Littlewood-Richardson Sequences Associated with C_0-Operators

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Abstract. We generalize the concept of the Littlewood-Richardson sequence associated with an invariant subspace of a nilpotent operator on a finite dimensional vector space to the context of C_0-contractions. The similarity invariants of nilpotent operators (decreasing sequences of sizes of the Jordan blocks) are replaced by the quasisimilarity invariants of C_0-contractions (sequences of inner functions).

0. INTRODUCTION

Let T be a linear operator on a finite dimensional Hilbert space $\mathcal{H}$ and let $\mathcal{M}$ be an invariant subspace of T. A natural but surprisingly difficult problem is to describe relationships between the similarity invariants for T, $T|\mathcal{M}$ (T restricted to $\mathcal{M}$), and the quotient map $\tilde{T} : \mathcal{H}/\mathcal{M} \to \mathcal{H}/\mathcal{M}$ (or, equivalently, $T_{\mathcal{H}\ominus\mathcal{M}}$, the compression of T to $\mathcal{H} \ominus \mathcal{M}$). This problem (and the more general one about p-modules) has been treated by the use of Littlewood-Richardson sequences (to be described below) first by Azenhas and de Sa [1] and Thijsse [12] (the case of groups was done earlier by Green [7] and Klein [8]). More recently, the finite matrix case and extensions of the problem to a certain class of operators on an infinite dimensional Hilbert space were studied in [6] and [9].

The present paper, which may be considered to be a continuation of [9], is concerned with relations between the quasisimilarity invariants for T, $T|\mathcal{M}$, and $T_{\mathcal{H}\ominus\mathcal{M}}$, where T is a C_0-operator on an infinite dimensional separable Hilbert space $\mathcal{H}$.

The paper is organized as follows. We recall some basic facts about operators of class C_0 in Section 1. In Section 2 we describe how to associate a Littlewood-Richardson sequence to a pair $(T, \mathcal{M})$ where T is an operator of class C_0 and $\mathcal{M}$ is an

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invariant subspace of T. Conversely, in Section 3, we show that, given a Littlewood-Richardson sequence, one can construct an operator T of class C_0 and a sequence of nested invariant subspaces $\{\mathcal{M}_k\}_{k=0}^{\infty}$ of T such that the Jordan models of the operators $\{T|\mathcal{M}_k\}$ correspond to the given Littlewood-Richardson sequence.

1. PRELIMINARIES

By an operator we always mean a bounded linear operator on a separable complex Hilbert space. Let $\mathcal{H}$ be a separable complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ be the set of all operators on $\mathcal{H}$. For $T \in \mathcal{L}(\mathcal{H})$, we denote by $\text{Lat}(T)$ the lattice of all (closed) invariant subspaces of T. For $x \in \mathcal{H}$, denote by $\mathcal{K}_T(x) = \vee\{T^n x : n \geq 0\}$ the invariant subspace of T generated by x, and similarly, $\mathcal{K}_T(x_1, \ldots, x_n)$ denotes the invariant subspace of T generated by the vectors $x_1, \ldots, x_n \in \mathcal{H}$. Let μ_T be the multiplicity of T, which is defined as the smallest cardinality of a subset $F \subset \mathcal{H}$ with the property that $\mathcal{H} = \vee\{T^n F : n \geq 0\}$. An operator of multiplicity one is also called multiplicity-free. For $T \in \mathcal{L}(\mathcal{H})$ and $\mathcal{M} \in \text{Lat}(T)$, we denote by $T|\mathcal{M}$ the restriction of T to $\mathcal{M}$. If $\mathcal{L}$ is any subspace of $\mathcal{H}$, the orthogonal projection of $\mathcal{H}$ onto $\mathcal{L}$ is denoted by $P_{\mathcal{L}}$. For $\mathcal{M}, \mathcal{N} \in \text{Lat}(T)$, and $\mathcal{N} \subset \mathcal{M}$, the compression of T to the semi-invariant subspace $\mathcal{M} \ominus \mathcal{N}$ is $T_{\mathcal{M} \ominus \mathcal{N}} = P_{\mathcal{M} \ominus \mathcal{N}} T | \mathcal{M} \ominus \mathcal{N}$.

If θ and ψ are inner functions, then we write $\theta|\psi$ if $\psi = u\theta$ for some inner function u, and $\theta \equiv \psi$ if and only if $\theta|\psi$ and $\psi|\theta$. Moreover, $\theta \wedge \psi$ is the greatest common inner divisor and $\theta \vee \psi$ is the least common inner multiple of θ and ψ, respectively.

We recall some facts from the theory of operators of class C_0. All results stated below without proof are proved either in [2] or in [11].

Denote by H^∞ the Banach algebra of all bounded analytic functions on the open unit disk $\mathbf{D}$. A completely nonunitary contraction T is of class C_0 if there exists a nonzero $u \in H^\infty$ such that $u(T) = 0$. For a C_0-contraction T there exists an inner function m_T (so called minimal function of T) such that $u(T) = 0$ implies $m_T | u$.

Next we will define the building blocks of C_0 operators. Let H^2 be the set of all analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in \mathbf{D}$ such that $\|f\|_2^2 = \sum_0^\infty |a_n|^2 < \infty$. The shift operator $S \in \mathcal{L}(H^2)$ is defined by $(Sf)(z) = zf(z) \quad (f \in H^2, z \in \mathbf{D})$. If ϕ is an inner function, then ϕH^2 is invariant for S, and so $H(\phi) := H^2 \ominus \phi H^2$ is invariant for

S^*. The Jordan block $S(\phi) \in \mathcal{L}(H(\phi))$ is defined by $S(\phi)^* = S^*|H(\phi)$, equivalently, $S(\phi) = P_{H(\phi)}S|H(\phi)$. The operator $S(\phi)$ is of class C_0 with minimal function ϕ. Some of the basic properties of Jordan blocks are listed below.

Proposition 1.1. *([2], p. 38) Let $\phi \in H^\infty$ be an inner function.*

(i) If θ is an inner divisor of ϕ then

$$\theta H^2 \ominus \phi H^2 = \operatorname{ran}\theta(S(\phi)) = \ker(\phi/\theta)(S(\phi)).$$

(ii) For any inner function $u \in H^\infty$, the operator $S(\phi)|\overline{\operatorname{ran} u(S(\phi))}$ is unitarily equivalent to $S(\phi/(u \wedge \phi))$.

Recall that a model function is a sequence of inner functions $\Phi = \{\phi_j : j \geq 1\}$ such that $\phi_{j+1}|\phi_j$ for all $j \geq 1$. For a model function Φ, set $H(\Phi) := \bigoplus_{j=1}^\infty H(\phi_j)$ and the Jordan model operator associated with the model function Φ is defined as $S(\Phi) := \bigoplus_{j=1}^\infty S(\phi_j)$ on $H(\Phi)$. We say that operators $T \in \mathcal{L}(\mathcal{H})$ and $T' \in \mathcal{L}(\mathcal{H}')$ are quasisimilar (shortly $T \sim T'$) if there exist quasiaffinities $X : \mathcal{H} \to \mathcal{H}'$ and $Y : \mathcal{H}' \to \mathcal{H}$ such that $XT = T'X$ and $YT' = TY$. All operators in the class C_0 can be classified up to quasisimilarity by Jordan model operators.

Theorem 1.2. *([4]) Every operator T of class C_0 is quasisimilar to a unique Jordan model operator.*

The unique Jordan model operator given above is called the Jordan model of T. In addition, if $T \sim S(\Phi)$, we will also call Φ to be the model function associated with T.

We need the following result about the relationship between multiplicity and the Jordan model of T.

Proposition 1.3. *(see [2], p.55) Let $T \in \mathcal{L}(\mathcal{H})$ be a C_0 operator with Jordan model $\bigoplus_{j=1}^\infty S(\phi_j)$. Then $\mu_T \leq n$ if and only if $\phi_{n+1} \equiv 1$. Furthermore, for each $j \geq 1$,*

$$\phi_j = \wedge\{u : \mu_{T|\overline{u(T)\mathcal{H}}} < j\}.$$

Next result is about the continuity of Jordan models relative to an increasing sequence of invariant subspaces (cf. [2, p. 195]).

Theorem 1.4. *Let $T \in \mathcal{L}(\mathcal{H})$ be an operator of class C_0 with model function $\Theta = \{\theta_j : j \geq 1\}$ and let $\{\mathcal{M}_k : k \geq 0\}$ be a sequence of invariant subspaces of T such that $\mathcal{M}_k \subset \mathcal{M}_{k+1}$ for all $k \geq 0$ and $\vee_{k=0}^{\infty}\mathcal{M}_k = \mathcal{H}$. Suppose that the model function associated with each $T|\mathcal{M}_k$ is $\Phi^{(k)} = \{\phi_j^{(k)} : j \geq 1\}$. Then $\theta_j = \vee\{\phi_j^{(k)} : k \geq 0\}$ for all j.*

We also need the following identities involving the model functions of T, the restriction of T to certain invariant subspace, and the compression of T to the orthogonal complement of the invariant subspace. Part (iii) is from [5].

Proposition 1.5. *Let T be an operator of class C_0 and $\mathcal{M} \in \mathrm{Lat}(T)$. Suppose that the Jordan models associated with $T|\mathcal{M}$, $T_{\mathcal{H}\ominus\mathcal{M}}$ and T are $\bigoplus_{j=1}^{\infty} S(\phi_j)$, $\bigoplus_{j=1}^{\infty} S(\psi_j)$, and $\bigoplus_{j=1}^{\infty} S(\theta_j)$, respectively. Then, for all $j, k \geq 1$,*

(i) $\phi_j|\theta_j$, $\psi_j|\theta_j$,

(ii) $(\theta_1\theta_2\cdots\theta_j)|(\phi_1\phi_2\cdots\phi_j\cdot\psi_1\psi_2\cdots\psi_j)$,

(iii) $(\phi_1\phi_2\cdots\phi_j\cdot\psi_1\psi_2\cdots\psi_k)|(\theta_1\theta_2\cdots\theta_{j+k})$,

(iv) $\left(\prod_{n=1}^{\infty}\phi_n\right)\cdot\left(\prod_{n=1}^{\infty}\psi_n\right) = \prod_{n=1}^{\infty}\theta_n$ *if* $\prod_{n=1}^{\infty}\theta_n$ *converges.*

Recall from [2] that an operator T of class C_0 with Jordan model $\bigoplus_{j=1}^{\infty} S(\theta_j)$ has property (P) if $\wedge\{\theta_j : j \geq 1\} \equiv 1$. In this case we also say that the model function $\Theta = \{\theta_j : j \geq 1\}$ has property (P). The following corollary is a direct consequence of Proposition 1.5.

Corollary 1.6. *Let $T \in \mathcal{L}(\mathcal{H})$ be an operator of class C_0 with model function $\Theta = \{\theta_j : j \geq 1\}$ and property (P). Let $\mathcal{M} \in \mathrm{Lat}(T)$ and let $\Phi = \{\phi_j : j \geq 1\}$ be the model function of $T|\mathcal{M}$. Assume that $T_{\mathcal{H}\ominus\mathcal{M}}$ is multiplicity-free. Then the Jordan model of $T_{\mathcal{H}\ominus\mathcal{M}}$ is $S\left(\prod_{j=1}^{\infty}\frac{\theta_j}{\phi_j}\right)$.*

Proof. Since $T_{\mathcal{H}\ominus\mathcal{M}}$ is multiplicity-free, we set the Jordan model of $T_{\mathcal{H}\ominus\mathcal{M}}$ to be $S(\alpha)$. From Proposition 1.5 (ii) and (iii), we immediately have, for each $j \geq 1$, $(\theta_1\cdots\theta_j)|(\phi_1\cdots\phi_j\cdot\alpha)$ and $(\alpha\cdot\phi_1\cdots\phi_j)|(\theta_1\cdots\theta_j\theta_{j+1})$. Thus $\left(\frac{\theta_1}{\phi_1}\cdots\frac{\theta_j}{\phi_j}\right)|\alpha$ and $\alpha|\theta_{j+1}\cdot\left(\frac{\theta_1}{\phi_1}\cdots\frac{\theta_j}{\phi_j}\right)$. Since $\wedge\{\theta_j : j \geq 1\} \equiv 1$, we have $\alpha \equiv \prod_{j=1}^{\infty}\frac{\theta_j}{\phi_j}$. Q.E.D.

The above corollary is false if T does not have property (P). Indeed, if $\wedge\{\theta_j : j \geq 1\} = \theta_0$ and $\theta_0 \not\equiv 1$, then $T \oplus S(\theta_0) \sim T$.

Finally, we need to recall some facts about maximal vectors.

Let $T \in \mathcal{L}(\mathcal{H})$ be of class C_0 and $\mathcal{M} \in \mathrm{Lat}(T)$. Recall that a vector x is said to be maximal for T if $u(T)x = 0$ implies $u(T) = 0$. For each nonzero vector $x \in \mathcal{H}$ write $h_T(x, \mathcal{M}) = \wedge\{u \in H^\infty : u \text{ is inner and } u(T)x \in \mathcal{M}\}$. A vector x is called $(T, \mathcal{M})$ - maximal if $h_T(y, \mathcal{M}) | h_T(x, \mathcal{M})$ for all $y \in \mathcal{H}$. Equivalently, $P_{\mathcal{H}\ominus\mathcal{M}}x$ is maximal for $T_{\mathcal{H}\ominus\mathcal{M}}$.

Our next result is a consequence of the "splitting principle"(cf. [2]); it can be also viewed as a special case of Proposition 1.17 of [3].

Proposition 1.7. *Let $T \in \mathcal{L}(\mathcal{H})$ be an operator of class C_0 and let $\bigoplus_{j=1}^\infty S(\theta_j)$ be the Jordan model of T. Suppose that $\{x_j : j \geq 1\}$ is a sequence of vectors in $\mathcal{H}$ satisfying the following two conditions:*

(i) x_1 is maximal for T,

(ii) for each $j \geq 2$, x_j is $(T, \mathcal{M}_{j-1})$-maximal where $\mathcal{M}_{j-1} = \mathcal{K}_T(x_1, \ldots, x_{j-1})$.

Then $\theta_1 = m_T$ and $\theta_j = h_T(x_j, \mathcal{M}_{j-1})$ for each $j \geq 2$.

For a given $\mathcal{M} \in \mathrm{Lat}(T)$, the set of all $(T, \mathcal{M})$-maximal vectors is a dense G_δ set in $\mathcal{H}$. This fact together with the Baire category theorem gives the first part of the next lemma; the second part is from [6].

Lemma 1.8. *Let $T \in \mathcal{L}(\mathcal{H})$ be an operator of class C_0 and $\{\mathcal{M}_\alpha\}_{\alpha\in\mathcal{A}}$ a collection of invariant subspaces of T. Suppose that either of the following two conditions is satisfied:*

(i) the set $\{\mathcal{M}_\alpha : \alpha \in \mathcal{A}\}$ is countable,

(ii) the set $\{\mathcal{M}_\alpha : \alpha \in \mathcal{A}\}$ is totally ordered by inclusion.

Then the set $\{x \in \mathcal{H} : x \text{ is } (T, \mathcal{M}_\alpha)\text{−maximal for all } \alpha \in \mathcal{A}\}$ is a dense G_δ set.

2. LITTLEWOOD-RICHARDSON SEQUENCES OF C_0 OPERATORS

Classical Littlewood-Richardson sequences are certain sequences of partitions where by a partition we mean a finite decreasing sequence of nonnegative integers. We refer the interested readers to I. Macdonal's book [10]. Here we will generalize Littlewood-Richardson sequences to sequences of model functions. If all the inner functions in the model functions are of the form $z \mapsto z^n$, our definition coincides with the classical one.

As in [9], we define Littlewood-Richardson sequences in terms of Littlewood-Richardson pairs and triples. This definition is equivalent to that in [6].

Definition 2.1. *Let $\Phi = \{\phi_j : j \geq 1\}$, $\Psi = \{\psi_j : j \geq 1\}$, and $\Theta = \{\theta_j : j \geq 1\}$ be model functions.*

(i) *(Φ, Ψ) is a Littlewood-Richardson pair if $\psi_{j+1}|\phi_j$ and $\phi_j|\psi_j$ for all $j \geq 1$.*

(ii) *(Φ, Ψ, Θ) is a Littlewood-Richardson triple if both (Φ,Ψ) and (Ψ,Θ) are Littlewood-Richardson pairs and*

$$\frac{\theta_1 \cdots \theta_j}{\psi_1 \cdots \psi_j} \Big| \frac{\psi_1 \cdots \psi_{j-1}}{\phi_1 \cdots \phi_{j-1}}, \qquad \text{for all} \qquad j \geq 1. \tag{2.1}$$

(In particular for $j = 1$ this means $\theta_1 = \psi_1$.)

(iii) *A sequence of model functions $(\Phi^{(k)})_{k=0}^{\infty}$ is a Littlewood-Richardson sequence if $(\Phi^{(k-1)}, \Phi^{(k)}, \Phi^{(k+1)})$ is a Littlewood-Richardson triple for each $k \geq 1$.*

Remark 2.2.

(i) *If (Φ, Ψ) is a Littlewood-Richardson pair, then $\prod_{j\geq 1}(\frac{\psi_j}{\phi_j})$ is an inner function and $\left(\prod_{j=1}^{\infty} \frac{\psi_j}{\phi_j}\right)|\psi_1$.*

(ii) *If $(\Phi^{(k)})_{k=0}^{\infty}$ is a Littlewood-Richardson sequence and $\Phi^{(k)} = \{\phi_j^{(k)} : j \geq 1\}$, then (ii) in Definition 2.1 implies that $\phi_k^{(i)} = \phi_k^{(k)}$ for all $i \geq k$.*

(iii) *If $(\Phi^{(k)})_{k=0}^{\infty}$ is a Littlewood-Richardson sequence and $\Phi^{(0)}$ has property (P) then $\Phi^{(k)}$ has property (P) for all k.*

Let $T \in \mathcal{L}(\mathcal{H})$ be an operator of class C_0 and let $\mathcal{M}$ be an invariant subspace of T. Our goal in this section is to associate a Littlewood-Richardson sequence with T and $\mathcal{M}$ in the following way: we construct a chain of invariant subspaces $\mathcal{M} = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \ldots$ such that $\vee_{k\geq 1}\mathcal{M}_k = \mathcal{H}$ and the model functions $\Phi^{(k)}$ of $T|\mathcal{M}_k$ form a Littlewood-Richardson sequence. Note that another, entirely different approach how to associate a Littlewood-Richardson sequence to a pair $(T, \mathcal{M})$ was given in [6]. Our approach here is analogous to that in [9].

Proposition 2.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be an operator of class C_0 with model function $\Psi = \{\psi_j : j \geq 1\}$. Let $\mathcal{M} \in \mathrm{Lat}(T)$, and let $\Phi = \{\phi_j : j \geq 1\}$ be the model function of $T|\mathcal{M}$. Suppose that $T_{\mathcal{H}\ominus\mathcal{M}}$ is multiplicity-free. Then (Φ, Ψ) is a Littlewood-Richardson pair.*

Moreover, if T has property (P) then the Jordan model of $T_{\mathcal{H}\ominus\mathcal{M}}$ is $S\left(\prod_{j=1}^{\infty}\frac{\psi_j}{\phi_j}\right)$.

Proof. Since $T_{\mathcal{H}\ominus\mathcal{M}}$ is multiplicity-free, there exists $x \in \mathcal{H}$ such that $\mathcal{H} = \mathcal{M} \vee \mathcal{K}_T(x)$. For every inner function u we have $\overline{u(T)\mathcal{H}} = \overline{u(T)\mathcal{M}} \vee \mathcal{K}_T(u(T)x)$ so that

$$\mu\big(T|\overline{u(T)\mathcal{M}}\big) \leq \mu\big(T|\overline{u(T)\mathcal{H}}\big) \leq \mu\big(T|\overline{u(T)\mathcal{M}}\big) + 1.$$

By Proposition 1.3, we have $\psi_j = \wedge\{u : \mu\big(T|\overline{u(T)\mathcal{H}}\big) < j\}$ and

$$\phi_j = \wedge\{u : \mu\big(T|\overline{u(T)\mathcal{M}}\big) < j\}.$$

Therefore $\phi_j|\psi_j$ and $\psi_{j+1}|\phi_j$ for all j.

If T has property (P) then the Jordan model of $T_{\mathcal{H}\ominus\mathcal{M}}$ is $S\left(\prod_{j=1}^{\infty}(\frac{\psi_j}{\phi_j})\right)$ by Corollary 1.6. Q.E.D.

Our next goal is to show that if $\mu_{T_{\mathcal{H}\ominus\mathcal{M}}} = 2$, then one can find $\mathcal{L} \in \mathrm{Lat}(T)$ such that $\mathcal{M} \subset \mathcal{L}$ and the model functions of $T|\mathcal{M}$, $T|\mathcal{L}$ and T form a Littlewood-Richardson triple.

For the rest of the section, fix an operator $T \in \mathcal{L}(\mathcal{H})$ of class C_0 with minimal function m_T. Write m_T as

$$m_T(z) = \gamma \prod_{\lambda\in\mathbf{D}} (b_\lambda(z))^{n(\lambda)} \exp\left(\int_{\mathbf{T}} \frac{z+\zeta}{z-\zeta} d\nu(\zeta)\right),$$

where $|\gamma| = 1$, $b_\lambda(z) = \frac{\bar{\lambda}}{\lambda}(\frac{\lambda - z}{1-\bar{\lambda}z})$ if $\lambda \neq 0$ and $b_0(z) = z$, $n : \mathbf{D} \to \{0, 1, 2 \ldots\}$ is the Blaschke function for θ: that is, n satisfies $\sum_{\lambda\in\mathbf{D}} n(\lambda)(1 - |\lambda|) < \infty$, and finally ν is a positive singular measure on $\mathbf{T} = \{z : |z| = 1\}$.

Let u be an inner divisor of m_T. Then

$$u(z) = \gamma_u \prod_{\lambda\in\mathbf{D}} (b_\lambda(z))^{n_u(\lambda)} \exp\left(\int_{\mathbf{T}} \frac{z+\zeta}{z-\zeta} d\nu_u(\zeta)\right),$$

where $|\gamma_u| = 1$, $0 \leq n_u(\lambda) \leq n(\lambda)$ $(\lambda \in \mathbf{D})$ and ν_u is a positive measure satisfying $0 \leq \nu_u \leq \nu$.

Thus we can associate with each inner divisor u of m_T the function $f_u : \overline{\mathbf{D}} \to [0, \infty)$ defined by

$$f_u|\mathbf{D} = n_u,$$

$$f_u|\mathbf{T} = \frac{d\nu}{d\nu_u} \qquad \text{(the Radon – Nikodym derivative)}.$$

The function f_u is integer-valued on $\mathbf{D}$, $\sum_{\lambda\in\mathbf{D}} f_u(\lambda)(1-|\lambda|)<\infty$ and $f_u|\mathbf{T}\in L^1(\nu)$, $0\le f_u|\mathbf{T}\le 1$; it is defined for all $\lambda\in\mathbf{D}$ and $a.e.(\nu)$ on $\mathbf{T}$.

If u and v are inner divisors of m_T then

$$u|v \iff f_u(z)\le f_v(z),$$

$$f_{uv}(z)=f_u(z)+f_v(z),$$

$$f_{u\wedge v}(z)=\min\{f_u(z),f_v(z)\}$$

$a.e.(\nu)$; by $a.e.(\nu)$ we mean that the relation is true for each $z\in\mathbf{D}$ and almost every $z\in\mathbf{T}$.

Let $b(z)=\prod_{\lambda\in\mathbf{D}}(b_\lambda(z))^{\min\{n(\lambda),1\}}$ and denote by $e(z)=\exp\left(\int_{\mathbf{T}}\frac{z+\zeta}{z-\zeta}d\nu(\zeta)\right)$ the singular part of m_T. Thus $(a.e.(\nu))$,

$$f_b(z)=\begin{cases}\min\{1,f_{m_T}\} & (z\in\mathbf{D}),\\ 0 & (z\in\mathbf{T}),\end{cases}$$

and

$$f_e(z)=\begin{cases}0 & (z\in\mathbf{D}),\\ 1 & (z\in\mathbf{T}).\end{cases}$$

Theorem 2.4. *Let $T\in\mathcal{L}(\mathcal{H})$ be an operator of class C_0 and let $\mathcal{M}\in\mathrm{Lat}(T)$ satisfy $\mu(T_{\mathcal{H}\ominus\mathcal{M}})=2$. Then there exists $\mathcal{L}\in\mathrm{Lat}(T)$, $\mathcal{M}\subset\mathcal{L}$, such that $T_{\mathcal{H}\ominus\mathcal{L}}$ and $T_{\mathcal{L}\ominus\mathcal{M}}$ are multiplicity-free and the model functions of $T|\mathcal{M}$, $T|\mathcal{L}$ and T form a Littlewood-Richardson triple.*

Proof. It follows from Lemma 1.4 that we can find a vector $x\in\mathcal{H}$ such that x is $(T,\overline{b^m(T)\mathcal{M}})$-maximal for all integers $m\ge 0$ and $(T,\overline{e^t(T)\mathcal{M}})$-maximal for all $t\in[0,1]$. Fix x with these properties. Set $\mathcal{L}=\mathcal{M}\vee\mathcal{K}_T(x)$. Since $\mu(T_{\mathcal{H}\ominus\mathcal{M}})=2$ and x is also $(T,\mathcal{M})$-maximal, we have immediately that both $T_{\mathcal{H}\ominus\mathcal{L}}$ and $T_{\mathcal{L}\ominus\mathcal{M}}$ are multiplicity-free.

Let $\Phi=\{\phi_j : j\ge 1\}$, $\Psi=\{\psi_j : j\ge 1\}$, and $\Theta=\{\theta_j : j\ge 1\}$ be the model functions associated with $T|\mathcal{M}$, $T|\mathcal{L}$, and T respectively. From Theorem 2.1, we have that (Φ,Ψ) and (Ψ,Θ) are Littlewood-Richardson pairs. To finish the proof, it suffices to show that, for each $j\ge 1$,

$$\frac{\theta_1\cdots\theta_j}{\psi_1\cdots\psi_j}\Big|\frac{\psi_1\cdots\psi_{j-1}}{\phi_1\cdots\phi_{j-1}},$$

i.e.,

$$\sum_{i=1}^{j}(f_{\theta_i}(\lambda)-f_{\psi_i}(\lambda))\le\sum_{i=1}^{j-1}(f_{\psi_i}(\lambda)-f_{\phi_i}(\lambda))\qquad(a.e.(\nu)).\tag{2.4}$$

We prove (2.4) in several steps. Fix $j \geq 1$.

Step I. Let g be either b^m or e^t for some integer $m \geq 0$ or $t \in [0,1]$. Let u and v be the minimal functions of $T_{\overline{g(T)\mathcal{L}} \ominus \overline{g(T)\mathcal{M}}}$ and $T_{\overline{g(T)\mathcal{H}} \ominus \overline{g(T)\mathcal{L}}}$, respectively. Then $u(T)g(T)x \in \overline{g(T)\mathcal{M}}$ and the maximality of x implies that $u(T)g(T)\mathcal{H} \subset \overline{g(T)\mathcal{M}} \subset \overline{g(T)\mathcal{L}}$, so that $v|u$.

Step II. Let g, u and v be as in Step I. It is easy to see (using Proposition 1.1) that the Jordan models of $T|\overline{g(T)\mathcal{M}}$, $T|\overline{g(T)\mathcal{L}}$, and $T|\overline{g(T)\mathcal{H}}$ are $\bigoplus_{i=1}^{\infty} S(\frac{\phi_i}{g \wedge \phi_i})$, $\bigoplus_{i=1}^{\infty} S(\frac{\psi_i}{g \wedge \psi_i})$ and $\bigoplus_{i=1}^{\infty} S(\frac{\theta_i}{g \wedge \theta_i})$, respectively. From Proposition 1.5, we have

$$\prod_{i=1}^{j} \frac{\theta_i}{g \wedge \theta_i} | v \cdot \prod_{i=1}^{j} \frac{\psi_i}{g \wedge \psi_i}$$

and

$$u \cdot \prod_{i=1}^{j-1} \frac{\phi_i}{g \wedge \phi_i} | \prod_{i=1}^{j} \frac{\psi_i}{g \wedge \psi_i}.$$

This, together with $v|u$, gives

$$\prod_{i=1}^{j} \Big(\frac{\theta_i}{\psi_i} \cdot \frac{g \wedge \psi_i}{g \wedge \theta_i} \Big) | \prod_{i=1}^{j-1} \Big(\frac{\psi_i}{\phi_i} \cdot \frac{g \wedge \phi_i}{g \wedge \psi_i} \Big) \cdot \frac{\psi_j}{g \wedge \psi_j}.$$

Thus, $a.e.(\nu)$,

$$\begin{aligned} &\sum_{i=1}^{j} \big(f_{\theta_i}(z) - f_{\psi_i}(z) + \min\{f_g(z), f_{\psi_i}(z)\} - \min\{f_g(z), f_{\theta_i}(z)\} \big) \\ \leq &\sum_{i=1}^{j-1} \big(f_{\psi_i}(z) - f_{\phi_i}(z) + \min\{f_g(z), f_{\phi_i}(z)\} - \min\{f_g(z), f_{\psi_i}(z)\} \big) \\ + &f_{\psi_j}(z) - \min\{f_g(z), f_{\psi_j}(z)\}. \end{aligned} \tag{2.5}$$

Step III. Let $z \in \mathbf{D}$, and let $g = b^{f_{\psi_j}(z)}$. Then $f_g(z) = f_{\psi_j}(z)$. Therefore, for $1 \leq i \leq j$ we have $f_{\psi_i}(z) \geq f_g(z)$ and $f_{\theta_i}(z) \geq f_g(z)$. Since (Φ, Ψ) is a Littlewood-Richardson pair, for each $1 \leq i \leq j-1$ we have $f_{\phi_i(z)} \geq f_g(z)$. Now (2.5) reduces to

$$\sum_{i=1}^{j} \big(f_{\theta_i}(z) - f_{\psi_i}(z) \big) \leq \sum_{i=1}^{j-1} \big(f_{\psi_i}(z) - f_{\phi_i}(z) \big)$$

so that we have (2.4) for $z \in \mathbf{D}$.

Step IV. Since $f_{e^s}(z) = s$ for $z \in \mathbf{T}$ $(a.e.(\nu))$, if we set $g = e^s$, then (2.5) reduces to

$$\begin{aligned}&\sum_{i=1}^{j}\big(f_{\theta_i}(z) - f_{\psi_i}(z) + \min\{s, f_{\psi_i}(z)\} - \min\{s, f_{\theta_i}(z)\}\big)\\ &\le \sum_{i=1}^{j-1}\big(f_{\psi_i}(z) - f_{\phi_i}(z) + \min\{s, f_{\phi_i}(z)\} - \min\{s, f_{\psi_i}(z)\}\big)\\ &+ f_{\psi_j}(z) - \min\{s, f_{\psi_j}(z)\}.\end{aligned} \tag{2.6}$$

a.e.(ν). Denote by A the set of all points $z \in \mathbf{T}$ for which (2.6) is true for all rational $s \in [0,1]$. Then $\nu(A) = \nu(\mathbf{T})$.

Fix $z \in A$. From the continuity in s we infer that (2.6) is true for all $s \in [0,1]$. In particular, for $s = f_{\psi_j(z)}$ we have

$$\sum_{i=1}^{j}\big(f_{\theta_i}(z) - f_{\psi_i}(z)\big) \le \sum_{i=1}^{j-1}\big(f_{\psi_i}(z) - f_{\phi_i}(z)\big)$$

for all $z \in A$, so that (2.4) is true. Q.E.D.

Theorem 2.5. *Let $\mathcal{M} \in \mathrm{Lat}(T)$. There exists a sequence of invariant subspaces $\mathcal{M} = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{H}$, such that $\vee_{k=0}^{\infty}\mathcal{M}_k = \mathcal{H}$ and the model functions $\Phi^{(k)} = \{\phi_j^{(k)} : j \ge 1\}$ of $T|\mathcal{M}_k$ form a Littlewood-Richardson sequence.*

Moreover, if T has property (P) then the Jordan model of $T_{\mathcal{H}\ominus\mathcal{M}}$ is

$$\bigoplus_{k=1}^{\infty} S\Big(\prod_{j=1}^{\infty}\frac{\phi_j^{(k)}}{\phi_j^{(k-1)}}\Big).$$

Proof. Let $\mathcal{G} = \{b^m : m = 0, 1 \ldots\} \cup \{e^t : t \in [0,1]\}$.

We construct the required sequence of invariant subspaces $\{\mathcal{M}_i\}$ inductively. As in Theorem 2.4, let x_1 be $(T, \overline{g(T)\mathcal{M}})$-maximal for all $g \in \mathcal{G}$ and $\mathcal{M}_1 = \mathcal{M} \vee \mathcal{K}_T(x_1)$. For $j \ge 2$, take x_j to be $(T, \overline{g(T)\mathcal{M}_{j-1}})$-maximal for all $g \in \mathcal{G}$ and define $\mathcal{M}_j = \mathcal{M}_{j-1} \vee \mathcal{K}_T(x_j)$. It follows immediately from Theorem 2.4 that $(\Phi^{(k)})$ is a Littlewood-Richardson sequence. Moreover, since each x_j can be chosen from a dense subset of $\mathcal{H}$, it is easy to achieve that $\vee_{k=0}^{\infty}\mathcal{M}_k = \mathcal{H}$.

The second statement follows from Proposition 1.7. Q.E.D.

3. CONSTRUCTION

The aim of this section is to construct an operator T of class C_0 and a chain of invariant subspaces $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \dots$ of T such that the model functions of $T|\mathcal{M}_k$ form a given Littlewood-Richardson sequence $(\Phi^{(k)})$.

To keep the description of the construction clear, we will construct operators similar to the Jordan model operators. It is clear that the Sz.-Nagy-Foias functional calculus can be extended to the operators that are similar to completely nonunitary operators. That is, if $T = XT'X^{-1}$ and T' is a completely nonunitary contraction, the map $u \mapsto u(T) = Xu(T')X^{-1}$ is a continuous algebra homomorphism. If T is similar to an operator T' in the class C_0, then we define the Jordan model of T to be the Jordan model of T'. Similarly, we extend the notions of multiplicity and maximal vectors.

We set up some notations that we will need throughout the section. Let $\hat{S} : \bigoplus_1^\infty H^2 \to \bigoplus_1^\infty H^2$ be the unilateral shift of infinite multiplicity. Recall that for a model function $\Phi = \{\phi_j : j \geq 1\}$ we write $H(\Phi) = \bigoplus_{j=1}^\infty (H^2 \ominus \phi_j H^2)$. The standard basis $\{e_j\}$ of $S(\Phi) := P_{\mathcal{H}(\Phi)}\hat{S}|\mathcal{H}(\Phi)$ is defined to be

$$e_j = P_{\mathcal{H}(\Phi)}\Big((\bigoplus_{i=1}^{j-1} 0) \oplus 1 \oplus (\bigoplus_{i=j+1}^{\infty} 0)\Big).$$

Let T be similar to $S(\Phi)$, say $T = XS(\Phi)X^{-1}$. The set of vectors $\{x_j = Xe_j : j = 1, 2, \dots\}$ is called a standard basis of T (induced by X). Clearly the vectors x_j determine the similarity X uniquely. Set $C_{\{x_j\}} = \|X\|\|X^{-1}\|$.

Our first step is to build an operator T and $\mathcal{M} \in \mathrm{Lat}(T)$ such that the model functions of $T|\mathcal{M}$ and T coincide with a given Littlewood-Richardson pair.

Proposition 3.1. *Let (Φ, Ψ) be a Littlewood-Richardson pair, $\phi = \{\Phi_j\}$, $\Psi = \{\Psi_j\}$, let Φ have property (P) and $\epsilon > 0$. Suppose that $T \in \mathcal{L}(\mathcal{M})$ is similar to $S(\Phi)$, with the standard basis $\{x_j : j \geq 1\}$. Then there exists an extension V of T (that is, $\mathcal{M} \in \mathrm{Lat}(V)$ and $V|\mathcal{M} = T$) such that V is similar to $S(\Psi)$, with a standard basis $\{y_j\}$, and:*

(i) $V_{\mathcal{H}\ominus\mathcal{M}}$ *is multiplicity-free with minimal function* $\prod_{j=1}^\infty \frac{\psi_j}{\phi_j}$,

(ii) $\vee\{\frac{\psi_j}{\phi_j}(V)y_j, x_j\} = \vee\{y_{j+1}, x_j\}$ *for all* $j = 1, 2, \dots$,

(iii) $C_{\{y_j\}} < (1 + \epsilon)C_{\{x_j\}}$.

Proof. Let $m = \psi_1$. We first consider the case when $\mathcal{M} = \bigoplus_{j=1}^{\infty}(\frac{m}{\phi_j}H^2 \ominus mH^2)$ and $T = P_{\mathcal{M}}\hat{S}|\mathcal{M}$. Clearly T is unitarily equivalent to $S(\Phi)$ and the vectors $x_j = P_{\mathcal{M}}(\frac{m}{\phi_j}e_j)$ $(j \geq 1)$ form a standard basis for T. Set $\mathcal{K} = \bigoplus_{n=1}^{\infty}(H^2 \ominus mH^2)$, $\hat{S}_{\mathcal{K}} = P_{\mathcal{K}}\hat{S}|\mathcal{K}$, and let a be a positive constant large enough so that $a > 2$ and $\frac{2}{a-2} < \epsilon$. Define

$$y_j = P_{\mathcal{K}}\Big(\bigoplus_{i=1}^{j-1} 0 \oplus \frac{m}{\psi_j} \oplus \bigoplus_{i=j+1}^{\infty} \frac{1}{a^{i-j}} \cdot \frac{m}{\psi_j} \cdot \frac{\phi_j \cdots \phi_{i-1}}{\psi_{j+1} \cdots \psi_i}\Big).$$

Let $\mathcal{H} = \vee\{\hat{S}_{\mathcal{K}}^n y_j : n \geq 0, j \geq 1\}$ and $V = \hat{S}_{\mathcal{K}}|\mathcal{H}$. It is obvious from the definition of y_j that

$$\frac{\psi_j}{\phi_j}(V)y_j - x_j = \frac{1}{a}y_{j+1}. \tag{3.1}$$

Thus (ii) is satisfied.

Also, (3.1) implies $\mathcal{M} \subset \mathcal{H}$, $\mathcal{M} \in \mathrm{Lat}(V)$, and $T = V|\mathcal{M}$. It is easy to show by induction on j that $y_j \in \mathcal{M} \vee \mathcal{K}_V(y_1)$, so that $\mathcal{H} = \mathcal{M} \vee \mathcal{K}_V(y_1)$ and $V_{\mathcal{H}\ominus\mathcal{M}}$ is multiplicity-free.

Consider the lower triangular operator matrix

$$B : \bigoplus_{j=1}^{\infty} H^2 \to \bigoplus_{j=1}^{\infty} H^2$$

defined by

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ \frac{1}{a}\frac{\phi_1}{\psi_2} & 1 & 0 & 0 & \dots \\ \frac{1}{a^2}\frac{\phi_1\phi_2}{\psi_2\psi_3} & \frac{1}{a}\frac{\phi_2}{\psi_3} & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Clearly B is a bounded operator, $\|B\| \leq \sum_{k=0}^{\infty} a^{-k} = \frac{1}{1-a^{-1}}$ and $\|B-I\| \leq \sum_{k=1}^{\infty} a^{-k} = \frac{1}{a-1} < 1$ so that B is invertible and $\|B^{-1}\| = \|\sum_{k=0}^{\infty}(I-B)^k\| \leq \sum_{k=0}^{\infty} \frac{1}{(a-1)^k} = \frac{a-1}{a-2}$. Thus $\|B\| \cdot \|B^{-1}\| < 1 + \epsilon$.

Let $\hat{B} : \mathcal{K} \to \mathcal{K}$ be the operator defined by $\hat{B} = P_{\mathcal{K}}B|\mathcal{K}$. Then $\hat{B}$ is an invertible operator and $\|\hat{B}\| \cdot \|\hat{B}^{-1}\| < 1 + \epsilon$.

Let $\mathcal{K}_0 = \bigoplus_{j=1}^{\infty}\big(\frac{m}{\psi_j}H^2 \ominus mH^2\big)$. Then $\hat{B}\mathcal{K}_0 = \mathcal{H}$ and $\hat{B}$ is a similarity between $P_{\mathcal{K}_0}\hat{S}|\mathcal{K}_0$ (which is unitarily equivalent to $S(\Psi)$) and V. From Corollary 1.6 we have immediately that the minimal function of $V_{\mathcal{H}\ominus\mathcal{M}}$ is $\prod_{j=1}^{\infty} \frac{\psi_j}{\phi_j}$. Further, $\hat{B}$ carries the standard basis to $\{y_j\}$, so that $C_{\{y_j\}} < 1 + \epsilon$. This finishes the proof for the case when T is unitarily equivalent to S_{Φ}.

The general case of T being only similar to $S(\Phi)$ follows immediately from the following lemma.

Lemma 3.2. *Let $\mathcal{H}, \mathcal{M}'$ be Hilbert spaces, let $V \in \mathcal{L}(\mathcal{H})$, $\mathcal{M} \in \mathrm{Lat}(V)$, $T = V|\mathcal{M}$, $T' \in \mathcal{L}(\mathcal{M}')$, and let $X : \mathcal{M} \to \mathcal{M}'$ be an invertible operator satisfying $XT = T'X$. Then there exist a Hilbert space $\mathcal{H}' \supset \mathcal{M}'$, $V' \in \mathcal{L}(\mathcal{H}')$ and an invertible operator $Y : \mathcal{H} \to \mathcal{H}'$ such that $\mathcal{M}' \in \mathrm{Lat}(V')$, $V'|\mathcal{M}' = T'$, $YV = V'Y$, and $\|Y\|\|Y^{-1}\| = \|X\|\|X^{-1}\|$.*

Proof. Let $\mathcal{N} = \mathcal{H} \ominus \mathcal{M}$ and $\mathcal{H}' = \mathcal{M} \oplus \mathcal{N}$. Define $Y : \mathcal{H} \to \mathcal{H}'$ by $Y = X \oplus I_{\mathcal{N}}$ and $V' \in \mathcal{L}(\mathcal{H}')$ by $V' = YVY^{-1}$. Then V' and Y satisfy all conditions required.

This finishes the proof of Lemma 3.2 and also of Theorem 3.1. Q.E.D.

Corollary 3.3. *Let Φ, Ψ be model functions. The following conditions are equivalent:*

(i) *(Φ, Ψ) is a Littlewood-Richardson pair.*

(ii) *There exist an operator T of class C_0 and $\mathcal{M} \in \mathrm{Lat}(T)$, such that $T_{\mathcal{H}\ominus\mathcal{M}}$ is multiplicity free and the model functions of $T|\mathcal{M}$ and T are Φ and Ψ, respectively.*

Now we are ready to construct an operator T similar to an operator in the class C_0 and associated with a given Littlewood-Richardson sequence in the sense of Theorem 2.5.

Theorem 3.4. *Let $\{\Phi^{(k)}\}_{k=0}^{\infty}$ be a Littlewood-Richardson sequence with $\wedge_{j=1}^{\infty}\phi_j^{(0)} = 1$. Then there exist $T \in \mathcal{L}(\mathcal{H})$, and a sequence of increasing invariant subspaces, $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \ldots \subset \mathcal{H}$ such that $\mathcal{H} = \vee_{k=0}^{\infty}\mathcal{M}_k$ and*

(i) *$T|\mathcal{M}_k$ is similar to $S(\Phi^{(k)})$,*

(ii) *$T_{\mathcal{M}_k\ominus\mathcal{M}_{k-1}}$ is multiplicity-free for all k,*

(iii) *T is similar to an operator of class C_0 with Jordan model $\bigoplus_{k=1}^{\infty} S(\phi_k^{(k)})$,*

(iv) *the Jordan model of $T_{\mathcal{H}\ominus\mathcal{M}}$ is $\bigoplus_{k=1}^{\infty} S\Big(\prod_{j=1}^{\infty} \frac{\phi_j^{(k)}}{\phi_j^{(k-1)}}\Big)$.*

Proof. Choose positive numbers $\epsilon_1, \epsilon_2, \ldots$ such that $\prod_{k=1}^{\infty}(1+\epsilon_k) < \infty$. Let $T \in \mathcal{L}(\mathcal{M}_0)$ be an operator unitarily equivalent to $S(\Phi^{(0)})$. Apply Proposition 3.1 inductively, so that we obtain an increasing sequence of subspaces $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \ldots$ and an extension of T defined in each $\mathcal{M}_k$, which we will still denote by T, such that:

(1) $T|\mathcal{M}_k$ is similar to $S(\Phi^{(k)})$,

(2) $T_{\mathcal{M}_k \ominus \mathcal{M}_{k-1}}$ is multiplicity-free,

(3) $\|T|\mathcal{M}_k\| \le (1+\epsilon_k)\|T|\mathcal{M}_{k-1}\|$ for $k = 1, 2, \ldots$,

(4) for each $k \ge 0$, $T|\mathcal{M}_k$ has a standard basis $\{x_j^{(i)} : i \le k, j \ge 1\}$, with the property that

$$\vee\Big\{\frac{\phi_j^{(k)}}{\phi_j^{(k-1)}}(T)x_j^{(k)}, x_j^{(k-1)}\Big\} = \vee\{x_{j+1}^{(k)}, x_j^{(k-1)}\}. \tag{3.2}$$

Let $\mathcal{H} = \vee_{k=0}^{\infty}\mathcal{M}_k$. Extend T to $\mathcal{H}$, and we still denote the extension by T. Thus $\|T\| \le \prod_{k=1}^{\infty}(1+\epsilon_k) < \infty$. It follows from Theorem 1.4 that the Jordan model function of T is $\{\vee_{k=0}^{\infty}\phi_1^{(k)}, \vee_{k=0}^{\infty}\phi_2^{(k)}, \ldots\} = \{\phi_1^{(1)}, \phi_2^{(2)}, \ldots\}$. Thus (i)-(iii) are satisfied.

It remains to prove (iv). To simplify the notation, we set $\beta_j^{(k)} = \frac{\phi_j^{(k)}}{\phi_j^{(k-1)}}$. Thus condition (2.1) in Definition 2.1 becomes

$$\beta_1^{(k)} \ldots \beta_j^{(k)} | \beta_1^{(k-1)} \ldots \beta_{j-1}^{(k-1)} \tag{3.3}$$

and (3.2) gives

$$\beta_j^{(k)}(T)x_j^{(k)} \in \vee\{x_{j+1}^{(k)}, x_j^{(k-1)}\} \qquad \text{and} \qquad x_{j+1}^{(k)} \in \vee\{\beta_j^{(k)}(T)x_j^{(k)}, x_j^{(k-1)}\}. \tag{3.4}$$

We divide the proof of (iv) into several steps.

Claim 1. *For all $k \ge 0$ and $j \ge 1$,*

$$x_j^{(k)} \in \mathcal{M}_0 + \big(\beta_1^{(k)} \cdots \beta_{j-1}^{(k)}\big)(T)\mathcal{M}_k \tag{3.5}$$

(we use the convention that $\beta_j^{(0)} \equiv 1$).

Obviously (3.5) holds for $k = 0$ or $j = 1$. We will prove Claim 1 by double induction, that is, if (3.5) holds for all (k', j') with $k' \le k$, $j' \le j$ and $(k', j') \ne (k, j)$, then we prove (3.5) for (k, j). Suppose that

$$x_j^{(k-1)} \in \mathcal{M}_0 + \big(\beta_1^{(k-1)} \cdots \beta_{j-1}^{(k-1)}\big)(T)\mathcal{M}_{k-1}$$

and

$$x_{j-1}^{(k)} \in \mathcal{M}_0 + \big(\beta_1^{(k)} \cdots \beta_{j-2}^{(k)}\big)(T)\mathcal{M}_k.$$

Using (3.4), we have

$$\begin{aligned} x_j^{(k)} &\in \vee\{x_{j-1}^{(k-1)}, \beta_{j-1}^{(k)}(T)x_{j-1}^{(k)}\} \\ &\subset \mathcal{M}_0 + \big(\beta_1^{(k-1)} \cdots \beta_{j-2}^{(k-1)}\big)(T)\mathcal{M}_{k-1} + \big(\beta_1^{(k)} \cdots \beta_{j-1}^{(k)}\big)(T)\mathcal{M}_k \\ &\subset \mathcal{M}_0 + \big(\beta_1^{(k)} \cdots \beta_{j-1}^{(k)}\big)(T)\mathcal{M}_k, \end{aligned}$$

since (3.3) and $\mathcal{M}_{k-1} \subset \mathcal{M}_k$.

This finishes the proof of Claim 1.

Claim 2. *For each* $j \geq 0$,

$$\big(\beta_1^{(k)} \cdots \beta_j^{(k)} \phi_{j+1}^{(k)}\big)(T)x_1^{(k)} \in \mathcal{M}_0 + \big(\beta_1^{(k)} \cdots \beta_j^{(k)} \phi_{j+1}^{(k)}\big)(T)\mathcal{M}_{k-1}.$$

Apply (3.4) repeatedly to obtain

$$\begin{aligned} &(\beta_1^{(k)} \ldots \beta_j^{(k)})(T)x_1^{(k)} \\ &\in \vee \{(\beta_2^{(k)} \ldots \beta_j^{(k)})(T)x_1^{(k-1)}, (\beta_2^{(k)} \ldots \beta_j^{(k)})(T)x_2^{(k)}\} \\ &\subset \vee \{(\beta_2^{(k)} \ldots \beta_j^{(k)})(T)x_1^{(k-1)}, (\beta_3^{(k)} \ldots \beta_j^{(k)})(T)x_2^{(k-1)}, (\beta_3^{(k)} \ldots \beta_j^{(k)})(T)x_3^{(k)}\} \subset \cdots \\ &\subset \vee \{(\beta_2^{(k)} \ldots \beta_j^{(k)})(T)x_1^{(k-1)}, (\beta_3^{(k)} \ldots \beta_j^{(k)})(T)x_2^{(k-1)}, \ldots \\ &\qquad (\beta_{i+1}^{(k)} \ldots \beta_j^{(k)})(T)x_i^{(k-1)}, \ldots, \beta_j^{(k)}(T)x_{j-1}^{(k-1)}, x_j^{(k-1)}, x_{j+1}^{(k)}\}. \end{aligned}$$

Using Claim 1 and (3.3), we have, for each $i = 1, \ldots, j-1$,

$$\begin{aligned} (\beta_{i+1}^{(k)} \ldots \beta_j^{(k)})(T)x_i^{(k-1)} &\in \mathcal{M}_0 + (\beta_1^{(k-1)} \ldots \beta_{i-1}^{(k-1)}\beta_{i+1}^{(k)} \ldots \beta_j^{(k)})(T)\mathcal{M}_{k-1} \\ &\subset \mathcal{M}_0 + (\beta_1^{(k)} \ldots \beta_{i-1}^{(k)}\beta_i^{(k)}\beta_{i+1}^{(k)} \ldots \beta_j^{(k)})(T)\mathcal{M}_{k-1}, \end{aligned}$$

and thus,

$$(\beta_1^{(k)} \ldots \beta_j^{(k)})(T)x_1^{(k)} \in \mathcal{M}_0 + \beta_1^{(k)} \ldots \beta_j^{(k)}(T)\mathcal{M}_{k-1} + \vee\{x_{j+1}^{(k)}\}.$$

Since $\phi_{j+1}^{(k)}(T)x_{j+1}^{(k)} = 0$, we have

$$(\beta_1^{(k)} \ldots \beta_j^{(k)} \cdot \phi_{j+1}^{(k)})(T)x_1^{(k)} \in \mathcal{M}_0 + (\beta_1^{(k)} \ldots \beta_j^{(k)} \cdot \phi_{j+1}^{(k)})(T)\mathcal{M}_{k-1},$$

which finishes the proof of Claim 2.

Set $\alpha^{(k)} = \prod_{j=1}^{\infty} \beta_j^{(k)}$.

Claim 3. $\alpha^{(k)}(T)\mathcal{M}_k \subset \mathcal{M}_0 + \alpha^{(k)}(T)\mathcal{M}_{k-1}$.

It is sufficient to show $\alpha^{(k)}(T)x_1^{(k)} \in \mathcal{M}_0 + \alpha^{(k)}(T)\mathcal{M}_{k-1}$ since $\mathcal{M}_k = \mathcal{M}_{k-1} \vee \mathcal{K}_T(x_1^{(k)})$. Clearly $\alpha^{(k)}|\beta_1^{(k)} \ldots \beta_j^{(k)} \cdot \phi_{j+1}^{(k)}$ for all $j \geq 0$. By Claim 2, $\beta_1^{(k)} \ldots \beta_j^{(k)} \cdot \phi_{j+1}^{(k)}(T)x_1^{(k)} \in \mathcal{M}_0 + \alpha^{(k)}(T)\mathcal{M}_{k-1}$. Furthermore, $\alpha^{(k)} = \wedge_{j\geq 0}(\beta_1^{(k)} \ldots \beta_j^{(k)} \cdot \phi_{j+1}^{(k)})$, hence $\alpha^{(k)}(T)x_1^{(k)} \in \mathcal{M}_0 + \alpha^{(k)}\mathcal{M}_{k-1}$.

Claim 4. *The Jordan model function of $T_{\mathcal{M}_k \ominus \mathcal{M}_0}$ is $\bigoplus_{i=1}^k S(\alpha^{(i)})$.*

Clearly the multiplicity of $T_{\mathcal{M}_k \ominus \mathcal{M}_0} \leq k$. Let $\bigoplus_{i=1}^k S(\gamma^{(i)})$ be the Jordan model of $T_{\mathcal{M}_k \ominus \mathcal{M}_0}$. Observe that

$$\prod_{i=1}^{k} \gamma^{(i)} = \prod_{i=1}^{\infty} \frac{\phi_i^{(k)}}{\phi_i^{(0)}} = \prod_{i=1}^{k} \alpha^{(i)}. \tag{3.6}$$

For $j \leq k$, $\alpha^{(k)}|\alpha^{(j)}$ and Claim 3 implies that

$$\begin{aligned} \alpha^{(j)}(T)\mathcal{M}_k &\subset \mathcal{M}_0 + \alpha^{(j)}(T)\mathcal{M}_{k-1} \\ &\subset \mathcal{M}_0 + \alpha^{(j)}(T)\mathcal{M}_{k-2} \subset \ldots \\ &\subset \mathcal{M}_0 + \alpha^{(j)}(T)\mathcal{M}_{j-1}. \end{aligned}$$

Thus, $\mu(T_{\mathcal{M}_k \ominus \mathcal{M}_0}|\overline{\operatorname{ran} \alpha^{(j)}(T_{\mathcal{M}_k \ominus \mathcal{M}_0})}) \leq j-1$. Consequently, $\gamma^{(j)}|\alpha^{(j)}$. Using (3.6), we have $\alpha^{(j)} \equiv \gamma^{(j)}$.

Finally, apply Theorem 1.4 to $T_{\mathcal{H} \ominus \mathcal{M}_0}$ with the increasing sequence of invariant subspaces $\{\mathcal{M}_k \ominus \mathcal{M}_0\}$, to establish (iv). Q.E.D.

Combining Theorem 2.2 and Theorem 3.4 we have the following characterization of all the possible Jordan models of $(T, T|\mathcal{M}, T_{\mathcal{H}\ominus\mathcal{M}})$ when T has property (P).

Corollary 3.5. *The following statements are equivalent:*

(i) There exist an operator $T \in \mathcal{L}(\mathcal{H})$ of class C_0 with property (P) and $\mathcal{M} \in \operatorname{Lat}(T)$ such that the Jordan models of $T|\mathcal{M}$, $T_{\mathcal{H}\ominus\mathcal{M}}$, and T are $S(\Phi), S(\Psi)$, and $S(\Theta)$, respectively.

(ii) There exists a Littlewood-Richardson sequence $(\Phi^{(0)}, \Phi^{(1)}, \ldots)$, $\Phi^{(k)} = \{\Phi_j^{(k)}\}_{j=1}^\infty$ such that $\Phi^{(0)} = \Phi$, $\theta_j = \phi_j^{(j)}$ and $\psi_j = \prod_{i=1}^\infty \frac{\Phi_i^{(j)}}{\Phi_i^{(j-1)}}$ for all j.

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