

Hypercyclic Behaviour of Operators in a Hypercyclic C_0 -Semigroup

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Abstract

Let $\{T_t\}_{t \geq 0}$ be a hypercyclic strongly continuous semigroup of operators. Then each T_t ($t > 0$) is hypercyclic as a single operator, and it shares the set of hypercyclic vectors with the semigroup. This answers in the affirmative a natural question concerning hypercyclic C_0 -semigroups. The analogous result for frequent hypercyclicity is also obtained.

Key words: Hypercyclic vectors, C_0 -semigroups, frequently hypercyclic operators, chaotic dynamics

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1 Introduction

A continuous linear operator $T : X \rightarrow X$ on a topological vector space X is said to be *hypercyclic* if there is a vector $x \in X$ (called a hypercyclic vector) whose orbit under T , $Orb(T, x) := \{T^n x : n \in \mathbb{N}\}$, is dense in X .

In [1] Ansari proved that all the powers of a hypercyclic operator T are also hypercyclic. Moreover, they share the same hypercyclic vectors with T . Recall that Ansari [2] and Bernal [7] showed that every infinite dimensional separable Banach space admits a hypercyclic operator. This result was also extended to the non-normable Fréchet case by Bonnet and Peris [11]. For more details about hypercyclic operators see the surveys [10,21,22].

In the continuous case, a one parameter family $\mathcal{T} = \{T_t\}_{t \geq 0}$ of continuous linear operators in $L(X)$ is a *strongly continuous semigroup* (or C_0 -semigroup) of operators in $L(X)$ if $T_0 = I$, $T_t T_s = T_{t+s}$ for all $t, s \geq 0$, and $\lim_{t \rightarrow s} T_t x = T_s x$ for all $s \geq 0$, $x \in X$. For further information about C_0 -semigroups we refer the reader to the books [20,29].

A C_0 -semigroup $\mathcal{T} = \{T_t\}_{t \geq 0}$ is said to be hypercyclic if $Orb(\mathcal{T}, x) := \{T_t x : t \geq 0\}$ is dense in X for some $x \in X$. The investigation of hypercyclic semigroups was initiated by Desch, Schappacher and Webb in [17]. So far, several specific examples of hypercyclic strongly continuous semigroups have been studied, see for example [17,16,19,24,28]. In [4] Bermúdez, Bonilla and Martín proved that every separable infinite dimensional Banach space admits a hypercyclic semigroup. This result was extended to Fréchet spaces in [12].

Given $T \in L(X)$, let us denote by $HC(T)$ the set of all hypercyclic vectors

of T , and analogously, denote by $HC(\mathcal{T})$ the set of hypercyclic vectors of a C_0 -semigroup \mathcal{T} . It is easy to see that if $\mathcal{T} = \{T_t\}_{t \geq 0}$ is a C_0 -semigroup and some operator T_t in the semigroup is hypercyclic, then the semigroup \mathcal{T} itself is hypercyclic.

When one analyzes the converse situation (from the continuous to the discrete case), as a consequence of an old result of Oxtoby and Ulam [27] it is possible to establish that, if $x \in HC(\mathcal{T})$ is hypercyclic, then there exists a residual set $G \subset \mathbb{R}_+$, such that $x \in HC(T_t)$ for all $t \in G$ (see, e.g., [13]). The point here is whether $G = \mathbb{R}_+$. That is, if $\mathcal{T} = \{T_t\}_{t \geq 0}$ is a hypercyclic C_0 -semigroup, is every operator T_t , $t > 0$, hypercyclic? This problem was explicitly stated in [4].

Our main result is the solution to this problem in the affirmative. To do this we will adapt an argument due to León and Müller [26] on rotations of hypercyclic operators. This approach is not new: Several authors have tried to use similar arguments to the ones in [26] for the C_0 -semigroups context without success (e.g., [14], [25] and [18]). The key point in the proof, proceeding by contradiction, is to construct a pair of continuous maps $f : HC(\mathcal{T}) \rightarrow \mathbb{T}$ and $g : \mathbb{D} \rightarrow HC(\mathcal{T})$ such that $f \circ g|_{\mathbb{T}}$ is homotopically nontrivial. Such a point has resisted previous attempts (notice that the homotopy in [18] does not yield any contradiction, which results in a serious gap), and it is finally solved here.

A new trend in hypercyclicity was recently open by the work of Bayart and Grivaux. Motivated by Birkhoff's ergodic Theorem, they introduced the notion of frequent hypercyclicity [8,6], by quantifying the frequency with which an orbit meets open sets. To be precise, let us define the *lower density* of a set $A \subset \mathbb{N}$ by $\underline{\text{dens}}(A) := \liminf_{N \rightarrow \infty} \#\{n \leq N : n \in A\}/N$. An operator

$T \in L(X)$ is said to be *frequently hypercyclic* if there exists $x \in X$ such that, for every non-empty open subset $U \subset X$, the set $\{n \in \mathbb{N} : T^n x \in U\}$ has positive lower density. Each such a vector x is called a *frequently hypercyclic vector* for T , and the set of all frequently hypercyclic vectors is denoted by $FHC(T)$.

Analogously, if we define the lower density of a measurable set $M \subset \mathbb{R}_+$ by $\underline{Dens}(M) := \liminf_{N \rightarrow \infty} \mu(M \cap [0, N])/N$, where μ is the Lebesgue measure on \mathbb{R}_+ , then a C_0 -semigroup $\mathcal{T} = \{T_t\}_{t \geq 0}$ in $L(X)$ is said to be frequently hypercyclic if there exists $x \in X$ such that for any non-empty open set $U \subset X$, the set $\{t \in \mathbb{R}_+ : T_t x \in U\}$ has positive lower density. As before, we denote by $FHC(\mathcal{T})$ the set of all hypercyclic vectors of \mathcal{T} . In both cases, frequent hypercyclicity is stronger than hypercyclicity. See also [23,9,5] For further details concerning frequently hypercyclic operators and semigroups.

We prove that, if a C_0 -semigroup $\mathcal{T} = \{T_t\}_{t \geq 0}$ is frequently hypercyclic, then every single operator $T_t \neq I$ is frequently hypercyclic.

From now on, X stands for an F -space over \mathbb{K} , where \mathbb{K} denotes the field of either real or complex numbers; by an F -space we mean that it is a metrizable and complete topological vector space. Let $\mathcal{U}_0(X)$ be a base of open balanced neighbourhoods of the origin in X . Within this context, any C_0 -semigroup $\mathcal{T} = \{T_t : X \rightarrow X\}_{t \geq 0}$ is *locally equicontinuous*, i.e., for any $t_0 > 0$, the family of operators $\{T_t : t \in [0, t_0]\}$ is equicontinuous. This fact will be used repeatedly throughout the paper. We would like to point out that there is no simplification in the proofs if we assume that X is a Banach space, and that our results remain valid for general topological vector spaces X if we assume that $\mathcal{T} = \{T_t\}_{t \geq 0}$ is locally equicontinuous.

2 Hypercyclic Operators and Semigroups

We begin this section with some technical results. The first one is an adaptation to F -spaces of a result of Costakis and Peris [15], using ideas of Wengenroth [30]. See also [12].

Lemma 2.1 *Let $\mathcal{T} = \{T_t\}_{t \geq 0}$ be a hypercyclic semigroup in $L(X)$. Then $T_t - \lambda I$ has dense range for all $t > 0$ and $\lambda \in \mathbb{K}$.*

Proof. Fix arbitrarily $\lambda \in \mathbb{K}$ and $t_0 > 0$. We assume $L := \overline{(T_{t_0} - \lambda I)(X)} \neq X$, and consider the quotient map $q : X \rightarrow X/L$, which satisfies $q \circ (T_{t_0} - \lambda I) = 0$. Inductively, this yields $q \circ T_{t_0}^n = \lambda^n q$ for all $n \in \mathbb{N}$. Consider $x \in HC(\mathcal{T})$, and define $M := q(\text{Orb}(\mathcal{T}, x)) = \{\lambda^n q(T_s x) : n \in \mathbb{N}_0, s \in [0, t_0]\}$, which is dense by the definition of q . Now we distinguish two cases:

$|\lambda| \leq 1$: Since $\{T_s x : s \in [0, t_0]\}$ is bounded in X , M must be bounded, so that it cannot be dense. A contradiction.

$|\lambda| > 1$: Fix an arbitrary $y \in L$ with $q(y) \neq 0$. There exists an $r > 0$ such that $q(T_r x) \neq 0$. We pick $U \in \mathcal{U}_0(X/L)$ satisfying $q(T_r x) \notin U$. The equicontinuity of $\{T_s : s \in [0, t_0]\}$ yields the existence of $V \in \mathcal{U}_0(X)$ such that $q(T_t(V)) \subset U$, $t \in [0, t_0]$. Fix $t' > r$ with $T_{t'} x \in V$. We write $t' = mt_0 - t + r$, for some $m \in \mathbb{N}$ and $t \in [0, t_0]$. Since $|\lambda| > 1$, we have $\lambda^m q(T_r x) \notin U$. On the other hand,

$$\lambda^m q(T_r x) = q(T_{mt_0+r} x) = q(T_t(T_{t'} x)) \in q(T_t(V)) \subset U,$$

which is a contradiction.

□

An easy consequence of the previous lemma is the following

Corollary 2.2 *Let $\mathcal{T} = \{T_t\}_{t \geq 0}$ be a hypercyclic semigroup in $L(X)$. If $t > 0$, $(\lambda_1, \lambda_2) \neq (0, 0)$ and $x \in HC(\mathcal{T})$, then $\lambda_1 x + \lambda_2 T_t x \in HC(\mathcal{T})$*

Theorem 2.3 *Let $\mathcal{T} = \{T_t\}_{t \geq 0}$ be a hypercyclic semigroup on $L(X)$, and let $x \in HC(\mathcal{T})$. Then $x \in HC(T_{t_0})$ for every $t_0 > 0$.*

Proof. Without loss of generality, we may assume that $t_0 = 1$. Indeed, we can consider the semigroup $\tilde{\mathcal{T}} = \{\tilde{T}_t\}_{t \geq 0}$ in $L(X)$, with $\tilde{T}_t := T_{tt_0}$ for every $t \geq 0$. Clearly, $x \in HC(\tilde{\mathcal{T}})$ and $\tilde{T}_1 = T_{t_0}$.

Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle, $\mathbb{D} := \{z \in \mathbb{C} : |z| \leq 1\}$ the closed unit disc, and let $\mathbb{R}_+ := \{t \in \mathbb{R} : t \geq 0\}$.

We define the map $\rho : \mathbb{R}_+ \rightarrow \mathbb{T}$ by $\rho(t) := e^{2\pi it}$. For every pair $u, v \in X$ let

$$F_{u,v} := \left\{ \lambda \in \mathbb{T} : \exists (t_n)_n \subset \mathbb{R} \text{ with } \lim_n t_n = \infty, \lim_n T_{t_n} u = v, \text{ and } \lim_n \rho(t_n) = \lambda \right\}.$$

Our proof is divided into several steps.

Step 1 *If $u \in HC(\mathcal{T})$, then $F_{u,v} \neq \emptyset$ for all $v \in X$. Since u is hypercyclic for \mathcal{T} , we can find an unbounded increasing sequence $\{t_k\}_k$ in \mathbb{R}_+ , such that $T_{t_k} u$ converges to v . By passing to a subsequence, if necessary, we may assume that $(\rho(t_k))_k$ is convergent. Its limit is an element of $F_{u,v}$.*

Step 2 *If $\lim_k v_k = v$, $\lambda_k \in F_{u,v_k}$, and $\lim_k \lambda_k = \lambda$, then $\lambda \in F_{u,v}$. (In particular, $F_{u,v}$ is a closed set for each $u, v \in X$.) Indeed, for each k we select $t_k > k$ such that $\lim_k (T_{t_k} u - v_k) = 0$ and $\lim_k |\rho(t_k) - \lambda_k| = 0$. It is easy to see that $\lim_k T_{t_k} u = v$ and that $\lim_k \rho(t_k) = \lambda$.*

Step 3 If $u, v, w \in X$, $\lambda \in F_{u,v}$, and $\mu \in F_{v,w}$, then $\lambda\mu \in F_{u,w}$. Given $U \in \mathcal{U}_0(X)$ and $\varepsilon > 0$, take $U' \in \mathcal{U}_0(X)$ such that $U' + U' \subset U$. Find t_1 such that $T_{t_1}v - w \in U'$ and $|\rho(t_1) - \mu| < \varepsilon$. Pick $V \in \mathcal{U}_0(X)$ and $t_2 > 0$ satisfying $T_{t_1}(V) \subset U'$, $T_{t_2}u - v \in V$, and $|\rho(t_2) - \mu| < \varepsilon$. Then

$$T_{t_1+t_2}u - w = T_{t_1}(T_{t_2}u - v) + (T_{t_1}v - w) \in T_{t_1}(V) + U' \subset U, \text{ and}$$

$$|\rho(t_1+t_2) - \lambda\mu| = |\rho(t_1)\rho(t_2) - \lambda\mu| \leq |\rho(t_1) - \mu| \cdot |\rho(t_2)| + |\mu| \cdot |\rho(t_2) - \lambda| < 2\varepsilon.$$

Hence $\lambda\mu \in F_{u,w}$.

Fix now $x \in HC(\mathcal{T})$. By Steps 1, 2 and 3, $F_{x,x}$ is a nonempty closed subsemigroup of \mathbb{T} with multiplication. Firstly, suppose that $F_{x,x} = \mathbb{T}$. Then, given $y \in X$, by Steps 1 and 3 we get $F_{x,y} = \mathbb{T}$. In particular $1 \in F_{x,y}$, which yields the existence of a sequence $(t_n)_n \subset \mathbb{R}_+$ tending to infinity such that $\lim_n T_{t_n}x = y$ and $\lim_n \rho(t_n) = 1$. Write t_n as $t_n = k_n + \varepsilon_n$ with $k_n \in \mathbb{N}$ and $\varepsilon_n \in (-1/2, 1/2]$. Then $\lim_n \varepsilon_n = 0$. Let $U \in \mathcal{U}_0(X)$. We fix $U', V \in \mathcal{U}_0(X)$ with $U' + U' \subset U$ and $T_s(V) \subset U'$, $0 \leq s \leq 2$. Let $n \in \mathbb{N}$ be large enough such that $T_{t_n}x - y \in V$ and $T_{1-\varepsilon_n}y - T_1y \in U'$. Then we have

$$\begin{aligned} T_{k_n+1}x - T_1y &= T_{1-\varepsilon_n}(T_{t_n}x - y) + (T_{1-\varepsilon_n} - T_1)y \\ &\in T_{1-\varepsilon_n}(V) + U' \subset U' + U' \subset U. \end{aligned}$$

Hence $T_1y \in \overline{Orb(T_1, x)}$. Since T_1 has dense range and $y \in X$ is arbitrary, then x is hypercyclic for T_1 .

For the rest of the proof we assume that $F_{x,x} \neq \mathbb{T}$, and we will show that it leads to a contradiction.

Step 4 There exists some $k \in \mathbb{N}$ such that, for each $y \in HC(\mathcal{T})$, there is $\lambda \in \mathbb{T}$ satisfying $F_{x,y} = \{\lambda z : z^k = 1\}$. It turns out that there is $k \in \mathbb{N}$ such that

$F_{x,x} = \{z \in \mathbb{T} : z^k = 1\}$. Indeed, given $z \in F_{x,x}$, the set $\{z^n : n \in \mathbb{N}\}$ is either dense in \mathbb{T} or finite. Since it is contained in the closed semigroup $F_{x,x} \neq \mathbb{T}$, it should be finite. Now, given $y \in HC(\mathcal{T})$, $\lambda \in F_{x,y}$, and $\mu \in F_{y,x}$, by Step 3, $\lambda F_{x,x} \subset F_{x,y}$, and $\mu F_{x,y} \subset F_{x,x}$, then $\#(F_{x,y}) = \#(F_{x,x})$. This implies that $F_{x,y} = \lambda F_{x,x}$

Step 5 *There is a continuous function $h : \mathbb{D} \rightarrow \mathbb{T}$, whose restriction to the unit circle is homotopically nontrivial. A contradiction.*

Let us recall that two maps $f, g : X \rightarrow Y$ are *homotopic* if there is a continuous map $H : X \times [0, 1] \rightarrow Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$, $x \in X$. f is *homotopically trivial* if it is homotopic to a constant map. If f is homotopically trivial, then so are all its restrictions. Any continuous map $f : \mathbb{D} \rightarrow Y$ is homotopically trivial. We say that a continuous map $g : \mathbb{T} \rightarrow \mathbb{T}$ has *index* n , ($n = 0, 1, 2, \dots$), if it is homotopic to the map $z \mapsto z^n$. Any continuous map $g : \mathbb{T} \rightarrow \mathbb{T}$ has some index, and it is homotopically trivial if and only if it has index 0. We refer the reader to, e.g., [3].

Consider the function $f : HC(\mathcal{T}) \rightarrow \mathbb{T}$ as $f(y) := \lambda^k$, where $\lambda \in F_{x,y}$. Clearly, by Steps 2 and 4, f is well defined and continuous. Besides, $f(x) = 1$ and, since $x \in HC(\mathcal{T})$, then $T_t x \in HC(\mathcal{T})$ for every $t \geq 0$ by Corollary 2.2. Therefore it easily follows that $e^{2\pi it} \in F_{x, T_t x}$ and $f(T_t x) = e^{2\pi itk}$ for every $t \geq 0$.

We will find $g : \mathbb{D} \rightarrow HC(\mathcal{T})$ such that $h := f \circ g$ is the desired function which will give the contradiction. We first define $g : \mathbb{T} \rightarrow HC(\mathcal{T})$, and then extend it to \mathbb{D} . To do this, since f is continuous at x , we find $U \in \mathcal{U}_0(X)$ such that $|f(y) - 1| < 1$ if $y \in HC(\mathcal{T})$ and $y - x \in U$. We now fix $t_0 > 1$

satisfying $T_{t_0}x - x \in U$. Let us define $g : \mathbb{T} \rightarrow HC(\mathcal{T})$ by

$$g(e^{2\pi it}) := \begin{cases} T_{2tt_0}x & \text{if } 0 \leq t < 1/2, \\ (2t - 1)x + (2 - 2t)T_{t_0}x & \text{if } 1/2 \leq t < 1. \end{cases}$$

Clearly, g is well defined and continuous. By Corollary 2.2, we have $g(\mathbb{T}) \subset HC(\mathcal{T})$. Since U is balanced, $g(e^{2\pi it}) - x \in U$, for $1/2 \leq t < 1$. This implies $|f(g(e^{2\pi it})) - 1| < 1$, $1/2 \leq t < 1$. Moreover $f(g(e^{2\pi it})) = e^{4\pi it t_0 k}$, $0 \leq t < 1/2$, which yields that the index of $f \circ g$ at 0 is between $[t_0]k$ and $([t_0] + 1)k$ (depending on the difference $t_0 - [t_0]$).

We extend the function g to \mathbb{D} by defining $g(z) := (1 - |z|)x + |z|g(z/|z|)$ for each $z \neq 0$, and $g(0) = x$. Clearly, this extension is also continuous on \mathbb{D} , and $g(z) \in HC(\mathcal{T})$ for every $z \in \mathbb{D}$ since $g(z)$ is a non-zero linear combination of x and $T_t x$, for some $0 < t \leq t_0$ (Corollary 2.2).

To sum up, we have a continuous function $h := f \circ g : \mathbb{D} \rightarrow \mathbb{T}$, such that its restriction to the unit circle is homotopically nontrivial, a contradiction.

□

3 Frequently Hypercyclic Operators and Semigroups

In this section we prove the analogous result for the stronger concept of frequent hypercyclicity. We first need a technical lemma concerning the frequently hypercyclic vectors of a C_0 -semigroup \mathcal{T} .

Lemma 3.1 *Let $\mathcal{T} = \{T_t\}_{t \geq 0}$ be a frequently hypercyclic semigroup on $L(X)$, and let $x \in FHC(\mathcal{T})$. For every $k \in \mathbb{N}$, $y \in X$, and $U \in \mathcal{U}_0(X)$*

$$\underline{Dens}(\{t \in \cup_{n \in \mathbb{N}} [n - 1/k, n) : T_t x - y \in U\}) > 0.$$

Proof. Clearly, $T_{j/k}x \in HC(\mathcal{T})$ for every $j = 0, \dots, k-1$, and even more, $T_{j/k}x \in HC(T_1)$ by Theorem 2.3. Fix $U, U' \in \mathcal{U}_0(X)$ such that $U' + U' \subset U$, and $y \in X$. Then there are some $n_j \in \mathbb{N}$ such that $T_{n_j+j/k}x - y \in U'$, $j = 0, \dots, k-1$. Besides, there is some $V \in \mathcal{U}_0(X)$ such that $T_s(V) \subset U'$ if $s \leq N_0 := \max\{n_j : j = 0, \dots, k-1\} + 1$.

Since $x \in FHC(\mathcal{T})$, we have $\underline{Dens}(\{t \in \mathbb{R}^+ : T_t x - x \in V\}) > 0$. So there are $C > 0$ and $N_1 \in \mathbb{N}$ such that $\mu(\{t \leq N : T_t x - x \in V\}) \geq CN$ for every $N \geq N_1$.

For every $N \in \mathbb{N}$, let us define $L := \{t \leq N : T_t x - x \in V\}$. In addition, for every $j = 0, \dots, k-1$, we define the sets $I_j := \cup_n [n + j/k, n + (j+1)/k)$, $L_j := L \cap I_j$, and the mapping $f_j : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ as $f_j(t) := t + n_{k-j-1} + (k-j-1)/k$. These mappings satisfy that $f_j(t) \in I_{k-1}$ for every $t \in L_j$, and

$$\begin{aligned} T_{f_j(t)}x - y &= T_{n_{k-j-1} + (k-j-1)/k}(T_t x - x) + (T_{n_{k-j-1} + (k-j-1)/k}x - y) \\ &\in T_{n_{k-j-1} + (k-j-1)/k}(V) + U' \subset U. \end{aligned}$$

Finally, for $N \geq N_0 + N_1$ we have

$$\begin{aligned} \mu(\{t \leq 2N : T_t x - y \in U \text{ and } t \in I_{k-1}\}) &\geq \mu(\cup_{j=0}^{k-1} f_j(L_j)) \\ &\geq \sum_{j=0}^{k-1} \mu(f_j(L_j))/k = \sum_{j=0}^{k-1} \mu(L_j)/k = \mu(L)/k \geq CN/k. \end{aligned}$$

Hence $\underline{Dens}(\{t \in I_{k-1} : T_t x - y \in U\}) > 0$, and we are done. □

Theorem 3.2 *Let $\mathcal{T} = \{T_t\}_{t \geq 0}$ be a frequently hypercyclic semigroup on $L(X)$, and let $x \in FHC(\mathcal{T})$. Then $x \in FHC(T_{t_0})$ for every $t_0 > 0$.*

Proof. Without loss of generality, we may again assume that $t_0 = 1$ as in the proof of Theorem 2.3. Fix $y \in X$, $U \in \mathcal{U}_0(X)$, and select $k \in \mathbb{N}$, $U' \in \mathcal{U}_0(X)$, such that $U' + U' \subset U$ and $T_t y - y \in U'$ for every $0 \leq t \leq 1/k$. Since \mathcal{T} is strongly continuous there is some $V \in \mathcal{U}_0(X)$ such that $T_t(V) \subset U'$ for every $0 \leq t \leq 1/k$. By the previous lemma, we know that $\underline{Dens}(\{t \in \cup_{n \in \mathbb{N}} [n - 1/k, n) : T_t x - y \in V\}) > 0$.

If $t \in [n - 1/k, n)$ for some $n \in \mathbb{N}$ and $T_t x - y \in V$, then we define $\eta_t := [t] + 1 - t$. Each η_t satisfies $0 < \eta_t \leq 1/k$, and $t + \eta_t \in \mathbb{N}$. So

$$T_{t+\eta_t} x - y = T_{\eta_t}(T_t x - y) + (T_{\eta_t} y - y) \in T_{\eta_t}(V) + U' \subset U.$$

Hence

$$\underline{dens}(\{n \in \mathbb{N} : T_n x - y \in U\}) \geq \underline{Dens}(\{t \in \cup_{n \in \mathbb{N}} [n - 1/k, n) : T_t x - y \in V\}) > 0$$

□

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