On orbit-reflexive operators

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Abstract. Let T be a bounded linear Banach space operator such that $\sum_{n=1}^{\infty} \frac{1}{\|T^n\|} < \infty$. Then T is orbit-reflexive. In particular, every Banach space operator with spectral radius different from 1 is orbit-reflexive. Better estimates are obtained for operators in Hilbert spaces.

We also exhibit a simple example of a non-orbit-reflexive Hilbert space operator and an example of a reflexive but non-orbit-reflexive operator (acting on ℓ_1).

1. Introduction

Let X be a Banach space. Denote by $B(X)$ the set of all bounded linear operators acting on X. All Banach spaces are considered to be complex unless it is stated otherwise.

The notion of orbit-reflexive operators on a Hilbert space was introduced and studied in [HNRR]. While the reflexivity of operators is connected to the invariant subspace problem, its natural analogue of orbit-reflexivity is in the same way connected to the problem of existence of closed invariant subsets.

We say that $T \in B(X)$ is reflexive if every $A \in B(X)$ belongs to the closure of $\{p(T) : p \text{ polynomial}\}$ in the strong operator topology, whenever $Au \in \{p(T)u : p \text{ polynomial}\}^-$ (the closure of the set $\{p(T)u : P \text{ polynomial}\}$) for each $u \in X$.

Analogously, T is orbit-reflexive if every $A \in B(X)$ belongs to the closure of the set $\{T^n : n \in \mathbb{N}\}$ in the strong operator topology, whenever $Au \in \{T^n u : n \in \mathbb{N}\}^-$ for each $u \in X$.

Many operators are known to be reflexive: e.g. subnormal operators on a Hilbert space [OT] (in particular, normal operators and isometries), or Hilbert space contractions with isometrical H^∞-calculus, see [BC].

The orbit-reflexivity of many classes of Hilbert space operators was shown in [HNRR], e.g. for normal operators, contractions, algebraic operators, weighted shifts and compact operators. Among others, each operator whose spectrum does not intersect the unit circle is orbit-reflexive.

In this paper, we improve this result and show that each Banach space operator T satisfying $\sum \|T^n\|^{-1} < \infty$ is orbit-reflexive. In particular, if the spectral radius of T is different from 1, then T is orbit-reflexive.

Better estimates are obtained for Hilbert space operators.

On the other hand, it is much more difficult to find operators that are not orbit-reflexive. In fact, up till recently the only known example of a non-orbit-reflexive operator was the Read operator [R]. The first example of a non-orbit-reflexive Hilbert

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space operator was given recently in [GR]. The operator is obtained by a modified Read-type construction and it is quite complicated. We exhibit a relatively simple example of a non-orbit-reflexive Hilbert space operator. Moreover, our operator $T \in B(H)$ satisfies $\inf_n \|T^n x\| = 0$ for each $x \in H$, but there are two points $u, v \in H$ with $\inf_n \max\{\|T^n u\|, \|T^n v\|\} > 0$. This is the first example of an operator with this property and it might be useful in other constructions. In particular, it gives a negative answer to Question 3 of [HNRR].

We also construct an operator acting on ℓ_1 which is not orbit-reflexive but in the same time is reflexive. Note that it is very easy to find an orbit-reflexive operator that is not reflexive, since all Hilbert space contractions are orbit-reflexive.

2. Orbit-reflexive operators

Our basic tool in this section will be the following solution to the plank problem.

Proposition 1. (K. Ball [1]) *Let X be a (real or complex) Banach space, $y \in X$ any vector and $f_1, f_2, \ldots \in X^*$ unit functionals. For each $n \in \mathbb{N}$, let $\alpha_n \geq 0$ be such that $\sum_{n=1}^{\infty} \alpha_n < 1$. Then there is a point $x \in X$ such that $\|x - y\| \leq 1$ and $|\langle x, f_n \rangle| \geq \alpha_n$ for every n.*

A stronger result is known for operators on a complex Hilbert space.

Proposition 2. (K. Ball [2]) *Let X be a complex Hilbert space and $f_1, f_2 \ldots \in X$ unit vectors. For each $n \in \mathbb{N}$, let $\alpha_n \geq 0$ be such that $\sum_{n=1}^{\infty} \alpha_n^2 < 1$. Then there is a point $x \in X$ such that $\|x\| = 1$ and $|\langle x, f_n \rangle)| \geq \alpha_n$ for every n.*

First we show that the conditions in Propositions 1 or 2 can be used for construction of orbits of operators tending to infinity. The next two theorems improve the results of [MV].

Theorem 3. *Let X be a (real or complex) Banach space and $S_n \in B(X)$, $n \in \mathbb{N}$. If*

$$\sum_{n=1}^{\infty} \frac{1}{\|S_n\|} < \infty,$$

then the set $\{x \in X : \|S_n x\| \to \infty\}$ is dense in X.

Proof. Let $u \in X$ and $\varepsilon > 0$. We are going to exhibit $x \in X$ such that $\|x - u\| \leq \varepsilon$ and $\|S_n x\| \to \infty$.

There exists a sequence of positive real numbers β_n $(n \in \mathbb{N})$ tending to infinity such that

$$s := \sum_{n=1}^{\infty} \frac{\beta_n}{\|S_n\|} < \infty.$$

Let

$$\alpha_n := \frac{1}{(s+1)} \frac{\beta_n}{\|S_n\|}.$$

Then $\sum_{n=1}^{\infty} \alpha_n < 1$.

For each $n \in \mathbb{N}$ there exists $y_n \in X^*$ such that $\|y_n\| \le 1$ and $\|S_n^* y_n\| \ge \frac{1}{2}\|S_n^*\| = \frac{1}{2}\|S_n\|$.

By Theorem 1, there is an $x' \in X$ with $\|x' - \frac{u}{\varepsilon}\| \le 1$ such that $\left|\left\langle x', \frac{S_n^* y_n}{\|S_n^* y_n\|}\right\rangle\right| \ge \alpha_n$ for every n. Let $x := \varepsilon x'$. Then $\|x - u\| \le \varepsilon$ and

$$\|S_n x\| \ge \varepsilon |\langle S_n x', y_n\rangle| = \varepsilon |\langle x', S_n^* y_n\rangle| \ge \varepsilon \alpha_n \|S_n^* y_n\| \ge \frac{\varepsilon \alpha_n \|S_n\|}{2} = \frac{\varepsilon \beta_n}{2(s+1)}$$

for all n. Hence $\|S_n x\| \to \infty$. □

The analogous assertion holds also for complex Hilbert spaces. However, the complex plank theorem (Theorem 2) is valid only for planks centered at the origin ("$y = 0$"), so that we don't obtain the density directly. To this end, we introduce one additional plank that places the obtained point z into the given ball.

Theorem 4. *Let X be a complex Hilbert space and $S_n \in B(X)$, $n \in \mathbb{N}$. If*

$$\sum_{n=1}^{\infty} \frac{1}{\|S_n\|^2} < \infty,$$

then the set $\{x \in X : \|S_n x\| \to \infty\}$ is dense in X.

Proof. Choose any point $u \in X$ with $\|u\| = 1$ and any number ε with $0 < \varepsilon < 1$. By linearity, it is sufficient to prove that there is an $x \in X$ such that $\|x - u\| \le \varepsilon$ and $\|S_n x\| \to \infty$.

Set $\delta := 1 - \frac{\varepsilon^2}{2}$. There is a sequence (β_n) of positive real numbers tending to infinity such that

$$s := \sum_{n=1}^{\infty} \frac{\beta_n}{\|S_n\|^2} < \infty.$$

Thus the sequence of coefficients

$$\alpha_n := \left(\frac{1-\delta^2}{s+1}\right)^{1/2} \frac{\beta_n^{1/2}}{\|S_n\|}$$

satisfies both

$$\delta^2 + \sum_{n=1}^{\infty} \alpha_n^2 < 1 \qquad \text{and} \qquad \alpha_n \|S_n\| \to \infty.$$

For each $n \in \mathbb{N}$ find $y_n \in X$ such that $\|y_n\| \le 1$ and $\|S_n^* y_n\| \ge \frac{1}{2}\|S_n^*\| = \frac{1}{2}\|S_n\|$. We apply now the complex plank theorem, using the points $u, \frac{S_1^* y_1}{\|S_1^* y_1\|}, \frac{S_2^* y_2}{\|S_2^* y_2\|}, \ldots$ as the functionals and numbers $\delta, \alpha_1, \alpha_2, \ldots$ as the coefficients. Thus, there is an $x' \in X$ with $\|x'\| = 1$ such that $|\langle x', u\rangle| \ge \delta$ and $|\langle x', S_n^* y_n\rangle| \ge \alpha_n \|S_n^* y_n\|$ for every n. Therefore

$$\|S_n x'\| \ge |\langle S_n x', y_n\rangle| = |\langle x', S_n^* y_n\rangle| \ge \alpha_n \|S_n^* y_n\| \ge \frac{\alpha_n}{2}\|S_n\| \to \infty,$$

as $n \to \infty$. Moreover, $|\langle x', u\rangle| \geq \delta$. Let $x := \frac{\langle u, x'\rangle}{|\langle x', u\rangle|} \cdot x'$. Then $\|S_n x\| \to \infty$ and

$$\langle x, u\rangle = \frac{\langle u, x'\rangle}{|\langle u, x'\rangle|} \langle x', u\rangle = |\langle x', u\rangle| \geq \delta,$$

and therefore

$$\|x - u\|^2 = \|x\|^2 + \|u\|^2 - 2\mathrm{Re}\,\langle x, u\rangle \leq 2 - 2\delta = \varepsilon^2.$$

Hence $\|x - u\| \leq \varepsilon$. □

Remark 5. (i) Note that in Theorems 3 and 4 we have proved the existence of a dense set of points $x \in X$ such that $\|S_n x\| \to \infty$ and $\inf_n \|S_n x\| \neq 0$.

(ii) Note that in general, the results proved in Theorems 3 and 4 are not true with bigger exponents, i.e., for Banach space operators satisfying $\sum \frac{1}{\|S_n\|^{1+\varepsilon}} < \infty$ or Hilbert space operators with $\sum \frac{1}{\|S_n\|^{2+\varepsilon}} < \infty$ for $\varepsilon > 0$, see [MV].

Let us turn now to the orbit-reflexivity. First, a simple observation shows that operators with spectral radius less than 1 are orbit-reflexive. In fact, we obtain more.

Theorem 6. *Let $T \in B(X)$. Then T is orbit-reflexive in any of the following cases:*
(i) *the orbit $\{T^n x : n = 0, 1, \dots\}$ is closed for each $x \in X$;*
(ii) *$\|T^n x\| \to \infty$ for all $x \in X$;*
(iii) *$\|T^n x\| \to 0$ for all $x \in X$.*

Proof. (i) Let $A \in B(X)$ satisfy $Au \in \{T^n u : n = 0, 1, \dots\}^- = \{T^n u : n = 0, 1, \dots\}$ for each $u \in X$. Then $Au = T^n u$ for some n and $\bigcup_{n=0}^{\infty} \ker(A - T^n) = X$. By the Baire category theorem, there exists m such that $\ker(A - T^m)$ has a nonempty interior. Since $\ker(A - T^m)$ is a linear subspace, we have $\ker(A - T^m) = X$, and so $A = T^m$.

(ii) follows from (i) and (iii) can be proved similarly. □

Theorem 7. *Suppose that $T \in B(X)$ satisfies $\sum_{n=1}^{\infty} \frac{1}{\|T^n\|} < \infty$. Then T is orbit-reflexive. In case X is a complex Hilbert space, then it is sufficient to assume that $\sum_{n=1}^{\infty} \frac{1}{\|T^n\|^2} < \infty$.*

Proof. Let $\sum_{n=1}^{\infty} \frac{1}{\|T^n\|} < \infty$. Let $A \in B(X)$ be such that $Au \in \{T^n u : n \in \mathbb{N}\}^-$ for each $u \in X$. There is no loss of generality in supposing that $A \neq T^n$ for all $n \in \mathbb{N}$. Observe that

$$\sum_{n=1}^{\infty} \frac{1}{\|T^n - A\|} < \infty.$$

Indeed, since $\|T^n\| \to \infty$ we have $\|T^n - A\| \geq \|T^n\| - \|A\| \geq \frac{1}{2}\|T^n\|$ for all n large enough. So for a certain $n_0 \in \mathbb{N}$ we have

$$\sum_{n=n_0}^{\infty} \frac{1}{\|T^n - A\|} \leq \sum_{n=n_0}^{\infty} \frac{1}{\|T^n\| - \|A\|} \leq \sum_{n=n_0}^{\infty} \frac{2}{\|T^n\|} < \infty.$$

Therefore, the operators $S_n := T^n - A$ satisfy the conditions in Theorem 3. So there exists (in fact a dense set of points) $x \in X$ with $\|(T^n - A)x\| > 0$ for all n and $\|(T^n - A)x\| \to \infty$, cf. Remark 5. Thus there is a constant $C > 0$ such that $\inf_n \|(T^n - A)x\| \geq C > 0$ and we have a contradiction with the assumption that $Ax \in \{T^n x : n \in \mathbb{N}\}^-$.

The second statement can be proved similarly by using Theorem 4 for the operators $T^n - A$. □

Corollary 8. *Every operator $T \in B(X)$ with $r(T) \neq 1$ is orbit-reflexive.*

Proof. If $r(T) < 1$ then $\lim_{n\to\infty} \|T^n\| = 0$. Now apply Theorem 6.

If $r(T) > 1$ then $\|T^n\| > n^2$ for all n large enough, since otherwise $r(T) = \inf_{n\to\infty} \|T^n\|^{1/n} \leq 1$. Now apply Theorem 7. □

Denote by $\{T\}'$ the commutant of an operator $T \in B(X)$, i.e., the set of all operators $S \in B(X)$ commuting with T. Denote by $\{T\}''$ the bicommutant of T, i.e., the set of all operators commuting with all operators in $\{T\}'$.

Proposition 9. *Let $T \in B(X)$. Suppose that there is a nonzero $x \in X$ such that the closure of its orbit $\{T^n x : n \in \mathbb{N}\}^-$ has cardinality less than continuum. Then either T has a nontrivial closed hyperinvariant subspace or each operator $A \in B(X)$ satisfying $Au \in \{T^n u : n \in \mathbb{N}\}^-$ for each $u \in X$ belongs to $\{T\}''$.*

Proof. Let $x \neq 0$ be a point such that the cardinality of the set $W := \{T^n x : n \in \mathbb{N}\}^-$ is less than 2^ω.

Set $M := \{Bx : B \in \{T\}'\}$. If $\overline{M}$ is a proper subspace of X, then it is a nontrivial closed hyperinvariant subspace.

Suppose that $\overline{M} = X$.

Let $A \in B(X)$ be such that $Au \in \{T^n u : n \in \mathbb{N}\}^-$ for every $u \in X$. Let $B \in \{T\}'$. We will prove that $BAx = ABx$.

Fix any $\alpha \in \varrho(B)$ (the resolvent set of B). According to our assumption on A, we have $A(\alpha I - B)x \in \{T^n(\alpha I - B)x : n \in \mathbb{N}\}^-$. But since $\alpha I - B$ commutes with T^n and is an invertible operator, we can rewrite the latter set as $(\alpha I - B)W$. In this way, we can assign to each $\alpha \in \varrho(B)$ a point $w_\alpha \in W$ for which $A(\alpha I - B)x = (\alpha I - B)w_\alpha$. Since the cardinality of W is smaller than the cardinality of $\varrho(B)$, there are two distinct complex numbers $\alpha, \beta \in \varrho(B)$ with $w_\alpha = w_\beta =: w$, i.e.,

$$A(\alpha x - Bx) = \alpha w - Bw,$$
$$A(\beta x - Bx) = \beta w - Bw,$$

which yields the identities

$$(\alpha - \beta)Ax = (\alpha - \beta)w,$$
$$(\beta - \alpha)ABx = (\beta - \alpha)Bw.$$

So $Ax = w$, $ABx = Bw$ and $BAx = Bw = ABx$.

Therefore, for each $C \in \{T\}'$ we have $ABCx = BCAx = BACx$. Since the set $\{Cx : C \in \{T\}'\}$ is dense in X, we have $AB = BA$ and so $A \in \{T\}''$. □

3. A non-orbit-reflexive Hilbert space operator

Denote by m the normalized Lebesgue measure on the unit circle $\mathbb{T}$. Denote by $\|\cdot\|_2$ and $\|\cdot\|_\infty$ the norms in the Hardy spaces $H^2(m)$ and $H^\infty(m)$, respectively.

Lemma 10. *Let p, q be polynomials, $\|p\|_2 \le 1$, $\|q\|_2 \le 1$ and let $0 < \varepsilon < 1/3$. Then there exist polynomials r, s such that $\|rp + sq\|_2 < \varepsilon$, $\|r\|_\infty \le 1$, $\|s\|_\infty \le 1$ and $\max\{\|r\|_2, \|s\|_2\} \ge 1/3$.*

Proof. Let $M_1 := \{z \in \mathbb{T} : |p(z)| \ge |q(z)|\}$, $M_2 = \mathbb{T} \setminus M_1$. Without loss of generality we can assume that $m(M_1) \ge 1/2$. Define functions $g, h : \mathbb{T} \to \mathbb{C}$ by

$$h(z) := \begin{cases} -1 & (z \in M_1) \\ 0 & (z \in M_2) \end{cases}$$

$$g(z) := \begin{cases} \frac{q(z)}{p(z)} & (z \in M_1) \\ 0 & (z \in M_2) \end{cases}$$

(if $p(z) = q(z) = 0$ then set $g(z) := 0$). Note that $\|g\|_\infty \le 1$, $\|h\|_\infty \le 1$ and $pg + qh = 0$.

Let $K = \max\{1, \|p\|_\infty, \|q\|_\infty\}$. There exist continuous functions $g_1, h_1 : \mathbb{T} \to \mathbb{C}$ such that $\|g_1 - g\|_2 < \frac{\varepsilon}{4K}$ and $\|h_1 - h\|_2 < \frac{\varepsilon}{4K}$.

Define $g_2, h_2 : \mathbb{T} \to \mathbb{C}$ by $g_2(z) := \frac{g_1(z)}{\max\{1, |g_1(z)|\}}$, $h_2(z) := \frac{h_1(z)}{\max\{1, |h_1(z)|\}}$. Clearly g_2, h_2 are continuous, $\|g_2\|_\infty \le 1$, $\|h_2\|_\infty \le 1$, $\|g_2 - g\|_2 < \frac{\varepsilon}{4K}$ and $\|h_2 - h\|_2 < \frac{\varepsilon}{4K}$.

There exist trigonometric polynomials g_3, h_3 such that $\|g_3 - g_2\|_\infty < \varepsilon/4K$, $\|h_3 - h_2\|_\infty < \varepsilon/4K$. Moreover, we may assume that $\|g_3\|_\infty \le 1$, $\|h_3\|_\infty \le 1$.

Choose $l \in \mathbb{N}$ such that $r := z^l g_3$ and $s := z^l h_3$ are polynomials. Then $\|r\|_\infty \le 1$, $\|s\|_\infty \le 1$ and

$$\begin{aligned} \|rp + qs\|_2 &= \|z^l g_3 p + z^l h_3 q\|_2 \le \|z^l gp + z^l hq\|_2 + \|z^l (g_3 - g)p\|_2 + \|z^l (h_3 - h)q\|_2 \\ &\le K\|g_3 - g\|_2 + K\|h_3 - h\|_2 \\ &\le K(\|g_3 - g_2\|_2 + \|g_2 - g\|_2) + K(\|h_3 - h_2\|_2 + \|h_2 - h\|_2) < \varepsilon. \end{aligned}$$

Finally,

$$\|s\|_2 = \|h_3\|_2 \ge \|h\|_2 - \|h_3 - h\|_2 \ge 1/2 - \varepsilon/2K \ge 1/3.$$

If $m(M_1) < 1/2$ then $m(M_2) \ge 1/2$ and we can proceed similarly. At the end we obtain $\|r\|_2 \ge 1/3$. □

Example 11. There exists a Hilbert space X and an operator $T \in B(X)$ such that

(i) $\inf_n \|T^n x\| = 0$ for all $x \in X$;

(ii) there are points $e_0, f_0 \in X$ such that $\inf_n \max\{\|T^n e_0\|, \|T^n f_0\|\} > 0$.

Consequently, T is not orbit-reflexive.

Construction. For $N = 1, 2, 3 \dots$ let $\varepsilon_N := N^{-1/3}$.

The underlying Hilbert space will be

$$X = Z \oplus \bigoplus_{k=1}^{\infty} Y_k,$$

where Z is the Hilbert space with an orthonormal basis $\{e_j, f_j : j = 0, 1, 2 \ldots\}$ and Y_k are finite-dimensional Hilbert spaces which will be determined in the construction.

We construct inductively integers k_N, $N = 0, 1, 2 \ldots$, integers a_k, spaces Y_k and elements $w_k \in Z$, $k = 1, 2, 3 \ldots$, in the following way. Set formally $k_0 := 0$ and $a_0 := 0$. Let $N \geq 1$ and suppose that the integers k_{N-1}, a_k, spaces Y_k and elements $w_k \in Z$ have already been defined for $1 \leq k \leq k_{N-1}$. Write for short $b_{N-1} := a_{k_{N-1}}$. Let $Z_N := \operatorname{Span}\{e_j, f_j : j = 0, \ldots, b_{N-1}\}$ and let $w_{k_{N-1}+1}, \ldots, w_{k_N}$ be an ε_N^2-net in the closed unit ball of Z_N.

For $k = k_{N-1} + 1, \ldots, k_N$ we can write $w_k = \sum_{i=0}^{b_{N-1}} (\alpha_i^{(k)} e_i + \beta_i^{(k)} f_i)$ with complex coefficients $\alpha_i^{(k)}$, $\beta_i^{(k)}$. We define numbers μ_i $(0 \leq i \leq b_{N-1})$ in the following way. If $1 \leq M \leq N-1$, $k_{M-1} < l < k_M$ and $a_l < i \leq 2a_l$ then set $\mu_i = \varepsilon_M^{-1}$. If $2a_l < i < 3a_l$ then $\mu_i = \varepsilon_M^{-(3a_l - i)/a_l}$. Set $\mu_i = 1$ otherwise.

Consider the polynomials p_k, q_k defined by $p_k(z) := \sum_{i=0}^{b_{N-1}} \mu_i \alpha_i^{(k)} z^i$ and $q_k(z) := \sum_{i=0}^{b_{N-1}} \mu_i \beta_i^{(k)} z^i$. We have $\|p_k\|_2 \leq \varepsilon_{N-1}^{-1}$ and $\|q_k\|_2 \leq \varepsilon_{N-1}^{-1}$.

By Lemma 10 for the polynomials $\varepsilon_{N-1} p_k$, $\varepsilon_{N-1} q_k$, there exist $m_k \in \mathbb{N}$ and polynomials $r_k(z) = \sum_{i=0}^{m_k} \gamma_i^{(k)} z^i$, $s_k(z) = \sum_{i=0}^{m_k} \delta_i^{(k)} z^i$ such that $\|r_k\|_\infty \leq 1$, $\|s_k\|_\infty \leq 1$, $\max\{\|r_k\|_2, \|s_k\|_2\} \geq 1/3$ and $\|r_k p_k + s_k q_k\|_2 < \varepsilon_N$.

Choose numbers a_k $(k_{N-1} + 1 \leq k \leq k_N)$ such that $a_{j+1} > a_j^2 + 3a_j + m_j$ $(j = k_{N-1}, \ldots, k_N - 1)$.

Let Y_k be the finite-dimensional Hilbert space with an orthonormal basis $u_{k,j}$ $(j = 0, \ldots, m_k + 2a_k - 1)$.

Using induction, we continue the construction in the above described way.

Now we define the operator $T \in B(X)$ by:

$$
\begin{aligned}
&Tu_{k,i} := u_{k,i+1} \qquad (k \in \mathbb{N}, 0 \leq i \leq m_k + 2a_k - 2),\\
&Tu_{k,m_k+2a_k-1} := 0,\\
&Te_{a_k} := \varepsilon_N e_{a_k+1} + \sum_{i=0}^{m_k} \gamma_i^{(k)} u_{k,i} \qquad (k_{N-1} < k \leq k_N),\\
&Tf_{a_k} := \varepsilon_N f_{a_k+1} + \sum_{i=0}^{m_k} \delta_i^{(k)} u_{k,i} \qquad (k_{N-1} < k \leq k_N),\\
&Te_j := \varepsilon_N^{-1/a_k} e_{j+1} \qquad (k_{N-1} < k \leq k_N, 2a_k \leq j < 3a_k),\\
&Tf_j := \varepsilon_N^{-1/a_k} f_{j+1} \qquad (k_{N-1} < k \leq k_N, 2a_k \leq j < 3a_k),\\
&Te_j := e_{j+1} \quad \text{and} \quad Tf_j = f_{j+1} \qquad \text{otherwise.}
\end{aligned}
$$

That is, T acts on the standard basis of Z as a pair of weighted shifts, up to the points of the form e_{a_k} and f_{a_k}. It is easy to see that T defines a bounded linear operator on X. It is easy to check that $\|T\| \leq 2$. Note also that for each $k \in \mathbb{N}$, we have $T^{a_k - a_{k-1}} e_{a_{k-1}} = e_{a_k}$ and $T^{a_k - a_{k-1}} f_{a_{k-1}} = f_{a_k}$.

Let $E := \operatorname{Span}\{e_i : i = 0, 1, \ldots\}$, $F := \operatorname{Span}\{f_i : i = 0, 1, \ldots\}$ and $Y := \bigoplus_{k=1}^\infty Y_k$. For a closed subspace $M \subset X$ denote by P_M the orthogonal projection onto M.

To prove (ii), let $j \in \mathbb{N}$. If $j \notin \bigcup_{k=1}^\infty \{a_k + 1, \ldots, 3a_k\}$ then $\|T^j e_0\| \geq \|P_Z T^j e_0\| =$

$\|e_j\| = 1$. So we may assume that $a_k + 1 \le j \le 3a_k$ for some k. Then

$$\begin{aligned}
&\max\{\|T^j e_0\|, \|T^j f_0\|\} \ge \max\{\|P_{Y_k} T^j e_0\|, \|P_{Y_k} T^j f_0\|\} \\
=&\max\{\|P_{Y_k} T^{j-a_k} e_{a_k}\|, \|P_{Y_k} T^{j-a_k} f_{a_k}\|\} = \max\{\|P_{Y_k} T e_{a_k}\|, \|P_{Y_k} T f_{a_k}\|\} \\
=&\max\Big\{\Big\|\sum_{i=0}^{m_k} \gamma_i^{(k)} u_{k,i}\Big\|, \Big\|\sum_{i=0}^{m_k} \delta_i^{(k)} u_{k,i}\Big\|\Big\} = \max\{\|r_k\|_2, \|s_k\|_2\} \ge 1/3.
\end{aligned}$$

So $\max\{\|T^j e_0\|, \|T^j f_0\|\} \ge 1/3$ for all j.

To prove (i), suppose that $x \in X$ is of norm 1 and $0 < \varepsilon < \frac{1}{2}$.

There exists $M \ge 1$ such that $\|(P_Z - P_{Z_M})x\| < \frac{\varepsilon}{18}$. There exists $N > M$ such that

$$\varepsilon_N^{1/2} < \frac{\varepsilon \cdot \varepsilon_M}{9},$$

$$\Big\|\sum_{k'=k_{N-1}+1}^{\infty} P_{Y_{k'}} x\Big\| < \frac{\varepsilon}{9},$$

$$\big\|P_{Z_{N+1}} x - P_{Z_N} x\big\| < \varepsilon_N^{3/2}.$$

Indeed, the first two conditions are satisfied for all N sufficiently large. Suppose on the contrary that $\|P_{Z_{N+1}} x - P_{Z_N} x\| \ge \varepsilon_N^{3/2}$ for all $N \ge N_0$. Then

$$1 = \|x\|^2 \ge \sum_{N=N_0}^{\infty} \big\|P_{Z_{N+1}} x - P_{Z_N} x\big\|^2 \ge \sum_{N=N_0}^{\infty} \varepsilon_N^3 = \infty,$$

a contradiction. Fix N with these properties.

Find k, $k_{N-1} < k \le k_N$ such that $\|P_{Z_N} x - w_k\| \le \varepsilon_N^2$. Set $j = 2a_k + 1$. We have

$$\begin{aligned}
\|T^j x\| \le &\Big\|\sum_{k'=1}^{k_{N-1}} T^j P_{Y_{k'}} x\Big\| + \Big\|\sum_{k'=k_{N-1}+1}^{\infty} T^j P_{Y_{k'}} x\Big\| + \|P_Z T^j P_{Z_M} x\| \\
&+ \|P_Z T^j (P_{Z_N} - P_{Z_M})x\| + \|P_Z T^j (P_{Z_{N+1}} - P_{Z_N})x\| + \|P_Z T^j (P_Z - P_{Z_{N+1}})x\| \\
&+ \big\|P_Y T^j (P_Z - P_{Z_{N+1}})x\big\| + \big\|P_Y T^j (P_{Z_{N+1}} - P_{Z_N})x\big\| \\
&+ \big\|P_Y T^j (P_{Z_N} x - w_k)\big\| + \big\|P_Y T^j w_k\big\|.
\end{aligned}$$

We estimate all the terms in the previous formula.

Since $k > k_{N-1}$ and $j > a_k > 2a_{k_{N-1}} + m_{k_{N-1}}$, we have $\sum_{k'=1}^{k_{N-1}} T^j P_{Y_{k'}} x = 0$.

For $k' > k_{N-1}$ we have $\|T^j|_{Y_{k'}}\| \le 1$, and so

$$\Big\|\sum_{k'=k_{N-1}+1}^{\infty} T^j P_{Y_{k'}} x\Big\| \le \Big\|\sum_{k'=k_{N-1}+1}^{\infty} P_{Y_{k'}} x\Big\| < \varepsilon/9.$$

It is easy to see that

$$\|P_Z T^j P_{Z_M}\| = \sup\{\|P_Z T^j e_i\| : i \le b_{M-1}\} \le \varepsilon_M^{-1} \varepsilon_N \varepsilon_N^{-i/a_k} < \varepsilon_M^{-1} \varepsilon_N^{1/2} < \frac{\varepsilon}{9},$$

and so $\|P_Z T^j P_{Z_M} x\| \le \frac{\varepsilon}{9}\|P_{Z_M} x\| \le \frac{\varepsilon}{9}$.

Similarly,

$$\|P_Z T^j (P_{Z_N} - P_{Z_M})\| = \max\{\|P_Z T^j e_i\| : b_{M-1} < i \le b_{N-1}\} \le 2,$$

$$\|P_Z T^j (P_{Z_{N+1}} - P_{Z_N})\| = \max\{\|P_Z T^j e_i\| : b_{N-1} < i \le b_N\} \le \varepsilon_N^{-1}$$

and

$$\|P_Z T^j (P_Z - P_{Z_{N+1}})\| = \max\{\|P_Z T^j e_i\| : b_N < i\} \le 2.$$

Thus

$$\|P_Z T^j (P_{Z_N} - P_{Z_M}) x\| \le 2\|(P_{Z_N} - P_{Z_M}) x\| < \frac{\varepsilon}{9},$$

$$\|P_Z T^j (P_{Z_{N+1}} - P_{Z_N}) x\| \le \varepsilon_N^{-1} \varepsilon_N^{3/2} = \varepsilon_N^{1/2} < \frac{\varepsilon}{9}$$

and

$$\|P_Z T^j (P_Z - P_{Z_{N+1}}) x\| \le 2\|(P_Z - P_{Z_{N+1}}) x\| < \frac{\varepsilon}{9}.$$

We show that $\|P_{Y_k} T^j P_Z\| \le 2\varepsilon_N^{-1}$. Clearly $\|P_{Y_k} T^j P_E\| = \|P_{Y_k} T^j P_{E_k}\|$ where $E_k = \operatorname{Span}\{e_0, \dots, e_{a_k}\}$. Let $y = \sum_{i=0}^{a_k} \lambda_i e_i$, $\|y\| = 1$. Note that the numbers μ_i mentioned in the construction satisfy $0 < \mu_i \le \varepsilon_N^{-1}$ $(0 \le i \le a_k)$ and $T^{a_k - i} e_i = \mu_i e_{a_k}$. We have

$$\|P_{Y_k} T^j y\| = \Big\| r_k(z) \cdot \sum_{i=0}^{m_k} \lambda_i \mu_i z^i \Big\|_2 \le \|r_k\|_\infty \cdot \Big\| \sum_{i=0}^{m_k} \lambda_i \mu_i z^i \Big\|_2$$

$$\le \Big(\sum_{i=0}^{m_k} |\lambda_i \mu_i|^2 \Big)^{1/2} \le \varepsilon_N^{-1} \Big(\sum_{i=0}^{m_k} |\lambda_i|^2 \Big)^{1/2} = \varepsilon_N^{-1}.$$

So $\|P_{Y_k} T^j P_E\| \le \varepsilon_N^{-1}$ and similarly, $\|P_{Y_k} T^j P_F\| \le \varepsilon_N^{-1}$. Hence

$$\|P_{Y_k} T^j P_Z\| \le \|P_{Y_k} T^j P_E\| + \|P_{Y_k} T^j P_F\| \le 2\varepsilon_N^{-1}.$$

It is easy to show that for $k' > k$ we have $\|P_{Y_{k'}} T^j P_Z\| \le 2$, and so $\|P_Y T^j P_Z\| = \sup_{k' \ge 1} \|P_{Y_{k'}} T^j P_Z\| \le 2\varepsilon_N^{-1}$. Furthermore,

$$\|P_Y T^j (P_Z - P_{Z_{N+1}})\| = \sup_{k' > k_N} \|P_{Y_{k'}} T^j (P_Z - P_{Z_{N+1}})\| \le 2.$$

So

$$\|P_Y T^j (P_Z - P_{Z_{N+1}}) x\| \le 2\|(P_Z - P_{Z_{N+1}}) x\| \le \frac{\varepsilon}{9},$$

$$\big\|P_Y T^j (P_{Z_{N+1}} - P_{Z_N}) x\big\| \le 2\varepsilon_N^{-1} \big\|(P_{Z_{N+1}} - P_{Z_N}) x\big\| < 2\varepsilon_N^{-1} \varepsilon_N^{3/2} = 2\varepsilon_N^{1/2} < \frac{\varepsilon}{9}$$

and

$$\big\|P_Y T^j (P_{Z_N} x - w_k)\big\| \le 2\varepsilon_N^{-1} \|P_{Z_N} x - w_n\| \le 2\varepsilon_N^{-1} \varepsilon_N^{3/2} = 2\varepsilon_N^{1/2} < \frac{\varepsilon}{9}.$$

Finally,

$$\|P_Y T^j w_k\| = \|r_k p_k + s_k q_k\|_2 \le \varepsilon_N < \frac{\varepsilon}{9}.$$

Hence $\|T^j x\| < \varepsilon$.

Consequently, T is not orbit-reflexive since the zero operator is not in the strong operator topology closure of polynomials of T but $0 \in \{T^n x : n \in \mathbb{N}\}^-$ for each $x \in X$.

4. An example of a reflexive operator that is not orbit-reflexive

In this section we construct a reflexive operator on ℓ_1 which is not orbit-reflexive. The construction is similar to the Hilbert space case which was considered in the previous section. However, since the norm in ℓ_1 is simpler to compute, we are able to prove the reflexivity of the operator (in the Hilbert space case we were not able to prove this because of technical difficulties).

Example 12. There exists a reflexive operator on ℓ_1 which is not orbit-reflexive.

Construction. For $N = 1, 2, 3\dots$ let $\varepsilon_N := 1/\sqrt{N}$. Let a_k, $k = 1, 2, 3\dots$, be an increasing sequence of positive integers such that $a_{k+1} > 6a_k^2$.

The underlying space will be the ℓ_1-direct sum

$$X = Z \oplus \bigoplus_{k=1}^{\infty} Y_k$$

where Z is the ℓ_1 space with standard basis $\{e_j, f_j : j = 0, 1, 2\dots\}$ and Y_k are the ℓ_1 spaces with standard bases $\{u_{k,i}, v_{k,i} : i = 1, 2, \dots, 5a_k^2\}$.

We construct inductively integers k_N, $N = 0, 1, 2\dots$, and elements $w_k \in Z$, $k = 1, 2, 3\dots$, in the following way. Set formally $k_0 := 0$ and $a_0 := 0$. Write for short $b_N := a_{k_N}$. Let $N \ge 1$ and suppose that the integer k_{N-1} and elements $w_1, \dots, w_{k_{N-1}}$ have already been defined. Let $Z_N := \mathrm{Span}\{e_j, f_j : j = 0, \dots, b_{N-1}\}$ and let $w_{k_{N-1}+1}, \dots, w_{k_N}$ be an ε_N^2-net in the closed unit ball of Z_N.

Using induction, we continue the construction in the above described way.

Now we define the operator $T \in B(X)$ by:

$$\begin{array}{lll}
Te_{a_k} := e_{a_k+1} + \frac{1}{a_k^2}\sum_{i=1}^{a_k^2} u_{k,i}, & Tf_{a_k} := f_{a_k+1} + \frac{1}{a_k^2}\sum_{i=1}^{a_k^2} v_{k,i}, & \\
Te_{a_k+3a_k^2} := \varepsilon_N e_{a_k+3a_k^2+1}, & Tf_{a_k+3a_k^2} := \varepsilon_N f_{a_k+3a_k^2+1} & (k_{N-1} < k \le k_N), \\
Te_j := \varepsilon_N^{-1/a_k^2} e_{j+1}, & Tf_j := \varepsilon_N^{-1/a_k^2} f_{j+1} & \\
\multicolumn{3}{c}{(k_{N-1} < k \le k_N,\ a_k + 3a_k^2 < j \le a_k + 4a_k^2),} \\
Te_j := e_{j+1}, & Tf_j := f_{j+1} & \text{otherwise.}
\end{array}$$

Thus T acts on the standard basis of Z as a pair of weighted shifts, up to the points of the form e_{a_k} and f_{a_k}.

Further, let

$$\begin{array}{lll}
Tu_{k,5a_k^2} := 0, & Tv_{k,5a_k^2} := 0, & \\
Tu_{k,i} := u_{k,i+1}, & Tv_{k,i} := v_{k,i+1} & (1 \le i < 2a_k^2 \text{ or } 2a_k^2 < i < 5a_k^2).
\end{array}$$

It remains to define T on $\mathrm{Span}\{u_{k,2a_k^2}, v_{k,2a_k^2}\}$. Since $w_k \in Z_N$ for $k_{N-1} < k \le k_N$, we have $w_k = \sum_{i=0}^{b_{N-1}} (\alpha_i^{(k)} e_i + \beta_i^{(k)} f_i)$ for some complex coefficients $\alpha_i^{(k)}, \beta_i^{(k)}$. For $i = 0, \dots, b_{N-1}$ we have $T^{a_k - i} e_i = \mu_i e_{a_k}$ and $T^{a_k - i} f_i = \mu_i f_{a_k}$ for some $\mu_i \in \mathbb{C}$

satisfying $|\mu_i| \le \varepsilon_N^{-1}$. Set $\alpha^{(k)} = \sum_{i=0}^{b_{N-1}} \mu_i \alpha_i^{(k)}$ and $\beta^{(k)} = \sum_{i=0}^{b_{N-1}} \mu_i \beta_i^{(k)}$. Without loss of generality we may assume that $|\alpha^{(k)}| \ne |\beta^{(k)}|$.

If $|\alpha^{(k)}| < |\beta^{(k)}|$ then set $Tu_{k,2a_k^2} := u_{k,2a_k^2+1}$ and $Tv_{k,2a_k^2} := -\frac{\alpha^{(k)}}{\beta^{(k)}} u_{k,2a_k^2+1}$. If $|\alpha^{(k)}| > |\beta^{(k)}|$ then set $Tv_{k,2a_k^2} := v_{k,2a_k^2+1}$ and $Tu_{k,2a_k^2} := -\frac{\beta^{(k)}}{\alpha^{(k)}} v_{k,2a_k^2+1}$. Note that in both cases we have $T(\alpha^{(k)} u_{k,2a_k^2} + \beta^{(k)} v_{k,2a_k^2}) = 0$.

Let $Y = \bigoplus_{k=1}^{\infty} Y_k$. Denote by P_Z, P_Y, P_{Z_N} and P_{Y_k} the natural projections onto the corresponding subspace of X.

It is easy to check that $\|T\| \le 2$. Note also that for each $k \in \mathbb{N}$, we have $T^{a_k - a_{k-1}} e_{a_{k-1}} = e_{a_k}$ and $T^{a_k - a_{k-1}} f_{a_{k-1}} = f_{a_k}$.

We prove that

$$\max\{\|T^n e_0\|, \|T^n f_0\|\} \ge 1$$

for all $n = 0, 1, 2 \ldots$, and for each $x \in X$ and $\varepsilon > 0$ there is a $j \in \mathbb{N}$ such that $\left\|T^j x\right\| < \varepsilon$. As in the previous section, this gives automatically that T is not orbit-reflexive.

To prove the first statement, let $n \in \mathbb{N}$. If $n \notin \bigcup_{k=1}^{\infty} \{a_k + 3a_k^2 + 1, \ldots, a_k + 4a_k^2\}$ then $P_Z T^n e_0 = e_n$, and so $\max\{\|T^n e_0\|, \|T^n f_0\|\} \ge \|P_Z T^n e_0\| = 1$.

Let $a_k + 3a_k^2 < n \le a_k + 4a_k^2$ for some k. Recall that $w_k = \sum_{i=0}^{b_{N-1}} (\alpha_i^{(k)} e_i + \beta_i^{(k)} f_i)$, $\alpha^{(k)} = \sum_{i=0}^{b_{N-1}} \mu_i \alpha_i^{(k)}$ and $\beta^{(k)} = \sum_{i=0}^{b_{N-1}} \mu_i \beta_i^{(k)}$, where $T^{a_k - i} e_i = \mu_i e_{a_k}$ and $T^{a_k - i} f_i = \mu_i f_{a_k}$. First suppose that $\left|\alpha^{(k)}\right| < \left|\beta^{(k)}\right|$ so that T is a shift on $u_{k,i}$. It is then easy to show that

$$P_{Y_k} T^n e_0 = \frac{1}{a_k^2} \sum_{i=n-a_k}^{n-a_k+a_k^2-1} u_{k,i},$$

and so $\|T^n e_0\| \ge 1$. If $\left|\alpha^{(k)}\right| > \left|\beta^{(k)}\right|$, then we obtain in the same way that $\|T^n f_0\| \ge 1$. Hence $\max\{\|T^n e_0\|, \|T^n f_0\|\} \ge 1$ for all n.

To prove the second statement, suppose that $x \in X$ is of norm 1 and $0 < \varepsilon < 1$.

There exists $M \ge 2$ such that $\|(P_Z - P_{Z_M})x\| < \frac{\varepsilon}{18}$. There exists $N > M$ such that

$$\begin{aligned} \varepsilon_N^{1/2} &< \frac{\varepsilon \cdot \varepsilon_M}{9}, \\ b_{N-1} \varepsilon_N &> \frac{18}{\varepsilon}, \\ \sum_{k'=k_{N-1}+1}^{\infty} \left\|P_{Y_{k'}} x\right\| &< \frac{\varepsilon}{9}, \\ \left\|P_{Z_{N+1}} x - P_{Z_N} x\right\| &< \varepsilon_N^2. \end{aligned} \tag{1}$$

Indeed, the first three conditions of (1) are satisfied for all N sufficiently large. Suppose on the contrary that $\|P_{Z_{N+1}} x - P_{Z_N} x\| \ge \varepsilon_N^2$ for all $N \ge N_0$. Then

$$1 = \|x\| \ge \sum_{N=N_0}^{\infty} \left\|P_{Z_{N+1}} x - P_{Z_N} x\right\| \ge \sum_{N=N_0}^{\infty} \varepsilon_N^2 = \infty,$$

a contradiction. Fix N with properties (1).

Find k, $k_{N-1} < k \le k_N$ such that $\|P_{Z_N}x - w_k\| \le \varepsilon_N^2$. Set $j = a_k + 3a_k^2 + 1$. We have

$$\|T^j x\| \le \Big\|\sum_{k'=1}^{k_{N-1}} T^j P_{Y_{k'}} x\Big\| + \Big\|\sum_{k'=k_{N-1}+1}^{\infty} T^j P_{Y_{k'}} x\Big\| + \|P_Z T^j P_{Z_M} x\|$$
$$+ \|P_Z T^j (P_{Z_N} - P_{Z_M})x\| + \|P_Z T^j (P_{Z_{N+1}} - P_{Z_N})x\| + \|P_Z T^j (P_Z - P_{Z_{N+1}})x\|$$
$$+ \big\|P_Y T^j (P_Z - P_{Z_{N+1}})x\big\| + \big\|P_Y T^j (P_{Z_{N+1}} - P_{Z_N})x\big\|$$
$$+ \big\|P_Y T^j (P_{Z_N}x - w_k)\big\| + \big\|P_Y T^j w_k\big\|.$$

Since $k > k_{N-1}$ and $j > a_k > 5a_{k_{N-1}}^2$, we have $\sum_{k'=1}^{k_{N-1}} T^j P_{Y_{k'}} x = 0$.

For $k' > k_{N-1}$ we have $\|T^j|_{Y_{k'}}\| \le 1$, and so

$$\Big\|\sum_{k'=k_{N-1}+1}^{\infty} T^j P_{Y_{k'}} x\Big\| \le \Big\|\sum_{k'=k_{N-1}+1}^{\infty} P_{Y_{k'}} x\Big\| < \frac{\varepsilon}{9}.$$

The folowing four terms can be estimated by $\varepsilon/9$ similarly as in the Hilbert space case. We omit the details.

We have

$$\|P_Y T^j (P_Z - P_{Z_{N+1}})\| = \max\{\|P_Y T^j e_i\|, \|P_Y T^j f_i\| : i > b_N\}$$
$$\le \max\{\|P_Z T^{j'} e_i\|, \|P_Z T^{j'} f_i\| : j' \le j, i > b_N\} \le \varepsilon_{N+1}^{-j/a_{k_{N+1}}^2} \le 2$$

and similarly

$$\|P_Y T^j P_{Z_{N+1}}\| \le \max\{\|P_Z T^{j'} e_i\|, \|P_Z T^{j'} f_i\| : j' \le j, i \le b_N\} \le \varepsilon_N^{-1}.$$

Thus

$$\|P_Y T^j (P_Z - P_{Z_{N+1}})x\| \le 2\|(P_Z - P_{Z_{N+1}})x\| \le \frac{\varepsilon}{9},$$
$$\big\|P_Y T^j (P_{Z_{N+1}} - P_{Z_N})x)\big\| \le \varepsilon_N^{-1}\,\big\|(P_{Z_{N+1}} - P_{Z_N})x\big\| < \varepsilon_N^{-1}\varepsilon_N^2 = \varepsilon_N < \frac{\varepsilon}{9}$$

and

$$\big\|P_Y T^j (P_{Z_N}x - w_k)\big\| \le \varepsilon_N^{-1}\,\|P_{Z_N}x - w_k\| \le \varepsilon_N^{-1}\varepsilon_N^2 < \frac{\varepsilon}{9}.$$

It remains to estimate $\|P_Y T^j w_k\|$. We have

$$\|P_Y T^j w_k\| = \|P_{Y_k} T^j w_k\|$$
$$= \Big\|T^{3a_k^2} \sum_{i=0}^{b_{N-1}} \Big(\frac{\mu_i \alpha_i^{(k)}}{a_k^2} \sum_{i'=1}^{a_k^2} u_{k,i+i'} + \frac{\mu_i \beta_i^{(k)}}{a_k^2} \sum_{i'=1}^{a_k^2} v_{k,i+i'}\Big)\Big\|$$
$$= \frac{1}{a_k^2}\Big\|T^{3a_k^2}\Big(\mu_0\alpha_0^{(k)} u_{k,1} + \mu_0\beta_0^{(k)} v_{k,1} + (\mu_0\alpha_0^{(k)} + \mu_1\alpha_1^{(k)})u_{k,2}$$
$$+ (\mu_0\beta_0^{(k)} + \mu_1\beta_1^{(k)})v_{k,2} + \cdots + \sum_{s=b_{N-1}+1}^{a_k^2} (\alpha^{(k)} u_{k,s} + \beta^{(k)} v_{k,s}) + \cdots$$
$$\cdots + \mu_{b_{N-1}}\alpha_{b_{N-1}}^{(k)} u_{k,a_k^2+b_{N-1}} + \mu_{b_{N-1}}\beta_{b_{N-1}}^{(k)} v_{k,a_k^2+b_{N-1}}\Big)\Big\|$$
$$\le \frac{1}{a_k^2}\cdot 2\varepsilon_N^{-1}(b_{N-1}+1)\|w_k\| \le \frac{2}{\varepsilon_N a_k} \le \frac{2}{\varepsilon_N b_{N-1}} < \frac{\varepsilon}{9}.$$

Hence $\|T^j x\| < \varepsilon$. This implies that T is not orbit-reflexive.

We show now that T is reflexive. Suppose that an operator $A \in B(X)$ leaves invariant all the closed subspaces which are invariant for T. Without loss of generality we may assume that $\|A\| = 1$. We have to show that A is a limit of polynomials of T in the strong operator topology.

Let $k \in \mathbb{N}$ and let $y \in Y_k$, $y \neq 0$. Let s satisfy $T^s y \neq 0$ and $T^{s+1} y = 0$. Since $\operatorname{Span}\{y, Ty, \dots, T^s y\}$ is invariant for A, there are numbers $\lambda_0, \dots, \lambda_s \in \mathbb{C}$ such that $Ay = \sum_{i=0}^{s} \lambda_i T^i y$.

Fix any natural numbers $l > k$ such that $|\alpha^{(l)}| < |\beta^{(l)}|$ (so that T is a shift on $u_{l,i}$; such a number certainly exists) and consider the spaces invariant for T generated by the vectors $u_{l,1}$ and $y + u_{l,1}$, respectively. Since these subspaces are invariant for A, there are complex numbers ξ_i and η_i such that

$$Au_{l,1} = \sum_{i=0}^{5a_l^2-1} \xi_i T^i u_{l,1}$$

and

$$A(y + u_{l,1}) = \sum_{i=0}^{5a_l^2-1} \eta_i T^i (y + u_{l,1}).$$

Thus

$$\sum_{i=0}^{s} \eta_i T^i y + \sum_{i=0}^{s} \eta_i T^i u_{l,1} + \sum_{i=s+1}^{5a_l^2-1} \eta_i T^i u_{l,1} = \sum_{i=0}^{s} \lambda_i T^i y + \sum_{i=0}^{s} \xi_i T^i u_{l,1} + \sum_{i=s+1}^{5a_l^2-1} \xi_i T^i u_{l,1}.$$

Since the vectors $T^i y \quad (0 \le i \le s)$ and $T^i u_{l,1} \quad (0 \le i \le 5a_l^2 - 1)$ are linearly independent, we have $\lambda_i = \xi_i = \eta_i \quad (0 \le i \le s)$ and $Ay = \sum_{i=0}^{5a_k^2-1} \xi_i T^i y$. Note that this equality does not depend on $y \in Y_k$. Note also that $\sum_{i=0}^{5a_k^2-1} |\xi_i| \le \left\| \sum_{i=0}^{5a_k^2-1} \xi_i T^i u_{l,1} \right\| \le \|Au_{l,1}\| \le \|A\| = 1$. Moreover, if $Ay = \sum_{i=0}^{5a_l^2-1} \xi'_i T^j y$ for all $y \in Y_l$ then $\xi_i = \xi'_i \quad (0 \le i \le 5a_k^2 - 1)$.

Thus there are numbers $\xi_0, \xi_1, \dots$ such that $\sum_{i=0}^{\infty} |\xi_i| \le 1$ and $Ay = \sum_{i=0}^{5a_j^2-1} \xi_i T^i y$ for all $j \in \mathbb{N}$ and $y \in Y_j$.

For $k \in \mathbb{N}$ let $p_k(z) := \sum_{i=0}^{5a_k^2-1} \xi_i z^i$. Then $\|p_k(T)|_Y\| \le 1$, and so we have $Ay = \lim_{k\to\infty} p_k(T) y$ for all $y \in Y$.

Let $E := \operatorname{Span}\{e_j : j \ge 0\}$ and $F := \operatorname{Span}\{f_j : j \ge 0\}$. Let $x_1, \dots, x_n \in E$ and $x_{n+1}, \dots, x_m \in F$ be unit vectors, $q \in \mathbb{N}$ and let $0 < \varepsilon < 1$. It is sufficient to show that there is a $k \ge q$ such that $\|p_k(T)x_i - Ax_i\| < \varepsilon \quad (i = 1, \dots, m)$. This will show that A belongs to the closure of polynomials of T in the strong operator topology.

As above, it is possible to show that there is an N such that

$$\begin{aligned} \varepsilon_N &< \frac{\varepsilon}{8}, \\ \sum_{j=k_N+1}^{k_{N+1}} |\xi_j| &< \varepsilon_N^2, \\ \|(I-P_{Z_{N+1}})x_i\| &< \frac{\varepsilon}{16} \qquad (i=1,\dots,m), \\ \|(P_{Z_{N+1}}-P_{Z_N})x_i\| &< \varepsilon_N^2 \qquad (i=1,\dots,m), \\ \Big\|\Big(I-P_{Z_N}-\sum_{k'=1}^{k_N} P_{Y_{k'}}\Big)Ax_i\Big\| &< \frac{\varepsilon}{4} \qquad (i=1,\dots,m). \end{aligned} \tag{2}$$

Set $k=k_N$. Fix $i\in\{1,\dots,n\}$ (for $n+1\le i\le m$ the proof will be similar). Let $x_i=\sum_{j=j_0}^\infty \gamma_j e_j$ with $\gamma_{j_0}\ne 0$. Clearly $j_0\le b_{N-1}$. Let $s=5a_k^2+a_k-j_0$. Let Q be the natural projection onto the space $\mathrm{Span}\{e_0,\dots,e_{5a_k^2+a_k},Y_{k'}\quad (k'\le k),v_{k+1,1},\dots,v_{k+1,s+1}\}$.

Consider the vectors x_i, $v_{k+1,1}$ and $x_i+v_{k+1,1}$. We have

$$QAv_{k+1,1}=\sum_{j=0}^{s}\xi_j T^j v_{k+1,1}$$

and there are complex numbers ν_j,η_j such that

$$QAx_i=Q\sum_{j=0}^{s}\nu_j T^j x_i$$

and

$$QA(x_i+v_{k+1,1})=Q\sum_{j=0}^{s}\eta_j T^j(x_i+v_{k+1,1}).$$

As above, we have $\nu_j=\xi_j=\eta_j\quad(0\le j\le s)$. So $QAx_i=Q\sum_{j=0}^s\xi_jT^jx_i$.

We have

$$\|(A-p_k(T))x_i\|\le\|(I-Q)Ax_i\|+\|Q(A-p_k(T))x_i\|+\|(I-Q)p_k(T)x_i\|.$$

By (2), $\|(I-Q)Ax_i\|<\varepsilon/4$ and

$$\begin{aligned} \|Q(A-p_k(T))x_i\| &= \Big\|Q\sum_{j=5a_k^2}^{s}\xi_jT^jx_i\Big\| \le \Big\|\sum_{j=5a_k^2}^{s}\xi_jT^jx_i\Big\| \\ &\le \sum_{j=5a_k^2}^{s}|\xi_j|\cdot\max\{\|T^j\|:5a_k^2\le j\le s\}\le\varepsilon_N^2\cdot 2\varepsilon_N^{-1}=2\varepsilon_N<\varepsilon/4. \end{aligned}$$

Furthermore, since $(I-Q)p_k(T)P_{Z_N}x_i=0$, we have

$$\begin{aligned} &\|(I-Q)p_k(T)x_i\| \\ &\le \|(I-Q)p_k(T)(I-P_{Z_{N+1}})x_i\|+\|(I-Q)p_k(T)(P_{Z_{N+1}}-P_{Z_N})x_i\| \\ &\le \|p_k(T)(I-P_{Z_{N+1}})x_i\|+\|p_k(T)(P_{Z_{N+1}}-P_{Z_N})x_i\|, \end{aligned}$$

where

$$\|p_k(T)(I-P_{Z_{N+1}})x_i\| = \Big\| \sum_{j=0}^{5a_k^2-1} \xi_j T^j (I-P_{Z_{N+1}})x_i \Big\|$$

$$\le \Big(\sum_{j=0}^{5a_k^2-1} |\xi_j| \Big) \max\{\|T^j(I-P_{Z_{N+1}})\| : 0 \le j \le 5a_k^2-1\} \cdot \|(I-P_{Z_{N+1}})x_i\| \le \frac{4\varepsilon}{16} = \frac{\varepsilon}{4}$$

and

$$\|p_k(T)(P_{Z_{N+1}}-P_{Z_N})x_i\| \le \|p_k(T)\| \cdot \|(P_{Z_{N+1}}-P_{Z_N})x_i\|$$
$$\le \max\{\|T^j\| : 0 \le j \le 5a_k^2-1\} \cdot \varepsilon_N^2 \le 2\varepsilon_N^{-1}\varepsilon_N^2 = 2\varepsilon_N < \varepsilon/4.$$

Hence $\|(A-p_k(T))x_i\| < \varepsilon$ for each i, $1 \le i \le n$, and similarly, for $n+1 \le i \le m$. This implies that A is a limit of polynomials of T in the strong operator topology and hence, T is reflexive.

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