $\operatorname{ran} T_V^*$ is weak–$*$ closed. The result now follows from Lemma 2.5 (c) and (d) of [4]; alternatively, one could argue as in the proof of [6, Prop 2.5.14]. □

Lemma 8. If U is open in $\mathbb{C}$ with $\{z : |z| \geq R\} \subset U$ for some $R \geq 0$, then $H(U, X) = H(\mathbb{C}, X) \oplus P(U_\infty, X)$, where $U_\infty = U \cup \{\infty\}$.

Proof. If $g \in H(U, X)$ and $z \in \mathbb{C}$, choose a contour γ_1 surrounding $\{z\} \cup (\mathbb{C} \setminus U)$ in the sense of Cauchy's theorem, and define $g_1(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{g(\omega)}{\omega - z} \, d\omega$. Then $g_1(z)$ is independent of the choice of γ_1, and so $g_1 \in H(\mathbb{C}, X)$. Similarly, for $z \in U$, let γ_2 be a contour surrounding $\mathbb{C} \setminus U$ in $\mathbb{C} \setminus \{z\}$ and define $g_2(z) = -\frac{1}{2\pi i} \int_{\gamma_2} \frac{g(\omega)}{\omega - z} \, d\omega$. Again, $g_2(z)$ is independent of the choice of γ_2; thus $g_2 \in H(U, X)$ and $|g_2(z)| \to 0$ as $z \to \infty$, so $g_2 \in P(U_\infty, X)$. If $z \in U$ and γ_1 and γ_2 are disjoint contours as above, then, since $\gamma_1 - \gamma_2$ is homotopic to zero in U, $g(z) = \frac{1}{2\pi i} \int_{\gamma_1 - \gamma_2} \frac{g(\omega)}{\omega - z} \, d\omega = g_1(z) + g_2(z)$. The mappings $g \mapsto g_j$ are clearly continuous with ranges $H(\mathbb{C}, X)$ and $P(U_\infty, X)$, respectively. If $g \in P(U_\infty, X)$ and γ_1 surrounds $\{z\} \cup (\mathbb{C} \setminus U)$, then $\frac{1}{2\pi i} \int_{\gamma_1} \frac{g(\omega)}{\omega - z} \, d\omega = 0$ by [12, Theorem 4.15]. □

Lemma 9. Let F_1 and F_2 be closed in $\mathbb{C}_\infty$ with $\infty \in F_1 \cap F_2$, and let $V_j = \mathbb{C} \setminus F_j$, $j = 1, 2$. Then the mapping $q : P(F_1, X) \oplus P(F_2, X) \to P(F_1 \cap F_2, X)$ given by $q([f_1] \oplus [f_2]) = [f_1 - f_2]$ is a continuous surjection. Consequently, its adjoint $q^* : H(V_1 \cup V_2, X^*) \to H(V_1, X^*) \oplus H(V_2, X^*)$, given by $q^* f = f|_{V_1} \oplus (-f|_{V_2})$, is injective with weak–$*$ closed range.

Proof. If $\infty \in F$ is closed and U is open with $F \subset U$, let $i_U : P(U, X) \to P(F, X)$ be defined by $i_U f = [f]$. Then a mapping S from $P(F, X)$ to an arbitrary topological vector space E is continuous if and only $S \circ i_U$ is continuous for every open neighborhood U of F. For $j = 1, 2$, let U_j be a neighborhood of F_j in $\mathbb{C}_\infty$, and let $W_j = U_j \cap \mathbb{C}$. Then the sequence $0 \to H(W_1 \cup W_2, X) \underset{f \mapsto f|_{W_1} \oplus f|_{W_2}}{\longrightarrow} H(W_1, X) \oplus H(W_2, X) \underset{f_1 \oplus f_2 \mapsto f_1 - f_2}{\longrightarrow} H(W_1 \cap W_2, X) \to 0$ is exact by [6, Proposition 2.1.7]. Suppose that $g \in P(U_1 \cap U_2, X)$ and $g|_{W_1 \cap W_2} = f_1 - f_2$ for some $f_j = f_{j,1} + f_{j,2} \in H(W_j, X) = H(\mathbb{C}, X) \oplus P(U_j, X)$ by the previous lemma. It follows that $f_{1,1} - f_{2,1} = 0$, and therefore, $g = f_{1,2} - f_{2,2}$. If $q_{U_1,U_2} : P(U_1, X) \oplus P(U_2, X) = P(U_1 \cap U_2, X)$ is defined by $q_{U_1,U_2}(f_1 \oplus f_2) = f_1 - f_2$, then it follows that q_{U_1,U_2} is a continuous surjective.

Define $q : P(F_1, X) \oplus P(F_2, X) \to P(F_1 \cap F_2, X)$ by $q([f_1] \oplus [f_2]) = [f_1 - f_2]$. We verify that q is well defined and continuous: $[f_1] \oplus [f_2] = [g_1] \oplus [g_2] \in P(F_1, X) \oplus P(F_2, X)$ if and only if there exists there exists a neighborhood G_j of F_j so that $f_j|_{G_j} = g_j|_{G_j}$, which implies that $((f_1 - g_1) - (f_2 - g_2))|_{G_1 \cap G_2} = 0$. In this case, $[(f_1 - g_1) - (f_2 - g_2)] = 0 \in P(F_1 \cap F_2, X)$. Also, $q \circ (i_{U_1} \oplus i_{U_2}) = i_{U_1 \cap U_2} \circ q_{U_1,U_2}$, and so q is continuous. The surjectivity of q follows from that of the mappings q_{U_1,U_2} since every open neighborhood of $F_1 \cap F_2$ has the form $U_1 \cap U_2$ for some open neighborhoods U_j of F_j. By the theorem of Köthe, [6, Prop. 2.5.9], $q^* : H(V_1 \cup V_2, X) \to H(V_1, X^*) \oplus H(V_2, X^*)$ is injective, with weak–$*$ closed range.

It remains to establish the formula for q^*. Let $f \in H(V_1 \cup V_2, X^*)$ and $g \in P(F_1 \cap F_2, X)$. Then g has representative $\tilde{g} \in P(U_1 \cap U_2, X)$ then $\tilde{g} = \tilde{g}_1 - \tilde{g}_2$ for some open neighborhoods U_j of F_j and $\tilde{g}_j \in P(U_j, X)$. Choose contours γ_j surrounding $\mathbb{C} \setminus U_j$ in V_j. Then

$$\begin{aligned}
\langle f, g\rangle &= \int_{\gamma_1} \langle f(z), \tilde{g}_1(z) - g_2(z)\rangle\, dz + \int_{\gamma_2} \langle f(z), \tilde{g}_1(z) - g_2(z)\rangle\, dz \\
&= \int_{\gamma_1} \langle f(z), \tilde{g}_1(z)\rangle\, dz - \int_{\gamma_1} \langle f(z), \tilde{g}_2(z)\rangle\, dz \\
&\qquad + \int_{\gamma_2} \langle f(z), \tilde{g}_1(z)\rangle\, dz - \int_{\gamma_2} \langle f(z), \tilde{g}_2(z)\rangle\, dz \\
&= \int_{\gamma_1} \langle f(z), \tilde{g}_1(z)\rangle\, dz - \int_{\gamma_2} \langle f(z), \tilde{g}_2(z)\rangle\, dz \\
&= \langle f|_{V_1} \oplus (-f|_{V_2}), g_1 \oplus g_2\rangle. \qquad \square
\end{aligned}$$

As a consequence of the Proposition 7 and Lemma 9, we obtain weak–∗ analogs of Theorems 3 and 4.

Theorem 10. There is a largest open set V on which $T^* \in B(X^*)$ has $(\mathrm{CR})^{\text{weak}-*}$.

Proof. Suppose that $T^* \in B(X^*)$ has $(\mathrm{CR})^{\text{weak}-*}$ on V_1 and V_2 and let Ω be an open subset of $V_1 \cup V_2$. Let $\mathcal{U}$ be an cover of Ω as in the proof of Proposition 2, and let $M = \bigcap_{D\in\mathcal{U}} \mathfrak{X}^*_{T^*}(\mathbb{C} \setminus D)$. By the previous proposition, for each $D \in \mathcal{U}$, $\mathfrak{X}^*_{T^*}(\mathbb{C} \setminus D)$ is weak–∗ closed and therefore M is also weak–∗ closed; in fact, $M \approx (X/{}^{\perp}M)^*$ and $X^*/M \approx ({}^{\perp}M)^*$. If $f \in \overline{\operatorname{ran} T^*_\Omega}^{\text{weak}-*}$, then by the previous lemma $f|_{\Omega_j} \in \overline{\operatorname{ran} T^*_{\Omega_j}}^{\text{weak}-*}$, and so, by assumption, there are $g_j \in H(\Omega_j, X^*)$ such that $f|_{\Omega_j} = T^*_{\Omega_j} g_j$ for $j = 1, 2$. We have $T^*_{\Omega_1\cap\Omega_2}(g_1 - g_2) = 0$, and so $(g_1 - g_2)(\Omega_1 \cap \Omega_2) \subset M$ by Lemma 1 (ii). If $\tilde{\varphi} := \varphi + M$ in X^*/M, then $\tilde{g}_1|_{(\Omega_1\cap\Omega_2)} = \tilde{g}_2|_{(\Omega_1\cap\Omega_2)}$ and we can define $h \in H(\Omega, X^*/M)$ by $h(z) = \tilde{g}_j(z)$ for $z \in \Omega_j$. We have $\tilde{f} = (T^*)\tilde{}_\Omega h$ and, by Gleason's theorem, there exists $g \in H(\Omega, X^*)$ such that $h = \tilde{g}$. Moreover, $f - T^*_\Omega g \in H(\Omega, M)$ and so $f - T^*_\Omega g = T^*_\Omega k$ for some $k \in H(\Omega, M)$. Hence $f = T^*_\Omega(g + k) \in \operatorname{ran} T^*_\Omega$. Thus $T^* \in B(X^*)$ has $(\mathrm{CR})^{\text{weak}-*}$ on $V_1 \cup V_2$.

To complete the argument, we adapt the proof of Theorem 4 similarly. The details are left to the reader. $\square$

Recall that $\operatorname{ran} T^*$ is weak–∗ closed in X^* if and only if $\operatorname{ran} T$ is closed in X, [6, A.1.10]. Also, $\sigma_K(T^*) = \sigma_K(T)$, [10, II.12 Theorem 11].

Theorem 11. Let $T \in B(X)$ and let $V \subset \mathbb{C}$ be an open set. Suppose that the set $\{z \in V : \operatorname{ran}(T - z) \text{ is not closed}\}$ is countable and that, for all $z \in V$ and $r_0 > 0$, there is an $r, 0 < r < r_0$ such that the space $\mathfrak{X}^*_{T^*}(\{z : |z| \geq r\})$ is weak–∗ closed. Then T^* has $(\mathrm{CR})^{\text{weak}-*}$ on V.

Proof. Since the conditions of the theorem are inherited by every open subset U of V, it suffices to show that T^*_V has weak–∗ closed range. Let $E := V \cap \sigma_K(T)$ and construct a covering $\mathcal{U} = \mathcal{U}_K \cup \mathcal{U}_E$ as in the proof of Theorem 4. Let $M = \bigcup_{D\in\mathcal{U}} \mathfrak{X}_T(\mathbb{C} \setminus D)$ and denote by $(T^*)\tilde{}$ the operator on X^*/M induced by T^*. Then Lemma 1 (iii) implies that $(T^* - z)M = M$ for all $z \in V$, and, as in the proof of

Theorem 4, $(T^*)\breve{} - z$ is bounded below for each $z \in V \setminus E$. The conclusion now follows from [9, Prop. 3.1], noting that, as in the proof of Theorem 4, it suffices that the set $E = V \cap \sigma_K(T)$ be countable rather than discrete. □

Corollary 12. Let $T \in B(X)$ and let $V \subset \mathbb{C}$ be an open set. Suppose that $K(T^* - z)$ is weak–$*$ closed for each $z \in V$ and that the set $\{z \in V : \operatorname{ran}(T - z) \text{ is not closed}\}$ is countable. Then T^* has $(\mathrm{CR})^{\text{weak}-*}$ on V. In particular, then T^* has $(\mathrm{CR})^{\text{weak}-*}$ on $\rho_{gk}(T)$.

Proof. The first statement follows from Theorem 10 just as Corollary follows from Theorem 4. If $T \in B(X)$ has generalized Kato decomposition (X_1, X_2), then $(X_2^\perp, X_1^\perp)$ is a generalized Kato decomposition for T^* consisting of weak–$*$ closed subspaces of X^*. Thus $\rho_{gk}(T) \subseteq \rho_{gk}(T^*)$. If $z \in \rho_{gk}(T)$, and (X_1, X_2) is a generalized Kato decomposition for T, then $K(T^* - z) = K((T^* - z)|_{X_2^\perp}) = R^\infty((T^* - z)|_{X_2^\perp}))$; in particular, $K(T^* - z)$ is weak–$*$ closed in X^*. Since $\rho_{gk}(T) \cap \sigma_K(T)$, is countable, the result follows. □

References

[1] P. Aiena, *Fredholm and Local Spectral Theory, with Applications to Multipliers*, Kluwer Academic Publ., Dordrecht, 2004.

[2] E. Albrecht and J. Eschmeier, *Analytic functional models and local spectral theory*, Proc. London Math. Soc. (3) **75** (1997), 323–348.

[3] E. Bishop *A duality theory for an arbitrary operator*, Pac. J. Math. **9** (1959), 379–397.

[4] J. Eschmeier, *Analytische Dualität und Tensorprodukte in der mehrdimensionalen Spektraltheorie*, Habilitationsschrift, Schriftenreihe des Mathematischen Instituts der Universität Münster, 2. Serie, Heft 42, Münster, 1987.

[5] J. Eschmeier, *On the essential spectrum of Banach space operators*, Proc. Edinburgh Math. Soc. (2) **43** (2000), 511–528.

[6] K. B. Laursen and M. M. Neumann, *An Introduction to Local Spectral Theory*, Clarendon Press, Oxford, 2000.

[7] T. L. Miller and V. G. Miller, *An operator satisfying Dunford's condition* (C) *but without Bishop's property* (β), with Vivien G. Miller, Glasgow Math. J., 40 (1998), 427–430.

[8] T. L. Miller, V. G. Miller and M. M. Neumann, *Localization in the spectral theory of operators on Banach spaces*, Proceedings of the Fourth Conference on Function Spaces at Edwardsville, Contemp. Math. **328**, Amer. Math. Soc., Providence, RI, 2003, 247–262.

[9] T. L. Miller, V. G. Miller and M. M. Neumann, *The Kato-type spectrum and local spectral theory*, Czech. Math. J., to appear.

[10] V. Müller, *Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras*, Birkhäuser Verlag, Basel, 2003.

[11] B. Nagy, *On S-decomposable operators,* J. Operator Theory **2** (1979), 277–286.

[12] F.-H. Vasilescu, *Analytic Functional Calculus and Spectral Decompositions*, Editura Academiei and D. Reidel Publishing Company, Bucharest and Dordrecht, 1982.

Dept. of Mathematics and Statistics, Mississippi State University, Drawer MA, Mississippi State, MS 39762

E-mail address: miller@math.msstate.edu

Mathematical Institute, Czech Academy of Sciences, Zitna 25, 115 67 Prague 1, Czech Republic

E-mail address: muller@math.cas.cz