On weak orbits of operators

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Abstract. Let T be a completely nonunitary contraction on a Hilbert space H with $r(T)=1$. Let $a_n>0$, $a_n\to 0$. Then there exists $x\in H$ with $|\langle T^nx,x\rangle|\ge a_n$ for all n. We construct a unitary operator without this property. This gives a negative answer to a problem of van Neerven.

Let X be a complex Banach space. Then each operator $T\in B(X)$ has orbits that are "large" in the following sense [M1], [B]:

Let (a_n) be a sequence of positive numbers such that $a_n\to 0$. Then there exists $x\in X$ such that $\|T^nx\|\ge a_n r(T^n)$ for all n. Moreover, for each $\varepsilon>0$ it is possible to find $x\in X$ with $\|x\|<\sup_n a_n+\varepsilon$.

The corresponding question for weak orbits $\langle T^nx,x^*\rangle$ was considered by J. van Neerven [N], see also [M3].

(1) Let $T\in B(X)$. Let (a_n) be a sequence of positive numbers such that $a_n\to 0$. Do there exist $x\in X$ and $x^*\in X^*$ such that $|\langle T^nx,x^*\rangle|\ge a_n r(T^n)$ for all n?

There are several interesting cases when the answer is positive. In [N], it was proved for positive operators on Banach lattices. In [M2] and [M4] the statement was shown for Banach space operators satisfying $T^n\to 0$ in the strong operator topology and $r(T)=1$.

In the present paper we consider Hilbert space operators and generalize this for operators satisfying $T^n\to 0$ in the weak operator topology. As a consequence, we get that (1) is true for any completely non-unitary contraction with $r(T)=1$.

Note that for unitary operators questions concerning weak orbits reduce to questions concerning Fourier coefficients of L^1 functions. We show that if μ is a Rajchman measure (in particular, an absolutely continuous measure) on the unit circle, then there is a positive function $f\in L^1(\mu)$ such that $|\hat f(n)|\ge a_{|n|}$ for all n (the statement is a folklore in case of the Lebesgue measure, see [K, p. 22 and 26]). However, the previous statement is not true in general. We construct an example of a Kronecker measure ν and a sequence (a_n) of positive numbers, $a_n\to 0$ such that there is no function $f\in L^1(\nu)$ with the above property. This also gives a negative answer to question (1) of van Neerven.

Let H be a complex Hilbert space and let $T\in B(H)$. We say that $T^n\to 0$ in the weak operator topology if $\langle T^nx,y\rangle\to 0$ for all $x,y\in H$.

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Theorem 1. Let $T \in B(H)$ satisfy $T^n \to 0$ in the weak operator topology, let $1 \in \sigma(T)$. Let $(a_n)_{n=0}^\infty$ be a sequence of positive numbers satisfying $a_n \to 0$ and let $\varepsilon > 0$. Then there exists $x \in H$ with $\|x\| < \sup_n a_n + \varepsilon$ such that

$$\mathrm{Re}\, \langle T^n x, x\rangle > a_n$$

for all $n \geq 0$.

Proof. Without loss of generality we may assume that $a_0 \geq a_1 \geq \cdots$. Indeed, we may replace the numbers a_n by $\sup_{j\geq n} a_j$ if necessary. Moreover, it is sufficient to show that if $1 > a_0 \geq a_1 \geq \cdots$ then there exists $x \in H$ of norm 1 such that $\mathrm{Re}\, \langle T^n x, x\rangle > a_n$ for all $n \geq 0$.

By the Banach-Steinhaus theorem, T is power bounded, i.e., $\sup_n \|T^n\| < \infty$. Let $K = \sup_n \|T^n\|$. Clearly $r(T) = 1$.

Suppose first that $1 \notin \sigma_e(T)$. Then 1 is an eigenvalue and the corresponding eigenvector x of norm 1 satisfies the required condition. Thus we may suppose that $1 \in \sigma_e(T)$. Hence $1 \in \partial\sigma_e(T)$, and by [HW], $T - I$ is not upper semi-Fredholm. Thus for every subspace $M \subset H$ of finite codimension and each $\varepsilon > 0$ there exists a vector $x \in M$ with $\|x\| = 1$ and $\|Tx - x\| < \varepsilon$. Moreover, given $k \in \mathbb{N}$, we can find a vector $u \in M$ of norm 1 such that $\|T^j u - u\| < \varepsilon$ for all $j \leq k$.

By [M2], there are positive numbers c_i $(i \geq 1)$ such that $\sum_{i=1}^\infty c_i^2 = 1$ and $\sum_{i=k+1}^\infty c_i^2 > 3Kc_k$ for all $k \geq 1$.

Set formally $\delta_0 = 0$. Let $\delta_1, \delta_2, \ldots$ be positive numbers satisfying $\delta_i < \frac{1-a_i}{2^i}$ and $\delta_i < \frac{K}{i^2 2^{i+2}} \min\{c_k : k = 1, \ldots, i+1\}$.

Find n_0 such that $a_{n_0} < \sum_{i=2}^\infty c_i^2 - 3Kc_1$. We construct an increasing sequence (n_k) of positive integers and a sequence (x_k) of unit vectors in H in the following way: Let $k \in \mathbb{N}$ and suppose that $x_i \in H$ and $n_i \in \mathbb{N}$ have already been constructed for $1 \leq i \leq k-1$. Find $x_k \in H$ of norm 1 such that

$$x_k \perp T^j x_t \qquad (0 \leq j \leq n_{k-1}, 1 \leq t \leq k-1)$$

and

$$\|T^j x_k - x_k\| < \delta_k \qquad (j \leq n_{k-1}).$$

Find $n_k > n_{k-1}$ such that

$$|\langle T^j x_t, x_s\rangle| < \delta_k \qquad (j \geq n_k, 1 \leq s, t \leq k)$$

and

$$a_{n_k} < \sum_{i=k+2}^\infty c_i^2 - 3Kc_{k+1}.$$

Let the sequences (x_k) and (n_k) have been constructed in the above described way. Set $x = \sum_{k=1}^\infty c_k x_k$. Since the vectors x_k are orthonormal, we have $\|x\| = (\sum_{k=1}^\infty c_k^2)^{1/2} = 1$.

For $j \leq n_0$ we have

$$\mathrm{Re}\, \langle T^j x, x\rangle = \mathrm{Re} \left\langle \sum_{s=1}^\infty c_s T^j x_s, x \right\rangle = \sum_{s=1}^\infty c_s \mathrm{Re} \left(\langle x_s, x\rangle - \langle x_s - T^j x_s, x\rangle\right)$$

$$\geq \sum_{s=1}^\infty c_s^2 - \sum_{s=1}^\infty c_s\delta_s \geq 1 - \sum_{s=1}^\infty \delta_s > 1 - \sum_{s=1}^\infty \frac{1-a_1}{2^s} = a_1 \geq a_j.$$

Let $k \geq 1$ and $n_{k-1} < j \leq n_k$. We have

$$
\begin{aligned}
\operatorname{Re}\langle T^j x, x\rangle &= \operatorname{Re}\Big\langle \sum_{s=1}^{k} c_s T^j x_s, x\Big\rangle + \operatorname{Re}\Big\langle \sum_{s=k+1}^{\infty} c_s T^j x_s, x\Big\rangle \\
&\geq \operatorname{Re}\Big\langle \sum_{s=1}^{k} c_s T^j x_s, \sum_{t=1}^{k} c_t x_t\Big\rangle + \operatorname{Re}\Big\langle \sum_{s=1}^{k} c_s T^j x_s, \sum_{t=k+1}^{\infty} c_t x_t\Big\rangle \\
&\quad + \sum_{s=k+1}^{\infty} c_s \operatorname{Re}\big(\langle x_s, x\rangle - \|T^j x_s - x_s\|\big) \\
&\geq \operatorname{Re}\Big\langle \sum_{s=1}^{k-1} c_s T^j x_s, \sum_{t=1}^{k-1} c_t x_t\Big\rangle + \operatorname{Re}\Big\langle \sum_{s=1}^{k-1} c_s T^j x_s, c_k x_k\Big\rangle + \operatorname{Re}\Big\langle c_k T^j x_k, \sum_{t=1}^{k} c_t x_t\Big\rangle \\
&\quad + \sum_{s=k+1}^{\infty} c_s^2 - \sum_{s=k+1}^{\infty} c_s \delta_s \\
&\geq -\sum_{s=1}^{k-1}\sum_{t=1}^{k-1} c_s c_t \delta_{k-1} - c_k \cdot \|T^j\| \Big\|\sum_{s=1}^{k-1} c_s x_s\Big\| - K c_k \Big\|\sum_{t=1}^{k} c_t x_t\Big\| + \sum_{s=k+1}^{\infty} c_s^2 - \sum_{s=k+1}^{\infty} \delta_s \\
&\geq \sum_{s=k+1}^{\infty} c_s^2 - 2Kc_k - (k-1)^2 \delta_{k-1} - \sum_{s=k+1}^{\infty} \delta_s \geq \sum_{s=k+1}^{\infty} c_s^2 - 3Kc_k > a_{n_{k-1}} \geq a_j.
\end{aligned}
$$

□

Recall that a contraction T acting on a Hilbert space H is called completely nonunitary if there is no subspace $H_0 \subset H$ reducing for T such that the restriction $T|_{H_0}$ is unitary.

Corollary 2. Let $T \in B(H)$ be a completely nonunitary contraction satisfying $1 \in \sigma(T)$. Let $(a_n)_{n=0}^{\infty}$ be a sequence of positive numbers satisfying $a_n \to 0$ and let $\varepsilon > 0$. Then there exists $x \in H$ with $\|x\| < \sup_n a_n + \varepsilon$ such that

$$\operatorname{Re}\langle T^n x, x\rangle > a_n$$

for all $n \geq 0$.

Proof. Let $U \in B(K)$ be the minimal unitary dilation of T. By [NF], Proposition II.1.4, there are subspaces $M_1, M_2 \subset K$ reducing for U such that $M_1 \vee M_2 = K$ and $U|M_1, U|M_2$ are bilateral shifts (of some multiplicity). It implies that $U^n \to 0$ in the weak operator topology, and consequently, $T^n \to 0$ in the weak operator topology. □

Corollary 3. Let $T \in B(H)$ satisfies $T^n \to 0$ in the weak operator topology and $r(T) = 1$. Let $(a_n)_{n=0}^{\infty}$ be a sequence of positive numbers satisfying $a_n \to 0$ and let $\varepsilon > 0$. Then there exists $x \in H$ with $\|x\| < \sup_n a_n + \varepsilon$ such that

$$|\langle T^n x, x\rangle| > a_n$$

for all $n \geq 0$. In particular, this is true for each completely nonunitary contraction T with $r(T) = 1$.

Better results can be obtained if we consider the Cesaro means. For $T \in B(H)$ and $n \geq 1$ write $A_n = \frac{1}{n}\sum_{i=0}^{n-1} T^i$.

Theorem 4. Let $T \in B(H)$ be a power bounded operator with $1 \in \sigma(T)$. Let $(a_n)_{n=0}^{\infty}$ be a sequence of positive numbers satisfying $a_n \to 0$ and $\varepsilon > 0$. Then there exists $x \in H$ with $\|x\| < \sup_n a_n + \varepsilon$ such that

$$\operatorname{Re}\langle A_n x, x\rangle > a_n$$

for all $n \geq 0$.

Proof. If 1 is in the point spectrum of T then it is sufficient to take a corresponding eigenvector of norm 1. If 1 is not in the point spectrum then $\|A_n y\| \to 0$ for each $y \in H$ by the ergodic theorem, see [Kr, p. 73]. The proof of Theorem 1 then works word by word if we replace T^n by A_n. □

We apply now the previous results to unitary operators. This gives statements about Fourier coefficients of L^1 functions.

Let μ be a non-negative finite Borel measure on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Recall that μ is called Rajchman if its Fourier transform $\hat{\mu}(n) = \int_{\mathbb{T}} e^{-2\pi int} d\mu(t)$ vanishes at infinity, i.e., $\lim_{|n|\to\infty} \hat{\mu}(n) = 0$. In particular, each absolutely continuous measure is Rajchman (the converse is not true).

Let U_μ be the operator on $L^1(\mu)$ defined by $(U_\mu f)(z) = zf(z) \quad (f \in L^1(\mu), z \in \mathbb{T})$. It is easy to see that μ is Rajchman if and only if $U_\mu^n \to 0$ in the weak operator topology.

Theorem 5. Let μ be a Rajchman measure. Let $(a_n)_{n=0}^{\infty}$ be a sequence of positive numbers satisfying $a_n \to 0$ and $\sup a_n < 1$. Then there exists $f \in L^1(\mu)$ of norm 1 such that $f \geq 0$ a.e. and $|\hat{f}(n)| > a_{|n|}$ for all integers n.

If $1 \in \operatorname{supp}\mu$ then it is possible to find $f \geq 0$ such that $\operatorname{Re}\hat{f}(n) > a_{|n|}$ for all non-zero n.

Proof. Let $H = L^2(\mu)$ and let $U : H \to H$ be the unitary operator defined by $(Uf)(z) = zf(z) \quad (f \in H, z \in \mathbb{T})$. Then $\lim_{n\to\infty}\langle U^n f, g\rangle = 0$ for all $f, g \in H$. By Corollary 3, there is a $g \in H$ such that $\|g\|_H = 1$ and $|\langle U^n g, g\rangle| > a_n$ for all $n \geq 1$. Set $f = |g|^2$. Then $f \in L^1(\mu)$, $\|f\|_1 = 1$ and $|\hat{f}(n)| \geq a_{|n|}$ for all nonzero integers n.

The second statement is similar. □

The previous theorem is not true for non-Rajchman measures. An example concerning the real parts is relatively simple. Recall that a set $E \subset \mathbb{T}$ is called independent if given $x_1, \ldots, x_r \in E$ and integers $m_1, \ldots, m_r$, $\prod_{j=1}^{r} x_j^{m_j} = 1$ implies $m_1 = \cdots = m_r = 0$.

Example 6. Let $(z_n) \subset \mathbb{T}$ be an independent sequence such that $z_n \to 1$. Let H be the Hilbert space with an orthonormal basis $\{e_1, e_2, \ldots\}$. Let $U \in B(H)$ be defined by $Ue_i = z_i e_i$. Clearly U is a unitary operator and $1 \in \sigma(T)$. We show that

$$\limsup_{n\to\infty} \operatorname{Re}\langle U^n x, x\rangle = \|x\|^2$$

and

$$\liminf_{n\to\infty} \operatorname{Re}\langle U^n x, x\rangle = -\|x\|^2$$

for each $x \in H$. Let $x \in H$ and $\varepsilon > 0$. Write $x = \sum_{j=1}^\infty \alpha_j e_j$ for some complex coefficients α_j. Then there exists an n_0 such that $\sum_{j=n_0+1}^\infty |\alpha_j|^2 < \varepsilon$. By the Kronecker theorem there are positive integers k_1, k_2 such that

$$|z_j^{k_1} - 1| < \varepsilon \qquad (j = 1, 2, \dots, n_0)$$

and

$$|z_j^{k_2} + 1| < \varepsilon \qquad (j = 1, 2, \dots, n_0).$$

Then

$$\operatorname{Re}\langle U^{k_1} x, x\rangle = \operatorname{Re}\sum_{j=1}^{\infty} z_j^{k_1}|\alpha_j|^2 \geq \operatorname{Re}\sum_{j=1}^{n_0} z_j^{k_1}|\alpha_j|^2 - \sum_{j=n_0+1}^{\infty}|\alpha_j|^2$$
$$\geq \sum_{j=1}^{n_0}(1-\varepsilon)|\alpha_j|^2 - \varepsilon \geq (1-\varepsilon)(\|x\|^2 - \varepsilon) - \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have $\limsup_{n\to\infty} \operatorname{Re}\langle U^n x, x\rangle = \|x\|^2$. Similarly,

$$\operatorname{Re}\langle U^{k_2} x, x\rangle \leq (-1+\varepsilon)(\|x\|^2 - \varepsilon) + \varepsilon,$$

and so $\liminf_{n\to\infty} \operatorname{Re}\langle U^n x, x\rangle = -\|x\|^2$.

Recall that a non-empty closed subset $E \subset \mathbb{T}$ is called Kronecker if for all continuous functions $f : \mathbb{T} \to \mathbb{T}$ and $\varepsilon > 0$ there is an $n \in \mathbb{Z}$ such that $\sup_{z\in E}|f(z) - z^n| < \varepsilon$. Note that it is possible to find $n > 0$ with this property.

By the Kronecker theorem, every finite independent set is Kronecker. Moreover, there are perfect Kronecker sets, i.e., Kronecker sets without isolated points, see [K, p. 184].

The next example will show that if $\operatorname{supp}\mu$ is a perfect Kronecker set then there is a sequence (a_n) of positive numbers with $a_n \to 0$ such that there is no function $f \in L^1(\mu)$ with $|\hat{f}(n)| \geq a_n \quad (n \geq 0)$. This gives also a negative answer to the van Neerven problem.

First we need a simple auxiliary lemma:

Lemma 7. Let $n \geq 2$ and let $a_1, \dots, a_n \in \mathbb{C}$ satisfy $\max_i |a_i| \leq \frac{1}{2}\sum_{i=1}^n |a_i|$. Then there are $\lambda_1, \dots, \lambda_n \in \mathbb{T}$ such that $\sum_{i=1}^n \lambda_i a_i = 0$.

Proof. We may assume that $a_i > 0$ for all i. The statement is clear for $n = 2$.

Let $n = 3$. Let $a_1 \geq a_2 \geq a_3$. The statement is clear if $a_1 = a_2 + a_3$. If $a_1 < a_2 + a_3$, then there is a triangle with sides a_1, a_2, a_3. Let $\alpha_1, \alpha_2, \alpha_3$ be the angle opposite the side a_1, a_2, a_3, respectively. It is easy to verify that $\lambda_1 = -1$, $\lambda_2 = \cos\alpha_3 + i\sin\alpha_3$ and $\lambda_2 = \cos(-\alpha_2) + i\sin(-\alpha_3)$ satisfy the required condition.

For $n \geq 4$ we prove the statement by induction. Let $n \geq 4$, $a_1, \dots, a_n > 0$ and $\max_i a_i \leq 1/2\sum_{i=1}^n a_i$. Without loss of generality we may assume that $a_1 \geq a_2 \geq \dots \geq$

a_n. Then $a_{n-1}+a_n \le \frac{1}{2}\sum_{i=1}^n |a_i|$ and by the induction assumption there are numbers $\lambda_1,\dots,\lambda_{n-1}\in\mathbb{T}$ such that

$$\lambda_1 a_1 + \cdots + \lambda_{n-2}a_{n-2} + \lambda_{n-1}(a_{n_1}+a_n) = 0.$$

So the statement is true for n. □

Example 8. Let $E\subset\mathbb{T}$ be a perfect Kronecker set. Then E is topologically homeomorphic to the Cantor discontinuum and there are finite families $\mathcal{P}_m \quad (m=1,2,\dots)$ of closed disjoint intervals such that $E=\bigcap_{m=1}^\infty P_m$, where $P_m=\bigcup\{I : I\in\mathcal{P}_m\}$, $P_{m+1}\subset P_m$ for all m and $\max\{\operatorname{diam}(I) : I\in\mathcal{P}_m\}\to 0$ as $m\to\infty$. Let μ be a positive measure such that $\operatorname{supp}\mu = E$ and $\mu(E)=1$. For each m, let F_m be the set of all functions

$$f:\mathcal{P}_m\to\{e^{2\pi i j/2^m} : j=0,1,\dots,2^m-1\}.$$

Clearly F_m is a finite set. Since E is Kronecker, for each $f\in F_m$ there is a positive integer n_f such that

$$\sup_{z\in E}\Big|z^{n_f} - \sum_{I\in\mathcal{P}_m} f(I)\cdot\chi_I\Big| < 2^{-m-1},$$

where χ_I denotes the characteristic function of I. Let $n_m=\max\{n_f : f\in F_m\}$. Define a sequence (a_j) by $a_j=2^{-m/2}$ for $n_m<j\le n_{m+1}$ (where we set formally $n_0=0$). We show that there is no function $g\in L^1(\mu)$ such that $|\hat{g}(j)|\ge a_j$ for all $j>0$. Let $g\in L^1(\mu)$, $g\ne 0$. Since the step functions are dense in $L^1(\mu)$, we can find m_0 such that

$$\sum_{I\in\mathcal{P}_{m_0}}\Big|\int_I g(z)d\mu\Big| > 0.9\|g\|_1.$$

Find $m_1\ge m_0$ such that

$$\sup_{I\in\mathcal{P}_{m_1}}\Big|\int_I g(z)d\mu\Big| \le 0.4\|g\|_1$$

and $\|g\|_1 < 2^{m_1/2}$. By Lemma 7, there are complex numbers $\lambda_I\in\mathbb{T} \quad (I\in\mathcal{P}_{m_1})$ such that

$$\sum_{I\in\mathcal{P}_{m_1}}\lambda_I\cdot\int_I g(z)d\mu = 0.$$

Let $f:\mathcal{P}_{m_1}\to\{e^{2\pi i j/2^{m_1}} : j=0,1,\dots,2^{m_1}-1\}$ be a function satisfying

$$|f(I)-\lambda_I| < 2^{-m_1-1} \qquad (I\in\mathcal{P}_{m_1}).$$

Then

$$\Big|\int z^{n_f} g d\mu\Big| \le \Big|\int (z^{n_f} - \sum_{I\in\mathcal{P}_{m_1}} f(I)\chi_I) g d\mu\Big| + \Big|\sum_{I\in\mathcal{P}_{m_1}}\int f(I)\chi_I g d\mu\Big|$$

$$\le 2^{-m_1-1}\cdot\|g\|_1 + \sum_{I\in\mathcal{P}_{m_1}} |\lambda_I - f(I)|\cdot\Big|\int_I g d\mu\Big| + \Big|\sum_{I\in\mathcal{P}_{m_1}}\lambda_I\int_I g d\mu\Big|$$

$$\le 2^{-m_1}\|g\|_1 < 2^{-m_1/2} \le a_{n_f}.$$

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