ON AN UPPER BOUND IN DIMENSION OF REFLEXIVITY CLOSURE

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ABSTRACT. Let $\mathcal{V}, \mathcal{W}$ be linear spaces over an algebraically closed field, and let $\mathcal{S}$ be an n–dimensional subspace of linear operators that map $\mathcal{V}$ into $\mathcal{W}$. We give a sharp upper bound for the dimension of the intersection of all images of nonzero operators from $\mathcal{S}$, namely $\dim\big(\bigcap_{A\in\mathcal{S}\setminus\{0\}} \operatorname{Im} A\big) \le \dim\mathcal{V} - n + 1$. As an application, we also give a sharp upper bound for the dimension of the reflexivity closure Ref $\mathcal{S}$ of $\mathcal{S}$, namely $\dim(\operatorname{Ref}\mathcal{S}) \le n(n+1)/2$.

1. Introduction and statements of the main results

1.1. **Intersection of images.** By definition, a field $\mathbb{F}$ is algebraically closed if every nonconstant polynomial with coefficient from $\mathbb{F}$ has a root if $\mathbb{F}$. A fundamental theorem of algebra states that $\mathbb{C}$, the field of complex numbers, is algebraically closed field. However, there are many others; one of the examples, important in number theory, is the field of algebraic numbers. This is in fact the smallest algebraically closed field that contains the integers.

In an algebraically closed field, every n–by–n matrix $A \in \mathcal{M}_n(\mathbb{F})$ has an eigenvalue. That is, at least one of the matrices $A + \lambda\,\mathrm{Id}$ is singular, as λ runs over all the scalars. There is an equivalent way of formulating this fact in terms of images of matrices: $\dim\bigcap_{\lambda\in\mathbb{F}} \operatorname{Im}(A+\lambda\,\mathrm{Id}) \le n-1$. We may even symmetrize the rôle of A and Id because $\operatorname{Im} X = \operatorname{Im}(\lambda_0 X)$ whenever $\lambda_0 \ne 0$ and because $\operatorname{Im}(X) \subseteq \operatorname{Im}(\mathrm{Id})$. So we derive yet another equivalent formulation:

$$\dim \bigcap_{(\lambda_0,\lambda)\in\mathbb{F}^2\setminus\{0\}} \operatorname{Im}(\lambda_0 A + \lambda\,\mathrm{Id}) \le n-1.$$

Moreover, we may replace the identity matrix Id by an arbitrary matrix B:

$$\dim \bigcap_{(\lambda_0,\lambda)\in\mathbb{F}^2\setminus\{0\}} \operatorname{Im}(\lambda_0 A + \lambda B) \le n-1.$$

This is clear if the matrix B is singular. If B is invertible, then the last formula for the pair (A, B) is equivalent to the formula for the pair $(B^{-1}A, \mathrm{Id})$.

The advantage of these formulas over the eigenvalue problem is that it allows us to work with rectangular, and not necessarily square matrices. Our main result below is the generalization of the above formula for more than two matrices.

1991 *Mathematics Subject Classification.* Primary 47L05 Secondary 15A03.

Key words and phrases. Determinantal varieties, Space of linear operators, Intersection of images, Reflexivity defect.

[1]Supported by grants Cncsis 54Gr/2007, Ancs CEx23/'05.

[1,2,3]Supported by a joint Czech-Slovene grant, BI-CZ/07-08-013.

[1,3]Supported by grant No. 201/06/0128 of GA CR.

Theorem 1.1. *Let $\mathbb{F} = \bar{\mathbb{F}}$ be algebraically closed field, let $n, m \geq 1$, and suppose $A_1, \dots, A_k \in \mathcal{M}_{m\times n}(\mathbb{C})$ are m–by–n matrices, with $1 \leq k \leq n+1$. Then,*

$$\dim \bigcap_{(\xi_1,\dots,\xi_k)\neq 0} \operatorname{Im}(\xi_1 A_1 + \dots + \xi_k A_k) \leq n - k + 1. \tag{1.1}$$

Remark 1.2. Although for m–by–n matrices, $\dim(\operatorname{Im} X) \leq \min\{m, n\}$ we cannot replace, in general, in (1.1) the right side with $\min\{m, n\} - k + 1$. We refer to the last section for more details.

Note that, if $A_1, \dots, A_k$ are linearly dependent, the formula (1.1) is automatically true, because the image of some linear combination of them is zero. Otherwise, $A_1, \dots, A_k$ span a k dimensional subspace of m–by–n matrices. So there is a more compact, but equivalent, version of Theorem 1.1:

Theorem 1.3. *Let $\mathbb{F} = \overline{\mathbb{F}}$, let $m, n \geq 1$ and $0 \leq k \leq n$. If $\mathcal{S} \subseteq \mathcal{M}_{m\times n}(\mathbb{F})$ is a subspace of dimension at least $(k+1)$ then*

$$\dim\Big(\bigcap_{A\in\mathcal{S}\setminus\{0\}} \operatorname{Im} A\Big) \leq n - k.$$

To prove Theorem 1.3, we will require a deep result from determinantal varieties. We state it in a form which resembles [4, Lemma 2.5]:

Lemma 1.4. *Let $\mathbb{F} = \bar{\mathbb{F}}$ be an algebraically closed field, let $0 \leq s \leq n-1$, and let $\mathcal{S} \subseteq \mathcal{M}_{(n-s+1)\times n}(\mathbb{F})$ be a linear subspace of dimension at least $(s+1)$. Then, $\mathcal{S}$ contains a nonzero matrix of rank at most $n - s$.*

Proof. Since $\mathcal{S} \setminus \{0\}$ is a projective space of dimension at least s, the statement follows by combining Proposition 11.4 and Proposition 12.2 of [8]. For an alternate proof we refer to Sylvester [11, Corollary I]; with his notation, one uses $m := n-s+1$. The conclusion is that the subspace of $(n-s+1)$–by–n matrices, such that each nonzero member has maximal rank, is of dimension at most $\ell(n-s+1, n-s+1, n) \leq n - (n-s+1) + 1 = s$. □

Proof of Theorem 1.3. We first make three reductions on $\mathcal{S}$ and then argue with contradiction. As for the first reduction, we may clearly assume $m \geq n$. Otherwise, when $m < n$, we would enlarge each member $A \in \mathcal{S}$, by adding zero rows, into $\hat{A} := \left(\begin{smallmatrix} A \\ 0 \end{smallmatrix}\right) \in \mathcal{M}_n(\mathbb{F})$. This procedure does not change the dimension of $\mathcal{S}$, nor does it change the dimension of the intersection of the images (because $\operatorname{Im} \hat{A} = (\operatorname{Im} A) \oplus 0_{n-m}$.)

As for the second reduction, we may also assume that $m = n$. Namely, if $m > n$ we will show that either $\bigcap_{A\in\mathcal{S}\setminus\{0\}} \operatorname{Im} A = 0$ or else we will construct a subspace $\mathcal{S}'$ of n–by–n matrices, with $\dim \mathcal{S}' = \dim \mathcal{S}$, but such that $\dim \bigcap_{A\in\mathcal{S}\setminus\{0\}} \operatorname{Im} A \leq \dim \bigcap_{A'\in\mathcal{S}'\setminus\{0\}} \operatorname{Im} A'$. To this end, we choose any nonzero member $A_0 \in \mathcal{S}$. Then, $r := \operatorname{rk} A_0 \leq \min\{m, n\}$ implies that there exists an invertible $S \in \mathcal{M}_m(\mathbb{F})$ (= a change of basis in $\mathbb{F}^m$) such that $S \operatorname{Im} A_0 = \mathbb{F}^r \oplus 0_{m-r} \subseteq \mathbb{F}^n \oplus 0_{m-n}$. Clearly, $S \operatorname{Im} A = \operatorname{Im}(SA)$, and together with invertibility of S we derive

$$S \bigcap_{A\in\mathcal{S}\setminus\{0\}} \operatorname{Im} A = \bigcap_{A\in\mathcal{S}\setminus\{0\}} \operatorname{Im}(SA) \subseteq \operatorname{Im}(SA_0) \subseteq \mathbb{F}^n \oplus 0. \tag{1.2}$$

We may identify $\mathbb{F}^n \oplus 0_{m-n}$ with $\mathbb{F}^n$. With this in mind, let $P : \mathbb{F}^m = \mathbb{F}^n \oplus \mathbb{F}^{m-n} \to \mathbb{F}^n$ be a projection along $0_n \oplus \mathbb{F}^{m-n}$. Then, in view of the last two containments in (1.2), and because the images of any function f obey $f\big(\bigcap_\omega \Omega_\omega\big) \subseteq \bigcap_\omega f(\Omega_\omega)$, we further have

$$\Big(\bigcap_{A\in\mathcal{S}\setminus\{0\}} \operatorname{Im}(SA)\Big) = P\Big(\bigcap_{A\in\mathcal{S}\setminus\{0\}} \operatorname{Im}(SA)\Big) \subseteq \bigcap_{A\in\mathcal{S}\setminus\{0\}} P\operatorname{Im}(SA) = \bigcap_{A\in\mathcal{S}\setminus\{0\}} \operatorname{Im}(PSA).$$

Now, if $\operatorname{Im}(PSA) = 0$ for some nonzero $A \in \mathcal{S}$, we are done. Otherwise, the map $A \mapsto PSA$ is a linear isomorphism from $\mathcal{S}$ onto $\mathcal{S}' := PS \cdot \mathcal{S} = \{PSA;\quad A \in \mathcal{S}\}$. Therefore, $\dim \mathcal{S}' = \dim \mathcal{S}$. Note also that $PSA : \mathbb{F}^n \to \mathbb{F}^n$ for $A \in \mathcal{S}$. So, if $m > n$ and the intersection of images is not zero, we would replace $\mathcal{S}$ by $\mathcal{S}'$. By doing so, we get the wanted subspace $\mathcal{S}' \subseteq \mathcal{M}_n(\mathbb{F})$, with dimension at least $k+1$, but such that

$$\dim \bigcap_{A\in\mathcal{S}\setminus\{0\}} \operatorname{Im} A = \dim \bigcap_{A\in\mathcal{S}\setminus\{0\}} \operatorname{Im}(SA) \le \dim \bigcap_{A'\in\mathcal{S}'\setminus\{0\}} \operatorname{Im} A'.$$

So it suffices to prove the theorem when $m = n$. We may further assume $k \ge 2$. Namely for $k = 0$ the claim is trivial, while for $k = 1$ Lemma 1.4 already implies that at least one nonzero member of $\mathcal{S} \subseteq \mathcal{M}_n(\mathbb{F})$ is not invertible (in fact it is a consequence of the nonemptiness of the spectrum, see the introduction). Hence, the dimension of its image is at most $n-1$ and we are done.

After these reductions we are ready to argue with contradiction. Assume on the contrary that a subspace $\mathcal{S} \subseteq \mathcal{M}_n(\mathbb{F})$, of dimension $\dim \mathcal{S} \le (k+1)$, satisfies $d := \dim \mathcal{W}' \ge n-k+1$, where

$$\mathcal{W}' := \bigcap_{A\in\mathcal{S}\setminus\{0\}} \operatorname{Im} A.$$

Using a similarity (that is, a change of basis) we may assume that $\mathcal{W}' = 0_{n-d} \oplus \mathbb{F}^d$. Then, $\mathcal{W}'' := \mathbb{F}^{k-1} \oplus 0_{n-k+1}$ is a subspace with $\mathcal{W}' + \mathcal{W}'' = \mathbb{F}^n$ (though the sum is direct only when $d = n-k+1$). Let $\mathsf{e}_1, \dots \mathsf{e}_{k-1} \in \mathcal{W}''$ be the standard basis of column vectors. We may now enlarge each member $A \in \mathcal{S}$ into an n–by–$(n+k-1)$ matrix

$$\widehat{A} := \Big[A \mid \mathsf{e}_1 \mid \cdots \mid \mathsf{e}_{k-1}\Big]$$

obtained by concatenating columns. Now, the decomposition $\mathbb{F}^n = \mathcal{W}'' \oplus \mathbb{F}^{n-k+1}$ gives

$$\widehat{A} = \begin{pmatrix} A' & \mathrm{Id}_{k-1} \\ A'' & 0 \end{pmatrix},$$

and so the rank of $\widehat{A}$ is equal to the rank of a reduced matrix

$$\begin{pmatrix} 0_{(k-1)\times n} & \mathrm{Id}_{k-1} \\ A'' & 0_{(n-k+1)\times(k-1)} \end{pmatrix}.$$

Note that $\mathcal{W}' \subseteq \operatorname{Im} A$ for every nonzero $A \in \mathcal{S}$. Since $\mathcal{W}' + \mathcal{W}'' = \mathbb{F}^n$, the columns of $\widehat{A}$ span the whole $\mathbb{F}^n$, and consequently, $\operatorname{rk} \widehat{A} = n$ for every nonzero $A \in \mathcal{S}$. In particular, this implies that $\operatorname{rk} A'' = n-(k-1)$ for every nonzero $A \in \mathcal{S}$. On one hand, since $n-(k-1) > 0$, the map $A \mapsto A''$ is a linear isomorphism from $\mathcal{S}$ onto a subspace $\mathcal{S}'' := \{A'';\quad A \in \mathcal{S}\} \subseteq \mathcal{M}_{(n-k+1)\times n}(\mathbb{F})$, so $\dim \mathcal{S}'' = \dim \mathcal{S} \ge (k+1)$. On the other hand, if $k = n$ then the space $\mathcal{S}''$ of 1–by–n matrices is at most n dimensional, and if $k < n$ then Lemma 1.4 implies that at least one nonzero member of $\mathcal{S}''$ must have rank $\le n-k$. A contradiction to $\operatorname{rk} A'' = n-(k-1)$. □

2. Application

Let $\mathcal{V}, \mathcal{W}$ be vector spaces over a field $\mathbb{F}$ and let $\mathcal{S} \subseteq \operatorname{Hom}(\mathcal{V}, \mathcal{W})$ be a subspace of $\mathbb{F}$-linear operators from $\mathcal{V}$ into $\mathcal{W}$. The reflexivity closure, $\operatorname{Ref}\mathcal{S}$ is the set of all linear operators $T \in \operatorname{Hom}(\mathcal{V}, \mathcal{W})$ such that for every $\mathbf{x} \in \mathcal{V}$ there exists some $S = S_{\mathbf{x}} \in \mathcal{S}$ with $T\mathbf{x} = S\mathbf{x}$. Equivalently, $T\mathbf{x} \in \mathcal{S}\mathbf{x} := \{S\mathbf{x},\ \ S \in \mathcal{S}\}$ for each $\mathbf{x}$. It is immediate that $\operatorname{Ref}\mathcal{S}$ is also a subspace and it contains $\mathcal{S}$. When $\dim \mathcal{S} < \infty$ we introduce the quantity $\operatorname{rd}\mathcal{S} := \dim(\operatorname{Ref}\mathcal{S}) - \dim \mathcal{S} = \dim((\operatorname{Ref}\mathcal{S})/\mathcal{S})$, which measures how much larger $\operatorname{Ref}\mathcal{S}$ is compared to $\mathcal{S}$. Following Delai [6], we call this integer a *reflexivity defect* of $\mathcal{S}$.

On the one extreme, it may happen that $\operatorname{Ref}\mathcal{S} = \mathcal{S}$. Such spaces are called reflexive, and have been extensively studied [3, 7, 9, 10]....

Example 2.1. If $\mathcal{S} = \operatorname{Lin}\{T\}$ is a one-dimensional subspace then the elementary exercise validates $\operatorname{Ref}\mathcal{S} = \mathcal{S}$, so $\operatorname{rd}\mathcal{S} = 0$. More generally, $\operatorname{rd}\mathcal{S} = 0$ whenever $\operatorname{Ref}\mathcal{S} = \mathcal{S}$, that is, whenever $\mathcal{S}$ is reflexive.

But on the other extreme, it may happen that the space is far from being reflexive.

Example 2.2. The ideal $\mathcal{F} \subseteq \mathcal{B}(\mathcal{X})$ of finite-rank operators on some Banach space $\mathcal{X}$ is transitive, that is, $\mathcal{F}\mathbf{x} = \mathcal{X}$ for every nonzero vector $\mathbf{x} \in \mathcal{X}$. It follows that $\operatorname{Ref}\mathcal{F} = \mathcal{B}(\mathcal{X})$. Inversely, if $\operatorname{Ref}\mathcal{S} = \mathcal{B}(\mathcal{X})$ for some subspace of operators, then $\mathcal{S}$ must be transitive. Therefore, transitive spaces are as far from being reflexive, as one can hope for.

However, there are also subspaces which are neither reflexive nor transitive.

Example 2.3. Let $\mathsf{e}_1, \dots, \mathsf{e}_n$ be a standard basis of column vectors in $\mathcal{V} := \mathbb{F}^n$. With respect to this basis, $\operatorname{Hom}(\mathbb{F}^n, \mathbb{F}^n)$ can be identified with $\mathcal{M}_n(\mathbb{F})$, the algebra of n–by–n matrices. Let $N \in \mathcal{M}_n(\mathbb{F})$ be an elementary upper-triangular Jordan nilpotent, and let $\mathcal{S} := \operatorname{Lin}\{\operatorname{Id}, N, \dots, N^{n-1}\} \subseteq \mathcal{M}_n(\mathbb{F})$. Then it follows from [5, Theorem 4.3], with $q(z) := z$ and $(n_1, n_2) = (n, 0)$, that $\operatorname{Ref}\mathcal{S}$ consists of all upper-triangular matrices. Thus, $\operatorname{rd}\mathcal{S} = n(n+1)/2 - n$.

It is our aim to show that, for algebraically closed fields, the reflexivity defect is always bounded above by $n(n+1)/2 - n$, where $n := \dim \mathcal{S}$ — see Theorem 2.7 below. Example 2.3 shows that this estimate is sharp. Before giving a proof, however, we introduce the following notation. Given any subspace $\mathcal{U} \subseteq \mathcal{V}$, and $\mathcal{S} \subseteq \operatorname{Hom}(\mathcal{V}, \mathcal{W})$, we let $\mathcal{S}|_{\mathcal{U}} := \{T|_{\mathcal{U}};\ \ T \in \mathcal{S}\}$ be the set of all restrictions of operators from $\mathcal{S}$. Recall that $T|_{\mathcal{U}} : \mathcal{U} \to \mathcal{W}$.

We start with a trivial observation.

Lemma 2.4. *If* $\mathcal{S} \subseteq \operatorname{Hom}(\mathcal{V}, \mathcal{W})$ *and* $\mathcal{U} \subseteq \mathcal{V}$ *is a subspace then* $(\operatorname{Ref}\mathcal{S})|_{\mathcal{U}} \subseteq \operatorname{Ref}(\mathcal{S}|_{\mathcal{U}})$.

Proof. Immediate. □

Lemma 2.5. *Suppose* $\mathcal{O} \subseteq \operatorname{Hom}(\mathcal{V}, \mathcal{W})$ *is a finite-dimensional subspace of linear operator from* $\mathcal{V}$ *into* $\mathcal{W}$. *If* $\mathcal{U} \subseteq \mathcal{V}$ *is a subspace then*

$$\dim \mathcal{O} = \dim(\mathcal{O}|_{\mathcal{U}}) + \dim\{T \in \mathcal{O};\ \ T|_{\mathcal{U}} = 0\}.$$

In addition, if $\mathcal{U} = \hat{\mathcal{U}} + \operatorname{Lin}\{\mathbf{x}\}$ *with* $\mathcal{O}|_{\hat{\mathcal{U}}} = 0$ *then*

$$\dim \mathcal{O} = \dim(\mathcal{O}\mathbf{x}) + \dim\{T \in \mathcal{O};\ \ T|_{\mathcal{U}} = 0\}.$$

Proof. Let $T_1, \dots, T_n \in \mathcal{O}$ be such that their restrictions $T_1|_{\mathcal{U}}, \dots, T_n|_{\mathcal{U}}$ form a basis for $\mathcal{O}|_{\mathcal{U}}$, and let $U_1, \dots, U_m \in \mathcal{O}$ be a basis of $\{T \in \mathcal{O};\ \ T|_{\mathcal{U}} = 0\}$. To prove the first part, it suffices to verify that $T_1, \dots, T_n, U_1, \dots, U_m$ form a basis of $\mathcal{O}$.

Pick any $T \in \mathcal{O}$, and consider its restriction to $\mathcal{U}$. There exist scalars $\lambda_1, \dots, \lambda_n$ such that $T|_{\mathcal{U}} = \lambda_1 T_1|_{\mathcal{U}} + \cdots + T_n|_{\mathcal{U}}$. Then, $T - \sum \lambda_i T_i \in \{T \in \mathcal{O};\ \ T|_{\mathcal{U}} = 0\}$. In particular, there exist scalars $\mu_1, \dots, \mu_m$ such that $T - \sum \lambda_i T_i = \mu_1 U_1 + \cdots + \mu_m U_m$. Wherefrom, $T = \sum \lambda_i T_i + \sum \mu_j U_j$.

It remains to show linear independence of the operators $T_1, \dots, T_n, U_1, \dots, U_m$. Let the scalars $\alpha_1, \dots, \alpha_n,\ \beta_1, \dots, \beta_m \in \mathbb{F}$ satisfy $\sum_{i=1}^n \alpha_i T_i + \sum_{j=1}^m \beta_j U_j = 0$. Then $\sum_{i=1}^n \alpha_i T|_{\mathcal{U}} = 0$, and so $\alpha_1 = \cdots = \alpha_n = 0$. Thus $\sum_{j=1}^m \beta_j U_j = 0$, and so $\beta_1 = \cdots = \beta_m = 0$ since the operators $U_1, \dots, U_m$ are linearly independent. This proves the first statement.

The second equality follows from the first one by noticing that $T|_{\mathcal{U}} \mapsto T\mathbf{x}$ is an isomorphism between $\mathcal{O}|_{\mathcal{U}}$ and $\mathcal{O}\mathbf{x}$. $\square$

We will also require the following general lemma:

Lemma 2.6. *Let $r \in \mathbb{N}$, let $\mathcal{X}, \mathcal{Y}$ be vector spaces over a commutative field $\mathbb{F}$ with $|\mathbb{F}| \ge r + 3$, and let $\mathcal{O} \subseteq \mathrm{Hom}(\mathcal{X}, \mathcal{Y})$ be a finite-dimensional subspace. Suppose the vectors $\mathbf{x}, \mathbf{x}' \in \mathcal{X}$ satisfy $r := \dim \mathcal{O}\mathbf{x} \ge \dim \mathcal{O}(\mathbf{x} + \lambda\mathbf{x}')$ for $\lambda \in \mathbb{F}$. If $\mathbf{y} \in \mathcal{O}(\mathbf{x} + \lambda\mathbf{x}')$ for each nonzero $\lambda \in \mathbb{F} \setminus \{0\}$ then also $\mathbf{y} \in \mathcal{O}\mathbf{x}$.*

Proof. There is nothing to prove when $r = 0$.

Assume $r \ge 1$. Let $S_1, \dots, S_n$ be a basis for $\mathcal{O}$. If necessary we re-index this basis such that the first r vectors $S_1\mathbf{x}, \dots, S_r\mathbf{x}$ are linearly independent, while $S_{r+1}\mathbf{x}, \dots, S_n\mathbf{x}$ are their linear combinations.

Clearly, the vectors $S_1\mathbf{x}, \dots, S_n\mathbf{x},\ S_1\mathbf{x}', \dots, S_n\mathbf{x}'$ span the finite-dimensional subspace $\mathcal{O}\mathbf{x} + \mathcal{O}\mathbf{x}' \subseteq Y$, and $\mathcal{O}(\mathbf{x} + \lambda\mathbf{x}') \subseteq \mathcal{O}\mathbf{x} + \mathcal{O}\mathbf{x}'$ for every λ. Choose and fix an arbitrary basis of $\mathcal{O}\mathbf{x} + \mathcal{O}\mathbf{x}'$. With respect to this basis, we may identify $\mathcal{O}\mathbf{x} + \mathcal{O}\mathbf{x}'$ with $\mathbb{F}^d$, for some $d \ge r$. By doing so, we may assume that $\mathbf{y},\ S_1\mathbf{x}, \dots, S_n\mathbf{x},\ S_1\mathbf{x}', \dots, S_n\mathbf{x}'$ are already column vectors from $\mathbb{F}^d$.

Construct a $d \times (n+1)$ matrix

$$\Xi(\lambda) := \Big[\mathbf{y} \mid S_1(\mathbf{x} + \lambda\mathbf{x}') \mid \cdots \mid S_n(\mathbf{x} + \lambda\mathbf{x}')\Big]$$

by concatenating the column vectors one after another. Now, by assumptions, $\mathbf{y} \in \mathcal{O}(\mathbf{x} + \lambda\mathbf{x}')$, so $\mathbf{y}$ is a linear combination of $S_1(\mathbf{x} + \lambda\mathbf{x}'), \dots, S_n(\mathbf{x} + \lambda\mathbf{x}')$, for every $\lambda \ne 0$. Moreover, due to $r = \dim \mathcal{O}\mathbf{x} \ge \dim \mathcal{O}(\mathbf{x} + \lambda\mathbf{x}')$, there are at most r linearly independent vectors among $S_1(\mathbf{x} + \lambda\mathbf{x}'), \dots, S_n(\mathbf{x} + \lambda\mathbf{x}')$.

Equivalently stated, $\mathrm{rk}\,\Xi(\lambda) \le r$ for every $\lambda \ne 0$. So, every $(r+1) \times (r+1)$ minor of $\Xi(\lambda)$ is identically zero, for $\lambda \ne 0$. But note that $S_i(\mathbf{x} + \lambda\mathbf{x}') = S_i\mathbf{x} + \lambda S_i\mathbf{x}'$ implies that these minors are polynomials in variable λ of degree at most $r+1$. Since they vanish for $\lambda \ne 0$, and the field $\mathbb{F}$ has at least $r+2$ nonzero elements, every $(r+1) \times (r+1)$ minor is a zero polynomial. Therefore, they also vanish at $\lambda = 0$, which gives $\mathrm{rk}\,\Xi(\lambda)|_{\lambda=0} \le r$.

By assumptions, $S_1\mathbf{x}, \dots, S_r\mathbf{x}$ are linearly independent, which means that the second, third, $\dots (r+1)$-th column of $\Xi(0)$ are also linearly independent. But then, $\mathrm{rk}\,\Xi(0) \le r$ implies that the first column of $\Xi(0)$, that is the vector $\mathbf{y}$, must be a linear combination of the vectors $S_1\mathbf{x}, \dots, S_r\mathbf{x}$. Equivalently, $\mathbf{y} \in \mathrm{Lin}\{S_1\mathbf{x}, \dots, S_r\mathbf{x}\} = \mathcal{O}\mathbf{x}$. $\square$

We can now prove our main result of this section.

Theorem 2.7. *Suppose that $\mathcal{V},\mathcal{W}$ are vector spaces over an algebraically closed field, and let $\mathcal{S}\subseteq \operatorname{Hom}(\mathcal{V},\mathcal{W})$ be a finite-dimensional subspace of operators from $\mathcal{V}$ to $\mathcal{W}$. Then,*

$$\dim(\operatorname{Ref}\mathcal{S})\le\frac{(\dim\mathcal{S})(1+\dim\mathcal{S})}{2}$$

Proof. To shorten the arguments we write $n:=\dim\mathcal{S}$. We first verify the claim for the restriction of $\mathcal{S}$ to finite-dimensional vector subspaces of $\mathcal{V}$.

So suppose $\mathcal{V}_k\subseteq\mathcal{V}$ is a subspace of dimension k, and consider $\mathcal{B}_0:=\operatorname{Ref}(\mathcal{S}|_{\mathcal{V}_k})$. Fix a vector $\mathbf{x}_1\in\mathcal{V}_k$ such that $\dim\mathcal{B}_0\mathbf{x}_1=\max_{\mathbf{x}\in\mathcal{V}_k}\dim\mathcal{B}_0\mathbf{x}$ is maximal. By the definition of reflexive closure, $\mathcal{B}_0\mathbf{x}\subseteq\mathcal{S}\mathbf{x}$ for every $\mathbf{x}\in\mathcal{V}_k$, giving $\dim\mathcal{B}_0\mathbf{x}\le n$. Now, let $\mathcal{B}_1:=\{A\in\mathcal{B}_0;\ \ A\mathbf{x}_1=0\}$. We next construct inductively vectors $\mathbf{x}_2,\mathbf{x}_3,\dots,\mathbf{x}_k\in\mathcal{V}_k$ and subspaces $\mathcal{B}_2,\dots,\mathcal{B}_k\subseteq\mathcal{B}_0$ such that $\dim\mathcal{B}_{i-1}\mathbf{x}_i=\max_{\mathbf{x}\in\mathcal{V}_k}\dim\mathcal{B}_{i-1}\mathbf{x}$ and $\mathcal{B}_i:=\{A\in\mathcal{B}_0;\ \ A\mathbf{x}_1=\dots=A\mathbf{x}_i=0\}$. Clearly we may assume that the vectors $\mathbf{x}_1,\dots,\mathbf{x}_k$ are linearly independent, so they form a basis of $\mathcal{V}_k$. Then we have $\mathcal{B}_0\supseteq\mathcal{B}_1\supseteq\dots\supseteq\mathcal{B}_k=\{0\}$. Moreover, the operators from $\mathcal{B}_0$ are determined by prescribing their values on basis elements of $\mathcal{V}_k$, so that

$$\dim\mathcal{B}_i\le\dim\mathcal{B}_0\le\dim\mathcal{V}_k\cdot\max_{\mathbf{x}\in\mathcal{V}_k}\dim\mathcal{S}\mathbf{x}\le kn<\infty.$$

We proceed by showing that $\mathcal{B}_{i-1}\mathbf{x}_i\subseteq\mathcal{B}_{i-2}\mathbf{x}_{i-1}$, $(i\ge 2)$. Let $i\ge 2$ and let $A\in\mathcal{B}_{i-1}$ be arbitrary. For each $\lambda\in\mathbb{F}\setminus\{0\}$ we have

$$A\mathbf{x}_i=A(\lambda^{-1}\mathbf{x}_{i-1}+\mathbf{x}_i)\in\mathcal{B}_{i-1}(\lambda^{-1}\mathbf{x}_{i-1}+\mathbf{x}_i)=\mathcal{B}_{i-1}(\mathbf{x}_{i-1}+\lambda\mathbf{x}_i)\subseteq\mathcal{B}_{i-2}(\mathbf{x}_{i-1}+\lambda\mathbf{x}_i);\qquad(\lambda\neq 0).$$

Since $\dim\mathcal{B}_{i-2}\mathbf{x}_{i-1}=\max_{\mathbf{x}\in\mathcal{V}_k}\dim\mathcal{B}_{i-2}\mathbf{x}$, and as algebraically closed fields have infinite cardinality, Lemma 2.6 for $\mathcal{O}:=\mathcal{B}_{i-2}$ and $\mathbf{y}:=A\mathbf{x}_i$ indeed gives $A\mathbf{x}_i\in\mathcal{B}_{i-2}\mathbf{x}_{i-1}$, as anticipated.

We now claim that

$$\mathcal{B}_{s-1}\mathbf{x}_s\subseteq\bigcap_{(\xi_1,\dots,\xi_s)\in\mathbb{F}^s\setminus\{0\}}\mathcal{S}\Big(\sum_{i=1}^{s}\xi_i\mathbf{x}_i\Big);\qquad(s\ge 1).\tag{2.1}$$

When $s=1$ this follows from the definition of reflexivity closure. So assume $s\ge 2$, and let $A\in\mathcal{B}_{s-1}$. Choose any nontrivial s-tuple $(\xi_1,\dots,\xi_s)\in\mathbb{F}^s\setminus\{0\}$, and let j be the last index with $\xi_j\neq 0$; so $\xi_{j+1}=0=\dots=\xi_s$. Now,

$$A\mathbf{x}_s\in\mathcal{B}_{s-1}\mathbf{x}_s\subseteq\mathcal{B}_{s-2}\mathbf{x}_{s-1}\subseteq\dots\subseteq\mathcal{B}_{j-1}\mathbf{x}_j=\mathcal{B}_{j-1}\Big(\mathbf{x}_j+\sum_{i=1}^{j-1}\xi_j^{-1}\xi_i\mathbf{x}_i\Big).$$

Since $\mathcal{B}_{j-1}\subseteq\operatorname{Ref}(\mathcal{S}|_{\mathcal{V}_k})$, we further have, by the definition of the reflexivity closure:

$$\mathcal{B}_{j-1}\Big(\mathbf{x}_j+\sum_{i=1}^{j-1}\xi_j^{-1}\xi_i\mathbf{x}_i\Big)\subseteq\mathcal{S}\Big(\mathbf{x}_j+\sum_{i=1}^{j-1}\xi_j^{-1}\xi_i\mathbf{x}_i\Big)=\mathcal{S}\Big(\sum_{i=1}^{j}\xi_i\mathbf{x}_i\Big),$$

and since $A\in\mathcal{B}_{s-1}$ was arbitrary, we deduce (2.1).

Fix a basis $S_1,\dots,S_n$ of $\mathcal{S}$. Then, $\widehat{\mathcal{W}}:=\mathcal{S}\mathcal{V}_k=\operatorname{Lin}\{S_i\mathbf{x}_j;\ \ 1\le i\le n,\ 1\le j\le k\}$ is a finite-dimensional subspace of $\mathcal{W}$. Actually, its dimension, $m:=\dim\widehat{\mathcal{W}}$ satisfies $m\le kn$. So we may identify $\widehat{\mathcal{W}}$ with $\mathbb{F}^m$ and associate to each vector $\mathbf{x}_j$ the m–by–n matrix, given by the columns

$$\mathfrak{S}_j:=\big[S_1\mathbf{x}_j|S_2\mathbf{x}_j|\cdots|S_n\mathbf{x}_j\big].\tag{2.2}$$

Given a vector $\mathbf{x} = \xi_1\mathbf{x}_1 + \cdots + \xi_j\mathbf{x}_j$, it is immediate that $\mathcal{S}\mathbf{x} = \mathrm{Im}(\xi_1\mathfrak{S}_1 + \cdots + \xi_j\mathfrak{S}_j)$. Consequently, we can restate (2.1) as

$$\dim \mathcal{B}_{s-1}\mathbf{x}_s \leq \dim \bigcap_{(\xi_1,\ldots,\xi_s)\in\mathbb{F}^s\setminus\{0\}} \mathcal{S}\Big(\sum_{i=1}^{s} \xi_i\mathbf{x}_i\Big) = \dim \bigcap_{(\xi_1,\ldots,\xi_s)\in\mathbb{F}^s\setminus\{0\}} \mathrm{Im}(\xi_1\mathfrak{S}_1 + \cdots + \xi_s\mathfrak{S}_s).$$

By Theorem 1.1,

$$\dim \mathcal{B}_{s-1}\mathbf{x}_s \leq \dim \bigcap_{(\xi_1,\ldots,\xi_s)\in\mathbb{F}^s\setminus\{0\}} \mathrm{Im}(\xi_1\mathfrak{S}_1 + \cdots + \xi_s\mathfrak{S}_s) \leq n - s + 1, \tag{2.3}$$

wherefrom, with a repeated use of Lemma 2.5 on a nest of subspaces $\mathcal{U}_j := \mathrm{Lin}\{\mathbf{x}_1, \ldots, \mathbf{x}_j\} \subseteq \mathcal{V}_k$:

$$\begin{aligned}
\dim \mathrm{Ref}(\mathcal{S}|_{\mathcal{V}_k}) &= \dim \mathcal{B}_0 \leq \dim(\mathcal{B}_0|_{\mathrm{Lin}\{\mathbf{x}_1\}}) + \dim \mathcal{B}_1 \\
&\leq \dim(\mathcal{B}_0|_{\mathrm{Lin}\{\mathbf{x}_1\}}) + \dim(\mathcal{B}_1|_{\mathrm{Lin}\{\mathbf{x}_1,\mathbf{x}_2\}}) + \dim \mathcal{B}_2 \leq \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots \\
&\leq \dim(\mathcal{B}_0|_{\mathrm{Lin}\{\mathbf{x}_1\}}) + \dim(\mathcal{B}_1|_{\mathrm{Lin}\{\mathbf{x}_1,\mathbf{x}_2\}}) + \cdots + \dim(\mathcal{B}_{k-1}|_{\{\mathbf{x}_1,\ldots,\mathbf{x}_k\}}) + \dim \mathcal{B}_k = \\
&\leq \dim \mathcal{B}_0\mathbf{x}_1 + \dim \mathcal{B}_1\mathbf{x}_2 + \cdots + \dim \mathcal{B}_{k-1}\mathbf{x}_k + \dim \mathcal{B}_k \\
&\leq \begin{cases} n + (n-1) + \cdots + (n-k+1) + 0; & k \leq n \\ n + (n-1) + \cdots + 1 + 0; & k \geq n \end{cases} \leq \frac{n(n+1)}{2}.
\end{aligned} \tag{2.4}$$

By Lemma 2.4, and in view of (2.4), $\dim\big((\mathrm{Ref}\,\mathcal{S})|_{\mathcal{V}_k}\big) \leq \dim \mathrm{Ref}(\mathcal{S}|_{\mathcal{V}_k}) \leq n(n+1)/2$ holds for every finite dimensional subspace $\mathcal{V}_k \subseteq \mathcal{V}$. Therefore, $\dim(\mathrm{Ref}\,\mathcal{S}) \leq n(n+1)/2$. □

3. Non-algebraically-closed fields

The estimate in Theorem 2.7 is not true for non-algebraically closed field.

Example 3.1. Consider the 2-dimensional subspace $\mathcal{S}$ of $\mathcal{M}_2(\mathbb{R})$ generated by the matrices

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is easy to check that for each nonzero vector $\mathbf{x}$ the vectors $A_1\mathbf{x}$ and $A_2\mathbf{x}$ are linearly independent and therefore span all $\mathbb{R}^2$. Hence $\mathrm{Ref}\,\mathcal{S} = \mathcal{M}_2(\mathbb{R})$ and $\dim \mathrm{Ref}\,\mathcal{S} = 4$ (in the complex case $\dim \mathcal{S} = 2$ implies $\dim \mathrm{Ref}\,\mathcal{S} \leq 3$, by Theorem 2.7).

We show that for any field we have $\dim \mathrm{Ref}\,\mathcal{S} \leq (\dim \mathcal{S})^2$. First we need the following reduction:

Lemma 3.2. *Let $n \in \mathbb{N}$, let $\mathbb{F}$ be a field satisfying $|\mathbb{F}| \geq n+3$. Let $\mathcal{V}, \mathcal{W}$ be vector spaces over $\mathbb{F}$, let $\mathcal{S}$ be an n-dimensional subspace of $\mathrm{Hom}(\mathcal{V}, \mathcal{W})$. Then there exist vector spaces $\mathcal{W}' \subseteq \mathcal{W}$ and $\mathcal{S}' \subseteq \mathrm{Hom}(\mathcal{V}, \mathcal{W}')$ such that $\dim \mathcal{W}' \leq n$, $\dim \mathcal{S}' \leq n$ and $\dim \mathrm{Ref}\,\mathcal{S}' \geq \dim \mathrm{Ref}\,\mathcal{S}$.*

Proof. Fix a vector $\mathbf{x} \in \mathcal{V}$ such that the dimension $\dim \mathcal{S}\mathbf{x}$ is maximal. Set $\mathcal{W}' = \mathcal{S}\mathbf{x}$. Clearly $\dim \mathcal{W}' \le n$.

Fix a projection $P : \mathcal{W} \to \mathcal{W}$ with $\operatorname{Im} P = \mathcal{W}'$. Let $\mathcal{S}' = P\mathcal{S} = \{PA;\ \ A \in \mathcal{S}\}$. Then $\mathcal{S}' \subseteq \operatorname{Hom}(\mathcal{V}, \mathcal{W}')$ and $\dim \mathcal{S}' \le \dim \mathcal{S} = n$.

Let $A \in \operatorname{Ref} \mathcal{S}$ and $A\mathbf{x} = 0$. We show that $\operatorname{Im} A \subseteq \mathcal{W}'$. Indeed, let $\mathbf{x}' \in \mathcal{V}$ be arbitrary. For each nonzero $\lambda \in \mathbb{F}$ we have

$$A\mathbf{x}' = A(\mathbf{x}' + \lambda^{-1}\mathbf{x}) \in \mathcal{S}(\mathbf{x}' + \lambda^{-1}\mathbf{x}) = \mathcal{S}(\mathbf{x} + \lambda\mathbf{x}').$$

By Lemma 2.6, we have $A\mathbf{x}' \in \mathcal{S}\mathbf{x} = \mathcal{W}'$.

We have just proved that $A \in \operatorname{Ref} \mathcal{S}$ and $A\mathbf{x} = 0$ imply $A = PA \in \mathcal{S}'$. Consequently,

$$\begin{aligned} \dim \operatorname{Ref} \mathcal{S} &= \dim \mathcal{S}\mathbf{x} + \dim\{A \in \operatorname{Ref} \mathcal{S};\ \ A\mathbf{x} = 0\} \\ &\le \dim \mathcal{S}'\mathbf{x} + \dim\{B \in \operatorname{Ref} \mathcal{S}';\ \ B\mathbf{x} = 0\} = \dim \operatorname{Ref} \mathcal{S}'. \end{aligned}$$

□

Theorem 3.3. *Let $n \in \mathbb{N}$, let $\mathbb{F}$ be a field satisfying $|\mathbb{F}| \ge n + 3$. Let $\mathcal{V}, \mathcal{W}$ be vector spaces over $\mathbb{F}$, let $\mathcal{S}$ be an n-dimensional subspace of* $\operatorname{Hom}(\mathcal{V}, W)$. *Then* $\dim \operatorname{Ref} \mathcal{S} \le n^2$.

Proof. Suppose on the contrary that $\dim \operatorname{Ref} \mathcal{S} > n^2$. By Lemma 3.2, we may assume that $\dim \mathcal{W} \le n$. Consider the space $\mathcal{S}^* = \{A^*;\ \ A \in \mathcal{S}\} \subseteq \operatorname{Hom}(\mathcal{W}^*, \mathcal{V}^*)$. Then $\dim \mathcal{S}^* = \dim \mathcal{S} = n$ and $\dim \operatorname{Ref} \mathcal{S}^* = \dim(\operatorname{Ref} \mathcal{S})^* > n^2$ (see [5, Proposition 2.1]). Also by the previous lemma, there exist subspaces $\mathcal{V}' \subseteq \mathcal{V}^*$ and $\mathcal{S}' \subseteq \operatorname{Hom}(\mathcal{W}^*, \mathcal{V}')$ such that $\dim \mathcal{S} \le n$ and $\dim \operatorname{Ref} \mathcal{S}' > n^2$. This is a contradiction since $\dim \operatorname{Hom}(\mathcal{W}^*, \mathcal{V}') = \dim \mathcal{W}^* \dim \mathcal{V}' \le n^2$. □

It is perhaps worth noting that the only place in the proof of Theorem 2.7, where we needed that the field is algebraically closed, was in the estimates (2.3) and (2.4). In all other places the arguments demand only $|\mathbb{F}| \ge 3 + \dim \mathcal{B}_{i-2}\mathbf{x}_{i-1}$ when invoking Lemma 2.6 to show that $\mathcal{B}_{i-1}\mathbf{x}_i \subseteq \mathcal{B}_{i-2}\mathbf{x}_{i-1}$. However, $\dim \mathcal{B}_{i-2}\mathbf{x}_{i-1} = \max_{\mathbf{x} \in \mathcal{V}_k} \dim \mathcal{B}_{i-2}\mathbf{x} \le \max_{\mathbf{x} \in \mathcal{V}_k} \dim \mathcal{B}_0\mathbf{x} = \dim \mathcal{S}\mathbf{x}_1 \le n$, so we only need $|\mathbb{F}| \ge n + 3$. To appreciate the extra information, we use the notation from the above proof, denote by $\vec{\mathbf{x}} := (\mathbf{x}_1, \ldots, \mathbf{x}_k)$ a basis for $\mathcal{V}_k$, and introduce subspaces

$$\overset{\vec{\mathbf{x}}}{\mathfrak{M}}_{s-1} := \bigcap_{(\alpha_1,\ldots,\alpha_s) \in \mathbb{F}^s \setminus \{0\}} \operatorname{Im}(\alpha_1 \mathfrak{S}_1 + \cdots + \alpha_s \mathfrak{S}_s), \qquad (s = 1, \ldots, k).$$

Recall that the m–by–n matrices $\mathfrak{S}_j$ were introduced in (2.2). We can now record the following corollary.

Corollary 3.4. *Let $n \in \mathbb{N}$ and let $\mathcal{V}, \mathcal{W}$ be finite-dimensional vector spaces over a field with $|\mathbb{F}| \ge n + 3$. Suppose $\mathcal{S} \subseteq \operatorname{Hom}(\mathcal{V}, \mathcal{W})$ is an n-dimensional subspace. Then, there exists a basis $\vec{\mathbf{x}} := (\mathbf{x}_1, \ldots, \mathbf{x}_k)$ of $\mathcal{V}$ such that*

$$\dim \operatorname{Ref} \mathcal{S} \le \dim \overset{\vec{\mathbf{x}}}{\mathfrak{M}}_0 + \dim \overset{\vec{\mathbf{x}}}{\mathfrak{M}}_1 + \cdots + \dim \overset{\vec{\mathbf{x}}}{\mathfrak{M}}_k \le \dim \mathcal{S} + \dim \overset{\vec{\mathbf{x}}}{\mathfrak{M}}_1 + \cdots + \dim \overset{\vec{\mathbf{x}}}{\mathfrak{M}}_k.$$

Corollary 3.5. *Under the notations from the previous corollary, suppose $\dim \overset{\vec{\mathbf{x}}}{\mathfrak{M}}_1 = 0$ for any basis vectors $\vec{\mathbf{x}}$ of $\mathcal{V}$. Then, the space $\mathcal{S}$ is reflexive.*

Proof. This is evident from the previous corollary plus the fact that $0 = \overset{\vec{\mathbf{x}}}{\mathfrak{M}}_1 \supseteq \overset{\vec{\mathbf{x}}}{\mathfrak{M}}_2 \supseteq \cdots \supseteq \overset{\vec{\mathbf{x}}}{\mathfrak{M}}_k$. □

Remark 3.6. Let $\mathcal{V}, \mathcal{W}$ be real vector spaces and $\mathcal{S} \subseteq \mathrm{Hom}(\mathcal{V}, \mathcal{W})$ a finite-dimensional subspace, $\dim \mathcal{S} = n$. By Theorem 3.3, $\dim \mathrm{Ref}\, \mathcal{S} \le n^2$. For $n = 2$ the estimate is the best possible, see Example 3.1. The same is true for $n = 4$ and $n = 8$ (the main reason is that in these cases there are n square matrices of order n such that each nontrivial linear combination of them is invertible, see [1] and [2]). However, $n = 2, 4, 8$ are the only cases when the estimate $\dim \mathrm{Ref}\, \mathcal{S} \le n^2$ is the optimal, since for other values of n such a system of n matrices does not exist. For example, for $n = 3$ each 3×3 matrix has an eigenvalue and it is easy to show that $\dim \mathrm{Ref}\, \mathcal{S} \le 7$.

Problem: What is the optimal estimate for $\dim \mathrm{Ref}\, \mathcal{S}$ in the real case?

Remark 3.7. We conclude with another question: What is the smallest possible transitive subspace in $\mathcal{M}_n(\mathbb{F})$? Our results for algebraically closed fields imply that it must have dimension at least $\dim \mathcal{S} \ge \frac{1}{2}\left(-1 + \sqrt{8n^2 + 1}\right)$— this follows because a subspace is transitive if $\mathrm{Ref}\, \mathcal{S} = \mathcal{M}_n(\mathbb{F})$. Comparing the dimensions gives $n^2 = \dim \mathrm{Ref}\, \mathcal{S} \le \dim \mathcal{S}(\dim \mathcal{S} + 1)/2$.

4. Examples

Here we provide several examples to illuminate our results. Firstly, it would be tempting to conjecture the more 'natural' formula $\dim \bigcap_{(\xi_1,\dots,\xi_k)\neq 0} \mathrm{Im}(\xi_1 A_1 + \cdots + \xi_k A_k) \le \min\{m, n\} - k + 1$ in place of (1.1). But this is wrong, in general.

Example 4.1. Consider the 2–by–4 matrices

$$A_0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad A_1 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to see that $\mathrm{Im}(\xi_0 A_0 + \xi_1 A_1) = \mathrm{Lin}\{(1,0)^t, (0,1)^t\}$ for every nonzero linear combination. Hence, the intersection of images has dimension 2. However, $\min\{m, n\} - k + 1 = 2 - 2 + 1 = 1$.

The next example shows that if $\mathcal{S} \subseteq \mathcal{M}_{m\times n}(\mathbb{C})$ and $\dim \mathcal{S} = k$, but $m \neq n - k + 1$, then in general there exists no nonzero matrix $A \in \mathcal{S}$ with rank $\le n - k + 1$ (c.f. Lemma 1.4).

Example 4.2. Let

$$\mathcal{A} = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ b & 0 & a \end{pmatrix} ;\ a, b, c \in \mathbb{C} \right\}.$$

Then every nonzero matrix in $\mathcal{A}$ has rank ≥ 2.

The estimates, provided in Theorem 1.1, respectively, in Theorem 1.3, are sharp. We show this in our next example.

Example 4.3. Let N be an n–by–n upper-triangular elementary Jordan nilpotent. Consider a subspace $\mathcal{S} := \mathrm{Lin}\{\mathrm{Id}, N, N^2, \dots, N^{k-1}\}$. It is easy to see that

$$\bigcap_{A\in\mathcal{S}\setminus\{0\}} \mathrm{Im}\, A = \bigcap_{(\xi_0,\dots,\xi_{k-1})\neq 0} \mathrm{Im}\big(\xi_0 N^0 + \cdots + \xi_{n-1} N^{k-1}\big) = \mathrm{Im}\, N^{k-1} = \mathbb{F}^{n-k+1} \oplus 0,$$

so for this subspace, the upper bound in Theorem 1.1 is achieved.

It would be tempting to conjecture that the inverse of the above statement is also true, up to multiplication by a fixed invertible matrix (that is, up to choosing a basis vectors). In order words, if the upper bound in (1.1) is achieved, is it always $\mathcal{S} = P\,\mathrm{Lin}\{\mathrm{Id}, N, N^2, \ldots, N^{k-1}\}Q$ for some invertible matrices P, Q? The answer is negative:

Example 4.4. Consider a subspace $\mathcal{S} \subseteq \mathcal{M}_3(\mathbb{C})$, spanned by the identity matrix A_1 and the nilpotents

$$A_2 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_3 := \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Thus, $n = 3 = k$. One easily sees that $\xi_2 A_2 + \xi_3 A_3$ is singular for every nonzero linear combination. So, $\mathrm{Im}(\xi_1 A_1 + \xi_2 A_2 + \xi_3 A_3) = \mathbb{C}^3$, unless $\xi_1 = 0$ in which case the image always contains the vector $(0, 1, 0)^t$. Hence,

$$\dim \bigcap_{\xi_1 \neq 0} \mathrm{Im}(\xi_1 A_1) = 3, \quad \dim \bigcap_{(\xi_1,\xi_2)\neq 0} \mathrm{Im}(\xi_1 A_1 + \xi_2 A_2) = 2, \quad \dim \bigcap_{(\xi_1,\xi_2,\xi_3)\neq 0} \mathrm{Im}(\xi_1 A_1 + \xi_2 A_2 + \xi_3 A_3) = 1.$$

and the upper bound in (1.1) is always achieved.

On the other hand, no invertible matrices P, Q would force $P\mathcal{S}Q$ to be upper–triangular. Namely otherwise, PA_1Q and PA_2Q are singular, hence they would have to be strictly upper-triangular. But two-dimensional subspace $\mathrm{Lin}\{PA_1Q, PA_2Q\}$ of 3–by–3 strictly upper triangular matrices necessarily contains a matrix of rank-one, a contradiction because every nonzero linear combination of A_1, A_2 is of rank-two. In particular, this shows that $P\mathcal{S}Q$ cannot be spanned by powers of a fixed nilpotent.

The inverse of Corollary 3.5 is not true, in general. It may happen that $\mathcal{S}$ is reflexive, yet $\dim \overset{\vec{\mathbf{x}}}{\mathfrak{M}}_1 \neq 0$.

Example 4.5. Consider a subspace $\mathcal{S} := \mathrm{Lin}\{A_1, A_2\}$ of 3–by–4 matrices, spanned by

$$A_1 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2 := \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $\vec{\mathbf{x}} := (\mathsf{e}_1, \ldots, \mathsf{e}_4)$ be the standard basis in $\mathbb{F}^4$. It follows from the definition (2.2) that $\mathfrak{S}_1 = \left[A_1\mathbf{e}_1 \mid A_2\mathsf{e}_1\right] = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\mathfrak{S}_2 = \left[A_1\mathbf{e}_2 \mid A_2\mathsf{e}_2\right] = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and so $\dim \bigcap_{(\alpha_1,\alpha_2)\neq 0} \mathrm{Im}(\alpha_1\mathfrak{S}_1 + \alpha_2\mathfrak{S}_2) = 1$. Despite this, $\mathcal{S}$ is reflexive. This can be computed directly, or else one uses the fact that every nonzero member from two-dimensional space $\mathcal{S}$ has rank 3, and then applies [10, Theorem 1.1].

This example also shows that the inequality $(\mathrm{Ref}\,\mathcal{S})|_{\mathcal{U}} \subseteq \mathrm{Ref}(\mathcal{S}|_{\mathcal{U}})$ in Lemma 2.4 can be strict — just use $\mathcal{U} := \mathrm{Lin}\{\mathsf{e}_1, \mathsf{e}_2\} \subseteq \mathbb{F}^4$.

Acknowledgement. The authors are especially thankful to Professor Bernik, for pointing them to Lemma 1.4, and making the paper [4] available to them. We also thank Professor Timotin for pointing us to the paper [11].

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