Hypercyclic sequences of operators

Fernando León-Saavedra and Vladimír Müller*

ABSTRACT. A sequence (T_n) of bounded linear operators between Banach spaces X, Y is said to be hypercyclic if there exists a vector $x \in X$ such that the orbit $\{T_n x\}$ is dense in Y. The paper gives a survey of various conditions that imply the hypercyclicity of (T_n) and studies relations among them. The particular case of $X = Y$ and mutually commuting operators T_n is analyzed. This includes the most interesting cases (T^n) and $(\lambda_n T^n)$, where T is a fixed operator and λ_n are complex numbers. We also study when a sequence of operators has a large (either dense or closed infinite dimensional) manifold consisting of hypercyclic vectors.

I. Introduction

Let X and Y be separable Banach spaces. Denote by $B(X,Y)$ the set of all bounded linear operators from X to Y. Let $(T_n) \subset B(X,Y)$ be a sequence of operators. A vector $x \in X$ is called hypercyclic for (T_n) if the set $\{T_n x\}$ is dense in Y. The sequence (T_n) is called hypercyclic if there is at least one vector hypercyclic for (T_n). We say that an operator $T : X \to X$ is hypercyclic if the sequence of its iterates (T^n) is hypercyclic.

Similarly, an operator T is said to be supercyclic if there exists a vector $x \in X$ such that the set $\{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}\}$ is dense; the vector x with this property is called supercyclic for T.

Usually it is not easy to verify whether a sequence (T_n) is hypercyclic or not. There are many criteria that have been studied by a number of authors implying the hypercyclicity of (T_n), see e.g. [K], [GS], [BG]. In the second section we give a survey of various conditions implying the hypercyclicity and study relations among them. A number of illustrative examples is given.

The third section concentrates on the situation when $Y = X$ and the operators $T_n : X \to X$ are mutually commuting. The relations among various conditions are much simpler in this case. The following section studies the case when $T_n = S_1 \cdots S_n$ where $S_j : X \to X$ are mutually commuting. This includes the most interesting cases (T^n) and $(\lambda_n T^n)$ where T is a fixed operator and λ_n complex numbers.

Sequences of operators with "many hypercyclic vectors"are very important in the hypercyclic theory. The interest in them (especially in the cases (T^n) and $(\lambda_n T^n)$) arises from the invariant subspace/subset problem.

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* Supported by the grant No. 201/03/0041 of GA ČR.

2000 Mathematics Subject Classification: Primary 47A16.

Keywords and phrases: Hypercyclic sequences, hypercyclic vectors, supercyclic vectors, hypercyclic subspaces.

There are two research lines in the literature. The first one, which was initiated by B. Beauzamy [Bea] and continued in [G],[GS],[He],[Bo] and recently [Gri], studies the existence of dense manifolds consisting of hypercyclic vectors. The second more recent line studies the existence of closed infinite dimensional subspaces whose all non-zero elements are hypercyclic, see [Mo], [LMo] and [GLMo]. The questions of this type will be studied in Section V.

II. Hypercyclicity of sequences of operators

Let X, Y be separable Banach spaces and let $(T_n) \subset B(X,Y)$ be a sequence of operators. It is well-known that the set of all hypercyclic vectors for (T_n) is a G_δ set. Indeed, $x \in X$ is hypercyclic for (T_n) if and only if $x \in \bigcap_U \bigcup_{n\in\mathbb{N}} T_n^{-1}U$, where U runs over a countable base of open subsets of Y; it is clear that $\bigcup_{n\in\mathbb{N}} T_n^{-1}U$ is open for each U.

Lemma 1. [GS] Let $(T_n) \subset B(X,Y)$ be a sequence of operators. The following conditions are equivalent:
(i) (T_n) has a dense subset of hypercyclic vectors;
(ii) the set of all hypercyclic vectors for (T_n) is residual (i.e., its complement is of the first category);
(iii) for all nonempty open subsets $U \subset X$, $V \subset Y$ there exists $n \in \mathbb{N}$ such that $T_nU \cap V \neq \emptyset$;
(iv) for all $x \in X$, $y \in Y$ and $\varepsilon > 0$ there exist $n \in \mathbb{N}$ and $u \in X$ such that $\|u - x\| < \varepsilon$ and $\|T_nu - y\| < \varepsilon$.

Denote by B_X the closed unit ball in a Banach space X.

The most useful criteria of hypercyclicity are the following two:

Definition 2. We say that a sequence $(T_n) \subset B(X,Y)$ satisfies condition (C) if there exist an increasing sequence of positive integers (n_k) and a dense subset $X_0 \subset X$ such that
(i) $\lim_{k\to\infty} T_{n_k}x = 0$ for all $x \in X_0$;
(ii) $\bigcup_k T_{n_k}B_X$ is dense in Y.

The second condition is similar:

Definition 3. We say that a sequence $(T_n) \subset B(X,Y)$ satisfies condition (C_{fin}) if there exist an increasing sequence of positive integers (n_k) and a dense subset $X_0 \subset X$ such that
(i) $\lim_{k\to\infty} T_{n_k}x = 0$ for all $x \in X_0$;
(ii) $\bigcup_k \underbrace{\left(T_{n_k}B_X \oplus \cdots \oplus T_{n_k}B_X\right)}_{j}$ is dense in $\underbrace{Y \oplus \cdots \oplus Y}_{j}$ for all $j \in \mathbb{N}$.

Clearly, condition (C_{fin}) implies (C). Condition (C) is the weakest known property which can be practically used to show the hypercyclicity of a sequence (T_n). Moreover, it implies the existence of a dense (and hence residual) set of hypercyclic vectors. Furthermore, under a reasonable additional condition it implies that there is a closed infinite dimensional subspace of hypercyclic vectors.

Condition (C_{fin}) has a number of equivalent formulations and it implies that there is a dense linear subspace consisting of hypercyclic vectors.

Theorem 4. Let $(T_n) \subset B(X,Y)$ be a sequence of operators. The following conditions are equivalent:

(i) (T_n) satisfies condition (C);

(ii) for all $j \in \mathbb{N}$ and nonempty open subsets $U_0, U_1, \dots, U_j \subset X$, $V_0, V \subset Y$ such that U_0 and V_0 contain the origins of X and Y, respectively, there exists $n \in \mathbb{N}$ such that $T_n U_i \cap V_0 \neq \emptyset \quad (i = 1, \dots, j)$ and $T_n U_0 \cap V \neq \emptyset$.

In particular, if (T_n) satisfies (C) then there is a dense (and hence residual) set of hypercyclic vectors for (T_n).

Proof. (i)$\Rightarrow$(ii): Clear.

(ii)$\Rightarrow$(i): Let $(x_n) \subset X$ and $(y_n) \subset Y$ be dense sequences. Set $u_{i,i} = x_i \quad (i \in \mathbb{N})$. By induction on k we construct an increasing sequence (n_k), and vectors $u_{i,k} \in X \quad (i = 1, \dots, k-1)$ and $v_k \in B_X$ such that

$$\begin{aligned} &\|T_{n_k} v_k - y_k\| < 2^{-k}, \\ &\|u_{i,k} - u_{i,k-1}\| < \frac{1}{2^k \max\{1, \|T_{n_1}\|, \dots, \|T_{n_{k-1}}\|\}}, \\ &\|T_{n_k} u_{i,k}\| < 2^{-k} \end{aligned}$$

for all i, k with $1 \le i < k$. For each i the sequence $(u_{i,k})_k$ is Cauchy. Let u_i be its limit. Then

$$\|u_i - x_i\| \le \sum_{k=i+1}^{\infty} \|u_{i,k} - u_{i,k-1}\| \le \sum_{k=i+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^i}.$$

Therefore the sequence (u_i) is dense in X.

Clearly the sequence $(T_{n_k} v_k)$ is dense, and so $\overline{\bigcup_k T_{n_k} B_X} = Y$. Further,

$$\begin{aligned} \lim_{k\to\infty} \|T_{n_k} u_i\| &\le \lim_{k\to\infty} \Big(\|T_{n_k} u_{i,k}\| + \sum_{j=k}^{\infty} \|T_{n_k}\| \cdot \|u_{i,j+1} - u_{i,j}\| \Big) \\ &\le \lim_{k\to\infty} \Big(\frac{1}{2^k} + \sum_{j=k}^{\infty} \frac{1}{2^{j+1}} \Big) = \lim_{k\to\infty} \frac{1}{2^{k-1}} = 0 \end{aligned}$$

for each i. Thus (T_n) satisfies (C).

To show that condition (C) implies the existence of a dense subset of hypercyclic vectors we use Lemma 1. Let $x \in X$, $y \in Y$ and $\varepsilon > 0$. By (ii), there are $x_0, x_1 \in X$ and $n \in \mathbb{N}$ such that $\|x_0\| < \varepsilon/2$, $\|T_n x_0 - y\| < \varepsilon/2$, $\|x_1 - x\| < \varepsilon/2$ and $\|T_n x_1\| < \varepsilon/2$. Then $\|(x_0 + x_1) - x\| \le \|x_0\| + \|x_1 - x\| < \varepsilon$ and $\|T_n(x_0 + x_1) - y\| \le \|T_n x_0 - y\| + \|T_n x_1\| < \varepsilon$. By Lemma 1, (T_n) has a dense subset of hypercyclic vectors. □

Theorem 5. Let $(T_n) \subset B(X,Y)$ be a sequence of operators. The following conditions are equivalent:

(i) (T_n) satisfies condition (C_{fin});
(ii) $(\underbrace{T_n \oplus \cdots \oplus T_n}_{j})$ satisfies condition (C) for all $j \in \mathbb{N}$;
(iii) $(\underbrace{T_n \oplus \cdots \oplus T_n}_{j})$ has a dense subset of hypercyclic vectors for all $j \in \mathbb{N}$;
(iv) for all $j \in \mathbb{N}$ and nonempty open subsets $U_1, \dots, U_j \subset X$, $V_1, \dots, V_j \subset Y$ there is an $n \in \mathbb{N}$ such that $T_n U_i \cap V_i \neq \emptyset \quad (i = 1, \dots, j)$;
(v) there is a subsequence (T_{n_k}) such that each its subsubsequence has a dense set of hypercyclic vectors;
(vi) there are dense subsets $X_0 \subset X$, $Y_0 \subset Y$, an increasing sequence $(n_k) \subset \mathbb{N}$ and mappings $S_i : Y_0 \to X \quad (i \in \mathbb{N})$ such that

$$\begin{aligned} T_{n_k} x &\to 0 \qquad (x \in X_0); \\ S_k y &\to 0 \qquad (y \in Y_0); \\ T_{n_k} S_k y &\to y \qquad (y \in Y_0); \end{aligned}$$

(vii) for each Banach space Z the sequence of operators $L_{T_n} : \overline{F(Z,X)} \to \overline{F(Z,Y)}$ defined by $L_{T_n} S = T_n S \quad (S \in \overline{F(Z,X)})$ has a dense set of hypercyclic vectors; here $F(Z,X)$ denotes the set of all finite rank operators from Z to X;
(viii) for each Banach space Z the sequence (L_{T_n}) satisfies condition (C).

Proof. The equivalences (vi)⇔(v)⇔(iii) were proved in [BG]. The implications (i)⇒(ii) and (vi)⇒(i) are obvious. The equivalence (iii)⇔(iv) follows from Lemma 1 and the implication (ii)⇒(iii) follows from Theorem 4. This implies the equivalence of the first six conditions.

(i)⇒(viii): Let X_0 be a dense subset of X and let (n_k) be a sequence satisfying $T_{n_k} x \to 0 \quad (x \in X_0)$ and $\overline{\bigcup (T_{n_k} B_X \oplus \cdots \oplus T_{n_k} B_X)} = Y \oplus \cdots \oplus Y$.

Let $\mathcal{M} \subset B(Z,X)$ be the set of all finite rank operators with the range included in the linear space generated by X_0. Clearly $\mathcal{M}$ is dense in $\overline{F(Z,X)}$. For $G \in \mathcal{M}$ we have $\lim_k L_{T_{n_k}} G = \lim_k T_{n_k} G = 0$.

Let $F \in F(Z,Y)$ and $\varepsilon > 0$. We can write $F = \sum_{i=1}^j z_i^* \otimes y_i$ for some $y_i \in Y$ and $z_i^* \in Z^*$. Since (T_n) satisfies condition (C_{fin}), there are vectors $u_i \in X \quad (i = 1, \dots, j)$ and $k \in \mathbb{N}$ such that $\|T_{n_k} u_i - y_i\| < \frac{\varepsilon}{j \max\{\|z_1^*\|, \dots \|z_j^*\|\}}$ and $\|u_i\| \leq \frac{1}{j \max\{\|z_1^*\|, \dots, \|z_j^*\|\}}$. Set $F_0 = \sum_{i=1}^j z_i^* \otimes u_i \in F(Z,X)$. Then $\|F_0\| \leq \sum_{i=1}^j \|z_i^*\| \cdot \|u_i\| \leq 1$ and

$$\begin{aligned} \|L_{T_{n_k}} F_0 - F\| = \|T_{n_k} F_0 - F\| &= \left\| \sum_{i=1}^j z_i^* \otimes T_{n_k} u_i - \sum_{i=1}^j z_i^* \otimes y_i \right\| \\ &= \left\| \sum_{i=1}^j z_i^* \otimes (T_{n_k} u_i - y_i) \right\| \leq \sum_{i=1}^j \|z_i^*\| \cdot \|T_{n_k} u_i - y_i\| < \varepsilon. \end{aligned}$$

(viii)⇒(vii): Follows from Theorem 4.

(vii)⇒(iii): Let $j \in \mathbb{N}$ and let Z be a j-dimensional Banach space. Then $\overline{F(Z,X)}$ is isomorphic to $\underbrace{X \oplus \cdots \oplus X}_{j}$ and $\overline{F(Z,Y)}$ to $\underbrace{Y \oplus \cdots \oplus Y}_{j}$. In the same way L_{T_n} can be identified with $T_n \oplus \cdots \oplus T_n$. □

For the sake of completeness we mention here also other conditions on a sequence $(T_n) \subset B(X,Y)$ that have been studied in the literature. In the diagram below we show the relations among them. The abbreviations there mean:

(HC) (Hypercyclicity criterion) There exist dense subsets $X_0 \subset X$, $Y_0 \subset Y$, an increasing sequence $(n_k) \subset \mathbb{N}$ and mappings $S_k : Y_0 \to X$ such that

$$\begin{aligned} T_{n_k} x &\to 0 && (x \in X_0); \\ S_k y &\to 0 && (y \in Y_0); \\ T_{n_k} S_k y &= y && (y \in Y_0, k \in \mathbb{N}). \end{aligned}$$

(hc) (T_n) is hypercyclic.

(dense hc) (T_n) has a dense set of hypercyclic vectors.

(4 nbhd) (4 neighbourhoods condition) for all nonempty open subsets $U, U_0 \subset X$, $V, V_0 \subset Y$ such that U_0 and V_0 contain the origins in X and Y, respectively, there exists $n \in \mathbb{N}$ such that $T_n U \cap V_0 \neq \emptyset$ and $T_n U_0 \cap V_n \neq \emptyset$ (when $X = Y$ this condition reduces to "the three open sets condition", which was introduced in [GS], section III).

(her hc) (hereditarily hypercyclic) There is a subsequence (T_{n_k}) such that each its subsubsequence is hypercyclic.

The relations among these conditions are given in the following diagram:

$$\begin{array}{ccccccc} \text{(HC)} & \longrightarrow & (\mathrm{C}_{fin}) & \longrightarrow & \text{(her hc)} & \longrightarrow & \text{(hc)} \\ & & \downarrow & & & & \uparrow \\ & & \text{(C)} & \longrightarrow & \text{(4 nbhd)} & \longrightarrow & \text{(dense hc)} \end{array}$$

Moreover, there are no other implications among the considered conditions.

The implications (HC) $\to$ (C_{fin}) and (C_{fin}) $\to$ (her hc) were proved in Theorem 5, the implication (C) $\to$ (4 nbhd) in Theorem 4. For the implication (4 nbhd) $\to$ (dense hc) see Proposition 6 below.

The remaining implications are trivial.

The negative results follow from the following examples. Note that it is sufficient to show $(\mathrm{C}_{fin}) \not\to$ (HC), (C) $\not\to$ (her hc), (her hc) $\not\to$ (dense hc), (dense hc) $\not\to$ (4nbhd) and (4nbhd) $\not\to$ (C). This will imply that there are no other implications in the diagram above.

Proposition 6. (cf. [GS]) If $(T_n) \subset B(X,Y)$ satisfies (4 nbhd), then there is a dense subset of hypercyclic vectors for (T_n).

Proof. The result was essentially proved already in the proof of Theorem 4. Let $x \in X$, $y \in Y$ and $\varepsilon > 0$. Then there are $n \in \mathbb{N}$, $u, v \in X$ such that $\|u - x\| < \varepsilon/2$, $\|T_n u\| < \varepsilon/2$, $\|v\| < \varepsilon/2$ and $\|T_n v - y\| < \varepsilon/2$. Set $x' = u + v$. Then $\|x - x'\| \leq \|x - u\| + \|v\| < \varepsilon$

and $\|T_n x' - y\| < \|T_n v - y\| + \|T_n u\| < \varepsilon$. By Lemma 1, this implies that (T_n) has a dense subset of hypercyclic vectors. $\square$

Example 7. Let X be a Hilbert space with an orthonormal basis $\{e_F : F \subset \mathbb{N}, \operatorname{card} F < \infty\}$. Let $Y = X$ and let the operators $T_n : X \to X$ be defined by

$$T_n e_F = \begin{cases} n e_{F \setminus \{n\}} & (n \in F), \\ 0 & (n \notin F). \end{cases}$$

It is easy to verify that the sequence (T_n) satisfies condition (C_{fin}) for the dense subspace $X_0 \subset X$ generated by the vectors $\{e_F : F \subset \mathbb{N}\}$. However, (T_n) does not satisfy (HC) since the operators T_n have not dense ranges, $T_n X = \bigvee\{e_F : n \notin F\}$.

Note that the operators T_n are even commuting.

Example 8. Let X and $T_n : X \to X$ be as in the previous example. Note that $e_{\{n\}} \perp T_n X$ for each n. Consider the operators $S_n : X \oplus \mathbb{C} \to X$ defined by

$$S_n(x \oplus \lambda) = T_n x + \lambda e_{\{n\}} \qquad (x \in X, \lambda \in \mathbb{C}).$$

Since the operators (T_n) satisfy condition (C_{fin}) and hence are hereditarily hypercyclic, it is easy to see that the sequence (S_n) is also hereditarily hypercyclic. On the other hand, the set of all vectors hypercyclic for (S_n) is not dense. Indeed, let $x \in X$, $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Then

$$\|S_n(x \oplus \lambda)\| = \|T_n x + \lambda e_{\{n\}}\| \geq |\lambda|$$

for each n, and so $x \oplus \lambda$ is not hypercyclic. This shows that (her hc) $\not\to$ (dense hc).

Example 9. Let X be a separable Hilbert space with an orthonormal basis $\{e_1, e_2, \dots\}$. Let $Y = \mathbb{C}^2$ and (y_n) be a dense sequence of elements of Y. Define $T_n : X \to Y$ by $T_n e_n = y_n$ and $T_n e_i = 0 \quad (i \neq n)$. Clearly $T_n x \to 0$ for each x that is a finite linear combination of the vectors $e_i \quad (i \in \mathbb{N})$. Further $\overline{\bigcup_n T_n B_X} \supset \{y_n : n \in \mathbb{N}\}^- = Y$. Thus (T_n) satisfies condition (C).

On the other hand, let (n_k) be any increasing sequence of positive integers such that (T_{n_k}) is hypercyclic. Let $U \subset Y$ be a nonempty open set such that $\overline{U} \neq Y$ and $\mathbb{C} \cdot U \subset U$. Choose a subsequence of those indices n_k for which $y_{n_k} \in U$. For such an n_k we have $T_{n_k} X = \mathbb{C} \cdot y_{n_k} \subset U$, and so (T_n) is not hereditarily hypercyclic. Consequently, (C) $\not\to$ (her hc).

Example 10. Let $\dim X = 1$ (i.e., $X = \mathbb{C}$) and let Y be a separable Hilbert space. Let (y_n) be a dense sequence in Y. Define $T_n : X \to Y$ by $T_n(\lambda) = \lambda y_n$. Clearly each non-zero $\lambda \in X$ is hypercyclic for (T_n). It is easy to see that (T_n) does not satisfy the condition (4 nbhd). Indeed, consider the neighbourhoods $U = \{z \in \mathbb{C} : |z| > 2\}$, $U_0 = \{z \in \mathbb{C} : |z| < 1\}$, $V = \{y \in Y : \|y\| > 2\}$ and $V_0 = \{y \in Y : \|y\| < 1\}$. Thus (dense hc)$\not\to$(4 nbhd).

Example 11. Let $X = \mathbb{C}^2$ and $\dim Y = \infty$. Let (y_n) be a dense sequence in Y and (x_n) dense in X. For each n find $u_n \in X$ linearly independent from x_n such that $\|u_n\| = 1/n$. For $m, n \in \mathbb{N}$ define $T_{m,n} : X \to Y$ by $T_{m,n} x_n = 0$ and $T_{m,n} u_n = y_m$.

Then $(T_{m,n})$ is a countable set of operators satisfying the condition (4 nbhd).

Let (T_{m_k,n_k}) be any subsequence such that $T_{m_k,n_k}x \to 0$ for all x in a dense subset of X. Then $T_{m_k,n_k} \to 0$ in the strong operator topology, and therefore this subsequence is bounded by the Banach-Steinhaus theorem. Thus $(T_{m,n})$ does not satisfy (C). Hence (4 nbhd) $\not\to$ (C).

III. Sequences of commuting operators

In this section we assume that $Y = X$ and $(T_n) \subset B(X)$ is a sequence of mutually commuting operators. The situation is much simpler in this case.

Theorem 12. Let $(T_n) \subset B(X)$ be a sequence of mutually commuting operators. The following conditions are equivalent:
- (i) (T_n) satisfies condition (C);
- (ii) (T_n) satisfies condition (C_{fin});
- (iii) (T_n) is hereditarily hypercyclic;
- (iv) (T_n) satisfies (4 nbhd); in fact in this case the 4 neighbourhoods condition reduces to the "3 neighbourhoods condition": for all nonempty open subsets $U, V, W \subset X$ with $0 \in W$ there exists n such that $T_n U \cap W \neq \emptyset$ and $T_n W \cap V \neq \emptyset$.

Proof. (i)⇒(ii): Let $X_0 \subset X$ be a dense subset and $(n_k) \subset \mathbb{N}$ an increasing sequence such that $T_{n_k}x \to 0 \quad (x \in X_0)$ and $\overline{\bigcup_k T_{n_k} B_X} = X$.

By Theorem 4, (T_{n_k}) is hypercyclic. Let $x \in X$ be a hypercyclic vector for (T_{n_k}).

Let $y_1, \dots, y_r \in X$ and $\varepsilon > 0$. Since x is hypercyclic, there are $k_1, \dots, k_r$ such that $\|T_{n_{k_i}}x - y_i\| < \varepsilon/2 \quad (i = 1, \dots, r)$. Further, there are $u \in X$, $\|u\| \le \max\{\|T_{n_{k_i}}\| : i = 1, \dots, r\}^{-1}$ and $s \in \mathbb{N}$ such that $\|T_{n_{k_s}}u - x\| < \frac{\varepsilon}{2\max\{\|T_{n_{k_1}}\|,\dots,\|T_{n_{k_r}}\|\}}$. Set $x_i = T_{n_{k_i}}u \quad (i = 1, \dots, r)$. Then $x_i \in B_X$ and

$$\|T_{n_{k_s}}x_i - y_i\| = \|T_{n_{k_s}}T_{n_{k_i}}u - y_i\| \le \|T_{n_{k_i}}(T_{n_{k_s}}u - x)\| + \|T_{n_{k_i}}x - y_i\| < \varepsilon$$

for all $i = 1, \dots, r$.

(ii)⇒(iii): Clear.

(iii)⇒(iv): Let $U, V, W \subset X$ be nonempty open sets, $0 \in W$. Let (n_k) be a sequence of positive integers such that each subsequence of (T_{n_k}) is hypercyclic. Let x be a hypercyclic vector for (T_{n_k}). Since each nonzero multiple of x is also hypercyclic, we can assume that $x \in W$. Consider the subsequence $(T_{n_k})_{k\in F}$ where $F = \{k : T_{n_k}x \in V\}$. Consequently, each $k \in F$ satisfies $T_{n_k}W \cap V \neq \emptyset$.

Let y be a vector hypercyclic for this subsequence. Thus there exists $k_0 \in F$ such that $T_{n_{k_0}}y \in U$. Moreover, we can choose increasing numbers $k_i \in F$ such that $T_{n_{k_i}}y \to 0 \quad (i \to \infty)$. Thus

$$\lim_{i\to\infty} T_{n_{k_i}}T_{n_{k_0}}y = \lim_{i\to\infty} T_{n_{k_0}}T_{n_{k_i}}y = 0$$

and there is an i with $T_{n_{k_i}}T_{n_{k_0}}y \in W$. Hence $T_{n_{k_i}}U \cap W \neq \emptyset$.

(iv)⇒(i): By Proposition 6, the sequence (T_n) has a dense subset of hypercyclic vectors.

Let $U_1, \dots, U_r, V, W \subset X$ be nonempty open subsets, $0 \in W$. Let x be a hypercyclic vector for the sequence (T_n). Find $n_1, \dots, n_r \in \mathbb{N}$ such that $T_{n_i} x \in U_i \quad (i = 1, \dots, r)$. Let $\varepsilon > 0$ satisfy $\{y : \|y - T_{n_i} x\| < \varepsilon\} \subset U_i \quad (i = 1, \dots, r)$ and $\{y : \|y\| < \varepsilon\} \subset W$. By assumption, there are $x' \in X$ and $n_0 \in \mathbb{N}$ such that $\|x' - x\| < \varepsilon \max\{\|T_{n_1}\|, \dots, \|T_{n_r}\|\}^{-1}$, $\|T_{n_0} x'\| < \varepsilon \max\{\|T_{n_1}\|, \dots, \|T_{n_r}\|\}^{-1}$ and $T_{n_0} W \cap V \neq \emptyset$. Then $\|T_{n_i} x' - T_{n_i} x\| \leq \|T_{n_i}\| \cdot \|x' - x\| < \varepsilon$, and so $T_{n_i} x' \in U_i \quad (i = 1, \dots, r)$. Further $\|T_{n_0} T_{n_i} x'\| = \|T_{n_i} T_{n_0} x'\| \leq \|T_{n_i}\| \cdot \|T_{n_0} x'\| < \varepsilon$, and so $T_{n_0} T_{n_i} x' \in W$. Hence $T_{n_0} U_i \cap W \neq \emptyset$ for all i, and so (T_n) satisfies condition (C). □

Thus for commuting operators $T_n : X \to X$ we have the following situation:

$$\text{(HC)} \longrightarrow \text{(C)} \longrightarrow \text{(dense hc)} \longrightarrow \text{(hc)}$$

A sequence (T_n) of commuting operators satisfying condition (C) but not (HC) was given in Example 7.

An example of commuting operators with a dense set of hypercyclic vectors but not satisfying condition (C) is the space $X = \mathbb{C}$ and operators $T_n \in B(X)$ defined by $T_n(\lambda) = r_n \lambda \quad (\lambda \in \mathbb{C})$ where (r_n) is a dense sequence in $\mathbb{C}$.

The existence of a hypercyclic sequence of commuting operators with a non-dense set of hypercyclic vectors is an open problem:

Problem 13. Let (T_n) be a hypercyclic sequence of mutually commuting operators acting on a Banach space X. Is the set of all vectors hypercyclic for (T_n) dense in X?

IV. Commuting chains of operators

The most important case of a sequence of operators is the sequence of powers (T^n) of a fixed operator $T \in B(X)$. Of importance are also sequences of the form $(\lambda_n T^n)$ where $T \in B(X)$ and λ_n are non-zero complex numbers. Hypercyclicity of these sequences is closely connected with the supercyclicity of the operator T. Indeed, an operator $T \in B(X)$ is supercyclic (i.e., there exists $x \in X$ such that the set $\{\lambda T^n x : \lambda \in \mathbb{C}, n \geq 0\}$ is dense) if and only if there are complex numbers (λ_n) such that the sequence $(\lambda_n T^n)$ is hypercyclic. In this way the problems concerning supercyclicity of operators can be reduced to the problems concerning hypercyclicity of sequences of operators.

It turns out that the most important property of the sequences (T^n) or $(\lambda_n T^n)$ is that they form a chain of commuting operators. We call a sequence $(T_n) \subset B(X)$ a chain of commuting operators if there are mutually commuting operators $S_j \in B(X)$ such that $T_n = S_1 \cdots S_n$ for all n.

For chains of commuting operators the situation is even simpler than for the sequences of commuting operators. A hypercyclic chain has always a dense subset of hypercyclic vectors and condition (C) is equivalent to (HC).

Proposition 14. Let $S_j \in B(X) \quad (j \in \mathbb{N})$ be mutually commuting operators and $T_n = S_1 \cdots S_n$. Suppose that the sequence (T_n) is hypercyclic. Then there exists a dense subset of vectors hypercyclic for (T_n).

Proof. Let $x \in X$ be a vector hypercyclic for (T_n). Clearly $T_1 X \supset T_2 X \supset \cdots$, and so T_n has dense range for all n. We show that $T_j x$ is hypercyclic for all j. We have $\{T_n T_j x : n \in \mathbb{N}\}^- \supset T_j \{T_n x : n \in \mathbb{N}\}^- = T_j X$, which is a dense subset of X. Hence $T_j x$ is hypercyclic for each j and the sequence (T_n) has a dense subset of hypercyclic vectors. □

Theorem 15. Let $S_j : X \to X \quad (j \in \mathbb{N})$ be mutually commuting operators and let $T_n = S_1 S_2 \cdots S_n$. Suppose that the sequence (T_n) satisfies condition (C_{fin}). Then it satisfies (HC).

Proof. Since any subsequence of (T_n) is again a chain of commuting operators, without loss of generality we can assume that (T_n) satisfies condition (C_{fin}) for all the sequence (T_n), i.e., that $T_n x \to 0$ for all x in a dense subset of X.

Note first that for all $k, j \in \mathbb{N}$ we have

$$\overline{\bigcup_{n>k} \left(S_{k+1} \cdots S_n B_X\right)^j} = Y^j. \tag{1}$$

Indeed, we have $T_k B_X \subset \|T_k\| B_X$, and so

$$\bigcup_{n>k} \left(S_{k+1} \cdots S_n B_X\right)^j \supset \|T_k\|^{-1} \bigcup_{n>k} \left(S_1 \cdots S_n B_X\right)^j = \|T_k\|^{-1} \bigcup_{n>k} \left(T_n B_X\right)^j,$$

which is dense in Y^j.

Let (x_k) be a sequence dense in X.

By induction on j we construct an increasing sequence n_j and vectors $u_{k,j} \in X$, $(k, j \in \mathbb{N}, j \geq k)$. Set formally $n_0 = 0$ and $u_{k,k} = x_k$.

Let $j \geq 1$ and suppose that n_{j-1} and $u_{k,j-1} \in X \quad (k \leq j-1)$ have already been constructed. By (1), we can find $n_j > n_{j-1}$ and vectors $u_{k,j} \in X \quad (k = 1, \ldots, j-1)$ such that

$$\left\| S_{n_{j-1}+1} \cdots S_{n_j} u_{k,j} - u_{k,j-1} \right\| < \frac{1}{2^{k+j} \prod_{i \leq n_{j-1}} \max\{1, \|S_i\|\}}$$

and

$$\|u_{k,j}\| < \frac{1}{2^{k+j}}.$$

Let $u_{k,j}$ be the vectors constructed in the above described way. Write for short $R_j = S_{n_{j-1}+1} \cdots S_{n_j}$. Then we have

$$\|R_j u_{k,j} - u_{k,j-1}\| < \frac{1}{2^{k+j} \prod_{i \leq j-1} \max\{1, \|R_i\|\}}$$

for all k, j, and $\|u_{k,j}\| < 2^{-(k+j)} \quad (k < j)$.

For fixed $k, j \in \mathbb{N}$ consider the sequence $(R_{j+1} \cdots R_m u_{k,m})_{m=j}^{\infty}$. Since

$$\begin{aligned} &\left\| R_{j+1} \cdots R_{m+1} u_{k,m+1} - R_{j+1} \cdots R_m u_{k,m} \right\| \\ \leq &\|R_{j+1} \cdots R_m\| \cdot \|R_{m+1} u_{k,m+1} - u_{k,m}\| \leq \frac{1}{2^{k+m+1}}, \end{aligned}$$

the sequence $(R_{j+1}\cdots R_m u_{k,m})_{m=j}^{\infty}$ is Cauchy. Denote by $v_{k,j}$ its limit.

For all k, j we have

$$R_{j+1}v_{k,j+1} = \lim_{m\to\infty} R_{j+1}R_{j+2}\cdots R_m u_{k,m} = v_{k,j}.$$

In particular, $T_{n_j}v_{k,j} = R_1\cdots R_j v_{k,j} = v_{k,0}$ for all k, j.

Furthermore,

$$\|v_{k,0} - x_k\| = \lim_{m\to\infty} \|R_1\cdots R_m u_{k,m} - u_{k,k}\|$$
$$\le \sum_{m=k}^{\infty} \|R_1\cdots R_{m+1}u_{k,m+1} - R_1\cdots R_m u_{k,m}\| \le \sum_{m=k}^{\infty} \frac{1}{2^{k+m+1}} = \frac{1}{2^k},$$

and so the sequence $(v_{k,0})$ is dense in X.

Finally, for $j > k$ we have

$$\|v_{k,j}\| = \lim_{m\to\infty} \|R_{j+1}\cdots R_m u_{k,m}\|$$
$$\le \|u_{k,j}\| + \sum_{m=j}^{\infty} \|R_{j+1}\cdots R_{m+1}u_{k,m+1} - R_{j+1}\cdots R_m u_{k,m}\|$$
$$\le \|u_{k,j}\| + \sum_{m=j}^{\infty} \|R_{j+1}\cdots R_m\| \cdot \|R_{m+1}u_{k,m+1} - u_{k,m}\|$$
$$\le \frac{1}{2^{k+j}} + \sum_{m=j}^{\infty} \frac{1}{2^{k+m+1}} = \frac{1}{2^{k+j-1}},$$

and so $\lim_{j\to\infty}\|v_{k,j}\| = 0$. Hence the sequence (T_n) satisfies condition (HC) for the sequence (n_j) and the dense set $\{v_{k,0} : k \in \mathbb{N}\}$. Indeed, it is sufficient to define $S_j v_{k.0} = v_{k,j}$. Then $T_{n_j}S_j v_{k,0} = v_{k,0}$ and $\lim_{j\to\infty} S_j v_{k,0} = \lim_{j\to\infty} v_{k,j} = 0$ for all k. □

Corollary 16. Let $T \in B(X)$ and let (λ_n) be a sequence of complex numbers. Then all the conditions (C), (C_{fin}), (HC), (her hc) and (4 nbhd) are equivalent for the sequence $(\lambda_n T^n)$.

If $(\lambda_n T^n)$ is hypercyclic then there is a dense subset of hypercyclic vectors.

Problem 17. Is there a chain of commuting operators (and in particular a sequence of the form (T^n)) which is hypercyclic but does not satisfy the Hypercyclicity criterion (or any of the equivalent conditions)?

V. Subspaces of hypercyclic vectors

In this section we study the existence of a dense (closed infinite dimensional, respectively) subspace consisting of hypercyclic vectors.

In case of a hypercyclic sequence (T^n) where $T \in B(X)$ is a fixed operator it is known that there is always a dense subspace consisting of hypercyclic vectors. The proof, however, uses special properties of the sequence (T^n).

Our first result gives the existence of a dense subspace consisting of hypercyclic vectors for any sequence $(T_n) \subset B(X,Y)$ satisfying condition C_{fin}.

Theorem 18. Let $(T_n) \subset B(X,Y)$ be a sequence of operators satisfying condition (C_{fin}). Then there exists a dense subspace $X_1 \subset X$ such that each non-zero vector in X_1 is hypercyclic for (T_n).

Proof. Let Z be any separable infinite dimensional Banach space. Let $x \in X$, $x \neq 0$ and $\varepsilon > 0$. Set $\mathcal{M} = \{V \in \overline{F(Z,X)} : \operatorname{dist}\{x, VZ\} < \varepsilon\}$. Clearly $\mathcal{M}$ is open. We show that it is dense in $\overline{F(Z,X)}$.

Let $W \in \overline{F(Z,X)}$ and $\delta > 0$. Then there exists a finite rank operator $W_1 : Z \to X$ such that $\|W - W_1\| < \delta/2$. Let $z \in \ker W_1$ and $z^* \in Z^*$ satisfy $\langle z, z^* \rangle = 1$. Set $W_2 = W_1 + \frac{\delta \cdot (z^* \otimes x)}{2\|x\| \cdot \|z^*\|}$. Then

$$\|W - W_2\| \leq \|W - W_1\| + \|W_1 - W_2\| < \delta$$

and $W_2 z = \frac{\delta x}{2\|x\| \cdot \|z^*\|}$. Thus $W_2 \in \mathcal{M}$ and $\mathcal{M}$ is dense in $\overline{F(Z,X)}$.

Let $(x_k) \subset X$ be a dense sequence of non-zero vectors. Clearly $V \in \overline{F(Z,X)}$ has dense range if and only if $\operatorname{dist}\{x_k, VZ\} < 1/k$ for all k. By the Baire category theorem, the set of all operators in $\overline{F(Z,X)}$ with dense range is residual.

By Theorem 5, the operators $L_{T_n} : \overline{F(Z,X)} \to \overline{F(Z,Y)}$ satisfy condition (C), and so there is a residual set of vectors hypercyclic for (L_{T_n}). Thus there exists an operator $V \in \overline{F(Z,X)}$ with dense range such that V is hypercyclic for (L_{T_n}).

It is easy to see that each nonzero vector in the range VZ is hypercyclic for the sequence (T_n). This completes the proof. □

Next we study the existence of a closed infinite dimensional subspace consisting of hypercyclic vectors for a sequence $(T_n) \subset B(X,Y)$. Such a subspace is known to exist (under a natural additional assumption) if (T_n) is hereditarily hypercyclic. We prove it now for sequences satisfying a more practical condition (C). Moreover, the proof is essentially simplified.

Note that a particularly simple argument is available in case of a sequence (T^n) satisfying the Hypercyclicity criterion (HC), see [ChT].

We say for short that a subspace $X_1 \subset X$ is a hypercyclic subspace for a sequence $(T_n) \subset B(X,Y)$ if each nonzero vector in X_1 is hypercyclic for (T_n).

Theorem 19. (cf. [Mo]) Let $(T_n) \subset B(X,Y)$ be a sequence of operators. Suppose that (n_k) is an increasing sequence of positive integers such that

(i) there exists a dense subset $X_0 \subset X$ such that $\lim_{k\to\infty} T_{n_k} x = 0 \quad (x \in X_0)$;

(ii) $\overline{\bigcup_{k\in\mathbb{N}} T_{n_k} B_X} = Y$;

(iii) there exists a closed infinite dimensional subspace $X_1 \subset X$ with the property that $\lim_{k\to\infty} T_{n_k} x = 0 \quad (x \in X_1)$.

Then there exists a closed infinite dimensional hypercyclic subspace for (T_n).

Proof. Without loss of generality we can assume that $\lim_{n\to\infty} T_n x = 0$ for all $x \in X_0 \cup X_1$.

Let $\{e_1, e_2, \dots\}$ be a normalized basic sequence in X_1. Let K be the corresponding basic constant and let $\varepsilon < \frac{1}{2K}$. Let (y_k) be a dense sequence in Y.

Let $\prec$ be an order on $\mathbb{N}\times(\mathbb{N}\cup\{0\})$ defined by $(i,j)\prec(i',j')$ if either $i+j<i'+j'$ or $i+j=i'+j'$ and $i<i'$.

Set $z_{i,0}=e_i \quad (i=1,2,\dots)$. By induction with respect to the order $\prec$ we construct vectors $z_{i,j}\in X_0 \quad (i,j\in\mathbb{N})$ and an increasing sequence $n_{i,j}\subset\mathbb{N}$.

Let $(i,j)\in\mathbb{N}\times\mathbb{N}$ and suppose that $z_{i',j'}\in X_0$ and $n_{i',j'}\in\mathbb{N}$ have already been constructed for all $(i',j')\prec(i,j)$. By definition, there exist $n_{i,j}>\max\{n_{i',j'}:(i',j')\prec(i,j)\}$ and $z_{i,j}\in X_0$ such that

$$\begin{aligned}
&\|T_{n_{i,j}}z_{i',j'}\|<\frac{\varepsilon}{2^{i'+j'+j}} \qquad ((i',j')\prec(i,j)),\\
&\|z_{i,j}\|<\frac{\varepsilon}{2^{i+j}\max\{1,2^{j'}\|T_{n_{i',j'}}\|:(i',j')\prec(i,j)\}},\\
&\|T_{n_{i,j}}z_{i,j}-y_j\|<\frac{\varepsilon}{2^{i+2j}}.
\end{aligned}$$

Construct vectors $z_{i,j}\in X_0$ and numbers $n_{i,j}$ inductively in this way.

Set $z_i=\sum_{j=0}^{\infty}z_{i,j} \quad (i\in\mathbb{N})$. Then

$$\|z_i-e_i\|\le\sum_{j=1}^{\infty}\|z_{i,j}\|<\sum_{j=1}^{\infty}\frac{\varepsilon}{2^{i+j}}=\frac{\varepsilon}{2^i}.$$

Hence $\sum_{i=1}^{\infty}\|z_i-e_i\|<\sum_{i=1}^{\infty}\frac{\varepsilon}{2^i}=\varepsilon$, and so (z_i) is a basic sequence.

Let $M=\bigvee\{z_i:i=1,2,\dots\}$. Let $z\in M$ be any non-zero vector. Then $z=\sum_{i=1}^{\infty}\alpha_i z_i$ for some complex coefficients α_i. We show that z is hypercyclic for (T_n).

Fix $k\in\mathbb{N}$ with $\alpha_k\neq0$. Since every non-zero scalar multiple of a hypercyclic vector is also hypercyclic, we can assume that $\alpha_k=1$. Let $r\in\mathbb{N}$ be arbitrary. Then

$$\begin{aligned}
&\|T_{n_{k,r}}z-y_r\|\le\sum_{i\neq k}|\alpha_i|\cdot\|T_{n_{k,r}}z_i\|+\|T_{n_{k,r}}z_k-y_r\|\\
&\le\sum_{i\neq k}\sum_{j=0}^{\infty}|\alpha_i|\cdot\|T_{n_{k,r}}z_{i,j}\|+\sum_{j\neq r}\|T_{n_{k,r}}z_{k,j}\|+\|T_{n_{k,r}}z_{k,r}-y_r\|\\
&\le\sum_{(i,j)\prec(k,r)}\max\{|\alpha_i|:i\in\mathbb{N}\}\cdot\|T_{n_{k,r}}z_{i,j}\|+\sum_{(k,r)\prec(i,j)}\max\{|\alpha_i|:i\in\mathbb{N}\}\cdot\|T_{n_{k,r}}z_{i,j}\|\\
&\qquad\qquad+\|T_{n_{k,r}}z_{k,r}-y_r\|\\
&<\sum_{(i,j)\prec(k,r)}\max\{|\alpha_i|:i\in\mathbb{N}\}\cdot\frac{\varepsilon}{2^{i+j+r}}+\sum_{(k,r)\prec(i,j)}\max\{|\alpha_i|:i\in\mathbb{N}\}\cdot\frac{\varepsilon}{2^{i+j+r}}+\frac{\varepsilon}{2^{k+2r}}\\
&\le\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\max\{|\alpha_i|:i\in\mathbb{N}\}\frac{\varepsilon}{2^{i+j+r}}\le\frac{K\varepsilon}{2^{r-1}}.
\end{aligned}$$

Hence z is hypercyclic for (T_n). □

Theorem 20. Let $(T_n)\subset B(X,Y)$ be a sequence of operators satisfying condition (C) for a subsequence (n_k). Suppose that there are infinite dimensional subspaces $M_1,M_2,\dots$ such that $X\supset M_1\supset M_2\supset\cdots$ and $\sup_k\|T_{n_k}|M_k\|<\infty$. Then there exists a closed infinite dimensional hypercyclic subspace for (T_n).

Proof. Without loss of generality we can assume that $T_n x \to 0$ for all x in a dense subset $X_0 \subset X$. It is sufficient to construct a closed infinite dimensional subspace $X_1 \subset X$ such that $T_n x \to 0 \quad (x \in X_1)$.

We can find a basic sequence (x_n) such that $x_i \in M_i$ for all i. Let K be the basic constant of this sequence. Let $\varepsilon < \frac{1}{2K}$ be a positive number. For each n find $e_n \in X_0$ such that $\|x_n - e_n\| < \frac{\varepsilon}{2^n \max\{1,\|T\|,\dots,\|T_n\|\}}$. Clearly (e_n) is a basic sequence with the basic constant $\le 2K$. Let (y_n) be a dense sequence in Y. Choose a subsequence (e_{n_k}) such that $\|T_{n_k} e_{n_i}\| < 2^{-(k+i)} \quad (i < k)$ and $\operatorname{dist}\{y_k, T_{n_k} B_X\} < 2^{-k}$. Set $X_1 = \bigvee\{e_{n_k} : k \in \mathbb{N}\}$. Let $e \in X_1$ be an arbitrary vector. We can write $e = \sum_{i=1}^{\infty} \alpha_i e_{n_i}$ for some complex coefficients α_i. We have

$$
\begin{aligned}
\|T_{n_k} e\| &\le \sum_{i=1}^{k-1} \|T_{n_k} \alpha_i e_{n_i}\| + \Big\|\sum_{i=k}^{\infty} T_{n_k} \alpha_i x_{n_i}\Big\| + \sum_{i=k}^{\infty} \|T_{n_k} \alpha_i (e_{n_i} - x_{n_i})\| \\
&\le 2K \sum_{i=1}^{k-1} \frac{1}{2^{i+k}} + \sup_n \|T_n|M_n\| \cdot \Big\|\sum_{i=k}^{\infty} \alpha_i x_{n_i}\Big\| + 2K \sum_{i=k}^{\infty} \frac{\varepsilon}{2^i} \\
&\le \frac{K}{2^{k-1}} + \sup_n \|T_n|M_n\| \cdot \Big\|\sum_{i=k}^{\infty} \alpha_i x_{n_i}\Big\| + \frac{K\varepsilon}{2^{k-2}} \to 0
\end{aligned}
$$

as $k \to \infty$. Further $\overline{\bigcup_j T_{n_j} B_X} = Y$, and so there is a closed infinite dimensional subspace consisting of hypercyclic vectors for (T_n). □

We give now a negative result — a condition implying that there is no closed infinite dimensional subspace consisting of hypercyclic vectors.

Recall the quantity $j_\mu(T) = \sup\big\{j(T|M) : M \subset X, \operatorname{codim} M < \infty\big\}$, where j denotes the minimum modulus, $j(S) = \inf\{\|Sx\| : \|x\| \le 1\}$. The number $j_\mu(T)$ can be called the essential minimum modulus of T.

Lemma 21. Let $T_1, \dots, T_k \in B(X,Y)$, let $X_1 \subset X$ be a closed infinite-dimensional subspace. Let $\varepsilon > 0$. Then there exists $x \in X_1$ of norm one such that $\|T_i x\| > j_\mu(T_i) - \varepsilon \quad (i = 1, \dots, k)$.

Proof. For $i = 1, \dots, k$ there is a subspace $M_i \subset X$ of finite codimension such that $j(T_i|M_i) > j_\mu(T_i) - \varepsilon$. Let x be any vector of norm one in $X_1 \cap \bigcap_{i=1}^k M_i$. Then

$$\|T_i x\| \ge j(T_i|M_i) > j_\mu(T_i) - \varepsilon$$

for all $i = 1, \dots, k$. □

Theorem 22. Let X, Y be Banach spaces, let $(T_n) \subset B(X,Y)$ be a sequence of operators, let (a_n) be a sequence of positive numbers such that $\lim_{i\to\infty} a_i = 0$ and let $X_1 \subset X$ be a closed infinite-dimensional subspace. Let $\delta > 0$. Then there exists a vector $x \in X_1$ with $\|x\| \le \sup_i a_i + \delta$ and $\|T_n x\| \ge a_n \cdot j_\mu(T_n)$ for all $n \in \mathbb{N}$.

Moreover, there is a subset X_2 dense in X_1 with the property that for each $x \in X_2$ there exists n_0 such that $\|T_n x\| \ge a_n j_\mu(T_n) \quad (n \ge n_0)$.

Proof. Without loss of generality we can assume that $a_1 \geq a_2 \geq \cdots$. Let $\varepsilon > 0$ satisfy $(1-\varepsilon)^2(a_1 + \frac{\delta}{2}) > a_1$. Find numbers $r_0 < r_1 < \cdots$ such that $a_{r_k} < \frac{(1-\varepsilon)^3\delta}{2^{k+3}}$. Find $x_0 \in X_1$ such that $\|x_0\| = a_1 + \delta/2$ and $\|T_n x_0\| > (1-\varepsilon)(a_1+\delta/2)j_\mu(T_n) \quad (n \leq r_0)$.

Let $k \geq 0$ and suppose that $x_0, \dots, x_k$ have already been constructed. Let $E_k = \bigvee\{T_n x_i : 0 \leq i \leq k, 1 \leq n \leq r_{k+1}\}$. Let M_k be a subspace of X of finite codimension such that

$$\|e + m\| \geq (1-\varepsilon)\max\{\|e\|, \|m\|/2\} \qquad (e \in E_k, m \in M_k),$$

see [M]. Since the space $L_k = \bigcap_{i=1}^k \bigcap_{n=1}^{r_{k+1}} T_n^{-1} M_i < \infty$ is of finite codimension, we can choose $x_{k+1} \in X_1 \cap L_k$ such that $\|x_{k+1}\| = \delta 2^{-(k+2)}$ and

$$\|T_n x_{k+1}\| \geq (1-\varepsilon)\delta 2^{-(k+2)} j_\mu(T_n) \qquad (1 \leq n \leq r_{k+1}).$$

Set $x = \sum_{i=0}^\infty x_i$. Then $x \in X_1$ and

$$\|x\| \leq \sum_{i=0}^\infty \|x_i\| \leq a_1 + \delta/2 + \sum_{i=1}^\infty \delta 2^{-(i+1)} = a_1 + \delta.$$

For $n = 1, \dots, r_0$ we have

$$\|T_n x\| = \left\| T_n x_0 + \sum_{i=1}^\infty T_n x_i \right\| \geq (1-\varepsilon)\|T_n x_0\| > a_1 j_\mu(T_n) \geq a_n j_\mu(T_n).$$

Let $k \geq 0$ and $r_k < n \leq r_{k+1}$. Then

$$\begin{aligned}\|T_n x\| &= \left\| \sum_{i=0}^\infty T_n x_i \right\| \geq (1-\varepsilon)\left\| \sum_{i=0}^{k+1} T_n x_i \right\| \\ &\geq \frac{(1-\varepsilon)^2}{2}\|T_n x_{k+1}\| \geq \frac{(1-\varepsilon)^3}{2} \cdot \frac{\delta}{2^{k+2}} j_\mu(T_n) \geq a_n \cdot j_\mu(T_n).\end{aligned}$$

Thus $\|T_n x\| \geq a_n j_\mu(T_n)$ for all $n \in \mathbb{N}$.

To show the second statement, let $u \in X_1$ and $\varepsilon > 0$. Find n_0 such that $a_n < \varepsilon$ for all $n \geq n_0$. As in the first part, taking $x_0 = u$, construct a vector $x \in X_1$ with $\|x - u\| \leq \varepsilon$ and $\|T_n x\| \geq a_n j_\mu(T_n) \quad (n \geq n_0)$. □

Corollary 23. Let X, Y be Banach spaces, let $(T_n) \subset B(X,Y)$ be a sequence of operators satisfying $\lim_{n\to\infty} j_\mu(T_n) = \infty$. Then there is no closed infinite dimensional hypercyclic subspace for (T_n).

Proof. Let M be a closed infinite dimensional subspace of X. By the previous result for the numbers $\alpha_n = \big(j_\mu(T_n)\big)^{-1/2}$, there exists $x \in M$ such that $\|T_n x\| \to \infty$. Therefore x is not hypercyclic for (T_n). □

We apply the previous results to the sequences of the form $(\lambda_n T^n)$ where $T \in B(X)$ and λ_n are complex numbers. Denote by $\sigma_e(T)$ the essential spectrum of T, $\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\}$.

Corollary 24. Let $T \in B(X)$ be an operator and let (λ_n) be a sequence of complex numbers. Suppose that $(\lambda_n T^n)$ satisfies condition (C) and $\sup_n |\lambda_n| d^n < \infty$ where $d = \text{dist}\,\{0, \sigma_e(T)\}$. Then there exists a closed infinite dimensional hypercyclic subspace for $(\lambda_n T^n)$.

Proof. Since $(\lambda_n T^n)$ satisfies condition (C), the range of T is dense. Without loss of generality we can assume that the numbers λ_n are non-zero.

Choose $\mu \in \sigma_e(T)$ with $|\mu| = d$. Thus $T - \mu$ is not Fredholm. We show that $T - \mu$ is not upper semi-Fredholm. This is clear if $d = 0$ since the range of T is dense. If $d > 0$ then $\mu \in \partial\sigma_e(T)$ and $T - \mu$ is not upper semi-Fredholm by [HW].

By [LS], there is a compact operator $K \in B(X)$ such that $\dim \ker(T - \mu - K) = \infty$. Set $M_0 = \ker(T - \mu - K)$. For each n we have $T^n = (T - K)^n + K_n$ for some compact operator K_n. Find subspaces $M'_n \subset X$ of finite codimension such that $\|K_n|M'_n\| \le |\lambda_n|^{-1}$. Set $M_n = M_0 \cap \bigcap_{i \le n} M'_i$. Then $M_1 \supset M_2 \supset \cdots$ and $\dim M_n = \infty$ for all n. For $z \in M_n$, $\|z\| = 1$ we have $(T - K)z = \mu z$ and

$$\|\lambda_n T^n z\| \le \|\lambda_n (T - K)^n z\| + \|\lambda_n K_n\| \le |\lambda_n \mu^n| + 1 = |\lambda_n| d^n + 1.$$

Thus $\sup_n \|\lambda_n T^n | M_n\| < \infty$. The statement now follows from the previous lemma. □

Corollary 25. Let $T : X \to X$, suppose that $(\lambda_n T^n)$ satisfies condition (C) and T is not Fredholm. Then there is an infinite dimensional closed hypercyclic subspace for $(\lambda_n T^n)$.

Proof. We have $d = \text{dist}\,\{0, \sigma_e(T)\} = 0$, and so the statement follows from the previous corollary. □

Corollary 26. Let $T \in B(X)$ and suppose that (T^n) satisfies condition (C). The following conditions are equivalent:

(i) there exists a closed infinite-dimensional hypercyclic subspace for (T^n);
(ii) the essential spectrum of T intersects the closed unit ball.

Proof. Write $d = \text{dist}\,\{0, \sigma_e(T)\}$.

(ii)⇒(i): If $d \le 1$ then Corollary 24 implies (i).

(i)⇒(ii): Let $d > 1$. Then T is Fredholm.

Recall the following standard construction from operator theory, see [S], [BHW]: let $\ell^\infty(X)$ be the space of all bounded sequences of elements of X; with the naturally defined algebraic operations and sup-norm it is a Banach space. Let $\tilde{X} = \ell^\infty(X)/m(X)$ where $m(X)$ is the subspace of all precompact sequences. Let $\tilde{T} : \tilde{X} \to \tilde{X}$ be the operator induced by T. It is well-known that $\tilde{T}$ is invertible and $\sigma(\tilde{T}) = \sigma_e(T)$. By the spectral radius formula we have $d = \text{dist}\,\{0, \sigma(\tilde{T})\} = r(\tilde{T}^{-1})^{-1} = \lim_{n\to\infty} \|\tilde{T}^{-n}\|^{-1/n} = \lim_{n\to\infty} j(\tilde{T}^n)^{1/n}$ where r denotes the spectral radius. By [F], $j_\mu(T^n) \le 2j(\tilde{T}^n) \le 4j_\mu(T^n)$ for all n. Thus $1 < d = \lim_{n\to\infty} j_\mu(T^n)^{1/n}$ and $\lim_{n\to\infty} j_\mu(T^n) = \infty$.

By Corollary 23, there is no closed infinite-dimensional hypercyclic subspace for (T^n). □

References

[Bea] B. Beauzamy, Un opèrateur sur l'espace de Hilbert, dont tous les polynômes sont hypercycliques, C.R. Acad. Sci. Paris Ser. I Math. 303 (1986), 923–927.

[BG] L. Bernal-González, K.-G. Große-Erdmann, The hypercyclicity criterion for sequences of operators

[Bo] P.S. Bourdon, Invariant manifolds of hypercyclic vectors, Proc. Amer. Math. Soc. 118 (1993), 845–847.

[BHW] J.J. Buoni, R. Harte, T. Wickstead, Upper and lower Fredholm spectra, Proc. Amer. Math. Soc. 66 (1977), 309–314.

[ChT] K.C. Chan, R.D. Taylor Jr., Hypercyclic subspaces of a Banach space, Int. Eq. Operator Theory 41 (2001), 381–388.

[F] A.S. Fainshtein, Measures of noncompactness of linear operators and analogues of the minimum modulus for semi-Fredholm operators, in: Spectral Theory of Operators 6, 182–195, Elm, Baku, 1985 (Russian).

[GLMo] M. González, F. León, A. Montes, Semi-Fredholm theory: Hypercyclic and Supercyclic Subspaces, Proc. London Math. Soc. 81 (2000) 169-189.

[G] G. Godefroy, Opèrateurs ayant un sous-espace dense de vecteurs hypercycliques, (Sèminaire d'Initiation à l'Analyse), Publ. Math. Univ. Pierre et Marie Curie, 94 (1989).

[GS] G. Godefroy, J.H. Shapiro, Operators with dense, invariant, cyclic vector manifolds, J. Funct. Anal. 98 (1991), 229–269.

[HW] R. Harte, T. Wickstead, Upper and lower Fredholm spectra II, Math. Z. 154 (1977), 253–256.

[H] D.A. Herrero, Limits of hypercyclic and supercyclic operators, J. Funct. Anal. 99 (1991), 179–190.

[K] C. Kitai, Invariant closed sets for linear operators, Dissertation, University of Toronto, Toronto, 1982.

[LS] A. Lebow, M. Schechter, Semigroups of operators and measures of noncompactness, J. Funct. Anal. 7 (1971), 1–26.

[LMo] F. León, A. Montes, Spectral theory and Hypercyclic Subspaces, Trans. Amer. Math. Soc. 353 (2000), 247–267.

A . Montes, Banach spaces of hypercyclic vectors, Michigan Math. J. 43 (1996), 419–436.

[M] V. Müller, Local behaviour of the polynomial calculus of operators, J. reine angew. Math. 430 (1992), 61–68.

[S] B.N. Sadovskii, Limit-compact and condensing operators, Uspekhi Mat. Nauk 27 (1972), 81–146 (Russian), English transl. Russian Math. Surveys 27 (1972), 85–155.

Dpto. de Matemáticas, Facultad
de Ciencias, Universidad de Cádiz
Pol. Rio San Pedro S/N
11500 Puerto Real, Spain

fernando.leon@uca.es

Mathematical Institute
Czech Academy of Sciences
Žitná 25, 115 67 Prague 1
Czech Republic

muller@math.cas.cz