AN OPERATOR IS A PRODUCT OF TWO QUASI-NILPOTENT OPERATORS IFF IT IS NOT SEMI-FREDHOLM

ROMAN DRNOVŠEK, VLADIMIR MÜLLER, AND NIKA NOVAK

Abstract. We prove that a (bounded linear) operator acting on an infinite-dimensional, separable, complex Hilbert space can be written as a product of two quasi-nilpotent operators if and only if it is not a semi-Fredholm operator. This solves the problem posed by Fong and Sourour in 1984. We also consider some closely related questions. In particular, we show that an operator can be expressed as a product of two nilpotent operators if and only if its kernel and co-kernel are both infinite-dimensional. This answers the question implicitly posed by Wu in 1989.

1. Introduction

Throughout the paper, let $\mathcal{H}$ denote a complex, separable, infinite dimensional Hilbert space. Denote by $\mathcal{B}(\mathcal{H})$ the algebra of all operators (i.e., bounded linear transformations) on $\mathcal{H}$ and by $\mathcal{K}(\mathcal{H})$ the ideal of all compact operators. The essential spectrum $\sigma_e(T)$, left essential spectrum $\sigma_{le}(T)$ and right essential spectrum $\sigma_{re}(T)$ of an operator $T \in \mathcal{B}(\mathcal{H})$ is defined as the spectrum, left spectrum and right spectrum of the class $T + \mathcal{K}(\mathcal{H})$ in the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is called upper semi-Fredholm if its range $T(\mathcal{H})$ is closed and $\dim \ker T < \infty$, and it is called lower semi-Fredholm if $\operatorname{codim} T(\mathcal{H}) < \infty$ (then the range $T(\mathcal{H})$ is closed automatically). One can show that T is not upper semi-Fredholm if and only if for each $\varepsilon > 0$ and each subspace $\mathcal{M} \subset \mathcal{H}$ of a finite codimension there exists a unit vector $x \in \mathcal{M}$ such that $\|Tx\| < \varepsilon$.

It is well known that $\sigma_e(T)$, $\sigma_{le}(T)$ and $\sigma_{re}(T)$ are the sets of all complex numbers λ such that $T - \lambda$ is not Fredholm, upper semi-Fredholm and lower semi-Fredholm, respectively. For the details of Fredholm theory we refer to the books [2] and [7].

In 1984, Fong and Sourour [4] considered the question which operators on $\mathcal{H}$ are products of two quasi-nilpotent operators. Their first observation was the following necessary condition for such operators.

1991 *Mathematics Subject Classification.* 47A65, 47A68, 47A53.

Key words and phrases. Quasi-nilpotent operators, factorization, semi-Fredholm operators.

Proposition 1.1. *If $T \in \mathcal{B}(\mathcal{H})$ is a product of (two) quasi-nilpotent operators, then $0 \in \sigma_e(T^*T) \cap \sigma_e(TT^*)$, or equivalently $0 \in \sigma_{le}(T) \cap \sigma_{re}(T)$.*

In other words, a necessary condition for such operators is that T is not a semi-Fredholm operator. Proposition 1.1 follows from the following assertion that is an easy consequence of the fact that a quasi-nilpotent element in a unital Banach algebra is a topological divisor of zero (see e.g. [5, section XXIX.3]).

Proposition 1.2. *Let t and q be elements in a unital Banach algebra. If q is quasi-nilpotent, then tq is not left invertible, and qt is not right invertible.*

Fong and Sourour found the following sufficient condition for an operator to be the product of two quasi-nilpotent operators (see [4, Theorems 5 and 6]).

Theorem 1.3. *Let $T \in \mathcal{B}(\mathcal{H})$. If $0 \in \sigma_e(T^*T + TT^*)$, then T is a product of two quasi-nilpotent operators. Moreover, if T is a compact operator, then T is a product of two compact quasi-nilpotent operators.*

The authors of [4] left open the question whether the necessary condition from Proposition 1.1 is also sufficient for an operator to be the product of two quasi-nilpotent operators. The main result of our paper, proved in Section 2, gives an affirmative answer to this question. In Section 3 we consider a question of finding common quasi-nilpotent factors for a given finite collection of operators. Finally, in Section 4 we study similar problems replacing quasi-nilpotent operators by nilpotent ones. We show that an operator T can be expressed as a product of two nilpotent operators if and only if $\dim\ker T = \dim\ker T^* = \infty$. This answers the question implicitly posed in the survey paper [8, p.55].

2. Products of two quasi-nilpotent operators

We start with the following description of a non-semi-Fredholm operator.

Proposition 2.1. *Suppose that $T \in \mathcal{B}(\mathcal{H})$ is not a semi-Fredholm operator. Then $\mathcal{H}$ can be decomposed as a direct sum of three infinite-dimensional closed subspaces, so that T is similar to an operator of the form*

$$\begin{pmatrix} 0 & A & 0 \\ K & C & D \\ 0 & L & 0 \end{pmatrix},$$

where K and L are compact operators.

Proof. Since neither T nor T^* is upper semi-Fredholm, we can find inductively an orthonormal sequence $f_1, g_1, f_2, g_2, \ldots$ in $\mathcal{H}$ such that $\|Tf_n\| \le \frac{1}{n}$ and $\|T^*g_n\| \le \frac{1}{n}$.

Let $\mathcal{M}$ be the closed linear span of the vectors $\{f_n\}$, and let $\mathcal{N}$ be the closed linear span of $\{g_n\}$. Then $\mathcal{H} = \mathcal{M} \oplus \mathcal{L} \oplus \mathcal{N}$, where $\mathcal{L} = (\mathcal{M} \oplus \mathcal{N})^\perp$. The matrix of T relative to this decomposition is

$$\begin{pmatrix} K_1 & A & B \\ K_2 & C & D \\ K_3 & K_4 & K_5 \end{pmatrix},$$

where $K_1, \ldots, K_5$ are Hilbert-Schmidt operators, and hence compact.

We now claim that the subspaces $\mathcal{M}$ and $\mathcal{N}$ in the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{L} \oplus \mathcal{N}$ can be reduced to suitable infinite-dimensional closed subspaces (and so the subspace $\mathcal{L}$ is enlarged) so that the corresponding operator B is equal to 0. To end this, we choose decompositions $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$ and $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ such that $\mathcal{N}_2$ and $\mathcal{M}_1$ are infinite-dimensional closed subspaces and $B(\mathcal{N}_2) \subseteq \mathcal{M}_2$. Then B as an operator from $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$ to $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ has the form $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$. Hence the operator T relative to the decomposition $\mathcal{H} = \mathcal{M}_1 \oplus (\mathcal{M}_2 \oplus \mathcal{L} \oplus \mathcal{N}_1) \oplus \mathcal{N}_2$ has the upper right corner equal to 0. This shows what we have claimed. So, we may assume that $B = 0$.

Using the same argument we can also show that there is no loss of generality in assuming that $K_3 = 0$.

In order to complete the proof, we apply [1, Theorem 2] to show that the operators K_1 and K_5 are similar to operators of the form $\begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$ and $\begin{pmatrix} * & * \\ * & 0 \end{pmatrix}$ respectively, where both zeroes act on infinite-dimensional Hilbert spaces. This fact easily implies the desired conclusion. □

As in [4], the following lemma is useful for our study.

Lemma 2.2. *If A and C are quasi-nilpotent operators and B is any operator, then* $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ *is quasi-nilpotent.*

Now we state the main result of this paper.

Theorem 2.3. *An operator $T \in \mathcal{B}(\mathcal{H})$ is a product of two quasi-nilpotent operators if and only if it is not a semi-Fredholm operator.*

Proof. By Proposition 1.1 the condition is necessary. To show that it is also sufficient, assume that T is not a semi-Fredholm operator. By Proposition 2.1, we may assume that

T can be represented as

$$\begin{pmatrix} 0 & A & 0 \\ K & C & D \\ 0 & L & 0 \end{pmatrix}$$

on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, where each $\mathcal{H}_j$ is isomorphic to $\mathcal{H}$, and K and L are compact operators.

Since $\mathcal{H}_1$ and $\mathcal{H}_3$ are infinite-dimensional, we can write each of them as an infinite sum of Hilbert spaces that are isomorphic to $\mathcal{H}$ so that in the obtained decomposition of the space $\mathcal{H}$ the operator T can be represented as

$$\begin{pmatrix}
 & & \vdots & \vdots & \vdots & & \\
 & & 0 & A_2 & 0 & & \\
\cdots & 0 & 0 & A_1 & 0 & 0 & \cdots \\
\cdots & K_2 & K_1 & \boxed{C} & D_1 & D_2 & \cdots \\
\cdots & 0 & 0 & L_1 & 0 & 0 & \cdots \\
 & & 0 & L_2 & 0 & & \\
 & & \vdots & \vdots & \vdots & &
\end{pmatrix},$$

where $\{K_n\}$ and $\{L_n\}$ are compact operators satisfying $\max\{\|K_n\|, \|L_n\|\} \le 4^{-n}$ for all $n \ge 2$. Now define the operators Q_1 and Q_2 on $\mathcal{H}$ by

$$Q_1 = \begin{pmatrix}
 & & & & \vdots & \vdots & \vdots & & \\
 & & & & 0 & A_3 & 0 & & \\
 & & & & 0 & A_2 & 0 & & \\
 & & & & 0 & A_1 & 0 & 0 & \cdots \\
\cdots & 2^{-2}I & 2^{-1}I & I & \boxed{0} & C & 0 & 0 & \cdots \\
 & & & & 0 & 0 & 2L_1 & & \\
 & & & & 0 & & 0 & 2^2 L_2 & \\
 & & & & 0 & & & 0 & \ddots \\
 & & & & \vdots & & & & \ddots
\end{pmatrix}$$

and

$$Q_2 = \begin{pmatrix}
\ddots & \ddots & & & \vdots & \vdots & & & \\
 & 0 & 2^2 K_2 & & 0 & 0 & & & \\
 & & 0 & 2K_1 & 0 & 0 & 0 & 0 & \cdots \\
 & & & 0 & 0 & D_1 & D_2 & D_3 & \cdots \\
\cdots & 0 & 0 & 0 & \boxed{0} & 0 & 0 & 0 & \cdots \\
 & & & & I & & & & \\
 & & & & 2^{-1}I & & & & \\
 & & & & 2^{-2}I & & & & \\
 & & & & \vdots & & & &
\end{pmatrix}.$$

To prove that the operator Q_1 (and similarly, Q_2) is quasi-nilpotent, it suffices, by Lemma 2.2, to show that the operator

$$R = \begin{pmatrix} 0 & 2L_1 & 0 & \cdots \\ 0 & 0 & 2^2 L_2 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is quasi-nilpotent. But this follows easily from the following estimate

$$\begin{aligned} \|R^n\| &= \sup\{\|(2^{j+1}L_{j+1})(2^{j+2}L_{j+2})\ldots(2^{j+n}L_{j+n})\| : j = 0,1,2,\ldots\} \\ &\le \max\{\|2^j L_j\| : j \in \mathbb{N}\} \cdot 2^{-(2+3+\ldots+n)} . \end{aligned}$$

Since a direct computation shows that $T = Q_1 Q_2$, the proof is complete. □

Theorem 2.3 has an interesting connection with the theory of integral operators.

Corollary 2.4. *Let T be an operator on $L^2(X,\mu)$, where (X,μ) is a σ-finite measure space which is not completely atomic. Then the following assertions are equivalent:*
(a) The operators T and T^* are unitarily equivalent to integral operators on $L^2(X,\mu)$.
(b) The operator T is a product of two quasi-nilpotent operators.
(c) The operator T is not a semi-Fredholm operator.

Proof. Recall [6, section 15] that an operator A on $L^2(X,\mu)$ is unitarily equivalent to an integral operator on $L^2(X,\mu)$ if and only if $0 \in \sigma_{re}(A)$. Therefore, the assertion (a) holds if and only if $0 \in \sigma_{le}(T) \cap \sigma_{re}(T)$, i.e., T is not a semi-Fredholm operator. Thus, the assertions (a) and (c) are equivalent. The equivalence of (b) and (c) was shown in Theorem 2.3. □

If in Corollary 2.4 the operators T and T^* are simultaneously unitarily equivalent to integral operators on $L^2(X,\mu)$, i.e., there exists a unitary operator U on $L^2(X,\mu)$ such that both UTU^* and UT^*U^* are integral operators, then we have $0 \in \sigma_e(TT^* + T^*T)$, so that even the assumption of Theorem 1.3 is satisfied. However, the unitary equivalence in Corollary 2.4 is not necessarily simultaneous. Namely, it is shown in [6, Example 15.10] that there exists an operator A such that both A and A^* are unitarily equivalent to integral operators, but $A + A^*$ is invertible. It follows that the operators A and A^* cannot be simultaneously unitarily equivalent to integral operators on $L^2(X,\mu)$.

3. Common quasi-nilpotent factors

In this section we consider the following related problem. Given operators $T_1, \ldots, T_n$ on $\mathcal{H}$, we are searching for quasi-nilpotent operators Q_1 and Q_2 and operators $S_1, \ldots, S_n$ such that $T_i = Q_1 S_i Q_2$ for all $i = 1, \ldots, n$. Using Proposition 1.2 it is easy to verify that a necessary condition for these factorizations is that $0 \in \sigma_e(\sum_{i=1}^n T_i T_i^*) \cap \sigma_e(\sum_{i=1}^n T_i^* T_i)$. Theorem 3.3 below shows that this condition is also sufficient.

We begin with a simple lemma, and first consider the case when $T_1, \ldots, T_n$ are compact operators.

Lemma 3.1. *Let A and K be positive operators in $\mathcal{B}(\mathcal{H})$ such that $A \leq K$. If K is a compact operator, then A is also compact.*

Proof. Let B be the positive square root of A, so that $A = B^2$. It is sufficient to show that B is compact. To prove this, let $\{x_n\}_{n\in\mathbb{N}}$ be an arbitrary bounded sequence in $\mathcal{H}$. Since K is compact, the sequence $\{Kx_n\}_{n\in\mathbb{N}}$ has a convergent subsequence $\{Kx_{n_j}\}_{j\in\mathbb{N}}$. If the inequality $\|Bx\|^2 = \langle Ax, x\rangle \leq \langle Kx, x\rangle \leq \|Kx\|\|x\|$ is applied to $x = x_{n_j} - x_{n_k}$ ($j, k \in \mathbb{N}$), we conclude easily that $\{Bx_{n_j}\}_{j\in\mathbb{N}}$ is a convergent sequence. This shows that B is a compact operator. □

Proposition 3.2. *For compact operators $K_1, \ldots, K_n$ on $\mathcal{H}$ the following assertions hold.*

(a) *There exist a quasi-nilpotent operator Q and compact operators $L_1, \ldots, L_n$ such that*

$$K_i = L_i Q\,, \quad i = 1, \ldots, n.$$

(b) *There exist a quasi-nilpotent operator Q and compact operators $L_1, \ldots, L_n$ such that*

$$K_i = Q L_i\,, \quad i = 1, \ldots, n.$$

(c) *There exist quasi-nilpotent operators Q_1 and Q_2 and compact operators $L_1, \ldots, L_n$ such that*

$$K_i = Q_1 L_i Q_2\,, \quad i = 1, \ldots, n.$$

Proof. First we prove part (a). Since $K = \sum_{i=1}^n K_i^* K_i$ is a positive compact operator, there exists an orthonormal basis $\{\varphi_j\}_{j\in\mathbb{N}}$ with respect to which K has a diagonal form diag $(\lambda_1, \lambda_2, \ldots)$, where $\{\lambda_j\}_{j\in\mathbb{N}}$ is a decreasing sequence of non-negative numbers converging to 0.

Define the operator Q on $\mathcal{H}$ by

$$Q\varphi_j = \sqrt[4]{\lambda_j}\,\varphi_{j+1} \text{ for all } j \in \mathbb{N}.$$

To show that Q is quasi-nilpotent, we simply notice that $\|Q^{2n}\| = \sqrt[4]{\lambda_1\lambda_2\ldots\lambda_{2n}} \leq (\lambda_1\lambda_{n+1})^{\frac{n}{4}}$, and therefore $\lim\limits_{n\to\infty} \|Q^n\|^{\frac{1}{n}} = 0$.

Now define linear transformations $L_1, \ldots, L_n$ on $\mathcal{H}$ by

$$L_i\varphi_1 = 0 \quad \text{ and } \quad L_i\varphi_j = \frac{1}{\sqrt[4]{\lambda_{j-1}}} K_i\varphi_{j-1} \text{ for } j > 1.$$

Since we have, for all $j > 1$,

$$\|L_i\varphi_j\|^2 = \frac{1}{\sqrt{\lambda_{j-1}}}\langle K_i^* K_i\varphi_{j-1}, \varphi_{j-1}\rangle \leq \frac{1}{\sqrt{\lambda_{j-1}}}\langle K\varphi_{j-1}, \varphi_{j-1}\rangle \leq \sqrt{\lambda_{j-1}},$$

we conclude that $L_1, \ldots, L_n$ belong to $\mathcal{B}(\mathcal{H})$. Define the operator L on $\mathcal{H}$ by

$$L = \sum_{i=1}^{n} L_i^* L_i = \operatorname{diag}(0, \sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots).$$

Since $\lim\limits_{j\to\infty} \lambda_j = 0$, the operator L is compact. By Lemma 3.1, the operators $L_i^* L_i$ are compact, an so are the square roots $(L_i^* L_i)^{1/2}$. Using the polar decomposition we conclude that the operators $L_1, \ldots, L_n$ are compact as well. Since $L_i Q = K_i$ for all i, the assertion (a) is proved.

For the proof of part (b) we apply part (a) for the operators $K_1^*, \ldots, K_n^*$ and then take the adjoints. Since part (c) is an immediate consequence of parts (a) and (b), the proof is complete. □

Part (c) in the following result can be considered as a generalization of [4, proposition 4].

Theorem 3.3. *For operators $T_1, \ldots, T_n$ on $\mathcal{H}$ the following assertions hold.*

(a) *If $0 \in \sigma_e(\sum_{i=1}^n T_i T_i^*)$, then there exist a quasi-nilpotent operator Q and operators $S_1, \ldots, S_n$ on $\mathcal{H}$ such that*

$$T_i = QS_i, \quad i = 1, \ldots, n.$$

(b) *If $0 \in \sigma_e(\sum_{i=1}^n T_i^* T_i)$, then there exist a quasi-nilpotent operator Q and operators $S_1, \ldots, S_n$ on $\mathcal{H}$ such that*

$$T_i = S_i Q, \quad i = 1, \ldots, n.$$

(c) *If* $0 \in \sigma_e(\sum_{i=1}^n T_iT_i^*) \cap \sigma_e(\sum_{i=1}^n T_i^*T_i)$, *then there exist quasi-nilpotent operators* Q_1 *and* Q_2 *and operators* $S_1, \dots, S_n$ *with* $0 \in \sigma_e(\sum_{i=1}^n S_iS_i^*) \cap \sigma_e(\sum_{i=1}^n S_i^*S_i)$ *such that*

$$T_i = Q_1 S_i Q_2, \quad i = 1, \dots, n.$$

Proof. Let $T = \sqrt{\sum_{i=1}^n T_iT_i^*}$. Since $0 \in \sigma_e(T)$, by [4, Proposition 2] there exists a quasi-nilpotent operator Q on $\mathcal{H}$ such that $T^2 = QQ^*$. Since $T_iT_i^* \le T^2 = QQ^*$ for $i = 1, \dots, n$, it follows from the well-known theorem of Douglas [3] that there exist operators $S_1, \dots, S_n$ such that $T_i = QS_i$ for $i = 1, \dots, n$. This completes the proof of part (a). Part (b) follows from part (a) by duality.

Let us now prove part (c).

We claim that there exist an invertible operator V on $\mathcal{H}$ and a decomposition of the space $\mathcal{H}$ such that

$$VT_iV^{-1} = \begin{pmatrix} 0 & A_i & 0 \\ K_i & C_i & D_i \\ 0 & L_i & 0 \end{pmatrix},$$

where K_i and L_i are compact operators for $i = 1, \dots, n$.

Let $A = \sqrt{\sum_{i=1}^n T_i^*T_i}$ and $B = \sqrt{\sum_{i=1}^n T_iT_i^*}$. Since the selfadjoint operators A and B are not Fredholm, they are not upper semi-Fredholm. Therefore we can construct inductively an orthonormal sequence $f_1, g_1, f_2, g_2, \dots$ in $\mathcal{H}$ such that $\|Af_k\| \le \frac{1}{k}$ and $\|Bg_k\| \le \frac{1}{k}$ for all k. Note that for $i = 1, \dots, n$ and $k \in \mathbb{N}$ we have $\|T_if_k\| \le \|Af_k\| \le \frac{1}{k}$ and $\|T_i^*f_k\| \le \|Bf_k\| \le \frac{1}{k}$.

Let $\mathcal{M}$ be the closed span of the vectors $f_1, f_2, \dots$, and $\mathcal{N}$ the closed span of the vectors $g_1, g_2, \dots$. Let $\mathcal{L}$ be the orthogonal complement of $\mathcal{M} \oplus \mathcal{N}$. Then in the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{L} \oplus \mathcal{N}$ the operators $\{T_i\}_{i=1}^n$ are of the form

$$T_i = \begin{pmatrix} K_i^{(1)} & A_i & B_i \\ K_i^{(2)} & C_i & D_i \\ K_i^{(3)} & K_i^{(4)} & K_i^{(5)} \end{pmatrix},$$

where $K_i^{(j)}$ are compact operators.

We now claim that the decomposition of the space can be chosen in such a way that all B_i are equal to 0. As in the proof of Proposition 2.1 we may assume that $B_1 = 0$. The subspaces $\mathcal{M}$ and $\mathcal{N}$ can be reduced to infinite-dimensional closed subspaces $\mathcal{M}_1$ and $\mathcal{N}_1$ so that in the decomposition of the space $\mathcal{H} = \mathcal{M}_1 \oplus \mathcal{L}_1 \oplus \mathcal{N}_1$ both operators B_1 and B_2 are equal to 0. After finitely many reductions of $\mathcal{M}$ and $\mathcal{N}$ we obtaine the desired decomposition.

Using the same argument as in Proposition 2.1 we can now find an invertible operator V on $\mathcal{H}$ and a decomposition of the space such that

$$VT_iV^{-1} = \begin{pmatrix} 0 & A_i & 0 \\ K_i & C_i & D_i \\ 0 & L_i & 0 \end{pmatrix},$$

where K_i and L_i are compact operators for $i = 1, \ldots, n$.

By Proposition 3.2 there exist quasi-nilpotent operators Q and R and compact operators $H_1, \ldots, H_n, M_1, \ldots, M_n$ such that $K_i = H_iQ$ and $L_i = RM_i$ for $i = 1, \ldots, n$.

Define operators Q'_1, Q'_2 and $\{S'_i\}_{i=1}^n$ on $\mathcal{H}$ by

$$Q'_1 = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ R & 0 & 0 \end{pmatrix}, \quad Q'_2 = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ Q & 0 & 0 \end{pmatrix} \text{ and } S'_i = \begin{pmatrix} M_i & 0 & 0 \\ A_i & 0 & 0 \\ C_i & D_i & H_i \end{pmatrix}.$$

Thus $VT_iV^{-1} = Q'_1S'_iQ'_2$ for all $i = 1, \ldots, n$. Since $Q'^3_1 = \mathrm{diag}\,(R, R, R)$, the operator Q'_1 is quasi-nilpotent. The same holds for the operator Q'_2.

Let $Q_1 = V^{-1}Q'_1V$, $Q_2 = V^{-1}Q'_2V$ and $S_i = V^{-1}S'_iV \quad (i = 1 \ldots, n)$. Clearly, the operators Q_1 and Q_2 are quasi-nilpotent, $0 \in \sigma_e(\sum_{i=1}^n S_iS_i^*) \cap \sigma_e(\sum_{i=1}^n S_i^*S_i)$, and $T_i = Q_1S_iQ_2$ for all $i = 1, \ldots, n$.

Note that we there exists an orthonormal sequence $(h_k) \subset \mathcal{H}$ such that $\|S'_ih_k\| \to 0 \quad (k \to \infty)$ for all $i = 1, \ldots, n$. This implies easily that $\left(\sum_{i=1}^n S_i^*S_i\right)V^{-1}h_k \to 0$ and so $0 \in \sigma_e(\sum_{i=1}^n S_i^*S_i)$. Similarly it is possible to show that $0 \in \sigma_e(\sum_{i=1}^n S_iS_i^*)$.

This completes the proof of (c). □

4. Case of nilpotent operators

In the last section we prove the nilpotent analogs of Theorems 2.3 and 3.3. We first characterize operators that are products of two nilpotent operators.

Theorem 4.1. *An operator $T \in \mathcal{B}(\mathcal{H})$ is a product of two nilpotent operators (with index of nilpotency at most 3) if and only if $\dim\ker T = \dim\ker T^* = \infty$.*

Proof. Suppose that we can write $T = MN$, where M and N are nilpotent operators. Since $\ker N \subset \ker T$ and $\ker M^* \subset \ker T^*$, and $\dim\ker N = \dim\ker M^* = \infty$, we have $\dim\ker T = \dim\ker T^* = \infty$.

Conversely, suppose that $\dim\ker T = \dim\ker T^* = \infty$. We can choose a decomposition of $\mathcal{H}$ as a direct sum of infinite-dimensional subspaces $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ such that $\mathcal{H}_1 \subseteq \ker T$

and $\mathcal{H}_3 \subseteq \ker T^*$. Then the matrix of T relative to this decomposition is of the form

$$\begin{pmatrix} 0 & A & B \\ 0 & C & D \\ 0 & 0 & 0 \end{pmatrix}.$$

Using the same argument as in the proof of Proposition 2.1 we can also achieve that $B = 0$. Define operators M and N on $\mathcal{H}$ by

$$M = \begin{pmatrix} 0 & 0 & A \\ I & 0 & C \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 0 & D \\ 0 & 0 & 0 \\ 0 & I & 0 \end{pmatrix}.$$

It is easy to verify that $T = MN$ and that M and N are nilpotent operators with index of nilpotency at most 3. $\square$

We now consider the nilpotent analog of Theorem 3.3. Given operators $T_1, \ldots, T_n$ on $\mathcal{H}$, we are seeking nilpotent operators N_1 and N_2 and operators $S_1, \ldots, S_n$ on $\mathcal{H}$ such that $T_i = N_1 S_i N_2$ for all $i = 1, \ldots, n$. It is easy to see that the necessary condition for these factorizations is infinite-dimensionality of both $\ker(\sum_{i=1}^n T_i T_i^*)$ and $\ker(\sum_{i=1}^n T_i^* T_i)$. Note that $\ker(\sum_{i=1}^n T_i T_i^*) = \bigcap_{i=1}^n \ker T_i$ and $\ker(\sum_{i=1}^n T_i^* T_i) = \bigcap_{i=1}^n \ker T_i^*$. The following result shows that this condition is also sufficient.

Theorem 4.2. *Let $T_1, T_2, \ldots, T_n$ be operators in $\mathcal{B}(\mathcal{H})$ such that*

$$\dim \ker \left(\sum_{i=1}^n T_i T_i^* \right) = \dim \ker \left(\sum_{i=1}^n T_i^* T_i \right) = \infty.$$

Then the following assertions hold:

(a) *There exist nilpotent operators $N, N_1, N_2, \ldots, N_n$ on $\mathcal{H}$ with index of nilpotency at most 3 such that $T_i = N N_i N$ for all $i = 1, 2, \ldots, n$.*
(b) *There exist nilpotent operators N_1 and N_2 on $\mathcal{H}$ with index of nilpotency at most 3 and operators $S_1, \ldots, S_n$ on $\mathcal{H}$ such that $T_i = N_1 S_i N_2$ for all $i = 1, 2, \ldots, n$, and the products $N_1 S_i$ and $S_i N_2$ are nilpotent operators for all $i = 1, 2, \ldots, n$.*

Proof. We can choose a decomposition of $\mathcal{H}$ as a direct sum of infinite-dimensional subspaces $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ such that $\mathcal{H}_1 \subseteq \ker(\sum_{i=1}^n T_i^* T_i)$ and $\mathcal{H}_3 \subseteq \ker(\sum_{i=1}^n T_i T_i^*)$. Since $\ker(\sum_{i=1}^n T_i^* T_i) \subseteq \ker T_j^* T_j = \ker T_j$ and $\ker(\sum_{i=1}^n T_i T_i^*) \subseteq \ker T_j T_j^* = \ker T_j^*$ for all $j = 1, 2, \ldots, n$, the operators $\{T_i\}_{i=1}^n$ are of the form

$$T_i = \begin{pmatrix} 0 & A_i & B_i \\ 0 & C_i & D_i \\ 0 & 0 & 0 \end{pmatrix}.$$

As in the proof of Theorem 3.3 we can also achieve that $B_i = 0$ for all $i = 1, \ldots, n$.

In order to show part (a), we define operators N and $\{N_i\}_{i=1}^n$ on $\mathcal{H}$ by

$$N = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and } N_i = \begin{pmatrix} 0 & 0 & 0 \\ A_i & 0 & 0 \\ C_i & D_i & 0 \end{pmatrix}.$$

Clearly, all of them are nilpotent operators with index of nilpotency at most 3. Since $T_i = NN_iN$ for all $i = 1, 2, \ldots, n$, the proof of (a) is complete.

For part (b) we define nilpotent operators N_1 and N_2 on $\mathcal{H}$ (with index of nilpotency at most 3) and operators $\{S_i\}_{i=1}^n$ on $\mathcal{H}$ by

$$N_1 = \begin{pmatrix} 0 & 0 & I \\ I & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; N_2 = \begin{pmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & I & 0 \end{pmatrix} \quad \text{and} \quad S_i = \begin{pmatrix} D_i & 0 & C_i \\ 0 & 0 & 0 \\ 0 & 0 & A_i \end{pmatrix}.$$

It is easy to check that $T_i = N_1 S_i N_2$ for all $i = 1, 2, \ldots, n$, and products $N_1 S_i$ and $S_i N_2$ are nilpotent operators for all $i = 1, 2, \ldots, n$. This completes the proof of the theorem. $\square$

Acknowledgement. The research was supported by grants No. 04-2003-04 and No. BI-CZ/03-04-4, Programme Kontakt of Czech and Slovenian Ministeries of Education. The second author was also partially supported by grant No. 201/03/0041 of GA ČR and by the Institutional Research Plan AV0Z 10190503.

References

1. H. H. Anderson, J. G. Stampfli, *Commutators and compressions*, Israel J. Math. **19** (1971), 433–441.
2. J.B. Conway, *A course in functional analysis*, Springer-Verlag, 1990.
3. R. G. Douglas, *On majorization, factorization, and range of inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), 413–415.
4. C. K. Fong, A. R. Sourour, *Sums and products of quasi-nilpotent operators*, Proc. Roy. Soc. Edinburgh, **99A** (1984), 193–200.
5. I. Gohberg, S. Goldberg, M. A. Kaashoek, *Classes of Linear Operators, Vol. II*, Birkhäuser Verlag, Basel, 1993.
6. P. R. Halmos, V. S. Sunder, *Bounded Integral Operators on L^2 Spaces*, Springer-Verlag, Berlin, Heidelberg, New York, 1978.
7. V. Müller, *Spectral theory of linear operators and spectral systems in Banach algebras*, Operator Theory: Advances and Applications 139, Birkhäuser Verlag, Basel, 2003.
8. P. Y. Wu, *The operator factorization problems*, Lin. Alg. Appl. **117** (1989), 35–63.

Department of Mathematics, University of Ljubljana, Jadranska 19, SI-1000 Ljubljana, Slovenia, e-mail : roman.drnovsek@fmf.uni-lj.si

Mathematical Institute, Czech Academy of Sciences, Žitná 25, 115 67 Prague 1, Czech Republic, e-mail : muller@math.cas.cz

Department of Mathematics, University of Ljubljana, Jadranska 19, SI-1000 Ljubljana, Slovenia, e-mail : nika.novak@fmf.uni-lj.si