Orbits of linear operators tending to infinity

Vladimír Müller and Jan Vršovský

Abstract

Let T be a bounded linear operator on a (real or complex) Banach space X satisfying $\sum_{n=1}^{\infty} \frac{1}{\|T^n\|} < \infty$. Then there is a unit vector $x \in X$ such that $\|T^n x\| \to \infty$. If X is a complex Hilbert space then it is sufficient to assume that $\sum_{n=1}^{\infty} \frac{1}{\|T^n\|^2} < \infty$. The above results are the best possible. We also show analogous results for weak orbits.

Mathematics Subject Classification (2000). Primary: 47A05, Secondary: 47A15, 47A16.
Keywords. Orbits of operators, plank problem.
The research was supported by grant No. 201/06/0128 of GA ČR.

1 Introduction

Let X be a real or complex Banach space. Denote by $\mathcal{L}(X)$ the set of all bounded linear operators on X. The *orbit* of a point $x \in X$ under an operator $T \in \mathcal{L}(X)$ is the sequence $(T^n x)_{n=1}^{\infty}$ of vectors. Analogously, the *weak orbit* of $x \in X$ and $x^* \in X^*$ is the sequence $(\langle T^n x, x^* \rangle)_{n=1}^{\infty}$ of real or complex numbers.

Orbits and weak orbits are closely connected with many fields of operator theory, for example local spectral theory, semigroups of operators and especially with the invariant subspace/subset problem. For a broader study, see Chapter III in [Be], or [M2].

It is still an open problem whether each operator on a Hilbert space (or more generally on a reflexive Banach space) has a nontrivial closed invariant subset (on ℓ_1 a negative solution to this problem was given by Read [R]). It is easy to see that an operator T has a nontrivial closed invariant subset if and only if there is a nonzero vector x such that its orbit is not dense.

The paper studies the existence of orbits tending to infinity, i.e. $\|T^n x\| \to \infty$ as $n \to \infty$. This is an easy way how to obtain a non-dense orbit and therefore a nontrivial closed invariant subset.

By the Banach-Steinhaus theorem, an operator $T \in \mathcal{L}(X)$ has unbounded orbits if and only if $\sup \|T^n\| = \infty$. With orbits tending to infinity the situation is not so simple. It is possible (cf. Example 4 or [Be], p.66) that $\|T^n\| \to \infty$ but there are no vectors x with $\|T^n x\| \to \infty$. However, if the sequence $\|T^n\|$ grows sufficiently fast, the desired orbit exists.

We show that if $\sum \frac{1}{\|T^n\|} < \infty$ then there are always orbits tending to infinity (and hence nontrivial closed invariant subsets). On the other hand, there is an operator T satisfying $\|T^n\| = n + 1$ for all n but without orbits tending to infinity.

For operators on complex Hilbert spaces it is possible to obtain better results. For the existence of an orbit tending to infinity it is sufficient to assume that $\sum \frac{1}{\|T^n\|^2} < \infty$. This result is also sharp.

These results improve Theorems III.2.A.7 and III.2.C.1 of Beauzamy [Be] (for Hilbert space operators) and the results of [M1] in the Banach space case. They also answer Problem 3.8 of [M2].

We study also the existence of weak orbits tending to infinity. For operators on complex Hilbert spaces the condition $\sum \frac{1}{\|T^n\|} < \infty$ implies even the existence of vectors x, y such that $|\langle T^n x, y\rangle| \to \infty$. On the other hand, there is a Hilbert space operator T satisfying $\|T^n\| = n + 1$ for all n, such that there are no x, y with $|\langle T^n x, y\rangle| \to \infty$. This improves the results of [M1].

2 Orbits tending to infinity

The key tool to show this are the following geometric theorems. We call them "plank theorems" since they solve several versions of the so-called plank problem.

Theorem 1. (K. Ball [B1]) *Let X be a (real or complex) Banach space and $f_1, f_2, \ldots \in X^*$ unit functionals. For each $n \in \mathbb{N}$, let $\alpha_n \geq 0$ be such that $\sum_{n=1}^{\infty} \alpha_n < 1$. Then there is a point $x \in X^*$ such that $\|x\| = 1$ and $|\langle x, f_n\rangle| \geq \alpha_n$ for every n.*

Theorem 2. (K. Ball [B2]) *Let X be a complex Hilbert space and $f_1, f_2 \in X$ unit vectors. For each $n \in \mathbb{N}$, let $\alpha_n \geq 0$ be such that $\sum_{n=1}^{\infty} \alpha_n^2 < 1$. Then there is a point $x \in X^*$ such that $\|x\| = 1$ and $|\langle x, f_n\rangle)| \geq \alpha_n$ for every n.*

For the proofs, we refer to [B1] and [B2], respectively. Using these deep results, we obtain our main theorem. In particular, if T_n is a sequence of powers of an operator, i.e. if $T_n := T^n$ for a fixed $T \in \mathcal{L}(X)$, then the following theorem gives the existence of an orbit tending to infinity.

Theorem 3. *Let X be a (real or complex) Banach space and $T_n \in \mathcal{L}(X)$, $n \in \mathbb{N}$. Suppose that one of the following conditions is satisfied:*

(i) either

$$\sum_{n=1}^{\infty} \frac{1}{\|T_n\|} < \infty,$$

(ii) or X is a complex Hilbert space and

$$\sum_{n=1}^{\infty} \frac{1}{\|T_n\|^2} < \infty.$$

Then there exists a point $x \in X$ such that $\|T_n x\| \to \infty$.

Proof. In case (i) set $r := 1$, in case (ii) set $r := 2$. Choose any sequence (β_n) of positive real numbers tending to infinity such that

$$s := \sum_{n=1}^{\infty} \frac{\beta_n}{\|T_n\|^r} < \infty.$$

The sequence of coefficients

$$\alpha_n := \frac{1}{(s+1)^{1/r}} \frac{\beta_n^{1/r}}{\|T_n\|}$$

satisfies both

$$\sum_{n=1}^{\infty} \alpha_n^r < 1 \qquad \text{and} \qquad \alpha_n \|T_n\| \to \infty.$$

Now consider the adjoint operators $T_n^* \in \mathcal{L}(X^*)$. For each $n \in \mathbb{N}$ find $g_n \in X^*$ such that $\|g_n\| \leq 1$ and $\|T_n^* g_n\| \geq \frac{1}{2}\|T_n^*\| = \frac{1}{2}\|T_n\|$. Finally, define unit functionals $f_n \in X^*$ by

$$f_n := \frac{T_n^* g_n}{\|T_n^* g_n\|}.$$

At this point, we are able to apply the plank theorems. Thus, there is an $x \in X$ with $\|x\| = 1$ such that $|\langle x, f_n \rangle| \geq \alpha_n$ for every n. Therefore

$$\|T_n x\| \;\geq\; \|T_n x\| \, \|g_n\|$$

$$
\begin{aligned}
&\geq |\langle T_n x, g_n\rangle| = |\langle x, T_n^* g_n\rangle| = |\langle x, f_n\rangle| \cdot \|T_n^* g_n\| \\
&\geq \frac{\alpha_n}{2}\|T_n\| \to \infty, \quad \text{as } n \to \infty.
\end{aligned}
$$

□

The exponent of $\|T^n\|$ in the above statements cannot be improved, as the following example shows. We show that there is a Banach space (complex or real) X and an operator $T \in \mathcal{L}(X)$ such that $\|T^n\| = n+1$ for all n (i.e., $\sum \frac{1}{\|T^n\|^{1+\varepsilon}} < \infty$ for each $\varepsilon > 0$) but there is no $x \in X$ with $\|T^n x\| \to \infty$. Similarly, there is a complex Hilbert space H and an operator $T \in \mathcal{L}(H)$ such that $\|T^n\| = (n+1)^{1/2}$ for all n (i.e., $\sum \frac{1}{\|T^n\|^{2+\varepsilon}} < \infty$ for each $\varepsilon > 0$) but there are no orbits $(T^n x)$ tending to infinity.

We construct these operators generally in ℓ^p spaces.

Example 4. *In the space $X = \ell^p$, $1 \leq p < \infty$, there is an operator $T \in \mathcal{L}(X)$ satisfying $\|T^n\| = (n+1)^{1/p}$ for all n, such that there is no $x \in X$ with $\|T^n x\| \to \infty$ as $n \to \infty$.*

Proof. Let $(e_k)_{k=1}^{\infty}$ be the standard basis in the space $X = \ell^p$ (real or complex). Let $T \in \mathcal{L}(X)$ be the weighted backward shift defined by

$$
Te_k := \begin{cases} \left(\frac{k}{k-1}\right)^{1/p} e_{k-1} & \text{for } k > 1, \\ 0 & \text{for } k = 1. \end{cases}
$$

Hence

$$
\|T^n\| = \prod_{k=2}^{n+1} \left(\frac{k}{k-1}\right)^{1/p} = (n+1)^{1/p}
$$

for all n. For the contradiction, suppose that there is an $x = \sum_{k=1}^{\infty} c_k e_k \in \ell^p$ such that $\|x\| = \left(\sum_{k=1}^{\infty} |c_k|^p\right)^{1/p} \leq 1$ but $\|T^n x\| \to \infty$ as $n \to \infty$. Consequently,

$$
\frac{1}{n}\sum_{j=n}^{2n-1} \left\|T^j x\right\|^p \to \infty, \qquad \text{as } n \to \infty.
$$

Let us estimate the above arithmetic mean. First we have

$$
\begin{aligned}
\left\|T^j x\right\|^p &= \left\| \sum_{k=j+1}^{\infty} \left(\frac{k}{k-j}\right)^{1/p} c_k e_{k-j} \right\|^p \\
&\leq \sum_{k=j+1}^{2j} |c_k|^p \frac{k}{k-j} + \sum_{k=2j+1}^{\infty} |c_k|^p \frac{k}{k-j},
\end{aligned}
$$

where the second sum can be estimated by $2\|x\|^p \leq 2$ since for $k > 2j$ we have $\frac{k}{k-j} < 2$. If we sum up the inequalities we get

$$\begin{aligned}
\sum_{j=n}^{2n-1} \left\|T^j x\right\|^p &\leq 2n + \sum_{j=n}^{2n-1} \sum_{k=j+1}^{2j} |c_k|^p \frac{k}{k-j} \\
&\leq 2n + \sum_{k=n+1}^{4n} |c_k|^p \sum_{i=1}^{k} \frac{k}{i} \\
&\leq 2n + \sum_{k=n+1}^{4n} |c_k|^p \, 4n(1+\ln 4n),
\end{aligned}$$

so that

$$2 + 4(1+\ln 4n) \sum_{k=n+1}^{4n} |c_k|^p \geq \frac{1}{n} \sum_{j=n}^{2n-1} \left\|T^j x\right\|^p \to \infty.$$

Hence, for all n large enough, the left hand side is greater than 6, i.e., if we write $s_n := \sum_{k=n+1}^{4n} |c_k|^p$ then

$$s_n \geq \frac{1}{1+\ln 4n}.$$

But this is a contradiction since for such n we have

$$\begin{aligned}
1 &= \sum_{k=1}^{\infty} |c_k|^p \geq s_n + s_{4n} + s_{4\cdot 4n} + s_{4\cdot 4\cdot 4n} + \dots \\
&\geq \sum_{j=1}^{\infty} \frac{1}{1+\ln 4^j n} = \sum_{j=1}^{\infty} \frac{1}{1+\ln n + j\ln 4} = \infty.
\end{aligned}$$

□

If we use a shift on ℓ^p with weights $\left(\frac{k+1\ln(k+1)}{k\ln k}\right)^{1/p}$ instead of $\left(\frac{k}{k-1}\right)^{1/p}$, a similar proof yields the same negative result concerning the operator $T \in \mathcal{L}(\ell^p)$ with even faster growth

$$\|T^n\| = \left(\frac{1}{2\ln 2}\right)^{1/p} ((n+2)\ln(n+2))^{1/p}.$$

In particular, there is an operator $T \in \mathcal{L}(\ell^1)$ with $\|T^n\| \sim n \ln n$ such there is no $x \in \ell^1$ with $\|T^n x\| \to \infty$.

Theorem 3 can be formulated also for semigroups of operators.

Corollary 5. *Let X be a Banach space, let $T(t)$ be a C_0-semigroup of operators on X. Suppose that $\int_0^\infty \frac{1}{\|T(t)\|} dt < \infty$. Then there exists $x \in X$ such that*

$$\lim_{t\to\infty} \|T(t)x\| = \infty.$$

If X is a complex Hilbert space then it is sufficient to assume that $\int_0^\infty \frac{1}{\|T(t)\|^2} dt < \infty$.

Proof. Let $C := \sup\{\|T(t)\| : 0 \le t \le 1\}$. Let $n \in \mathbb{N}$. For $t \in (n-1, n\rangle$ we have $T(t)T(n-t) = T(n)$, and so $\|T(t)\| \ge \|T(n)\| \cdot \|T(n-t)\|^{-1} \ge C^{-1}\|T(n)\|$. Thus the condition $\int_0^\infty \frac{1}{\|T(t)\|} dt < \infty$ implies that $\sum_{n=1}^\infty \frac{1}{\|T(n)\|} < \infty$.

By Theorem 3, there exists $x \in X$ such that $\lim_{n\to\infty} \|T(n)x\| = \infty$. For $t \in (n-1, n\rangle$ we have $\|T(n)x\| \le \|T(n-t)\| \cdot \|T(t)x\|$ and so $\|T(t)x\| \ge C^{-1}\|T(n)x\|$. Thus $\lim_{t\to\infty} \|T(t)x\| = \infty$.

The statement for Hilbert spaces is similar. □

On the other hand, for each p, $1 \le p < \infty$, there is a C_0-semigroup $T(t)$ on $X = L^p(1,\infty)$ such that $\|T(t)\| = (t+1)^{1/p}$ but such that there is no $f \in X$ with $\lim_{t\to\infty} \|T(t)f\| \to \infty$. Let the semigroup $T(t)$ be defined on $L^p(1,\infty)$ as a weighted backward shift by

$$(T(t)f)(z) := \left(\frac{z+t}{z}\right)^{1/p} f(z+t),$$

for $f \in X$, $t \ge 0$ and $z \ge 1$, so that $\|T(t)\| = (t+1)^{1/p}$. If there is an $f \in X$ with $\|T(t)f\| \to \infty$ then it is possible to use an argument analogous to that in Example 4 to get a contradiction.

3 Weak orbits

We study also the question when there are weak orbits $(\langle T^n x, x^*\rangle)$ tending to infinity.

Theorem 6. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Suppose that one of the following conditions is satisfied:*

(i) either

$$\sum_{n=1}^\infty \frac{1}{\|T^n\|^{1/2}} < \infty,$$

(ii) or X is a complex Hilbert space and

$$\sum_{n=1}^{\infty} \frac{1}{\|T^n\|} < \infty,$$

Then there exist a point $x \in X$ and a functional $x^* \in X^*$ such that $|\langle T^n x, x^* \rangle| \to \infty$.

But first, let us formulate a dual version of the plank theorem, see [B1].

Proposition 7. *Let X be a (real or complex) Banach space and $x_1, x_2, \ldots$ unit vectors in X. For each $n \in \mathbb{N}$, let $\alpha_n \geq 0$ be such that $\sum_{n=1}^{\infty} \alpha_n < 1$. Then there is a functional $f \in X^*$ such that $\|f\| = 1$ and $|\langle x_n, f \rangle| \geq \alpha_n$ for every n.*

Using Proposition 7, we succeed with a proof analogous to that of Theorem 3.

Proof of Theorem 6. Again, in case (i) set $r := 1$, in case (ii) set $r := 2$. Choose any sequence (β_n) of positive real numbers tending to infinity such that

$$s := \sum_{n=1}^{\infty} \frac{\beta_n}{\|T_n\|^{r/2}} < \infty.$$

The sequence of coefficients

$$\alpha_n := \frac{1}{(s+1)^{1/r}} \frac{\beta_n^{1/r}}{\|T_n\|^{1/2}}$$

satisfies both

$$\sum_{n=1}^{\infty} \alpha_n^r < 1 \qquad \text{and} \qquad \alpha_n \|T_n\|^{1/2} \to \infty.$$

For each $n \in \mathbb{N}$ find $x_n \in X$ such that $\|x_n\| \leq 1$ and $\|T_n x_n\| \geq \frac{1}{2} \|T_n\|$.

Consider the unit vectors $\frac{T_n x_n}{\|T_n x_n\|}$. In case (i) we apply Proposition 7, in case (ii) apply Theorem 2. In both cases we obtain a functional $x^* \in X^*$ with $\|x^*\| = 1$ such that $|\langle T_n x_n, x^* \rangle| \geq \alpha_n \|T_n x_n\|$. If we apply again the plank theorems to the functionals $\frac{T_n^* x^*}{\|T_n^* x^*\|}$ we obtain a point $x \in X$ with $\|x\| = 1$ such that $|\langle x, T_n^* x^* \rangle| \geq \alpha_n \|T_n^* x^*\|$. Therefore

$$\begin{aligned} |\langle T_n x, x^* \rangle| &= |\langle x, T_n^* x^* \rangle| \geq \alpha_n \|T_n^* x^*\|, \\ &\geq \alpha_n |\langle x_n, T_n^* x^* \rangle| = \alpha_n |\langle T_n x_n, x^* \rangle|, \\ &\geq \alpha_n^2 \|T_n x_n\| \geq \frac{\alpha_n^2}{2} \|T_n\| \to \infty, \quad \text{as } n \to \infty. \end{aligned}$$

$\square$

Example 8. *There are a Hilbert space H (real or complex) and an operator $T \in \mathcal{L}(H)$ satisfying $\|T^n\| = (n+1)$ for each n, such that there is no pair $x, y \in H$ with $|\langle T^n x, y\rangle| \to \infty$ as $n \to \infty$.*

Proof. Let H be the Hilbert space with an orthonormal basis $\{e_{k,j} \colon k \in \mathbb{N}, 1 \le j \le k\}$. Let $T \in \mathcal{L}(H)$ be defined by

$$Te_{k,j} := \begin{cases} \left(\frac{j+1}{j} \cdot \frac{k-j+1}{k-j}\right)^{1/2} e_{k,j+1} & \text{for } j < k, \\ 0 & \text{for } j = k. \end{cases}$$

We have $T^n e_{k,j} = \left(\frac{j+n}{j} \cdot \frac{k-j+1}{k-j+1-n}\right)^{1/2} e_{k,j+n}$ for $j \le k-n$. It is easy to see that $\left(\frac{j+n}{j} \cdot \frac{k-j+1}{k-j+1-n}\right)^{1/2} \le n+1$. Moreover, $T^n e_{n+1,1} = (n+1)e_{n+1,n+1}$, and so $\|T^n\| = n+1$ for each n.

Let $x, y \in H$, $x = \sum_{k,j} \alpha_{k,j} e_{k,j}$, $y = \sum_{k,j} \beta_{k,j} e_{k,j}$ with real or complex coefficients $\alpha_{k,j}, \beta_{k,j}$. Suppose on the contrary that $|\langle T^n x, y\rangle| \to \infty$. Without loss of generality we may assume that $\|x\| = 1 = \|y\|$.

For each n large enough we have

$$\sum_{r=n}^{2n-1} |\langle T^r x, y\rangle| \ge 7n.$$

On the other hand, we have

$$\begin{aligned} \sum_{r=n}^{2n-1} |\langle T^r x, y\rangle| &= \sum_{r=n}^{2n-1} \sum_{k=r+1}^{\infty} \sum_{j=1}^{k-r} \Big(\frac{j+r}{j} \cdot \frac{k-j+1}{k-j+1-r}\Big)^{1/2} |\alpha_{k,j}\beta_{k,j+r}| \\ &\le A + B + C + D, \end{aligned}$$

where

$$\begin{aligned} A &= \sum_{r=n}^{2n-1} \sum_{k=r+1}^{4n} \sum_{j=1}^{k-r} \Big(\frac{j+r}{j} \cdot \frac{k-j+1}{k-j+1-r}\Big)^{1/2} |\alpha_{k,j}\beta_{k,j+r}|, \\ B &= \sum_{k=4n+1}^{\infty} \sum_{r=n}^{2n-1} \sum_{j=n+1}^{k-r-n} \Big(\frac{j+r}{j} \cdot \frac{k-j+1}{k-j+1-r}\Big)^{1/2} |\alpha_{k,j}\beta_{k,j+r}|, \\ C &= \sum_{k=4n+1}^{\infty} \sum_{r=n}^{2n-1} \sum_{j=1}^{n} \Big(\frac{j+r}{j} \cdot \frac{k-j+1}{k-j+1-r}\Big)^{1/2} |\alpha_{k,j}\beta_{k,j+r}|, \end{aligned}$$

$$D \;=\; \sum_{k=4n+1}^{\infty} \sum_{r=n}^{2n-1} \sum_{j=k-r-n+1}^{k-r} \Big(\frac{j+r}{j} \cdot \frac{k-j+1}{k-j+1-r}\Big)^{1/2} |\alpha_{k,j}\beta_{k,j+r}|.$$

We have:

$$\begin{aligned}
A &\leq \sum_{k=n+1}^{4n} \sum_{j=1}^{k}\sum_{i=1}^{k} \frac{k}{\sqrt{ij}} |\alpha_{k,j}\beta_{k,k-i+1}| \leq \frac{4n}{2} \sum_{k=n+1}^{4n} \sum_{i,j=1}^{k} \Big(\frac{|\alpha_{k,j}|}{\sqrt{i}}\Big)^2 + \Big(\frac{|\beta_{k,k-i+1}|}{\sqrt{j}}\Big)^2 \\
&\leq 2n(1+\ln(4n)) \sum_{k=n+1}^{4n} \sum_{j=1}^{k} (|\alpha_{k,j}|^2 + |\beta_{k,j}|^2). \\
B &\leq \sum_{k=4n+1}^{\infty} \sum_{r=n}^{2n-1} \sum_{j=n+1}^{k-r-n} 3|\alpha_{k,j}\beta_{k,j+r}| \leq \frac{3}{2} \sum_{k=4n+1}^{\infty} \sum_{r=n}^{2n-1} \sum_{j=n+1}^{k-r-n} (|\alpha_{k,j}|^2 + |\beta_{k,j+r}|^2) \\
&\leq \frac{3n}{2} \sum_{k=4n+1}^{\infty} \sum_{j=1}^{k} (|\alpha_{k,j}|^2 + |\beta_{k,j}|^2) \leq \frac{3n}{2}. \\
C &\leq \sum_{k=4n+1}^{\infty} \sum_{r=n}^{2n-1} \sum_{j=1}^{n} \Big(\frac{3n-1}{j} \cdot 3\Big)^{1/2} |\alpha_{k,j}\beta_{k,j+r}| \\
&\leq 3\sqrt{n} \sum_{k=4n+1}^{\infty} \sum_{j=1}^{n} \frac{|\alpha_{k,j}|}{\sqrt{j}} \sum_{i=n+1}^{3n-1} |\beta_{k,i}| \\
&\leq 3\sqrt{n} \cdot \sqrt{2n} \sum_{k=4n+1}^{\infty} \sum_{j=1}^{n} \frac{|\alpha_{k,j}|}{\sqrt{j}} \Big(\sum_{i=n+1}^{3n-1} |\beta_{k,i}|^2\Big)^{1/2} \\
&\leq \frac{3n}{\sqrt{2}} \sum_{k=4n+1}^{\infty} \sum_{j=1}^{n} \Big(|\alpha_{k,j}|^2 + \sum_{i=n}^{3n} \frac{|\beta_{k,i}|^2}{j}\Big) \\
&\leq \frac{3n}{\sqrt{2}} + \frac{3n}{\sqrt{2}}(1+\ln n) \sum_{k=4n+1}^{\infty} \sum_{i=n}^{3n} |\beta_{k,i}|^2.
\end{aligned}$$

Since the terms C and D are symmetrical, we have

$$D \leq \frac{3n}{\sqrt{2}} + \frac{3n}{\sqrt{2}}(1+\ln n) \sum_{k=4n+1}^{\infty} \sum_{i=k-3n}^{k-n} |\alpha_{k,i}|^2.$$

Thus for n large enough we have

$$\begin{aligned}
7n &\leq \sum_{r=n}^{2n-1} |\langle T^r x, y\rangle| \\
&\leq \frac{3n}{2} + 3n\sqrt{2} + 2n(1+\ln(4n)) \sum_{k=n+1}^{4n} \sum_{j=1}^{k} (|\alpha_{k,j}|^2 + |\beta_{k,j}|^2)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{3n}{\sqrt{2}}(1+\ln n)\sum_{k=4n+1}^{\infty}\sum_{j=n}^{3n}(|\beta_{k,j}|^2+|\alpha_{k,k-j}|^2)\\
\leq\ & 6n+n(1+\ln(4n))\Big(2\sum_{k=n+1}^{4n}\sum_{j=1}^{k}(|\alpha_{k,j}|^2+|\beta_{k,j}|^2)\\
&+3\sum_{k=4n+1}^{\infty}\sum_{j=n}^{3n}(|\beta_{k,j}|^2+|\alpha_{k,k-j}|^2)\Big).
\end{aligned}
$$

Thus for all $n \geq n_0$ we have

$$
2\sum_{k=n+1}^{4n}\sum_{j=1}^{k}(|\alpha_{k,j}|^2+|\beta_{k,j}|^2)+3\sum_{k=4n+1}^{\infty}\sum_{j=n}^{3n}(|\beta_{k,j}|^2+|\alpha_{k,k-j}|^2)\geq\frac{1}{1+\ln(4n)}.
$$

In particular, for $n = 4^s n_0 \quad (s = 1, 2, \ldots)$ we have

$$
\begin{aligned}
&10=5\sum_{k=1}^{\infty}\sum_{j=1}^{k}(|\alpha_{k,j}|^2+|\beta_{k,j}|^2)\\
\geq\ & 2\sum_{s=1}^{\infty}\sum_{k=4^s n_0+1}^{4^{s+1}n_0}\sum_{j=1}^{k}(|\alpha_{k,j}|^2+|\beta_{k,j}|^2)+3\sum_{s=1}^{\infty}\sum_{k=4^s n_0+1}^{\infty}\sum_{j=4^s n_0}^{3\cdot 4^s n_0}(|\beta_{k,j}|^2+|\alpha_{k,k-j}|^2)\\
\geq\ & \sum_{s=1}^{\infty}\frac{1}{1+\ln 4^{s+1}n_0}=\sum_{s=1}^{\infty}\frac{1}{1+\ln n_0+(s+1)\ln 4}=\infty,
\end{aligned}
$$

a contradiction. Hence there are no $x, y \in H$ with $|\langle T^n x, y\rangle| \to \infty$. $\square$

4 Summary

We summarize the results in the following way. We consider the classes of all complex (real) Banach spaces and of complex (real) Hilbert spaces. For each such class $\mathcal{X}$, denote by $\alpha_{\mathcal{X}}$ the supremum of all positive t such that the condition $\sum \frac{1}{\|T^n\|^t} < \infty$ (for an operator T on a space from the class) implies that there is an orbit $(T^n x)$ tending to infinity. Similarly, denote by $\beta_{\mathcal{X}}$ the supremum of all t such that $\sum \frac{1}{\|T^n\|^t} < \infty$ implies that there are x, x^* with $|\langle T^n x, x^*\rangle| \to \infty$.

The known results are summarized in the following tableau (note that the supremum is in fact maximum whenever the exact value is known):

$\mathcal{X}$	$\alpha_{\mathcal{X}}$	$\beta_{\mathcal{X}}$
complex Banach spaces	1	$1/2 \le \beta_{\mathcal{X}} \le 1$
complex Hilbert spaces	2	1
real Banach spaces	1	$1/2 \le \beta_{\mathcal{X}} \le 1$
real Hilbert spaces	$1 \le \alpha_{\mathcal{X}} \le 2$	$1/2 \le \beta_{\mathcal{X}} \le 1$

References

[B1] K.M. Ball, *The plank problem for symmetric bodies*, Invent. Math. **10** (1991), 535–543.

[B2] K.M. Ball, *The complex plank problem*, Bull. London Math. Soc. **33** (2001), 433–442.

[Be] B. Beauzamy, *Introduction to Operator Theory and Invariant Subspaces*, North-Holland, Amsterdam–New York–Oxford–Tokyo, 1988.

[M1] V. Müller, *Orbits, Weak Orbits and Local Capacity of Operators*, Integral Equations Operator Theory **41** (2001), 230–253.

[M2] V. Müller, *Orbits of operators*, in: Advanced Courses of Mathematical Analysis I, Cadiz, pp. 53–79, 2004.

[R] C.J. Read, *The invariant subspace problem for a class of Banach spaces II. Hypercyclic operators*, Israel J. Math. **63** (1988), 1–40.

Institute of Mathematics AV ČR
Žitná 25, 115 67 Praha 1
Czech republic

muller@math.cas.cz
vrsovsky@math.cas.cz