Orbits of operators

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Abstract

The aim of this paper is to give a survey of results and ideas concerning orbits of operators and related notions of weak and polynomial orbits. These concepts are closely related to the invariant subspace/subset problem. Most of the proofs are not given in full details, we rather try to indicate the basic ideas. The central problems in the field are also formulated.

Mathematics Subject Classification: primary 47A05, 47A15, 47A16, secondary 47A11.
Keywords: orbits, invariant subspace problem, hypercyclic vectors, weak orbits, capacity, Scott Brown technique.

1 Introduction

Denote by $B(X)$ the algebra of all bounded linear operators acting on a complex Banach space X. Let $T \in B(X)$. By an orbit of T we mean a sequence $\{T^n x : n = 0, 1, \dots\}$ where $x \in X$ is a fixed vector.

The concept of orbits comes from the theory of dynamical systems. In the context of operator theory the notion was first used by Rolewicz [Ro]. Orbits of operators are closely connected with the local spectral theory, the theory of semigroups of operators [N], and especially, with the invariant subspace problem, see e.g. [B2].

The invariant subspace problem is the most important open problem of operator theory. Recall that a subset $M \subset X$ is invariant with respect to an operator $T \in B(X)$ if $TM \subset M$. The set M is nontrivial if $\{0\} \neq M \neq X$.

Problem 1.1. (invariant subspace problem) Let T be an operator on a Hilbert space H of dimension ≥ 2. Does there exist a nontrivial closed subspace invariant with respect to T?

It is easy to see that the problem has sense only for separable infinite-dimensional spaces. Indeed, if H is nonseparable and $x \in H$ any nonzero vector, then the vectors $x, Tx, T^2x, \dots$ span a nontrivial closed subspace invariant with respect to T.

*Supported by grant No. 201/03/0041 of GA ČR.

If $\dim H < \infty$, then T has at least one eigenvalue and the corresponding eigenvector generates an invariant subspace of dimension 1. Note that the existence of eigenvalues is equivalent to the fundamental theorem of algebra that each nonconstant complex polynomial has a root. Thus the invariant subspace problem is nontrivial even for finite-dimensional spaces.

Examples of Banach space operators without nontrivial closed invariant subspaces were given by Enflo [E], Beuzamy [B1] and Read [R1]. Read [R2] also gave an example of an operator T with a stronger property that T has no nontrivial closed invariant subset.

It is not known whether such an operator exists on a Hilbert space. The following "invariant subset problem" may be easier than Problem 1.1.

Problem 1.2. (invariant subset problem) Let T be an operator on a Hilbert space H. Does there exist a nontrivial closed subset invariant with respect to T?

Both Problems 1.1 and 1.2 are also open for operators on reflexive Banach spaces. More generally, the following problem is open:

Problem 1.3. Let T be an operator on a Banach space X. Does T^* have a nontrivial closed invariant subset/subspace?

It is easy to see that an operator $T \in B(X)$ has no nontrivial closed invariant subspace if and only if all orbits corresponding to nonzero vectors span all the space X (i.e., each nonzero vector is cyclic).

Similarly, $T \in B(X)$ has no nontrivial closed invariant subset if and only if all orbits corresponding to nonzero vectors are dense, i.e., all nonzero vectors are hypercyclic.

Thus orbits provide the basic information about the structure of an operator.

Typically, the behaviour of an orbit $\{T^n x : n = 0, 1, \dots\}$ depends much on the initial vector $x \in X$. An operator can have some orbits very regular and other orbits extremely irregular.

Example 1.4. Let H be a separable Hilbert space with an orthonormal basis $\{e_0, e_1, \dots\}$. Let S be the backward shift, i.e., S is defined by $Se_0 = 0$ and $Se_i = e_{i-1} \quad (i \geq 1)$. Consider the operator $T = 2S$. Then:

(i) there is a dense subset $M_1 \subset H$ such that $\|T^n x\| \to 0 \quad (x \in M_1)$;

(ii) there is a dense subset $M_2 \subset H$ such that $\|T^n x\| \to \infty \quad (x \in M_2)$;

(iii) there is a residual subset $M_3 \subset H$ (i.e., $H \setminus M_3$ is of the first cetegory) such that the set $\{T^n x : n = 0, 1, \dots\}$ is dense in H for all $x \in M_3$.

As the set M_1 it is possible to take the set of all finite linear combinations of the basis vectors e_i. Properties (ii) and (iii) follow from general results that will be discussed in the subsequent sections.

The paper is organized as follows. In the following section we study regular orbits. Of particular interest are the orbits satisfying $\|T^n x\| \to \infty$. It is easy to see that if an operator T has such an orbit, then $\{T^n x : n = 0, 1, \dots\}^-$ is a nontrivial closed invariant subset for T.

In the third section we study the other extreme — hypercyclic vectors, i.e., the vectors with very irregular orbits.

In the subsequent sections we study weak and polynomial orbits. A weak orbit of T is a sequence $\{\langle T^n x, x^* \rangle : n = 0, 1, \dots\}$ and a polynomial orbit of T is a set of the form $\{p(T)x : p \text{ polynomial}\}$, where $x \in X$ and $x^* \in X^*$.

Polynomial orbits are closely related with the notions of capacity and local capacity of an operator. These concepts are studied in Section 6.

In the last section we discuss the Scott Brown technique which is the most efficient method of constructing invariant subspaces of operators. In an illustrative example we show the basic ideas of the method, which are closely connected with orbits.

For simplicity we consider only complex Banach spaces. However, some results concerning orbits remain true also for real Banach spaces. In particular, all results based on the Baire category theorem remain unchanged for real Banach spaces.

Although the invariant subspace problem is usually formulated for complex Hilbert spaces, the corresponding question for real spaces (of dimension ≥ 3) is also open; it is very easy to find an operator on a 2-dimensional real Hilbert space without nontrivial invariant subspaces.

2 Regular orbits

Let X be a complex Banach space and let $T \in B(X)$. If T is power bounded (i.e., $\sup_n \|T^n\| < \infty$) then all orbits are bounded. The converse follows from the Banach-Steinhaus theorem.

Theorem 2.1. Let $T \in B(X)$. Then T is power bounded if and only if $\sup_n \|T^n x\| < \infty$ for all $x \in X$.

A more precise statement is given by the following theorem. Recall that a subset $M \subset X$ is called residual if its complement $X \setminus M$ is of the first category. Equivalently, M is residual if and only if it contains a dense G_δ subset.

Theorem 2.2. Let $T \in B(X)$ and let (a_n) be a sequence of positive numbers such that $\lim_{n\to\infty} a_n = 0$. Then the set of all points $x \in X$ with the property that $\|T^n x\| \geq a_n \|T^n\|$ for infinitely many n is residual.

Proof. The statement is trivial if T is nilpotent. In the following we assume that $T^n \neq 0$ for all n.

For $k \in \mathbb{N}$ set

$$M_k = \Big\{ x \in X : \text{ there exists } n \geq k \text{ such that } \|T^n x\| > a_n \|T^n\| \Big\}.$$

Clearly M_k is an open set. We prove that M_k is dense. Let $x \in X$ and $\varepsilon > 0$. Choose $n \geq k$ such that $a_n \varepsilon^{-1} < 1$. There exists $z \in X$ of norm one such that $\|T^n z\| > a_n \varepsilon^{-1} \|T^n\|$. Then

$$2a_n \|T^n\| < \|T^n(2\varepsilon z)\| \leq \|T^n(x + \varepsilon z)\| + \|T^n(x - \varepsilon z)\|,$$

and so either $\|T^n(x + \varepsilon z)\| > a_n \|T^n\|$ or $\|T^n(x - \varepsilon z)\| > a_n \|T^n\|$. Thus either $x + \varepsilon z \in M_k$ or $x - \varepsilon z \in M_k$, and therefore $\operatorname{dist}\{x, M_k\} \leq \varepsilon$. Since x and ε were arbitrary, the set M_k is dense.

By the Baire category theorem, the intersection $\bigcap_{k=1}^{\infty} M_k$ is a dense G_δ set, hence it is residual. Clearly each $x \in \bigcap_{k=1}^{\infty} M_k$ satisfies $\|T^n x\| \geq a_n \|T^n\|$ for infinitely many n. $\square$

Denote by $r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}$ the spectral radius of an operator $T \in B(X)$. By the spectral radius formula we have $r(T) = \lim_{n \to \infty} \|T^n\|^{1/n} = \inf_n \|T^n\|^{1/n}$. Recall that $r(T^n) = \big(r(T)\big)^n$ for all n.

For $x \in X$ let $r_x(T)$ denote the local spectral radius defined by $r_x(T) = \limsup_{n \to \infty} \|T^n x\|^{1/n}$ (the limit $\lim_{n \to \infty} \|T^n x\|^{1/n}$ in general does not exist). The local spectral radius plays an important role in the local spectral theory. Note that the resolvent $z \mapsto (z - T)^{-1} = \sum_{n=0}^{\infty} \frac{T^n}{z^{n+1}}$ is analytic on the set $\{z : |z| > r(T)\}$. Similarly, the local resolvent $z \mapsto (z - T)^{-1} x = \sum_{n=0}^{\infty} \frac{T^n x}{z^{n+1}}$ can be analytically extended to the set $\{z : |z| > r_x(T)\}$.

It is easy to see that $r_x(T) \leq r(T)$ for all $x \in X$.

Corollary 2.3. cf. [V] Let $T \in B(X)$. Then the set $\{x \in X : r_x(T) = r(T)\}$ is residual.

Proof. Let $a_n = n^{-1}$. By Theorem 2.2, there is a residual subset $M \subset X$ such that for each $x \in M$ we have $\|T^n x\| \geq n^{-1} \|T^n\|$ for infinitely many n. Thus

$$r_x(T) = \limsup_{n \to \infty} \|T^n x\|^{1/n} \geq \limsup_{n \to \infty} \Big(\frac{\|T^n\|}{n}\Big)^{1/n} = r(T)$$

for all $x \in M$. $\square$

As we have seen, it is relatively easy to construct vectors x such that infinitely many powers $T^n x$ are large. It is much more difficult to construct orbits such that all powers $T^n x$ are large in the norm. The result (and many other results concerning orbits) is based on the spectral theory and therefore it is valid only for complex spaces. For real Banach spaces see Remark 2.14.

Denote by $\sigma_e(T)$ the essential spectrum of $T \in B(X)$, i.e., the spectrum of $\rho(T)$ in the Calkin algebra $B(X)/\mathcal{K}(X)$, where $\mathcal{K}(X)$ is the ideal of all compact operators on X and $\rho : B(X) \longrightarrow B(X)/\mathcal{K}(X)$ is the canonical projection. Equivalently, $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}$. Let $r_e(T)$ denote the essential spectral radius, $r_e(T) = \max\{|\lambda| : \lambda \in \sigma_e(T)\}$.

If X is an infinite dimensional Banach space and $T \in B(X)$ then $\sigma_e(T)$ is a nonempty compact subset of $\sigma(T)$. Moreover, the difference $\sigma(T) \setminus \sigma_e(T)$

is equal to the union of some bounded components of $\mathbb{C} \setminus \sigma_e(T)$ and of at most countably many isolated points. In particular, if $\lambda \in \sigma(T)$ belongs to the unbounded component of $\mathbb{C} \setminus \sigma_e(T)$ then λ is an isolated point of the spectrum $\sigma(T)$, it is an eigenvalue of finite multiplicity and the corresponding spectral subspace is finite dimensional.

Denote further by $\sigma_{\pi e}(T)$ the essential approximate point spectrum of T, i.e., $\sigma_{\pi e}(T)$ is the set of all complex numbers λ such that

$$\inf\bigl\{\|(T-\lambda)x\| : x \in M, \|x\| = 1\bigr\} = 0$$

for every subspace $M \subset X$ with $\operatorname{codim} \mathrm{M} < \infty$.

It is easy to see that $\lambda \notin \sigma_{\pi e}(T)$ if and only if $\dim \operatorname{Ker}(\mathrm{T}-\lambda) < \infty$ and $T-\lambda$ has closed range, i.e., if $T-\lambda$ is upper semi-Fredholm. It is known [HW] that $\sigma_{\pi e}(T)$ contains the topological boundary of the essential spectrum $\sigma_e(T)$. In particular, $\sigma_{\pi e}(T)$ is a nonempty compact subset of the complex plane for each operator T on an infinite dimensional Banach space X.

The elements of the essential approximate point spectrum $\sigma_{\pi e}(T)$ are very useful for the study of orbits. For each $\lambda \in \sigma_{\pi e}(T)$ there are "approximate eigenvectors" — vectors $x \in X$ of norm 1 such that $\|(T-\lambda)x\|$ is arbitrarily small. Moreover, the approximate eigenvectors can be chosen in an arbitrary subspace of finite codimension. This property is particularly useful in various inductive constructions.

The following result was proved in [M1]; for Hilbert space operators see [B2].

Theorem 2.4. Let T be an operator on a Banach space X, let $\varepsilon > 0$ and let (a_n) be a sequence of positive numbers such that $\lim_{n\to\infty} a_n = 0$. Then:

(i) there exists a vector $x \in X$ such that $\|x\| < \sup_n a_n + \varepsilon$ and $\|T^n x\| \geq a_n \cdot r(T^n)$ for all n;

(ii) there exists a dense subset of points $x \in X$ such that $\|T^n x\| \geq a_n r(T^n)$ for all but a finite number of n.

Outline of the proof. Let $\lambda \in \sigma(T)$ satisfy $|\lambda| = r(T)$. We distinguish two cases:

(a) Suppose that $r(T) > r_e(T)$. Then λ is an eigenvalue of T. The corresponding eigenvector x of norm one satisfies $\|T^n x\| = \|\lambda^n x\| = r(T^n)$ for all n.

Moreover, λ is an isolated point of the spectrum of T and the spectral subspace X_0 corresponding to λ is finite dimensional. Let $u \in X \setminus X_0$. It is easy to verify that there is a positive constant $c = c(u)$ such that $\|T^j u\| \geq c \cdot r(T^j)$ for all j. Thus in this case the set of all points satisfying (ii) is even residual.

(b) Let $r(T) = r_e(T)$. Since $\lambda \in \sigma_{\pi e}(T)$, for all $\varepsilon > 0$, $n \in \mathbb{N}$ and $M \subset X$ of finite codimension there exists $x \in M$ such that $\|x\| = 1$ and $\|(T^j - \lambda^j)x\| < \varepsilon \quad (j \leq n)$. These "approximate eigenvectors" are basic building stones used in the construction of the vector with the required properties.

We indicate the proof for Hilbert space operators.

Without loss of generality it is possible to assume that $1 > a_1 > a_2 > \cdots$, $r(T) = 1$ and $\lambda = 1$. For $k \geq 0$ set $r_k = \min\{j : a_j < 2^{-k}\}$.

We construct inductively vectors $x_k \in X$ of norm 1 such that $T^j x_k$ is approximately equal to x_k for $j \leq r_k$ and

$$T^j x_k \perp T^j x_i \quad (i < k, j \leq r_k)$$

(note that the subspace $\{u : T^j u \perp T^j x_i \quad (i < k, j \leq r_k)\}$ is of finite codimension).

Set $x = \sum_{i=1}^{\infty} 2^{-i+1} x_i$.

Let $r_{k-1} < j \leq r_k$. Since $T^j x_i \perp T^j x_k \quad (i \neq k)$, we have

$$\|T^j x\| = \Big\|\sum_{i=1}^{\infty} 2^{-i+1} T^j x_i\Big\| \geq \|2^{-k+1} T^j x_k\|.$$

The last term is approximately equal to 2^{-k+1}, which is greater than a_j. Thus x satisfies $\|T^j x\| \geq a_j$ for all j.

The full statement of Theorem 2.4 can be obtained by a modification of this argument; we omit the details. □

For Banach spaces the proof is a little bit more complicated. The basic idea is to use instead of the orthogonal complement of a finite dimensional subspace (which was in fact used here) the following lemma.

Lemma 2.5. Let X be a Banach space, let $F \subset X$ be a finite dimensional subspace and let $\varepsilon > 0$. Then there exists a subspace $M \subset X$ of finite codimension such that

$$\|m + f\| \geq (1 - \varepsilon) \max\{\|f\|, \|m\|/2\}$$

for all $f \in F$ and $m \in M$.

An immediate consequence of Theorem 2.4 is the following corollary.

Corollary 2.6. Let $T \in B(X)$. Then:
(i) the set $\{x \in X : \liminf_{n\to\infty} \|T^n x\|^{1/n} = r(T)\}$ is dense;
(ii) the set $\{x \in X : \limsup_{n\to\infty} \|T^n x\|^{1/n} = r(T)\}$ is residual;
(iii) the set of all $x \in X$ such that the limit $\lim_{n\to\infty} \|T^n x\|^{1/n}$ exists (and is equal to $r(T)$) is dense.

As another corollary we get that the infimum and the supremum in the spectral radius formula

$$r(T) = \inf_k \|T^k\|^{1/k} = \inf_k \sup_{\|x\|=1} \|T^k x\|^{1/k}$$

can be exchanged.

Corollary 2.7. Let $T \in B(X)$. Then

$$r(T) = \sup_{\|x\|=1} \inf_k \|T^k x\|^{1/k}.$$

Example 2.8. Let H be a separable Hilbert space with an orthonormal basis $\{e_j : j = 0, 1, \ldots\}$ and let S be the backward shift, $Se_0 = 0$, $Se_j = e_{j-1} \quad (j \geq 1)$. Then the set $\{x \in H : \liminf_{n\to\infty} \|S^n x\|^{1/n} = 0\}$ is residual.

Since $r(S) = 1$ and the set $\{x \in H : \limsup_{n\to\infty} \|S^n x\|^{1/n} = 1\}$ is also residual, we see that the set $\{x \in H : \text{ the limit } \lim_{n\to\infty} \|S^n x\|^{1/n} \text{ exists}\}$ is of the first category (but it is always dense by Corollary 2.6).

Proof. For $k \in \mathbb{N}$ let

$$M_k = \Big\{x \in X : \text{ there exists } n \geq k \text{ such that } \|S^n x\| < k^{-n}\Big\}.$$

Clearly M_k is an open subset of X. Further, M_k is dense in X. To see this, let $u \in X$ and $\varepsilon > 0$. Let $u = \sum_{j=0}^\infty \alpha_j e_j$ and choose $n \geq k$ such that $\sum_{j=n}^\infty |\alpha_j|^2 < \varepsilon^2$. Set $y = \sum_{j=0}^{n-1} \alpha_j e_j$. Then $\|y - u\| < \varepsilon$ and $S^n y = 0$. Thus $y \in M_k$ and M_k is a dense open subset of X.

By the Baire category theorem, the set $M = \bigcap_{k=0}^\infty M_k$ is a dense G_δ subset of X, hence it is residual.

Let $x \in M$. For each $k \in \mathbb{N}$ there is an $n_k \geq k$ such that $\|S^{n_k} x\| < k^{-n_k}$, and so $\liminf_{n\to\infty} \|S^n x\|^{1/n} = 0$. □

It is also possible to combine conditions of Theorems 2.2 and 2.4 and to obtain points $x \in X$ with $\|T^n x\| \geq a_n \cdot \|T^n\|$ for all n; in this case, however, there is a restriction on the sequence (a_n).

Theorem 2.9. Let $T \in B(X)$, let (a_n) be a sequence of positive numbers such that $\sum_n a_n^{2/3} < \infty$. Then there exists $x \in X$ such that $\|T^n x\| \geq a_n \|T^n\|$ for all n. There is a dense subset $L \subset X$ such that for each $x \in L$ there is a $k \in \mathbb{N}$ with the property that

$$\|T^n x\| \geq a_n \|T^n\| \qquad (n \geq k).$$

Outline of the proof: Fix $k \in \mathbb{N}$. We indicate the construction of a vector x satisfying $\|T^j x\| \geq a_j \|T^j\| \quad (j \leq k)$. The vector satisfying this relations for all n can be then obtained by a limit procedure.

For $j = 1, 2, \ldots, k$ fix a vector $z_j \in X$ of norm one which almost attains the norm of T^j, i.e., $\|T^j x\| \doteq \|T^j\|$ (we omit the exact calculations).

Let $\Lambda = \{\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k : |\lambda_j| \leq a_j^{2/3} \text{ for all } j\}$. For $\lambda \in \Lambda$ let $u_\lambda = \sum_{j=1}^k \lambda_j z_j$. Consider the Lebesgue measure μ on Λ.

For $j = 1, \ldots, k$ let $\Lambda_j = \{\lambda \in \Lambda : \|T^j u_\lambda\| < a_j \|T^j\|\}$. The basic idea of the proof is to show that $\mu(\Lambda \setminus \bigcup_{j=1}^k \Lambda_j) > 0$, which means that there exists $\lambda \in \Lambda$ such that $\|T^j u_\lambda\| \geq a_j \|T^j\|$ for all $j = 1, \ldots, k$. For details see [M5]. $\square$

A better estimate can be obtained if we replace the norm $\|S\|$ of an operator $S \in B(X)$ by the quantity $\|S\|_\mu = \inf\bigl\{\|S|M\| : M \subset X, \operatorname{codim} M < \infty\bigr\}$.

If S is an operator on a separable Hilbert space H then $\|S\|_\mu$ coincides with the essential norm $\|S\|_e = \inf\{\|S + K\| : K \in \mathcal{K}(H)\}$.

For the proof of the next result see [M5].

Theorem 2.10. Let $T \in B(X)$. Let (a_n) be a sequence of positive numbers satisfying $\sum_n a_n < \infty$. Then there exists $x \in X$ such that $\|T^n x\| \geq a_n \|T^n\|_\mu$ for all n.

The results of Theorems 2.9 and 2.10 can be improved for Hilbert space operators, see [B2].

Theorem 2.11. Let T be an operator on a Hilbert space H, let (a_n) be a sequence of positive numbers.

(i) if $\sum_n a_n < \infty$ then there exists $x \in H$ such that $\|T^n x\| \geq a_n \|T^n\|$ for all $n \in \mathbb{N}$;

(ii) if $\sum_n a_n^2 < \infty$ then there exists $x \in H$ such that $\|T^n x\| \geq a_n \|T^n\|_e$ for all $n \in \mathbb{N}$.

The following result is true for Hilbert space operators; in Banach spaces it is false.

Theorem 2.12. [B2] Let T be a non-nilpotent operator on a Hilbert space H. Then the set $\Bigl\{x \in H : \sum\limits_n \frac{\|T^n x\|}{\|T^n\|} = \infty\Bigr\}$ is residual.

Example 2.13. Let X be the ℓ_1 space with the standard basis $\{e_i : i = 0, 1 \ldots\}$. Let $T \in B(X)$ be defined by $Te_0 = 0$ and $Te_n = (\frac{n+1}{n})^2 e_{n-1} \quad (n \geq 1)$. Then $\sum\limits_n \frac{\|T^n x\|}{\|T^n\|} < \infty$ for all $x \in X$.

This can be verified by a direct calculation, see [M5].

Remark 2.14. Some results from this section remain true for real Banach spaces as well, see [M5]. This is true for Theorem 2.2.

Theorem 2.4 can be reformulated as follows: if $a_n > 0$, $a_n \to 0$, then there exists a dense subset $L \subset X$ such that for each $x \in L$ there is a constant $c > 0$ with $\|T^n x\| > c a_n r(T)^n$ for all n.

Theorem 2.9 can be modified in the following way: let T be an operator on a real Banach space X, let (a_n) be a sequence of positive numbers such that

$\sum_n a_n^{1/2} < \infty$. Then there exists $x \in X$ such that $\|T^n x\| \geq a_n \|T^n\|$ for all n. There is a dense subset $L \subset X$ such that for each $x \in L$ there is a $k \in \mathbb{N}$ with the property that

$$\|T^n x\| \geq a_n \|T^n\| \qquad (n \geq k).$$

3 Hypercyclic vectors

Vectors with extremely irregular orbits are called hypercyclic. More precisely, a vector $x \in X$ is called hypercyclic for an operator $T \in B(X)$ if the set $\{T^n x : n = 0, 1, \dots\}$ is dense in X. An operator T is called hypercyclic if there is at least one vector hypercyclic for T.

Recall also that a vector $x \in X$ is called cyclic for $T \in B(X)$ if the set $\{p(T)x : p \text{ polynomial}\}$ is dense in X, and supercyclic for T if $\{\lambda T^n x : \lambda \in \mathbb{C}, n = 0, 1 \dots\}^- = X$.

These notions make sense only for separable Banach spaces. It is easy to see that an operator in a non-separable Banach space can not have cyclic (supercyclic, hypercyclic) vectors. Moreover, it is not difficult to show that there are no hypercyclic operators on finite-dimensional Banach spaces (this follows from the fact that T^* has eigenvalues, cf. the proof of Theorem 3.2 below). In the rest of this section we assume that all Banach spaces are infinite dimensional and separable.

It is easy to find an operator that is not hypercyclic. For example, any contraction (or more generally, a power bounded operator) is not hypercyclic. On the other hand, the existence of hypercyclic operators is not so obvious. The first example of a hypercyclic vector was given by Rolewicz [Ro]. In the last years, hypercyclic vectors have been studied intensely by a number of authors. For a survey of results with an extensive bibliography see [Gr].

It turns out that hypercyclic vectors are not so exceptional that they may seem at the first glance. The first result shows that if an operator has at least one hypercyclic vector then almost all vectors are hypercyclic.

Theorem 3.1. Let $T \in B(X)$ be a hypercyclic operator. Then there is a residual set of vectors hypercyclic for T.

Proof. Let $x \in X$ be hypercyclic for T. For each $k \in \mathbb{N}$ the vector $T^k x$ is hypercyclic for T, and so the set of all hypercyclic vectors is dense.

Let (U_j) be a countable base of open subsets in X. It is easy to see that the set of all vectors hypercyclic for T is equal to $\bigcap_j \bigcup_n T^{-n} U_j$, which is a G_δ subset. $\square$

Theorem 3.2. [Bo] Let $T \in B(X)$ be a hypercyclic operator. Then there exists a dense linear manifold $L \subset X$ such that each nonzero vector in L is hypercyclic for T.

Proof. We show first that T^* has no eigenvalues. Suppose on the contrary that there are $\lambda \in \mathbb{C}$ and a nonzero vector $x^* \in X^*$ such that $T^* x^* = \lambda x^*$.

Let $x \in X$ be a hypercyclic vector for T. Then

$$\mathbb{C} = \big\{\langle T^n x, x^*\rangle : n = 0, 1, \dots\big\}^- = \langle x, x^*\rangle \cdot \big\{\lambda^n : n = 0, 1, \dots\big\}^-.$$

It is easy to see that the last set can not be dense in $\mathbb{C}$. Thus λ is not an eigenvalue of T^*, and so $(T - \lambda)X$ is dense in X for each $\lambda \in \mathbb{C}$.

Let x be a hypercyclic vector for T. We show that $p(T)x$ is also hypercyclic for each nonzero polynomial p. Write $p(z) = \alpha(z - \lambda_1) \cdots (z - \lambda_n)$ where $\alpha \neq 0$, $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. Then

$$\big\{T^n p(T)x : n = 0, 1, \dots\big\} = \alpha(T - \lambda_1) \cdots (T - \lambda_n)\{T^n x : n = 0, 1, \dots\}.$$

The last set is dense in X since x is hypercyclic for T, and the operators $T - \lambda_j$ have dense ranges for each j. Thus $p(T)x$ is hypercyclic for T. □

The following criterion provides a simple way of constructing hypercyclic vectors, see [K], [GS]. It also implies property (iii) in Example 1.4.

Theorem 3.3. Let $T \in B(X)$. Suppose that there is an increasing sequence of positive integers (n_k) such that:

(i) there is a dense subset $X_0 \subset X$ such that $\lim_{k\to\infty} T^{n_k} x \to 0$ for all $x \in X_0$;

(ii) $\bigcup_k T^{n_k} B_X$ is dense in X, where B_X denotes the closed unit ball in X.

Then T is hypercyclic. By Theorem 3.1, this means that the set of all hypercyclic vectors is residual.

Conversely, suppose that T is hypercyclic. Then it is not difficult to show that T satisfies both conditions (i) and (ii), but not necessarily for the same subsequence (n_k). Thus the conditions in Theorem 3.3 are close to the notion of hypercyclicity (cf. Problem 3.12).

A similar criterion may be used to construct closed infinite dimensional subspaces consisting of hypercyclic vectors, see [Mo], [GLMo] and [LMo].

Theorem 3.4. Let $T \in B(X)$. Suppose that T satisfies the conditions of Theorem 3.3 and that the essential spectrum $\sigma_e(T)$ intersects the closed unit ball. Then there is a closed infinite dimensional subspace $M \subset X$ such that each nonzero vector in M is hypercyclic for T.

Theorems 3.1 – 3.4 indicate that hypercyclic vectors and operators are quite common, that in some sense it is a typical behavior of an orbit.

Similarly as in Theorem 3.1 it is possible to show that the set of all hypercyclic operators on a Banach space X is a G_δ set. It is not dense since the operators with $\|T\| < 1$ can not be hypercyclic. Thus the set of all hypercyclic operators is a residual subset of its closure.

By [He], it is possible to characterize the closure of hypercyclic operators on a separable Hilbert space. For Banach spaces such a characterization is not known.

Theorem 3.5. Let H be a separable Hilbert space, let $T \in B(H)$. Then T belongs to the closure of hypercyclic operators if and only if the following conditions are satisfied:

(i) the set $\sigma_W(T) \cup \{z \in \mathbb{C} : |z| = 1\}$ is connected;

(ii) $\sigma_0(T) = \emptyset$;

(iii) $\operatorname{ind}(\lambda - T) \geq 0$ for all $\lambda \in \mathbb{C}$ such that $\lambda - T$ is semi-Fredholm.

Here $\sigma_W(T)$ denotes the Weyl spectrum of T, $\sigma_W(T) = \bigcap_{K \in \mathcal{K}(H)} \sigma(T+K)$. Equivalently, $\lambda \notin \sigma_W(T)$ if and only if $T - \lambda$ is Fredholm and $\operatorname{ind}(T - \lambda) = 0$.

Recall that an operator S is called semi-Fredholm if it has closed range and either $\dim \ker S < \infty$ or $\operatorname{codim} SX < \infty$. The index of a semi-Fredholm operator S is defined by $\operatorname{ind} S = \dim \ker S - \operatorname{codim} SX$.

Furthermore, $\sigma_0(T)$ denotes the set of all isolated points of $\sigma(T)$ such that the corresponding spectral subspace is finite dimensional.

Theorem 3.6. Let $T \in B(X)$ be an operator and let $x \in X$ be hypercyclic for T. Then:

(i) x is hypercyclic for T^n for each $n \in \mathbb{N}$;

(ii) x is hypercyclic for λT for each $\lambda \in \mathbb{C}, |\lambda| = 1$;

(iii) if T is invertible then T^{-1} is hypercyclic.

The first two statements of Theorem 3.6 are quite deep, the third one is an easy consequence of the observation that T is hypercyclic if and only if for all nonempty open subsets $U, V \subset X$ there exists $n \in \mathbb{N}$ such that $T^n U \cap V \neq \emptyset$.

Statement (i) was proved by Ansari [A] and it in fact started the serious study of hypercyclic operators. For (ii) see [LM].

Although it is relatively easy to construct an operator with a residual set of hypercyclic vectors (see Example 1.4), it is extremely difficult to construct an operator with all nonzero vectors hypercyclic (recall that such an operator gives a counterexample to the invariant subspace problem). The first example of this type was constructed by Read [R2] on the space ℓ_1. Equivalently, such an operator has no nontrivial closed invariant subset. It is an open problem whether this can happen in Hilbert spaces, cf. Problems 1.2 and 1.3.

It follows from the previous results that such an operator must satisfy certain rather narrow conditions on the norms $\|T^n\|$.

Theorem 3.7. Let T be an operator on a Banach space X which has no nontrivial closed invariant subsets. Then $r(T) = r_e(T) = 1$, $\sup_n \|T^n\| = \infty$, $\sum_n \|T^n\|^{-2/3} < \infty$ and $\sum_n \|T^n\|_\mu^{-1} < \infty$.

If X is a Hilbert space then $\sum_n \|T^n\|^{-1} < \infty$ and $\sum_n \|T^n\|_e^{-1/2} < \infty$.

Indeed, if T does not satisfy the conditions above, then either T is power bounded or there exists a vector $x \in X$ with $\|T^n x\| \to \infty$, see Theorems 2.4, 2.9, 2.10 and 2.11. Hence $\{T^n x : n = 0, 1 \ldots\}^-$ is a nontrivial closed invariant subset with respect to T.

Thus it is a very interesting question for which operators there are orbits satisfying $\|T^n x\| \to \infty$.

Problem 3.8. What are the best exponents in Theorem 3.7?

Example 3.9. There is an operator T on a Hilbert space H such that $\|T^n\| \to \infty$ and there is no $x \in H$ with $\|T^n x\| \to \infty$, see [B2]. As an example it is possible to take a unilateral weighted shift with suitable weights; the operator satisfies $\|T^n\| = (\ln n)^{1/2}$.

It is also possible to construct an operator $T \in B(H)$ such that $\inf_n \|T^n x\| = 0$ and $\sup_n \|T^n x\| = \infty$ for all nonzero vectors $x \in H$.

We mention now some other open problems.

Problem 3.10. Let T be a Hilbert space operator such that $\lim_{n\to\infty} \|T^n\| = \infty$ and the norms $\|T^n\|$ form a nondecreasing sequence. Does there exist a vector $x \in H$ such that $\|T^n x\| \to \infty$?

Problem 3.11. Is the characterization of the closure of hypercyclic operators (Theorem 3.5) true also for Banach spaces?

Problem 3.12. Does there exist a hypercyclic operator $T \in B(X)$ that does not satisfy conditions of Theorem 3.3?

There are other equivalent formulations of this problem. The most interesting reformulation is: does there exist a hypercyclic operator T such that $T \oplus T$ is not hypercyclic, see [BP], [He]?

We finish this section with a remark about weakly hypercyclic operators that have been introduced in [Fe].

An operator $T \in B(X)$ is called weakly hypercyclic if there exists a vector $x \in X$ such that the orbit $\{T^n x : n = 0, 1, \ldots\}$ is weakly dense in X.

Note that the corresponding notion of weakly cyclic vectors makes no sense since a closed linear manifold is automatically weakly closed by the Hahn-Banach theorem. However, it is possible that a weakly dense orbit is not dense, cf. [Fe]. Also, there is a weakly hypercyclic operator that is not hypercyclic, see [ChS].

4 Weak orbits

Weak orbits were introduced and first studied by van Neerven [N]. Many results for orbits of operators can be modified also for weak orbits. For a survey of results see e.g. [N], [M5].

The following three results are analogous to the corresponding statements for orbits.

Theorem 4.1. Let $T \in B(X)$ and let (a_n) be a sequence of positive numbers such that $\lim_{n\to\infty} a_n = 0$. Then the set of all pairs $(x, x^*) \in X \times X^*$ with the property that $|\langle T^n x, x^*\rangle| > a_n \|T^n\|$ for infinitely many n is a residual subset of $X \times X^*$.

Theorem 4.2. Let T be an operator on a Banach space X, let (a_n) be a sequence of positive numbers such that $\sum_n a_n^{1/3} < \infty$. Then there exist $x \in X$ and $x^* \in X^*$ such that $|\langle T^n x, x^*\rangle| \ge a_n \|T^n\|$ for all n.

Theorem 4.3. Let $T \in B(X)$. Then:
(i) the set $\big\{(x, x^*) \in X \times X^* : \liminf\limits_{n\to\infty} |\langle T^n x, x^*\rangle|^{1/n} = r(T)\big\}$ is dense;
(ii) the set $\big\{(x, x^*) \in X \times X^* : \limsup\limits_{n\to\infty} |\langle T^n x, x^*\rangle|^{1/n} = r(T)\big\}$ is residual;
(iii) the set of all pairs $(x, x^*) \in X \times X^*$ such that the limit $\lim\limits_{n\to\infty} |\langle T^n x\rangle|^{1/n}$ exists (and is equal to $r(T)$) is dense.

The statement analogous to Theorem 2.12 for weak orbits is not true:

Example 4.4. There exists an operator T on a Hilbert space H such that $\sum\limits_n \frac{|\langle T^n x, y\rangle|}{\|T^n\|} < \infty$ for all $x, y \in H$.

As an example it is possible to take the operator $T = \bigoplus_{k=1}^\infty S_k$, where S_k is the shift operator on a $(k+1)$-dimensional Hilbert space, see [M5].

The statement analogous to Theorem 2.4 for weak orbits is an open problem:

Problem 4.5. Let $T \in B(X)$, let (a_n) be a sequence of positive numbers satisfying $\lim_{n\to\infty} a_n = 0$. Do there exist $x \in X$ and $x^* \in X^*$ such that $|\langle T^n x, x^*\rangle| \ge a_n r(T^n)$ for all n?

The statement is false for real Banach spaces. A partial positive answer is given in the following case which is important from the point of view of the invariant subspace problem. Some other partial results were given in [N].

Theorem 4.6. Let T be an operator on a Hilbert space H such that $1 \in \sigma(T)$ and $T^n x \to 0$ for all $x \in H$. Let (a_n) be a sequence of positive numbers satisfying $\lim_{n\to\infty} a_n = 0$. Then there exists $x \in H$ such that $\mathrm{Re}\,\langle T^n x, x\rangle > a_n$ for all n.

Using Theorem 4.6 and techniques of [LM] it is possible to obtain the following result.

Theorem 4.7. Let T be a power bounded operator on a Hilbert space H satisfying $r(T) = 1$. Then there is a nonzero vector $x \in H$ such that x is not supercyclic. Moreover, if $1 \in \sigma(T)$ then T has a nontrivial closed invariant positive cone, i.e., there is a nontrivial closed subset $M \subset H$ such that $TM \subset M$, $M + M \subset M$ and $tM \subset M \quad (t \geq 0)$.

It is a natural question whether the previous result can be improved in order to obtain an invariant real subspace.

Problem 4.8. Let T be a power bounded operator on a Hilbert space such that $1 \in \sigma(T)$. Does T have a nontrivial closed invariant real subspace, i.e., does there exists a nontrivial closed subset $M \subset H$ such that $TM \subset M$, $M+M \subset M$ and $tM \subset M \quad (t \in \mathbb{R})$?

Problem 4.9. Is Theorem 4.7 true for operators on reflexive Banach spaces?

5 Polynomial orbits

If x is an eigenvector of T, $Tx = \lambda x$ for some complex λ, then $p(T)x = p(\lambda)x$ for every polynomial p, and so we have complete information about the polynomial orbit $\{p(T)x : p \text{ polynomial}\}$. Unfortunately, operators on infinite dimensional Banach spaces have usually no eigenvalues. The proper tool appears to be the notion of the essential approximate point spectrum $\sigma_{\pi e}(T)$.

The following result is an analogue of Theorem 2.4.

Theorem 5.1. [M3] Let T be an operator on a Banach space X, let $\lambda \in \sigma_{\pi e}(T)$. Let (a_n) be a sequence of positive numbers with $\lim_{n\to\infty} a_n = 0$. Then:
(i) there exists $x \in X$ such that

$$\|p(T)x\| \geq a_{\deg p} \cdot |p(\lambda)|$$

for every polynomial p;
(ii) let $u \in X$, $\varepsilon > 0$. Then there exists $y \in X$ and a positive constant $C = C(\varepsilon)$ such that $\|y - u\| \leq \varepsilon$ and

$$\|p(T)y\| \geq C \cdot a_{\deg p} \cdot |p(\lambda)|$$

for every polynomial p.

In the previous theorem we expressed the estimate of $\|p(T)x\|$ by means of $|p(\lambda)|$ where λ was a fixed element of $\sigma_{\pi e}(T)$. Next we are looking for an estimate in terms of $\max\{|p(\lambda)| : \lambda \in \sigma_{\pi e}(T)\}$. Since $\partial\sigma_e(T) \supset \sigma_{\pi e}(T)$, by the spectral mapping theorem for the essential spectrum σ_e we have

$$\max_{\lambda\in\sigma_{\pi e}(T)} |p(\lambda)| = \max_{\lambda\in\sigma_e(T)} |p(\lambda)| = \max\{|z| : z \in \sigma_e(p(T))\} = r_e(p(T)).$$

An important tool for the results in this section is the following classical lemma of Fekete [F]. It enables to estimate the maximum of a polynomial on a (in general very complicated) compact set $\sigma_{\pi e}(T)$ by means of its values at finitely many points.

Lemma 5.2. Let K be a non-empty compact subset of the complex plane and let $k \geq 1$. Then there exist points $u_0, u_1, \dots, u_k \in K$ such that

$$\max\{|p(z)| : z \in K\} \leq (k+1) \cdot \max_{0 \leq i \leq k} |p(u_i)|$$

for every polynomial p with $\deg p \leq k$. Moreover, we have

$$\max\{|p(z)| : z \in K\} \leq (k+1)^{1/2} \Big(\sum_{i=0}^{k} |p(u_i|^2 \Big)^{1/2}$$

for all polynomials p with $\deg p \leq k$.

By using the previous lemma we can get ([M2], [M4])

Theorem 5.3. Let T be an operator on a Banach space X, let $\varepsilon \geq 0$ and $k \geq 1$. Then:
(i) if $\operatorname{card} \sigma_{\pi e}(T) \geq k+1$ then there exists $x \in X$ with $\|x\| = 1$ and

$$\|p(T)x\| \geq \frac{1-\varepsilon}{k+1}\, r_e(p(T))$$

for every polynomial p with $\deg p \leq k$.
(ii) let $x \in X$ and $\varepsilon > 0$. Then there exists $y \in X$ and a positive constant $C = C(\varepsilon)$ such that $\|y - x\| \leq \varepsilon$ and

$$\|p(T)y\| \geq C \cdot (1 + \deg p)^{-(1+\varepsilon)}\; r_e(p(T))$$

for every polynomial p.

The proof of Theorem 5.3 is much simpler for operators on Hilbert spaces. The same result for Banach space operators can be obtained by a Dvoretzky's theorem type argument. A simpler proof based on Lemma 2.5 is available for weaker estimates $\|p(T)x\| \geq \frac{1-\varepsilon}{2(k+1)^2} r_e(p(T))$ and $\|p(T)x\| \geq C \cdot (1 + \deg p)^{-(2+\varepsilon)} r_e(p(T))$, respectively.

The estimates in Theorem 5.3 (i) are the best possible.

Example 5.4. [M4] Let $k \in \mathbb{N}$. There exists a Banach space X and an operator $T \in B(X)$ such that for each $x \in X$ of norm one there is a polynomial p of degree $\leq k$ with $\|p(T)x\| \leq (k+1)^{-1} r_e(p(T))$.

6 Capacity

The notion of capacity of an operator (or more generally, of a Banach algebra element) was introduced and studied by Halmos [H]. If $T \in B(X)$ then

$$\operatorname{cap} T = \lim_{k\to\infty} (\operatorname{cap}_k T)^{1/k} = \inf_k (\operatorname{cap}_k T)^{1/k},$$

where

$$\operatorname{cap}_k T = \inf\left\{\|p(T)\| : p \in \mathcal{P}_k^1\right\}$$

and $\mathcal{P}_k^1$ is the set of all monic (i.e., with leading coefficient equal to 1) polynomials of degree k.

This is a generalization of the classical notion of capacity (sometimes also called Tshebyshev constant) of a nonempty compact subset K of the complex plane:

$$\operatorname{cap} K = \lim_{k\to\infty} (\operatorname{cap}_k K)^{1/k} = \inf_k (\operatorname{cap}_k K)^{1/k}$$

where

$$\operatorname{cap}_k K = \inf\left\{\|p\|_K : p \in \mathcal{P}_k^1\right\} \qquad \text{and} \qquad \|p\|_K = \sup\{|p(z)| : z \in K\}.$$

The classical capacity $\operatorname{cap} K$ is equal to the capacity of the identical function $f(z) = z$ considered as an element of the Banach algebra of all continuous functions on K with the sup-norm.

Another connection between these two notions is given by the following main result of [H].

Theorem 6.1. $\operatorname{cap} T = \operatorname{cap} \sigma(T)$ for each operator $T \in B(X)$.

Let $x \in X$. The local capacity of T at x can be defined analogously. We define

$$\operatorname{cap}_k(T, x) = \inf\left\{\|p(T)x\| : p \in \mathcal{P}_k^1\right\}$$

and

$$\operatorname{cap}(T, x) = \limsup_{k\to\infty} \operatorname{cap}_k(T, x)^{1/k}$$

(in general the limit does not exist).

It is easy to see that $\operatorname{cap}(T, x) \le \operatorname{cap} T$ for every $x \in X$.

Note that there is an analogy between the spectral radius and the capacity of an operator:

$$\begin{aligned}
r(T) &= \lim_{k\to\infty} \|T^k\|^{1/k} = \inf \|T^k\|^{1/k}, \\
r_x(T) &= \limsup_{k\to\infty} \|T^k x\|^{1/k}, \\
\operatorname{cap} T &= \lim_{k\to\infty} (\operatorname{cap}_k T)^{1/k} = \inf(\operatorname{cap}_k T)^{1/k}, \\
\operatorname{cap}(T, x) &= \limsup_{k\to\infty} \left(\operatorname{cap}_k(T, x)\right)^{1/k}.
\end{aligned}$$

Furthermore, $\operatorname{cap} T \le r(T)$ and $\operatorname{cap}(T,x) \le r_x(T)$ for all $x \in X$.

Theorem 6.2. Let $T \in B(X)$. Then:

(i) the set $\{x \in X : \liminf_{k\to\infty} \operatorname{cap}_k(T,x)^{1/k} = \operatorname{cap} T\}$ is dense in X;

(ii) the set $\{x \in X : \operatorname{cap}(T,x) = \operatorname{cap} T\}$ is residual in X;

(iii) the set $\{x \in X : \lim_{k\to\infty} \operatorname{cap}_k(T,x)^{1/k} = \operatorname{cap} T\}$ is dense in X.

Outline of the proof. By Theorem 5.3, there is a dense subset of vectors $x \in X$ with the property that $\|p(T)x\| \ge C \cdot (1+\deg p)^{-2} r_e(p(T))$ for all polynomials p. Thus we have

$$\begin{aligned}\operatorname{cap}_k(T,x) &= \inf\bigl\{\|p(T)x\| : p \in \mathcal{P}_k^1\bigr\}\\ &\ge C\cdot(1+k)^{-2}\inf\bigl\{r_e(p(T)) : p \in \mathcal{P}_k^1\bigr\} = C(1+k)^{-2}\operatorname{cap}_k \sigma_e(T).\end{aligned}$$

Hence

$$\liminf_{k\to\infty}(\operatorname{cap}_k T)^{1/k} \ge \liminf_{k\to\infty}(\operatorname{cap}_k \sigma_e(T))^{1/k} = \operatorname{cap}\sigma_e(T).$$

Using the general relations between $\sigma(T)$ and $\sigma_e(T)$, it is possible to see that $\operatorname{cap}\sigma_e(T) = \operatorname{cap}\sigma(T)$. Hence, by Theorem 6.1, $\liminf_{k\to\infty}(\operatorname{cap}(T,x))^{1/k} = \operatorname{cap} T$ for all x in a dense subset of X.

The second statement requires a more refined arguments, see [M5]. $\square$

An operator $T \in B(X)$ is called quasialgebraic if and only if $\operatorname{cap} T = 0$. Similarly, T is called locally quasialgebraic if $\operatorname{cap}(T,x) = 0$ for every $x \in X$.

It follows from Theorem 6.2 that these two notions are equivalent.

Corollary 6.3. An operator is quasialgebraic if and only if it is locally quasialgebraic.

Corollary 6.3 is an analogy to the well-known result of Kaplansky: an operator is algebraic (i.e. $p(T) = 0$ for some non-zero polynomial p) if and only if it is locally algebraic (i.e., for every $x \in X$ there exists a polynomial $p_x \ne 0$ such that $p_x(T)x = 0$).

7 Scott Brown technique

The Scott Brown technique is an efficient way of constructing invariant subspaces. It was first used for subnormal operators but later it was adapted to contractions on Hilbert spaces and, more generally, to polynomially bounded operators on Banach spaces. Some results are also known for n-tuples of commuting operators.

We are going to give an illustrative example showing how this method works, but first we need some preliminary results.

The basic idea of the Scott Brown technique is to construct a weak orbit $\{\langle T^n x, x^* \rangle : n = 0, 1, \dots\}$ which behaves in a precise way. Typically, vectors $x \in X$ and $x^* \in X^*$ are constructed such that

$$\langle T^n x, x^* \rangle = \begin{cases} 0 & n \geq 1; \\ 1 & n = 0. \end{cases}$$

Equivalently,

$$\langle p(T)x, x^* \rangle = p(0) \tag{1}$$

for all polynomials p. Then T has a nontrivial closed invariant subspace. Indeed, either $Tx = 0$ (and x generates a 1-dimensional invariant subspace) or the vectors $\{T^n x : n \geq 1\}$ generate a nontrivial closed invariant subspace.

The vectors x and x^* satisfying the above described conditions are constructed as limits of sequences that satisfy (1) approximately.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc in the complex plane and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle. Denote by $\mathcal{P}$ the normed space of all polynomials with the norm $\|p\| = \sup\{|p(z)| : z \in \mathbb{D}\}$. Let $\mathcal{P}^*$ be its dual with the usual dual norm.

Let $\phi \in \mathcal{P}^*$. By the Hahn-Banach theorem, ϕ can be extended without changing the norm to a functional on the space of all continuous function on $\mathbb{T}$ with the sup-norm. By the Riesz theorem, there exists a Borel measure μ on $\mathbb{T}$ such that $\|\mu\| = \|\phi\|$ and $\phi(p) = \int p \, d\mu$ for all polynomials p. Clearly, the measure is not unique.

Let L^1 be the Banach space of all complex integrable functions on $\mathbb{T}$ with the norm $\|f\|_1 = (2\pi)^{-1} \int_{-\pi}^{\pi} |f(e^{it})| dt$.

Of particular interest are the following functionals on $\mathcal{P}$:

(i) Let $\lambda \in \mathbb{D}$. Denote by $\mathcal{E}_\lambda$ the evaluation at the point λ, i.e., $\mathcal{E}_\lambda$ is defined by $\mathcal{E}_\lambda(p) = p(\lambda) \quad (p \in \mathcal{P})$.

(ii) Let $f \in L^1$. Denote by $M_f \in \mathcal{P}^*$ the functional defined by

$$M_f(p) = (2\pi)^{-1} \int_{-\pi}^{\pi} p(e^{it}) f(e^{it}) dt \qquad (p \in \mathcal{P}).$$

Then $\|M_f\| \leq \|f\|_1$.

The evaluation functionals $\mathcal{E}_\lambda$ are also of this type. Indeed, for $\lambda \in \mathbb{D}$ we have $\mathcal{E}_\lambda = M_{P_\lambda}$, where $P_\lambda(e^{it}) = \frac{1-|\lambda|^2}{|\lambda - e^{it}|^2}$ is the Poisson kernel. In particular, if $g = 1$ then $M_g(p) = p(0)$ for all p, and so M_g is the evaluation at the origin.

(iii) Let $k > 0$ and let $T : X \to X$ a polynomially bounded operator with constant k, i.e., T satisfies the condition $\|p(T)\| \leq k\|p\|$ for all polynomials p. Fix $x \in X$ and $x^* \in X^*$. Let $x \otimes x^* \in \mathcal{P}^*$ be the functional defined by

$$(x \otimes x^*)(p) = \langle p(T)x, x^* \rangle \qquad (p \in \mathcal{P}).$$

Since T is polynomially bounded, $x \otimes x^*$ is a bounded functional and we have $\|x \otimes x^*\| \leq k\|x\| \cdot \|x^*\|$.

Of course the definition of $x \otimes x^*$ depends on the operator T but since we are going to consider only one operator T, this can not lead to a confusion.

By the von Neumann inequality, any contraction on a Hilbert space is polynomially bounded with constant 1. More generally, every operator on a Hilbert space which is similar to a contraction is polynomially bounded. Recall that there are polynomially bounded Hilbert space operators that are not similar to a contraction. This was shown recently by Pisier [P] who gave thus a negative answer to a well-known longstanding open problem given by Halmos.

Denote by L^∞ the space of all bounded measurable functions on $\mathbb{T}$ with the sup-norm. Since $\mathcal{P} \subset L^\infty = (L^1)^*$, the space $\mathcal{P}$ inherits the w^*-topology from L^∞.

Of particular importance for the Scott Brown technique are those functionals on $\mathcal{P}$ that are w^*-continuous, i.e., that are continuous functions from $(\mathcal{P}, w^*)$ to $\mathbb{C}$. Equivalently, these functionals can be represented by absolutely continuous measures.

The next result summarizes the basic facts about w^*-continuous functionals on $\mathcal{P}$.

Theorem 7.1.
(i) Let $(p_n) \subset \mathcal{P}$ be a sequence of polynomials. Then $p_n \xrightarrow{w^*} 0$ if and only if (p_n) is a Montel sequence, i.e., $\sup_n \|p_n\| < \infty$ and $p_n(z) \to 0 \quad (z \in \mathbb{D})$;

(ii) The w^* closure of $\mathcal{P}$ in L^∞ is the Hardy space H^∞ of all bounded functions analytic on $\mathbb{D}$.

(iii) $\psi \in \mathcal{P}^*$ is w^*-continuous if and only if it can be represented by an absolutely continuous measure. By the F. and M. Riesz theorem, in this case each measure representing ψ is absolutely continuous. By the Radon-Nikodym theorem, there exists $f \in L^1$ such that $\|f\|_1 = \|\psi\|$ and $\psi = M_f$.

(iv) Let $\psi \in \mathcal{P}^*$ be w^*-continuous. Let $\Lambda \subset \mathbb{D}$ be a dominant subset, i.e., $\sup_{\lambda \in \Lambda} |f(\lambda)| = \|f\|$ for all $f \in H^\infty$. Let $\varepsilon > 0$. Then there are numbers $\lambda_1, \dots, \lambda_n \in \Lambda$ and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ such that $\sum_{i=1}^n |\alpha_i| \le \|\psi\|$ and $\left\|\psi - \sum_{i=1}^n \alpha_i \mathcal{E}_{\lambda_i}\right\| < \varepsilon$.

Let $T \in B(X)$ be a polynomially bounded operator such that $\|T^n u\| \to 0$ for all $u \in X$. Then all the functionals $x \otimes x^*$ can be represented by absolutely continuous measures. Equivalently, these functionals are w^*-continuous, i.e., they are continuous on the space $(\mathcal{P}, w^*)$. These results can be shown using classical results from measure theory.

We summarize the results in the following theorem. Conditions (iii) and (iv) are not necessary for our purpose, we include them only for the sake of completeness.

Theorem 7.2. Let T be a polynomially bounded operator on a Banach space X. Suppose that $\|T^n u\| \to 0$ for all $u \in X$. Then:

(i) $x \otimes x^*$ can be represented by an absolutely continuous measure for all $x \in X$ and $x^* \in X^*$. Equivalently, $x \otimes x^*$ is w^*-continuous;

(ii) the set $\{p(T)x : p \in \mathcal{P}, \|p\| \leq 1\}$ is precompact for all $x \in X$;

(iii) the functional $p \mapsto \langle p(T)x, x^* \rangle$ extends to the w^*-closure of $\mathcal{P}$ in L^∞, i.e., to the Hardy space H^∞ of all bounded functions analytic on $\mathbb{D}$;

(iv) it is possible to define the H^∞-functional calculus, i.e., an algebraic homomorphism $\Phi : H^\infty \to B(X)$ such that $\Phi(1) = I$ and $\Phi(z) = T$. Moreover, Φ is (w^*, SOT) continuous, i.e., the mapping $h \to \Phi(h)x$ is a continuous function from (H^∞, w^*) to X for each $x \in X$.

Now we are able to give an illustrative example how the Scott Brown technique works.

Theorem 7.3. Let T be a contraction on a Hilbert space H such that $\sigma(T) \cap \mathbb{D}$ is dominant in $\mathbb{D}$ and $\|T^n x\| \to 0$ for all $x \in H$. Then T has a nontrivial closed invariant subspace.

Outline of the proof. Without loss of generality we may assume that neither T nor T^* has eigenvalues. In particular, $\sigma_{\pi e}(T) = \sigma(T)$.

The first step in the proof is that we can approximate (with an arbitrary precision) the evaluation functionals $\mathcal{E}_\lambda$ for $\lambda \in \sigma_{\pi e}(T)$ by the functionals of the type $x \otimes x$ with $x \in H$.

(a) Let $\lambda \in \sigma_{\pi e}(T)$, $\varepsilon > 0$, let $x \in H$, $\|x\| = 1$ and $\|(T - \lambda)x\| < \varepsilon$. Then

$$\|x \otimes x - \mathcal{E}_\lambda\| < \frac{2k\varepsilon}{1 - |\lambda|}.$$

Indeed, we have

$$\big\|x \otimes x - \mathcal{E}_\lambda\big\| = \sup_{\|p\|=1} \big|\langle p(T)x, x\rangle - p(\lambda)\big|.$$

For $p \in \mathcal{P}$, $\|p\| = 1$ write $q(z) = \frac{p(z) - p(\lambda)}{z - \lambda}$. Then $\|q\| \leq \frac{2\|p\|}{1 - |\lambda|} = \frac{2}{1 - |\lambda|}$. Thus

$$\big|\langle p(T)x, x\rangle - p(\lambda)\big| = \big|\langle q(T)(T - \lambda)x, x\rangle\big| \leq \|q(T)\| \cdot \|(T - \lambda)x\| \leq \frac{2k\varepsilon}{1 - |\lambda|}.$$

For the approximation procedure we need a stronger version of (a).

(b) Let $u_1, \dots, u_n \in H$ be given. Let $\lambda \in \sigma_{\pi e}(T)$ and $\varepsilon > 0$. Then there exists $x \in H$ of norm 1 such that $x \perp \{u_1, \dots, u_n\}$ and

$$\begin{aligned} &\|x \otimes x - \mathcal{E}_\lambda\| < \varepsilon, \\ &\|x \otimes u_i\| < \varepsilon \qquad (i = 1, \dots, n), \\ &\|u_i \otimes x\| < \varepsilon \qquad (i = 1, \dots, n). \end{aligned}$$

Indeed, since $\lambda \in \sigma_{\pi e}(T)$, we can choose a vector $x \perp \{u_1, \dots, u_n\}$ such that $\|(T - \lambda)x\|$ is small enough. Thus the inequality $\|x \otimes x - \mathcal{E}_\lambda\| < \varepsilon$ follows from (a). Using the same estimates it is possible to obtain also that $\|x \otimes u_i\| < \varepsilon \quad (i = 1, \dots, n)$.

For the last inequality (note that the second and third inequalities are not symmetrical!) it is possible to use the compactness of the set $\{p(T)u_i : \|p\| \le 1, i = 1, \dots, n\}^-$, see Theorem 7.2 (ii). Indeed, it is possible to choose x "almost orthogonal" to all vectors of the form $p(T)u_i$ where $\|p\| \le 1$ and $i = 1, \dots, n$.

In the following we use the fact that any w^*-continuous functional can be approximated by convex linear combinations of the evaluations at points of $\sigma_{\pi e}(T)$, see Theorem 7.1 (iv). We show that if $\psi \in \mathcal{P}^*$ is any w^*-continuous functional and $x \otimes y$ its approximation, then it is possible to find a better approximation $x' \otimes y'$ of ψ that is not too far from $x \otimes y$.

(c) Let $\psi \in \mathcal{P}^*$ be a w^*-continuous functional, let $x, y \in H$ and $\varepsilon > 0$. Then there are $x', y' \in H$ such that

$$\begin{aligned} &\|x' \otimes y' - \psi\| < \varepsilon, \\ &\|x' - x\| \le \|x \otimes y - \psi\|^{1/2}, \\ &\|y' - y\| \le \|x \otimes y - \psi\|^{1/2}. \end{aligned}$$

Indeed, by Theorem 7.1 (iv) there are elements $\lambda_1, \dots, \lambda_n \in \sigma_{\pi e}(T)$ and nonzero complex numbers $\alpha_1, \dots, \alpha_n$ such that $\sum_{i=1}^n |\alpha_i| \le \|x \otimes y - \psi\|$ and

$$\Big\| x \otimes y - \psi + \sum_{i=1}^n \alpha_i \mathcal{E}_{\lambda_i} \Big\| < \varepsilon/2.$$

Let δ be a sufficiently small positive number.

By (b), we can find inductively mutually orthogonal unit vectors $u_1, \dots, u_n \in H$ such that

$$\begin{aligned} &\|x \otimes u_i\| < \delta, \\ &\|u_i \otimes y\| < \delta, \\ &\|u_i \otimes u_j\| < \delta \qquad (i \ne j), \\ &\|u_i \otimes u_i - \mathcal{E}_{\lambda_i}\| < \delta. \end{aligned}$$

Set $x' = x + \sum_{i=1}^n \frac{\alpha_i}{|\alpha_i|^{1/2}} u_i$ and $y' = y + \sum_{i=1}^n |\alpha_i|^{1/2} u_i$. Since the vectors $u_1, \dots, u_n$ are orthonormal, we have $\|x' - x\|^2 = \sum_{i=1}^n |\alpha_i| \le \|x \otimes y - \psi\|$, and similarly, $\|y' - y\|^2 \le \|x \otimes y - \psi\|$. Furthermore,

$$\begin{aligned} \|x' \otimes y' - \psi\| \le &\Big\| x \otimes y - \psi + \sum_{i=1}^n \alpha_i \mathcal{E}_{\lambda_i} \Big\| + \Big\| \sum_{i=1}^n \alpha_i \big(u_i \otimes u_i - \mathcal{E}_{\lambda_i}\big) \Big\| \\ + &\sum_{i=1}^n |\alpha_i|^{1/2} \|u_i \otimes y\| + \sum_{i=1}^n |\alpha_i|^{1/2} \|x \otimes u_i\| + \sum_{i \ne j} |\alpha_i|^{1/2} \cdot |\alpha_j|^{1/2} \cdot \|u_i \otimes u_j\| \\ \le &\varepsilon/2 + \delta \Big(\sum_{i=1}^n |\alpha_i| + 2 \sum_{i=1}^n |\alpha_i|^{1/2} + \sum_{i \ne j} |\alpha_i|^{1/2} |\alpha_j|^{1/2} \Big) < \varepsilon \end{aligned}$$

provided δ is sufficiently small.

(d) There are $x, y \in H$ such that $x \otimes y = \mathcal{E}_0$. As it was shown above, this implies that T has a nontrivial invariant subspace.

Set $x_0 = 0 = y_0$. Using (c) it is possible to construct inductively vectors $x_j, y_j \in H \quad (j \in \mathbb{N})$ such that

$$\begin{aligned}
&\|x_j \otimes y_j - \mathcal{E}_0\| \le 2^{-2j}, \\
&\|x_{j+1} - x_j\| \le \|x_j \otimes y_j - \mathcal{E}_0\|^{1/2} \le 2^{-j} \quad \text{and} \\
&\|y_{j+1} - y_j\| \le 2^{-j}.
\end{aligned}$$

Clearly the sequences (x_j) and (y_j) are Cauchy. Let x and y be their limits. It is easy to verify that $x \otimes y = \mathcal{E}_0$. □

The condition that $T^n x \to 0$ for all $x \in H$ can be omitted by a standard reduction argument.

Theorem 7.4. [BCP1] Let T be a contraction on a Hilbert space H such that the spectrum $\sigma(T) \cap \mathbb{D}$ is dominant in $\mathbb{D}$. Then T has a nontrivial invariant subspace.

Outline of the proof. Let $M_1 = \{x \in H : T^n x \to 0\}$. It is easy to see that M_1 is a closed subspace of H invariant with respect to T. If $M_1 = H$ then T has a nontrivial invariant subspace by Theorem 7.3. Thus we can assume without loss of generality that $M_1 = \{0\}$.

Since a subspace $M \subset H$ is an invariant subspace for T if and only if $M^\perp$ is an invariant subspace for T^*, we can do all the previous considerations also for T^* instead of T. Thus we can also assume that $M_2 = \{x \in H : T^{*n} x \to 0\} = \{0\}$.

Contractions $T \in B(H)$ satisfying $M_1 = \{0\} = M_2$ are called contractions of class C_{11} in [NF]. It is proved there that such T is quasisimilar to a unitary operator (i.e., there is a Hilbert space K, a unitary operator $U \in B(K)$ and injective operators $A : H \to K$, $B : K \to H$ with dense ranges such that $UA = AT$ and $BU = TB$). Consequently, T has many invariant subspaces, see [NF]. □

Theorem 7.4 is a classical application of the Scott Brown technique. By refined methods it is possible to obtain the following much deeper result.

Theorem 7.5. [BCP2] Let T be a contraction on a Hilbert space H such that $\sigma(T)$ contains the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Then T has a nontrivial closed invariant subspace.

Theorem 7.5 can be also generalized to the Banach space setting.

Theorem 7.6. [AM] Let T be a polynomially bounded operator on a Banach space X such that $\sigma(T)$ contains the unit circle. Then T^* has a nontrivial invariant subspace. In particular, if X is reflexive then T has an invariant subspace.

Note that Theorem 7.6 is stronger than Theorem 7.5 even for Hilbert space operators.

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