

# Invertible extensions and growth conditions

## Extensions inversibles et conditions de croissance

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### Abstract

We study invertible extensions of Banach and Hilbert space bounded linear operators with prescribed growth conditions for the norm of inverses. In particular, the solutions of some open problems are obtained. *To cite this article : C. Badea, V.Müller, C. R. Acad. Sci. Paris, Ser. I ??? (200?).*

### Résumé

Nous étudions les extensions inversibles des opérateurs linéaires et bornés sur un espace de Banach ou de Hilbert avec des conditions de croissance données pour les normes des inverses. Nous obtenons en particulier la réponse à plusieurs problèmes ouverts formulés dans la littérature. *Pour citer cet article : C. Badea, V.Müller, C. R. Acad. Sci. Paris, Ser. I ??? (200?).*

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### Version française abrégée

Un thème classique en théorie des opérateurs (linéaires et bornés) est l'existence des dilatations et extensions ayant des bonnes propriétés spectrales. Dans le cas des opérateurs agissant sur l'espace de Hilbert, un exemple célèbre est le théorème de dilatation de B. Sz.-Nagy, [20], qui affirme que chaque contraction a une dilatation unitaire. Notons aussi, [20], que chaque contraction a une extension co-isométrique. Un autre exemple hilbertien est la notion d'opérateur sous-normal. Un exemple banachique, [6], est le théorème suivant dû à R.G. Douglas : chaque isométrie sur un espace de Banach admet comme extension une isométrie surjective. La construction de Douglas est hilbertienne : si l'isométrie agit sur un espace de Hilbert, alors son extension, *a posteriori* un opérateur unitaire, agit aussi sur un espace de Hilbert. Dans le cadre d'une algèbre de Banach commutative  $\mathcal{A}$ , d'après un théorème de R.F. Arens, [1], un élément

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$u \in \mathcal{A}$  est inversible dans une algèbre de Banach commutative contenant  $\mathcal{A}$  si et seulement si  $u$  n'est pas un diviseur topologique de zéro.

Nous nous intéressons ici aux extensions inversibles des opérateurs avec des conditions de croissance données pour les normes des inverses. On considère notamment la condition de croissance de type polynomial ( $(P(s))$ ), la condition  $(B)$  de Beurling, et la condition de croissance de type exponentiel ( $(E(s))$ ) (les conditions sont définies dans la version anglaise). Si  $T$  vérifie une de ces trois conditions, alors  $T$  a une extension inversible,  $S$ , agissant sur un espace plus grand, et qui vérifie le même type de condition de croissance. Toutes les constructions sont hiberniennes. On obtient ainsi la réponse à trois problèmes ouverts formulés dans [12, Problem 6.1.15] et [7,13,14].

Cette Note d'annonce ne contient aucune preuve. Celles-ci seront publiées ultérieurement [3].

## 1. Introduction

We introduce some notation and terminology.

**Operators.** In this paper  $X$  (and  $Y$ ) will denote complex Banach spaces and  $H$  (and  $K$ ) will denote Hilbert spaces. Denote by  $B(X)$  the algebra of all bounded linear operators on the Banach space  $X$ . By an operator we always mean a bounded linear operator.

For an operator  $T \in B(X)$  acting on a Banach space  $X$ , we denote  $m(T) = \inf\{\|Tx\| : x \in X, \|x\| = 1\}$ . This quantity is called the *minimum modulus* of  $T$  ([4]) or the *lower bound* of  $T$  ([12]). We write for short  $v_n(T) = \max\{\|T^n\|, m(T^n)^{-1}\}$  ( $n \geq 0$ ).

We denote by  $\sigma(T)$  and  $\sigma_{ap}(T)$  the spectrum and the approximate point spectrum of a bounded linear operator  $T \in B(X)$ , respectively. The latter is given by

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : \inf\{\|(T - \lambda)x\| : \|x\| = 1\} = 0\}.$$

Note that  $m(T) > 0$  if and only if  $T \in B(X)$  is one-to-one and of closed range. If  $T$  is a Hilbert space operator, then  $\sigma_{ap}(T)$  coincides with the left spectrum and  $m(T) > 0$  if and only if  $T$  is left invertible.

We say that  $S \in B(Y)$  is an extension of  $T \in B(X)$  if there is an isometry  $\pi : X \rightarrow Y$  such that  $S\pi = \pi T$ . We can also consider  $X$  as a subspace of  $Y$  and write  $T = S|_X$ .

**Banach spaces of class  $SQ_p$ .** Let  $p \geq 1$  be a real number. A Banach space  $E$  is said to be a  $SQ_p$ -space if it is a quotient of a subspace of an  $L_p$ -space. Let  $X$  be a Banach space. A Banach space  $E$  is said to be a  $SQ_p(X)$ -space if it is (isometric to) a quotient of a subspace of an ultraproduct of spaces of the form  $L_p(\Omega, \mu, X)$ , for some measure spaces  $(\Omega, \mu)$ . Since ultraproducts of  $L_p$ -spaces are  $L_p$ -spaces, the latter definition is consistent with the former one. Note that any Banach space is isometric to a subspace (resp. a quotient) of an  $L_\infty$ -space (resp. an  $L_1$ -space). Also, if  $H$  is a Hilbert space, then each  $SQ_2(H)$ -space is a Hilbert space too. Note also that  $SQ_p$ -spaces are precisely the  $p$ -spaces in the sense of Herz, [10]. We refer to [11] for more information.

**Growth conditions.** We consider the following growth conditions for the operator  $T$ :

( $P(s)$ ) (polynomial growth condition) there are  $C > 0$  and  $s \geq 0$  such that  $v_n(T) \leq Cn^s$  ( $n \geq 1$ );

( $B$ ) (Beurling-type condition)  $\sum_{n=1}^{\infty} \frac{\log v_n(T)}{n^2} < \infty$ ;

( $E(s)$ ) (Exponential growth) there are  $C > 0$  and  $0 < s < 1$  such that  $v_n(T) \leq Ce^{n^s}$  ( $n \geq 1$ ).

Note that condition ( $P(s)$ ) implies ( $E(s')$ ) (for any  $s' > 0$ ), which implies ( $B$ ). Also, if  $T$  satisfies ( $B$ ) and  $T$  is invertible, then  $\sigma(T) = \sigma_{ap}(T) \subset \mathbb{T}$ . Here  $\mathbb{T} = \{z : |z| = 1\}$ . If  $T$  satisfies ( $B$ ) and  $0 \in \sigma(T)$ , then  $\sigma_{ap}(T) = \mathbb{T}$  and  $\sigma(T) = \{z : |z| \leq 1\}$ .

We consider in this Note the following problems. Let  $T$  be an operator satisfying one of the above three conditions. Then  $T$  has an invertible extension  $S$  satisfying the same type of growth condition. Note that  $m(S^n)^{-1} = \|S^{-n}\|$  for invertible operators  $S$  and so the growth conditions for invertible  $S$  are in fact

growth conditions for the norm of iterates of  $S$  and  $S^{-1}$ . The existence of such invertible extensions solves several open problems mentioned in the literature, as we explain below. Proofs will be given elsewhere [3].

## 2. $\mathcal{E}(\mathbb{T})$ -subscalar operators

We denote by  $\mathcal{E}(\mathbb{C}) = C^\infty(\mathbb{C})$  the usual Fréchet algebra of all  $C^\infty$ -functions on  $\mathbb{C}$  with the topology of uniform convergence of derivatives of all orders on compact subsets of  $\mathbb{C}$ . An operator  $S \in B(X)$  is said to be *generalized scalar* (or  $\mathcal{E}(\mathbb{C})$ -scalar) if there is a continuous algebra homomorphism  $\Phi : \mathcal{E}(\mathbb{C}) \rightarrow B(X)$  for which  $\Phi(1) = I$  and  $\Phi(z) = S$  (see [5,12]). Such a homomorphism may be interpreted as an operator-valued distribution with compact support ([12]). A bounded linear operator is  $\mathcal{E}(\mathbb{C})$ -subscalar if it is similar to the restriction of a  $\mathcal{E}(\mathbb{C})$ -scalar operator to one of its closed invariant subspaces. According to a result by J. Eschmeier and M. Putinar (see [8, Sect. 6.4]), a Banach space operator  $T$  is  $\mathcal{E}(\mathbb{C})$ -subscalar if and only if  $T$  has property  $(\beta)_\mathcal{E}$ , i.e., for every open set  $U \subset \mathbb{C}$ , the operator  $T_U$  on  $\mathcal{E}(U, X)$  (the space of  $C^\infty$ -functions from  $U$  into  $X$ ), defined by  $T_U(f)(z) = (T - z)f(z)$ , is injective and has closed range.

The following statements are equivalent (see [5]) :

- (1)  $T$  is  $\mathcal{E}(\mathbb{T})$ -scalar, i.e., it has a continuous functional calculus on the Fréchet algebra  $\mathcal{E}(\mathbb{T})$  of smooth functions on the unit circle  $\mathbb{T}$  ;
- (2)  $T$  is generalized scalar with  $\sigma(T) \subset \mathbb{T}$  ;
- (3)  $T$  is invertible, and there exist constants  $C > 0$  and  $s \geq 0$  such that

$$\|T^n\| \leq C(1 + |n|)^s \quad (n \in \mathbb{Z}).$$

K.B. Laursen and M.M. Neumann [12, Problem 6.1.15] and M. Didas [7] asked if  $\mathcal{E}(\mathbb{T})$ -subscalar operators are characterized by the polynomial growth condition  $(P(s))$  above. One implication is easy. We refer to [7,12,13,14,15,16] for several partial results. By [6] the hard implication holds for  $s = 0$  and  $C = 1$ .

Since condition  $(P(s))$  implies that  $\sigma_{ap}(T) \subset \mathbb{T}$ , it follows ([17,18]) that  $T$  has an invertible extension  $S$  such that  $\sigma(S) = \sigma_{ap}(T) \subset \mathbb{T}$ . By [19], if  $T$  acts on a Hilbert space, then  $S$  acts also on a Hilbert space. However, no control on the norms of inverses is guaranteed by this method.

The following result gives a complete positive answer.

**Theorem 2.1** (1) *An operator  $T \in B(X)$  is  $\mathcal{E}(\mathbb{T})$ -subscalar if and only if there exist constants  $C > 0$  and  $s \geq 0$  such that*

$$(P(s)) \quad \frac{1}{Cn^s} \|x\| \leq \|T^n x\| \leq Cn^s \|x\| \quad (x \in X; n \in \mathbb{N}).$$

Moreover, given  $p \geq 1$ , there exist a  $SQ_p(X)$ -space  $Y$ , an invertible  $\mathcal{E}(\mathbb{T})$ -scalar operator  $S$  on  $Y$  and a closed subspace  $M \subset Y$  invariant with respect to  $S$  such that  $T$  is similar to the restriction  $S|_M$ . We also have  $\sigma(S) = \sigma_{ap}(T)$ .

For  $p = 1$  the operator  $S$  is an extension of  $T$ .

(2) *If the Hilbert space operator  $T \in B(H)$  verifies*

$$(P(s)) \quad \frac{1}{Cn^s} \|h\| \leq \|T^n h\| \leq Cn^s \|h\| \quad (h \in H; n \in \mathbb{N}),$$

then there exists a Hilbert space  $K$  and a  $\mathcal{E}(\mathbb{T})$ -scalar extension  $S \in B(K)$  with  $\sigma(S) = \sigma_{ap}(T)$ .

## 3. Operators with Bishop's property $(\beta)$

Recall that an equivalent definition of decomposable operators is the following :  $T \in B(X)$  is *decomposable* if for every open cover  $\mathbb{C} = U \cup V$ , there are closed invariant (for  $T$ ) subspaces  $Y$  and  $Z$  of  $X$  such that

$X = Y + Z$  and  $\sigma(T|Y) \subset U$ ,  $\sigma(T|Z) \subset V$ . We refer for instance to [5] and [12]. An operator  $T \in B(X)$  has *Bishop's property* ( $\beta$ ) if, for every open set  $U \subset \mathbb{C}$ , the operator  $T_U$  defined by  $T_U(f)(z) = (T - z)f(z)$  on the set  $\mathcal{O}(U, X)$  of holomorphic functions from  $U$  into  $X$  is injective and has closed range. According to a result by E. Albrecht and J. Eschmeier (see [12,8]),  $T \in B(X)$  is *subdecomposable* (i.e.,  $T$  is similar to the restriction of a decomposable operator) if and only if  $T$  has Bishop's property ( $\beta$ ).

It was proved in [5, 5.3.2] that an invertible operator  $S \in B(X)$  is decomposable provided that

$$\sum_{n=-\infty}^{\infty} \frac{\log \|S^n\|}{1+n^2} < \infty.$$

The following result answers in the affirmative a question from [13] and [14].

**Theorem 3.1** *Let  $T \in B(X)$  be a Banach space operator such that*

$$\sum_{n=1}^{\infty} \frac{\log \max(\|T^n\|, m(T^n)^{-1})}{n^2} < \infty.$$

*Then there exists a Banach space  $Y \supset X$  and an invertible operator  $S \in B(Y)$  such that  $T = S|_X$  and  $S$  satisfies*

$$\sum_{n=-\infty}^{\infty} \frac{\log \|S^n\|}{1+n^2} < \infty.$$

*In particular,  $T$  has Bishop's property ( $\beta$ ). Moreover,  $\sigma(S) = \sigma_{ap}(T) = \sigma(T) \cap \mathbb{T}$ .*

*If  $X = H$  is a Hilbert space, then  $Y = K$  can be chosen to be a Hilbert space too.*

#### 4. Condition ( $E(s)$ )

The following result answers an open question from [13].

**Theorem 4.1** *Let  $T \in B(X)$  satisfy ( $E(s)$ ). Then there exist a Banach space  $Y \supset X$  and an invertible operator  $S$  on a larger space such that  $T$  is a restriction of  $S$  and  $S$  satisfies ( $E(s')$ ) for suitable  $s' < 1$ . The construction is hilbertian.*

The construction is hilbertian means that if  $X = H$  is Hilbert, then  $Y = K$  can be chosen a Hilbert space too.

#### 5. Applications

We obtain the following consequences.

##### 5.1. Operators with countable spectrum

Using the above results and [21] we obtain the following characterization of operators which are similar to unitaries with a countable spectrum. Recall that a Hilbert space operator  $T \in B(H)$  is said to be similar to a unitary if there is an invertible operator  $L \in B(H)$  such that  $L^{-1}TL$  is a unitary operator.

**Corollary 5.1** *Let  $T \in B(H)$  be a Hilbert space operator such that  $\sup_{n \geq 1} \|T^n\| < \infty$ . Suppose that there are positive constants  $C$  and  $s < \frac{1}{2}$  such that*

$$m(T^n)^{-1} \leq Ce^{n^s} \quad (n \geq 1)$$

*and that  $\sigma(T)$  is countable. Then  $T$  is invertible and it is similar to a unitary operator.*

We note that a classical similarity criterion of Sz.-Nagy states that an invertible Hilbert space operator  $T$  is similar to a unitary if and only if  $\sup_{m \in \mathbb{Z}} \|T^m\| < \infty$ .

### 5.2. Contractions with spectrum a Carleson set

Recall that a closed set  $E$  of  $\mathbb{T}$  is said to be a *Carleson set* if

$$\int_0^{2\pi} \log \left( \frac{2}{\text{dist}(e^{it}, E)} \right) dt < +\infty.$$

Using the above results and [9] we obtain the following consequence.

**Corollary 5.2** *Let  $T \in B(H)$  be a Hilbert space contraction such that  $\sigma_{ap}(T) \subset \mathbb{T}$  is a Carleson set. Suppose that there exist  $C > 0$  and  $s \geq 0$  such that  $m(T^n)^{-1} \leq Cn^s$ . Then  $T$  is an isometry.*

### 5.3. A hilbertian counterpart of Arens' result.

We also obtain the following hilbertian counterpart of Arens' result. We refer to [17,18] for versions of Arens' result in the case of the (non-commutative) Banach algebra  $B(X)$ .

**Corollary 5.3** *Let  $T \in B(H)$  be an operator on Hilbert space with  $m(T) > 0$ . Then there exist a Hilbert space  $K$ , an isometric embedding  $\pi : H \mapsto K$  and an invertible operator  $S \in B(K)$  such that  $S\pi = \pi T$ ,  $\|S^j\| \leq \|T^j\|$  ( $j \geq 1$ ),  $\|S^{-1}\| \leq 2/m(T)$  and*

$$\left\| \sum_{j=0}^N S^{-j} \pi(x_j) \right\|^2 \leq 2 \sum_{j=0}^N \left( \frac{\sqrt{2}}{m(T)} \right)^{2j} \|x_j\|^2$$

for every  $N \in \mathbb{N}$  and all  $x_j \in H$ .

The last condition says, in the terminology of [2], that  $S^{-1}$  is quadratically near the null operator modulo  $\pi(H)$ .

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## References

- [1] Arens, R. Inverse-producing extensions of normed algebras, *Trans. Amer. Math. Soc.* 88(1958), 536-548.
- [2] Badea, C. Perturbations of operators similar to contractions and the commutator equation, *Studia Math.* 150(2002), 273-293.
- [3] Badea, C ; Müller, V. Growth conditions and inverse producing extensions, Preprint.
- [4] Gindler, H. A. ; Taylor, A. E. The minimum modulus of a linear operator and its use in spectral theory, *Studia Math.* 22(1962/1963), 15-41.

- [5] Colojoară, I. ; Foiaş, C. *Theory of generalized spectral operators*, Mathematics and its Applications, Vol. 9. Gordon and Breach, Science Publishers, New York-London-Paris, 1968.
- [6] Douglas, R.G. On extending commutative semigroups of isometries, *Bull. London Math. Soc.* 1 (1969), 157–159.
- [7] Didas, M.  $\mathcal{E}(\mathbb{T}^n)$ -subscalar  $n$ -tuples and the Cesaro operator on  $HP$ , *Ann. Univ. Sarav. Ser. Math.* 10 (2000), no. 2, pp. i–iii and 284–335.
- [8] Eschmeier, J. ; Putinar, M. *Spectral decompositions and analytic sheaves*, London Mathematical Society Monographs. New Series, 10. The Clarendon Press, Oxford University Press, New York, 1996.
- [9] Esterle, J. Uniqueness, strong forms of uniqueness and negative powers of contractions, in: *Functional analysis and operator theory (Warsaw, 1992)*, 127–145, Banach Center Publ., 30, Polish Acad. Sci., Warsaw, 1994.
- [10] Herz, C. The theory of  $p$ -spaces with an application to convolution operators. *Trans. Amer. Math. Soc.* 154(1971) 69–82.
- [11] Kwapien, S. On operators factorizable through  $L_p$  space, Actes du Colloque d'Analyse Fonctionnelle de Bordeaux (Univ. de Bordeaux, 1971), pp. 215–225. *Bull. Soc. Math. France, Mem. No. 31–32*, Soc. Math. France, Paris, 1972.
- [12] Laursen, K.B. ; Neumann, M.M. *An introduction to local spectral theory*, London Mathematical Society Monographs. New Series, 20. The Clarendon Press, Oxford University Press, New York, 2000.
- [13] Miller, T.L. ; Miller, V. ; Neumann, M.M. Growth conditions and decomposable extensions, in: *Proceedings of the Conference on Trends in Banach Spaces and Operator Theory*, to appear.
- [14] Miller, T.L. ; Miller, V. ; Neumann, M.M. Spectral subspaces of subscalar and related operators, *Proc. Amer. Math. Soc.*, to appear.
- [15] Miller, T.L. ; Miller, V. ; Neumann, M.M. Local spectral properties of weighted shifts, *J. Operator Theory*, to appear.
- [16] Miller, T.L. ; Miller, V. ; Neumann, M.M. Localization in the spectral theory of operators on Banach spaces, in: *Proceedings of the Fourth Conference on Function Spaces at Edwardsville*, to appear.
- [17] Müller, V. Adjoining inverses to noncommutative Banach algebras and extensions of operators, *Studia Math.* 91 (1988), 73–77.
- [18] Read, C. J. Spectrum reducing extension for one operator on a Banach space, *Trans. Amer. Math. Soc.* 308 (1988), 413–429.
- [19] Read, C. J. Extending an operator from a Hilbert space to a larger Hilbert space, so as to reduce its spectrum, *Israel J. Math.* 57 (1987), 375–380.
- [20] Sz.-Nagy, B. ; Foiaş, C. *Harmonic analysis of operators on Hilbert space*, Translated from the French and revised, North-Holland, Amsterdam, 1970.
- [21] Zarrabi, M. Contractions à spectre dénombrable et propriétés d'unicité des fermés dénombrables du cercle, *Ann. Inst. Fourier (Grenoble)* 43 (1993), 251–263.