Corrigendum an Addendum: "On the axiomatic theory of spectrum II"

J.J. Koliha, M. Mbekhta, V. Müller*, Pak Wai Poon

The main purpose of this paper is to correct the proof of Theorem 15 of [4], concerned with the stability of the class of quasi-Fredholm operators under finite rank perturbations, and to answer some open questions raised there.

Recall some notations and terminology from [4].

For closed subspaces M, L of a Banach space X we write $M \overset{e}{\subset} L$ $\big(M$ is essentially contained in $L\big)$ if there exists a finite-dimensional subspace $F \subset X$ such that $M \subset L + F$. Equivalently, $\dim M/(M \cap L) = \dim(M + L)/L < \infty$. Similarly we write $M \overset{e}{=} L$ if $M \overset{e}{\subset} L$ and $L \overset{e}{\subset} M$.

For a (bounded linear) operator $T \in \mathcal{L}(X)$ write $R^\infty(T) = \bigcap_{n=0}^\infty R(T^n)$ and $N^\infty(T) = \bigcup_{n=0}^\infty N(T^n)$.

An operator $T \in \mathcal{L}(X)$ is called semi-regular (essentially semi-regular) if $R(T)$ is closed and $N(T) \subset R^\infty(T)$ $\big(N(T) \overset{e}{\subset} R^\infty(T)$, respectively$\big)$. Further, T is called quasi-Fredholm if there exists $d \geq 0$ such that $R(T^{d+1})$ is closed and $R(T) + N(T^d) = R(T) + N^\infty(T)$ (equivalently, $N(T) \cap R(T^d) = N(T) \cap R^\infty(T)$).

The proof of Theorem 15 of [4] relies on the following statement (where d is an integer whose existence is postulated in the definition of quasi-Fredholm operators):

if T is quasi-Fredholm and F of rank 1 then $N(T) \cap R(T^d) \subset R^\infty(T + F)$.

This, however, need not be satisfied.

Counterexample. Let H be the Hilbert space with an orthonormal basis $\{e_1, e_2, \dots\}$. Define $T, F \in \mathcal{L}(H)$ by

$$Te_1 = 0, Te_n = e_{n-1} \ \ (n \geq 2), \qquad Fe_2 = -e_1, \ Fe_n = 0 \ \ (n \neq 2).$$

Then T is quasi-Fredholm (with $d = 0$) and is surjective, F has rank 1, and $T + F$ is given by

$$(T + F)e_1 = (T + F)e_2 = 0, \quad (T + F)e_n = e_{n-1} \ (n \geq 3).$$

It follows that $R^\infty(T + F) = R(T + F)$ is equal to the linear span of $\{e_2, e_3, \dots\}$, and $N(T)$ to the one-dimensional space spanned by e_1. Thus $N(T) \not\subset R^\infty(T + F)$.

We proceed now to give a correct proof of Theorem 15 of [4].

Theorem. Let $T \in \mathcal{L}(X)$ be a quasi-Fredholm operator and let $F \in \mathcal{L}(X)$ be a finite-rank operator. Then $T + F$ is also quasi-Fredholm.

Proof. Clearly it is sufficient to consider only the case of $\dim R(F) = 1$. Thus there exist $z \in X$ and $\varphi \in X^*$ such that $Fx = \varphi(x)z$ $(x \in X)$.

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Since $R((T+F)^n) \stackrel{e}{=} R(T^n)$ for all n by Observation 8 following Table 1 in [4], $R((T+F)^n)$ is closed if and only if $R(T^n)$ is closed, and hence it is sufficient to show only the algebraic condition in the definition of quasi-Fredholm operators for $T+F$.

Since T is quasi-Fredholm, there exists $d \ge 0$ such that $N(T) \cap R(T^d) \subset R^\infty(T)$ and $R(T^d), R(T^{d+1})$ are closed. Set $M = R(T^d)$ and $T_1 = T|M$. Then $N(T_1) = N(T) \cap R(T^d) \subset R^\infty(T) = R^\infty(T_1)$ and the range $R(T_1) = R(T^{d+1})$ is closed. Thus T_1 is semi-regular.

It is sufficient to show that $N(T_1) \stackrel{e}{\subset} R^\infty(T+F)$. Indeed, then we have

$$N(T+F) \cap R((T+F)^d) \stackrel{e}{=} N(T) \cap R(T^d) = N(T_1) \stackrel{e}{\subset} R^\infty(T+F)$$

so that $N(T+F) \cap R((T+F)^d) \stackrel{e}{=} N(T+F) \cap R^\infty(T+F)$.

This means that $N(T+F) \cap R((T+F)^n) = N(T+F) \cap R^\infty(T+F)$ for some $n \ge d$ and $T+F$ is quasi-Fredholm.

To prove $N(T_1) \stackrel{e}{\subset} R^\infty(T+F)$ we distinguish two cases:

A. $N^\infty(T_1) \subset \ker \varphi$.

Let $x_0 \in N(T_1)$. Since T_1 is semi-regular, there exist vectors $x_1, x_2, \ldots \in R^\infty(T_1)$ such that $Tx_i = x_{i-1}$ for all i. By the assumption $\varphi(x_i) = 0$, so that $Fx_i = 0$ for all i. For $n \in \mathbf{N}$ we have

$$(T+F)^n x_n = (T+F)^{n-1} x_{n-1} = \ldots = (T+F)x_1 = x_0,$$

so that $x_0 \in R((T+F)^n)$. Since x_0 and n were arbitrary, we have $N(T_1) \subset R^\infty(T+F)$.

B. $N^\infty(T_1) \not\subset \ker \varphi$.

There exists $k \ge 1$ such that $N(T_1^k) \not\subset \ker \varphi$. Choose the minimal k with this property so that $N(T_1^{k-1}) \subset \ker \varphi$ and there exists $u \in N(T_1^k)$ with $\varphi(u) = 1$.

Set

$$Y = \{x \in N(T_1) : \text{ there is } y \in M \text{ with } T^{k-1}y = x \text{ and } T^i y \in \ker \varphi \ \ (i = 0, \ldots, k-1)\}.$$

We show that $\dim N(T_1)/Y \le k$. Indeed, let $x^{(1)}, \ldots, x^{(k+1)} \in N(T_1)$. Since T_1 is semi-regular, there are $y^{(1)}, \ldots, y^{(k+1)} \in M$ such that $T^{k-1}y^{(j)} = x^{(j)} \quad (j = 1, \ldots, k+1)$. Then there exists a nontrivial linear combination $y = \sum_{j=1}^{k+1} \alpha_j y_j$ such that $T^i y \in \ker \varphi$ for all $i = 0, \ldots, k-1$. Consequently $\sum_{j=1}^{k+1} \alpha_j x^{(j)} \in Y$ and $\dim N(T_1)/Y \le k$. Hence $Y \stackrel{e}{=} N(T_1)$ and it is sufficient to show $Y \subset R^\infty(T+F)$.

Let $x \in Y$. We prove by induction on n the following statement:

$$\text{There exists } x_n \in M \text{ such that } T^n x_n = x \text{ and } T^i x_n \in \ker \varphi \ \ (i = 0, \ldots, n). \tag{1}$$

Clearly (1) for $n = 0, \ldots, k-1$ follows from the definition of Y.

Suppose that (1) is true for some $n \ge k-1$, i.e., there is $x_n \in M$ such that $T^n x_n = x$ and $T^i x_n \in \ker \varphi \quad (i = 0, \ldots, n)$. Since T_1 is semi-regular, we can find $x'_{n+1} \in M$ such that $Tx'_{n+1} = x_n$. Set $x_{n+1} = x'_{n+1} - \varphi(x'_{n+1})u$. Then

$$T^{n+1} x_{n+1} = T^n x_n - \varphi(x'_{n+1}) T^{n+1} u = x.$$

Clearly $\varphi(x_{n+1}) = 0$. For $1 \le i \le k-1$ we have $\varphi(T^i x_{n+1}) = \varphi(T^{i-1} x_n) - \varphi(x'_{n+1})\varphi(T^i u) = 0$ since $T^i u \in N(T_1^{k-1}) \subset \ker \varphi$. For $k \le i \le n$ we have $T^i u = 0$ so that $\varphi(T^i x_{n+1}) = \varphi(T^{i-1} x_n) = 0$ by the induction assumption.

Thus (1) is true for all n and $(T+F)^n x_n = (T+F)^{n-1} T x_n = \ldots = T^n x_n = x$. Thus $x \in R((T+F)^n)$ for all n and consequently $Y \subset R^\infty(T+F)$.

This finishes the proof of the theorem.

As a corollary we obtain the corresponding result for essentially semi-regular operators, see [2]. Recall the numbers $k_n(T)$ defined for an operator $T \in \mathcal{L}(X)$ and $n \geq 0$ by

$$\begin{aligned} k_n(T) &= \dim[R(T) + N(T^{n+1})]/[R(T) + N(T^n)] \\ &= \dim[N(T) \cap R(T^n)]/[N(T) \cap R(T^{n+1})], \end{aligned}$$

see [4] and [1].

Corollary. If $T, F \in \mathcal{L}(X)$, T is essentially semi-regular and F of finite rank then $T + F$ is essentially semi-regular.

Proof. By the previous theorem $T + F$ is quasi-Fredholm so that $k_i(T + F) = 0$ for all i sufficiently large. Also $k_i(T) < \infty$ implies $k_i(T + F) < \infty$ for all i. Thus $T + F$ is essentially semi-regular.

This finishes the 'corrigendum' part of the paper. For the 'addendum' part, we give counterexamples that will complete Table 2 of [4] answering thus question posed in that paper.

Recall the classes defined in [4]:

$$\begin{aligned} R_{11} &= \{T \in \mathcal{L}(X) : T \text{ is semi-regular}\}, \\ R_{12} &= \{T \in \mathcal{L}(X) : T \text{ is essentially semi-regular}\}, \\ R_{13} &= \{T \in \mathcal{L}(X) : R(T) \text{ is closed and } k_n(T) < \infty \text{ for all } n \in \mathbf{N}\}, \\ R_{14} &= \{T \in \mathcal{L}(X) : T \text{ is quasi-Fredholm}\}, \\ R_{15} &= \{T \in \mathcal{L}(X) : \text{ there is } d \in \mathbf{N} \text{ with } R(T^{d+1}) \text{ closed and } k_n(T) < \infty \ (n \geq d)\}. \end{aligned}$$

Further, for $i = 11, \ldots, 15$, set $\sigma_i(T) = \{\lambda \in \mathbf{C} : T - \lambda \notin R_i\}$.

Example 1. In general, σ_{13} and σ_{15} are not closed. Consequently, R_{13} is not stable under small commuting perturbations:

Consider the operator defined in Example 14 of [4],

$$S = \bigoplus_{n=1}^{\infty} S_n$$

where $S_n \in \mathcal{L}(H_n)$, H_n is an n-dimensional Hilbert space with an orthonormal basis $e_{n1}, \ldots, e_{nn}$ and S_n is the shift operator, that is, $S_n e_{n1} = 0$, $S_n e_{ni} = e_{n,i-1}$ $(2 \leq i \leq n)$. Then $S \in R_{13} \subset R_{15}$, see Example 14 of [4].

Let $\varepsilon \neq 0, |\varepsilon| < 1$. Then $S_n - \varepsilon$ is invertible for all $n \in \mathbf{N}$ so that $S - \varepsilon$ is injective.

For $n \in \mathbf{N}$ set $x_n = \sum_{i=1}^{n} \varepsilon^{i-1} e_{ni}$. Then $\|x_n\| \geq 1$ and

$$\|(S - \varepsilon)x_n\| = \| - \varepsilon^n e_{nn}\| = |\varepsilon^n|.$$

Thus $S - \varepsilon$ is not bounded below and $R(S - \varepsilon)$ is not closed. Hence $S - \varepsilon \notin R_{13}$ and $\sigma_{13}(S)$ is not closed.

Further, for each $k \in \mathbf{N}$, we have

$$\|(S-\varepsilon)^k x_n\| = |\varepsilon^n| \cdot \|(S-\varepsilon)^{k-1} e_{nn}\| \le |\varepsilon^n| \cdot \|(S-\varepsilon)^{k-1}\| \le |\varepsilon^n| \cdot (1+|\varepsilon|)^{k-1}$$

so that $\lim_{n\to\infty} \|(S-\varepsilon)^k x_n\| = 0$ for all $k \in \mathbf{N}$ and $R((S-\varepsilon)^k)$ is not closed. Consequently, $S - \varepsilon \notin R_{15}$ and $\sigma_{15}(S)$ is not closed.

Example 2. The class R_{13} is not stable under commuting compact perturbations:

Consider the operator S from the previous example and let $K = \bigoplus_{n=1}^{\infty} (1/n) I_n$ where I_n denotes the identity operator on H_n. Clearly K is compact, $KS = SK$, $S + K$ is injective and, as above, $S + K$ is not bounded below. Thus $R(S+K)$ is not closed and $S + K \notin R_{13}$.

Example 3. R_{13} is not stable under commuting quasinilpotent perturbations:

For $k \in \mathbf{N}$ let $H^{(k)}$ be the Hilbert space with an orthonormal basis $e_{ni}^{(k)}$ $(n \in \mathbf{N}, i = 1, \dots, \max\{k, n\})$. Let $S^{(k)} \in \mathcal{L}(H^{(k)})$ be the shift to the left,

$$S^{(k)} e_{ni}^{(k)} = \begin{cases} e_{n,i-1}^{(k)} & (i \ge 2), \\ 0 & (i = 1). \end{cases}$$

Set $S = \bigoplus_{k=1}^{\infty} S^{(k)}$. Clearly S is a direct sum of finite-dimensional shifts where n-dimensional shift appears $(2n-1)$-times (once in each $S^{(1)}, \dots, S^{(n-1)}$ and n times in $S^{(n)}$). Thus $S \in R_{13}$.

Define $Q^{(k)} \in \mathcal{L}(H^{(k)})$ by $Q^{(k)} e_{ni}^{(k)} = (1/n) e_{n+1,i}^{(k)}$ for all n, i. Let $Q = \bigoplus_{k=1}^{\infty} Q^{(k)}$. Clearly $SQ = QS$ and Q is a quasinilpotent since $\|Q^j\|^{1/j} = (1/j!)^{1/j} \to 0$.

We prove that $S - Q \notin R_{13}$. Set

$$x^{(k)} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} e_{nn}^{(k)} \in H^{(k)}.$$

Then

$$(S-Q)x^{(k)} = \sum_{n=2}^{\infty} \frac{1}{(n-1)!} e_{n,n-1}^{(k)} - \sum_{n=1}^{\infty} \frac{1}{n!} e_{n+1,n}^{(k)} = 0.$$

Further $x^{(k)} \notin R(S^{(k)}) + R(Q^{(k)})$ so that $x^{(k)} \notin R\big(S^{(k)} - Q^{(k)}\big)$. It is easy to see that each linear combination of $x^{(k)}$'s has the same property with respect to S and Q so that these vectors are linearly independent modulo $R(S-Q)$. Thus

$$k_0(S-Q) = \dim N(S-Q)/\big(N(S-Q) \cap R(S-Q)\big) = \infty$$

and $S - Q \notin R_{13}$.

Consequently, the complete version of Table 2 of [4] is:

	(A) $\sigma_i \neq \emptyset$	(B) $\sigma_i$closed	(C) small commut. perturbations	(D) finite dim. perturbations	(E) commut.comp. perturbations	(F) commut. quasinilp. pert.
R_{11} semi-reg	yes	yes	yes	no	no	yes
R_{12} ess.s-reg.	yes	yes	yes	yes	yes	yes
R_{13}	yes	no	no	yes	no	no
R_{14} $q\varphi$	no	yes	no	yes	no	no
R_{15}	no	no	no	yes	no	no

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Department of Mathematics
The University of Melbourne
Parkville VIC 3052 Australia
email: j.koliha@ms.unimelb.edu.au

Université de Lille
U.F.R. de Mathématiques
F-59655 - Villeneuve d'Ascq Cedex
France

Institute of Mathematics AV ČR
Žitna 25, 115 67 Prague 1
Czech Republic
e-mail: muller@math.cas.cz

Department of Mathematics
The University of Melbourne
Parkville VIC 3052 Australia