HYPERREFLEXIVITY OF FINITE-DIMENSIONAL SUBSPACES

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Abstract. We show that each reflexive finite-dimensional subspace of operators is hyperreflexive. This gives a positive answer to a problem of Kraus and Larson. We also show that each n–dimensional subspace of Hilbert space operators is $[\sqrt{2n}]$–hyperreflexive.

1. Introduction

Let X be a complex Banach space and let $B(X)$ be the algebra of all bounded linear operators on X. For an algebra $\mathcal{W} \subset B(X)$ with identity, let $\operatorname{Alg}\operatorname{Lat}\mathcal{W}$ denote the set of all operators which leave invariant all (closed) subspaces of X, which are invariant for all operators from $\mathcal{W}$. The algebra $\mathcal{W}$ is called *reflexive* if $\mathcal{W} = \operatorname{Alg}\operatorname{Lat}\mathcal{W}$.

The definition was introduced for the first time in [16] and further studied by a number of authors. The concept of reflexivity is interesting even if the underlying space is finite dimensional. For example, the algebra $\left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \oplus [a] : a, b \in \mathbb{C} \right\}$ is reflexive, but the algebra $\left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} : a, b \in \mathbb{C} \right\}$ is not reflexive (the former example will be used later).

The definition of reflexivity was extended to subspaces of operators in [13]. Let X, Y be Banach spaces and let $\mathcal{M}$ be a norm-closed subspace of $B(X, Y)$ — the space of all bounded linear operators from X into Y. Write

$$\operatorname{ref}\mathcal{M} = \left\{ T \in B(X,Y) \colon Tx \in \overline{\mathcal{M}x} \text{ for all } x \in X \right\},$$

where $\mathcal{M}x = \{Sx : S \in \mathcal{M}\}$. The subspace $\mathcal{M}$ is called *reflexive* if $\mathcal{M} = \operatorname{ref}\mathcal{M}$. For algebras with identity both definition coincide.

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A stronger concept of hyperreflexivity was introduced for algebras in [1] and extended for subspaces of operators in [10]. Denote by $\operatorname{dist}(\cdot,\cdot)$ the usual distance in Y; we use also the same notation for the distance in $B(X,Y)$. Let $\mathcal{M}\subset B(X,Y)$ be a norm-closed subspace and $T\in B(X,Y)$. Write

$$\alpha(T,\mathcal{M})=\sup\big\{\operatorname{dist}(Tx,\mathcal{M}x)\colon x\in X,\|x\|=1\big\}. \tag{1}$$

We always have $\alpha(T,\mathcal{M})\leqslant\operatorname{dist}(T,\mathcal{M})$. The subspace $\mathcal{M}$ is called *hyperreflexive* if there is a constant $C>0$ such that for all $T\in B(X,Y)$, we have

$$\operatorname{dist}(T,\mathcal{M})\leqslant C\,\alpha(T,\mathcal{M}). \tag{2}$$

The smallest constant C fulfilling (1) is called the *hyperreflexive constant* and denoted by $\kappa_{\mathcal{M}}$.

Let us observe that if $\mathcal{M}$ is reflexive and $T\in\operatorname{ref}\mathcal{M}$, then $\alpha(T,\mathcal{M})=0$. Hence $\operatorname{dist}(T,\mathcal{M})=0$ and, since $\mathcal{M}$ is norm closed, we have $T\in\mathcal{M}$. Thus each hyperreflexive subspace is also reflexive. On the other hand there are reflexive non-hyperreflexive subspaces (see [9]). However, if both spaces X and Y are finite dimensional then each reflexive subspace is also hyperreflexive. Namely, as we have observed above the reflexivity of a norm-closed subspace $\mathcal{M}$ is equivalent to the condition:

$$\alpha(T,\mathcal{M})=0\quad\Longleftrightarrow\quad\operatorname{dist}(T,\mathcal{M})=0.$$

Thus, for the whole conclusion, it is enough to note that all norms on the finite dimensional space $B(X,Y)/\mathcal{M}$ are equivalent.

In [10, Problem 3.9], Kraus and Larson posed the question whether the concepts of reflexivity and hyperreflexivity are equivalent for finite-dimensional subspaces of operators on infinite dimensional spaces. The problem was considered also in [6].

In [10] it was shown that each one-dimensional subspace is hyperreflexive. By [14], the hyperreflexive constant is equal to 1.

The aim of this paper is to give a positive answer to the problem of Kraus and Larson. The main result of the paper is

Main Theorem. *Let $\mathcal{M}\subset B(X,Y)$ with $\dim\mathcal{M}<\infty$. If $\mathcal{M}$ is reflexive, then $\mathcal{M}$ is hyperreflexive.*

In [12] it was shown that each n-dimensional subspace of Hilbert space operators is $[\sqrt{2n}]$-reflexive, where $[\sqrt{2n}]$ is the integer part of $\sqrt{2n}$. Using our main result we show in Section 3 that each n-dimensional subspace is even $[\sqrt{2n}]$-hyperreflexive (for definitions see Section 3).

Remark. Many authors (including [10]) considered the reflexivity and hyperreflexivity only for subspaces of operators on a Hilbert space.

They use a different definition of the distance $\alpha(T,\mathcal{M})$:

$$\alpha(T,\mathcal{M}) = \sup\bigl\{\|QTP\| : P, Q \text{ are projections and } Q\mathcal{M}P = 0\bigr\}.$$

To see the equivalence of both definitions of the distance $\alpha(\cdot,\cdot)$, note that (see [3, Proposition 58.1]) both distances are equal to

$$\alpha(T,\mathcal{M}) = \sup\bigl\{|(Tx,y)| \colon \|x\| = \|y\| = 1,\ (Sx,y) = 0 \text{ for all } S \in \mathcal{M}\bigr\}. \tag{3}$$

It is easy to see that the definitions of reflexivity and hyperreflexivity used in this paper also agree with the more general definitions introduced in [5].

2. Main theorem

Let X, Y be Banach spaces. Denote by $F(X,Y)$ the set of all finite-rank operators from X to Y and by $F_k(X,Y)$ the set of all operators in $B(X,Y)$ of rank smaller or equal to k. Denote by $S_X = \{x \in X : \|x\| = 1\}$ the unit sphere in X.

Let $n \geq 1$ and let $A_1,\dots,A_n \in B(X,Y)$. Denote by $\operatorname{span}\{A_i : i = 1,\dots,n\}$ the closed linear space generated by $A_1,\dots,A_n$. Write

$$s_0(A_1,\dots,A_n) = \inf\Bigl\{\Bigl\|\sum_{i=1}^n \lambda_i A_i\Bigr\| : \lambda_1,\dots,\lambda_n \in \mathbb{C}, \max|\lambda_i| = 1\Bigr\}.$$

More generally, for $k \in \mathbb{N}$ set

$$s_k(A_1,\dots,A_n) = \inf\bigl\{s_0(A_1|_M,\dots A_n|_M) : M \subset X, \operatorname{codim} M \leqslant k\bigr\}.$$

The following lemma summarizes the properties of the quantities s_k.

Lemma 2.1. *Let $A_1,\dots,A_n \in B(X,Y)$. Then:*

1. $s_0(A_1,\dots,A_n) = \inf\Bigl\{\Bigl\|\sum_{i=1}^n \lambda_i A_i\Bigr\| : \max|\lambda_i| \geqslant 1\Bigr\}$;
2. $s_0(A_1) = \|A_1\|$;
3. $s_0(A_1,\dots,A_n) \leqslant \min\{\|A_i\| : i = 1,\dots,n\}$;
4. $s_0(A_1,\dots,A_n) > 0$ *if and only if the operators* $A_1,\dots,A_n$ *are linearly independent;*
5. $s_k(A_1,\dots,A_n) \geqslant s_l(A_1,\dots,A_n)$ *for* $k \leqslant l$;
6. $s_k(A_1,\dots,\widehat{A_j},\dots,A_n) \geqslant s_k(A_1,\dots,A_n)$ *for* $j = 1,\dots,n$, *where the hat denotes the omitted term;*
7. *if M is a subspace of X and* $\operatorname{codim} M \leqslant k$ *then for any l we have* $s_l(A_1|_M,\dots,A_n|_M) \geqslant s_{l+k}(A_1,\dots,A_n)$;
8. $\operatorname{dist}(A_j, \operatorname{span}\{A_i : i \neq j\}) \geqslant s_0(A_1,\dots,A_n)$ *for* $j = 1,\dots,n$;
9. *if $k \in \mathbb{N}$ and no non-trivial linear combination of $A_1,\dots,A_n$ belongs to $F_k(X,Y)$, then* $s_k(A_1,\dots,A_n) > 0$.

Proof. The statements (1)–(7) are trivial. To see (8), fix j and observe that

$$\operatorname{dist}(A_j, \operatorname{span}\{A_i : i \neq j\}) = \inf\Big\{\Big\|\sum_{i=1}^n \lambda_i A_i\Big\| : |\lambda_j| = 1\Big\}$$
$$\geqslant \inf\Big\{\Big\|\sum_{i=1}^n \lambda_i A_i\Big\| : \max|\lambda_i| \geqslant 1\Big\} = s_0(A_1, \dots, A_n).$$

To see (9), let us fix $k \geqslant 0$. Let $Z = \big\{\sum_{i=1}^n \lambda_i A_i : \max|\lambda_i| = 1\big\}$. Since Z is compact and $F_k(X,Y)$ closed, we have $\operatorname{dist}(Z, F_k(X,Y)) > 0$.

Let $M \subset X$, $\operatorname{codim} M \leqslant k$. Let $P \in B(X)$ be a projection onto M such that $\|P\| \leqslant k+2$, see [4, Exercise 5.24]. Let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, $\max|\lambda_i| = 1$. Then

$$\operatorname{dist}\Big(\sum_{i=1}^n \lambda_i A_i, F_k(X,Y)\Big) \leqslant \Big\|\sum_{i=1}^n \lambda_i A_i - \sum_{i=1}^n \lambda_i A_i(I-P)\Big\|$$
$$\Big\|\sum_{i=1}^n \lambda_i A_i P\Big\| \leqslant \Big\|\sum_{i=1}^n \lambda_i A_i|_M\Big\| \cdot \|P\| \leqslant (k+2)\Big\|\sum_{i=1}^n \lambda_i A_i|_M\Big\|.$$

Thus

$$\Big\|\sum_{i=1}^n \lambda_i A_i|_M\Big\| \geqslant \frac{1}{k+2} \operatorname{dist}\Big(\sum_{i=1}^n \lambda_i A_i, F_k(X,Y)\Big),$$

and so

$$s_k(A_1, \dots, A_n) \geqslant \frac{1}{k+2} \operatorname{dist}(Z, F_k(X,Y)) > 0.$$

□

The following lemma is a quantitative version of [15, Lemma 1]. Note that for Hilbert spaces it is possible to take $M = F^\perp$.

Lemma 2.2. *Let $F \subset X$, $\dim F = n < \infty$, let $\varepsilon > 0$. Then there exists a subspace $M \subset X$ such that* $\operatorname{codim} M \leqslant (4n\varepsilon^{-1} + 3)^{2n}$ *and*

$$\|f + m\| \geqslant (1-\varepsilon)\max\{\|f\|, \|m\|/2\}$$

for all $m \in M$, $f \in F$.

In particular, there is a subspace $M_0 \subset X$ with $\operatorname{codim} M_0 \leqslant (12n + 3)^{2n}$ *such that*

$$\|f + m\| \geqslant \tfrac{1}{3}\max\{\|f\|, \|m\|\}$$

for all $f \in F$ and $m \in M_0$.

Proof. By the Auerbach lemma there are vectors $x_1,\dots,x_n\in F$ and $x_1^*,\dots,x_n^*\in F^*$ of norm one such that $\langle x_j,x_k^*\rangle=\delta_{j,k}$ (the Kronecker symbol) for all j,k. Thus for all $\gamma_1,\dots,\gamma_n\in\mathbb{C}$ we have

$$\Big\|\sum_{j=1}^n\gamma_jx_j\Big\|\geqslant\max_k\Big|\Big\langle\sum_{j=1}^n\gamma_jx_j,x_k^*\Big\rangle\Big|=\max_k|\gamma_k|.$$

In particular, the vectors $x_1,\dots,x_n$ are linearly independent and therefore form a basis of F. Let

$$Z=\Big\{\sum_{j=1}^n\Big(\frac{k_j\varepsilon}{2n}+i\frac{l_j\varepsilon}{2n}\Big)x_j:k_j,l_j\text{ integers },|k_j|,|l_j|\leqslant 2n\varepsilon^{-1}+1\Big\}.$$

Then $\operatorname{card}Z\leqslant(4n\varepsilon^{-1}+3)^{2n}$.

Let $u\in F$, $\|u\|=1$. Write $u=\sum_{j=1}^n(t_j+is_j)x_j$ for real t_j,s_j. Clearly $|t_j|,|s_j|\leqslant 1$ and we can find integers k_j,l_j such that $|\frac{k_j\varepsilon}{2n}-t_j|\leqslant\frac{\varepsilon}{4n}$ and $|\frac{l_j\varepsilon}{2n}-s_j|\leqslant\frac{\varepsilon}{4n}$. Thus

$$\Big\|u-\sum_{j=1}^n\Big(\frac{k_j\varepsilon}{2n}+i\frac{l_j\varepsilon}{2n}\Big)x_j\Big\|\leqslant\sum_{j=1}^n\Big(\Big|\frac{k_j\varepsilon}{2n}-t_j\Big|+\Big|\frac{l_j\varepsilon}{2n}-s_j\Big|\Big)\leqslant\tfrac{\varepsilon}{2}.$$

So $\operatorname{dist}(u,Z)\leqslant\frac{\varepsilon}{2}$. For $z\in Z$ choose $z^*\in X^*$ such that $\|z^*\|=1$ and $\langle z,z^*\rangle=\|z\|$. Let $M=\bigcap_{z\in Z}\ker z^*$. Clearly $\operatorname{codim}M\leqslant\operatorname{card}Z\leqslant(4n\varepsilon^{-1}+3)^{2n}$.

Let $f\in F$, $\|f\|=1$ and $m\in M$. Then there exists $z\in Z$ such that $\|z-f\|\leqslant\frac{\varepsilon}{2}$. Thus $\|z\|\geqslant 1-\frac{\varepsilon}{2}$. Let $z^*\in X^*$ be the functional considered above. Then we have

$$\begin{aligned}\|f+m\|&\geqslant|\langle f+m,z^*\rangle|=|\langle f,z^*\rangle|\\&\geqslant|\langle z,z^*\rangle|-|\langle f-z,z^*\rangle|\geqslant\|z\|-\tfrac{\varepsilon}{2}\geqslant 1-\varepsilon.\end{aligned}$$

Hence $\|f+m\|\geqslant(1-\varepsilon)\|f\|$ for all $f\in F$, $m\in M$.

Furthermore,

$$\begin{aligned}\|f+m\|&\geqslant\tfrac12(1-\varepsilon)\tfrac{2-\varepsilon}{1-\varepsilon}\|f+m\|=\tfrac12(1-\varepsilon)\big(\|f+m\|+\tfrac{1}{1-\varepsilon}\|f+m\|\big)\\&\geqslant\tfrac12(1-\varepsilon)\big(\|m\|-\|f\|+\|f\|\big)=\tfrac12(1-\varepsilon)\|m\|.\end{aligned}$$

In particular, for $\varepsilon=\frac13$ we get that there exists a subspace $M_0\subset X$ with $\operatorname{codim}M_0\leqslant(12n+3)^{2n}$ such that

$$\|f+m\|\geqslant\tfrac13\max\{\|f\|,\|m\|\}$$

for all $f\in F$ and $m\in M_0$. □

For simplicity we write $r(n)=(12n+3)^{2n}$ for the codimension of the space M_0 in the previous lemma.

Theorem 2.3. *There are increasing sequences of nonnegative integers $h(n)$, $g(n)$ and sequences of positive numbers c_n, c'_n with the following properties:*

(a) *if $A_1,\dots,A_n\in B(X,Y)$ satisfy $\|A_j\|\leqslant 1$ for $j=1,\dots,n$ and no non-trivial linear combination of $A_1,\dots,A_n$ belongs to $F(X,Y)$, then there exists a unit vector $u\in X$ such that*

$$\Big\|\sum_{i=1}^n \lambda_i A_i u\Big\|\geqslant c_n\, s^n_{h(n)}(A_1,\dots,A_n)\cdot\max\{|\lambda_i|:i=1,\dots,n\}$$

for all $\lambda_1,\dots,\lambda_n\in\mathbb{C}$;

(b) *if $T,A_1,\dots,A_n\in B(X,Y)$ satisfy $\|A_1\|\leqslant 1$ for $j=1,\dots,n$ and no non-trivial linear combination of $A_1,\dots,A_n$ belongs to $F(X,Y)$, then*

$$\alpha(T,\operatorname{span}\{A_1,\dots,A_n\})\geqslant c'_n\, s^n_{g(n)}(A_1,\dots,A_n)\cdot\operatorname{dist}(T,\operatorname{span}\{A_1,\dots,A_n\}).$$

Proof. We prove both statements by induction on n.

Let $n=1$ and let $A_1\in B(X,Y)$ satisfy $\|A_1\|\leqslant 1$. Set $c_1=\frac12$ and $h(1)=0$. There is a vector $u\in X$ such that $\|u\|=1$ and $\|A_1u\|\geqslant\frac12\|A_1\|$. Thus $\|\lambda_1A_1u\|\geqslant\frac12|\lambda_1|\cdot\|A_1\|=\frac12|\lambda_1|\,s_0(A_1)$ for all $\lambda_1\in\mathbb{C}$. This proves statement (a) for $n=1$.

$(a)_n\Rightarrow(b)_n$: Let $g(n)=h(n)+2n+2+(n+1)\,r\big((2n+2)(n+1)\big)$ and $c'_n=\Big(\frac{12n}{c_n}+6\Big)^{-1}$.

Let $T\in B(X,Y)$. Write for short $\varepsilon=\alpha(T,\operatorname{span}\{A_1,\dots,A_n\})$ and $\varepsilon'=\frac{\varepsilon}{c'_n s^n_{g(n)}(A_1,\dots,A_n)}$. For $x\in X$ with $\|x\|=1$ and $\delta>0$ set

$$D_{x,\delta}=\Big\{(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n:\Big\|Tx-\sum_{j=1}^n\lambda_jA_jx\Big\|\leqslant\delta\Big\}.$$

Clearly $D_{x,\delta}$ is a closed convex set. By the definition of the distance α, $D_{x,\varepsilon}\neq\emptyset$ for all $x\in X$, $\|x\|=1$.

To show property $(b)_n$, we must prove that

$$\bigcap_{x\in S_X}D_{x,\varepsilon'}\neq\emptyset. \tag{4}$$

Indeed, for $(\gamma_1,\dots,\gamma_n)\in\bigcap_{x\in S_X} D_{x,\varepsilon'}$ we have $\left\|Tx-\sum_{j=1}^n\gamma_jA_jx\right\|\leqslant\varepsilon'$ for all $x\in X$, $\|x\|=1$, and so $\left\|T-\sum_{j=1}^n\gamma_jA_j\right\|\leqslant\varepsilon'$. Therefore $\operatorname{dist}(T,\operatorname{span}\{A_1,\dots,A_n\})\leqslant\varepsilon'$, and so statement (b) for n is fulfilled.

By $(a)_n$ and Lemma 2.1(9), there exists a vector $x_0\in X$ with $\|x_0\|=1$ and a constant $c>0$ such that $\left\|\sum_{i=1}^n\lambda_iA_ix_0\right\|\geq c\max|\lambda_i|$ for all $(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n$. Therefore the set $D_{x_0,\varepsilon'}$ is bounded. Thus (4) is equivalent to

$$\bigcap_{x\in S_X}(D_{x,\varepsilon'}\cap D_{x_0,\varepsilon'})\neq\emptyset, \tag{5}$$

where the sets $D_{x,\varepsilon'}\cap D_{x_0,\varepsilon'}$ are convex compact subsets of $\mathbb{C}^n\sim\mathbb{R}^{2n}$. By the classical Helly theorem (see [7]), it is sufficient to show that

$$\bigcap_{i=0}^{2n+1}D_{x_i,\varepsilon'}\neq\emptyset$$

for all $(2n+1)$-tuples of unit vectors $x_1,\dots,x_{2n+1}\in X$.

Fix $x_1,\dots,x_{2n+1}\in X$ of norm one. Let $F_1=\operatorname{span}\{x_i : i=0,\dots,2n+1\}$ and let $M_1\subset X$ be a subspace such that $X=F_1\oplus M_1$. Then $\operatorname{codim}M_1\leq 2n+2$ and $F_1\cap M_1=\emptyset$. Let

$$F_2=\operatorname{span}\{Tx_i,A_jx_i : i=0,\dots,2n+1, j=1,\dots,n\}.$$

Then $\dim F_2\leqslant(2n+2)(n+2)$. By Lemma 2.2, there is a subspace $M_2\subset Y$ with $\operatorname{codim}M_2\leqslant r((2n+2)(n+1))$ such that $\|f+m\|\geqslant\frac13\max\{\|f\|,\|m\|\}$ for all $f\in F_2$, $m\in M_2$.

Let $M=M_1\cap T^{-1}M_2\cap\bigcap_{j=1}^n A_j^{-1}M_2$. Then $\operatorname{codim}M\leqslant\operatorname{codim}M_1+(n+1)\operatorname{codim}M_2$, and so $h(n)+\operatorname{codim}M\leqslant g(n)$. By the induction assumption $(a)_n$ and by Lemma 2.1(7), (5), there exists a vector $u\in M$, $\|u\|=1$ such that

$$\begin{aligned}\left\|\sum_{i=j}^n\lambda_jA_ju\right\|&\geqslant c_n\,s^n_{h(n)}(A_1|_M,\dots,A_n|_M)\cdot\max_j|\lambda_j|\\&\geqslant c_n\,s^n_{g(n)}(A_1,\dots,A_n)\cdot\max_j|\lambda_j|\end{aligned}\tag{6}$$

for all $\lambda_1,\dots,\lambda_n\in\mathbb{C}$.

Claim 1. $D_{x_i,6\varepsilon}\cap D_{u,6\varepsilon}\neq\emptyset$ for $i=0,1,\dots,2n+1$.

Proof. Fix $i\in\{0,1,\dots,2n+1\}$. Note that $x_i\in F_1$, $u\in M\subset M_1$, and so $x_i+u\neq0$. Set $v=\frac{x_i+u}{\|x_i+u\|}$. Suppose on the contrary that

$D_{x_i,6\varepsilon} \cap D_{u,6\varepsilon} = \emptyset$. For $\lambda_1,\dots,\lambda_n \in \mathbb{C}$ we have

$$\Big\|Tv - \sum_{j=1}^n \lambda_j A_j v\Big\| = \frac{1}{\|x_i+u\|}\Big\|Tx_i - \sum_{j=1}^n \lambda_j A_j x_i + Tu - \sum_{j=1}^n \lambda_j A_j u\Big\|$$
$$\geqslant \tfrac{1}{6}\max\Big\{\Big\|Tx_i - \sum_{j=1}^n \lambda_j A_j x_i\Big\|, \Big\|Tu - \sum_{j=1}^n \lambda_j A_j u\Big\|\Big\},$$

since $Tx_i - \sum_{j=1}^n \lambda_j A_j x_i \in F_2$ and $Tu - \sum_{j=1}^n \lambda_j A_j u \in M_2$.

Since either $(\lambda_1,\dots,\lambda_n) \notin D_{x_i,6\varepsilon}$ or $(\lambda_1,\dots,\lambda_n) \notin D_{u,6\varepsilon}$, at least one of the two terms is greater than 6ε. Thus

$$\Big\|Tv - \sum_{j=1}^n \lambda_j A_j v\Big\| > \varepsilon$$

for all $\lambda_1,\dots,\lambda_n \in \mathbb{C}$. Hence $D_{v,\varepsilon} = \emptyset$, a contradiction.

Claim 2. Let $(\lambda_1,\dots,\lambda_n),(\mu_1,\dots,\mu_n) \in D_{u,6\varepsilon}$. Then

$$\max_j |\lambda_j - \mu_j| \leqslant \frac{12\varepsilon}{c_n s^n_{g(n)}(A_1,\dots,A_n)}.$$

Proof. Let $(\lambda_1,\dots,\lambda_n),(\mu_1,\dots,\mu_n) \in D_{u,6\varepsilon}$. Then

$$\Big\|Tu - \sum_{j=1}^n \lambda_j A_j u\Big\| \leqslant 6\varepsilon \quad \text{and} \quad \Big\|Tu - \sum_{j=1}^n \mu_j A_j u\Big\| \leqslant 6\varepsilon.$$

Hence

$$\Big\|\sum_{j=1}^n (\lambda_j - \mu_j) A_j u\Big\| \leqslant 12\varepsilon,$$

and so, by (6) we have

$$\max_j |\lambda_j - \mu_j| \leqslant \frac{\Big\|\sum_{j=1}^n (\lambda_j - \mu_j) A_j u\Big\|}{c_n\, s^n_{g(n)}(A_1,\dots,A_n)} \leqslant \frac{12\varepsilon}{c_n\, s^n_{g(n)}(A_1,\dots,A_n)}.$$

Claim 3. Let $i \in \{0,1,\dots,2n+1\}$. Then $D_{x_i,\varepsilon'} \supset D_{u,6\varepsilon}$.

Proof. Let $(\lambda_1,\dots,\lambda_n)\in D_{u,6\varepsilon}\cap D_{x_i,6\varepsilon}$. Let $(\mu_1,\dots,\mu_n)\in D_{u,6\varepsilon}$ be arbitrary. Then $\max\limits_j|\lambda_j-\mu_j|\leqslant\frac{12\varepsilon}{c_n\, s^n_{g(n)}(A_1,\dots,A_n)}$ and

$$\Big\|Tx_i-\sum_{j=1}^n\mu_jA_jx_i\Big\|\leqslant\Big\|Tx_i-\sum_{j=1}^n\lambda_jA_jx_i\Big\|+\Big\|\sum_{j=1}^n(\lambda_j-\mu_j)A_jx_i\Big\|$$
$$\leqslant 6\varepsilon+\frac{12\varepsilon n}{c_ns^n_{g(n)}(A_1,\dots,A_n)}\leqslant\frac{\varepsilon}{s^n_{g(n)}(A_1,\dots,A_n)}\big(6+\tfrac{12n}{c_n}\big)\varepsilon'.$$

Thus $(\mu_1,\dots,\mu_n)\in D_{x_i,\varepsilon'}$.

Hence $\bigcap\limits_{i=0}^{2n+1}D_{x_i,\varepsilon'}\supset D_{u,6\varepsilon}\neq\emptyset$, and (4) is fulfilled. This proves statement (b) for n.

$(b)_{n-1}\Rightarrow(a)_n$: Let $n\geqslant 2$ and suppose that property (b) holds for $n-1$. Set $c_n=\frac{c'_{n-1}}{18n}$ and $h(n)=g(n-1)+n^2\cdot r(n(n-1))$.

We construct inductively vectors $u_1,\dots,u_n\in X$ of norm one in the following way. Let $k\in\{1,\dots,n\}$ and suppose that the vectors u_j, $j=1,\dots,k-1$ have already been constructed. Let

$$F_k=\operatorname{span}\{A_iu_j: i=1,\dots,n,\ j=1,\dots,k-1\}.$$

Then $\dim F_k\leqslant n(k-1)\leqslant n(n-1)$. By Lemma 2.2, there is a subspace $M_k\subset Y$ such that $\operatorname{codim}M_k\leqslant r(n(n-1))$ and $\|f+m\|\geqslant\frac13\max\{\|f\|,\|m\|\}$ for all $f\in F_k$, $m\in M_k$. Let $M'_k=\bigcap\limits_{j=1}^{k}\bigcap\limits_{i=1}^{n}A_i^{-1}M_j$. Then $\operatorname{codim}M'_k\leqslant n^2\cdot\operatorname{codim}M_k$, and so $g(n-1)+\operatorname{codim}M'_k\leqslant h(n)$. By property $(b)_{n-1}$, there is a vector $u_k\in M'_k$ of norm one such that

$$\operatorname{dist}\big(A_ku_k,\operatorname{span}\{A_iu_k\ : i\neq k\}\big)$$
$$\geqslant\tfrac12\,c'_{n-1}s^{n-1}_{g(n-1)}(A_1|_{M'_k},\dots,\widehat{A_k|_{M'_k}},\dots,A_n|_{M'_k})$$
$$\cdot\operatorname{dist}\big(A_k|_{M'_k},\operatorname{span}\{A_i|_{M'_k}: i\neq k\}\big)$$
$$\geqslant\tfrac12\,c'_{n-1}s^{n-1}_{h(n)}(A_1,\dots,A_n)\cdot s_0(A_1|_{M'_k},\dots,A_n|_{M'_k})$$
$$\geqslant\tfrac12\,c'_{n-1}s^{n}_{h(n)}(A_1,\dots,A_n),$$

where the hat denotes the omitted term; in the estimates we used Lemma 2.1(6),(8) and (5).

Let $u_1,\dots,u_n\in S_X$ be constructed in the above described way. Set $v=\sum\limits_{j=1}^n u_j$. For $\lambda_1,\dots,\lambda_n\in\mathbb{C}$ and $k\in\{1,\dots,n\}$, since $\sum\limits_{j=1}^k\sum\limits_{i=1}^n\lambda_iA_iu_j\in F_{k+1}$, $\sum\limits_{j=k+1}^n\sum\limits_{i=1}^n\lambda_iA_iu_j\in M_{k+1}$, $\sum\limits_{j=1}^{k-1}\sum\limits_{i=1}^n\lambda_iA_iu_j\in F_k$ and $\sum\limits_{i=1}^n\lambda_iA_iu_k\in M_k$, we have

$$\Big\|\sum_{i=1}^n \lambda_i A_i v\Big\| = \Big\|\sum_{j=1}^n\sum_{i=1}^n \lambda_i A_i u_j\Big\| \geqslant \tfrac{1}{3}\Big\|\sum_{j=1}^k\sum_{i=1}^n \lambda_i A_i u_j\Big\|$$

$$\geqslant \tfrac{1}{9}\Big\|\sum_{i=1}^n \lambda_i A_i u_k\Big\| \geqslant \tfrac{1}{9}\,|\lambda_k|\cdot \operatorname{dist}\big(A_k u_k, \operatorname{span}\{A_i u_k : i\neq k\}\big)$$

$$\geqslant \tfrac{1}{18}\, c'_{n-1}\, s^n_{g(n)}(A_1,\dots,A_n)\cdot|\lambda_k|,$$

(if $k=n$ then the first inequality is trivial). In particular, $v\neq 0$, by Lemma 2.1(9). Hence the vector $u=\frac{v}{\|v\|}$ satisfies $\|u\|=1$ and

$$\Big\|\sum_{i=1}^n \lambda_i A_i u\Big\| \geqslant \tfrac{1}{18\,\|v\|}\, c'_{n-1}\, s^n_{g(n)}(A_1,\dots,A_n)\cdot \max_k|\lambda_k|$$

$$\geqslant c_n\, s^n_{g(n)}(A_1,\dots,A_n)\cdot \max_k|\lambda_k|.$$

This finishes the proof. □

Corollary 2.4. *Let $\mathcal{M}\subset B(X,Y)$ be a finite-dimensional subspace which contains no non-zero finite rank operators. Then $\mathcal{M}$ is hyperreflexive.*

Proof. Choose a basis $A_1,\dots,A_n$ of $\mathcal{M}$. The proof follows from the previous theorem, property (b). □

Now we are ready to prove the main theorem.

Theorem 2.5. *Let $\mathcal{M}\subset B(X,Y)$, $\dim\mathcal{M}<\infty$. Then $\mathcal{M}$ is hyperreflexive if and only if $\mathcal{M}$ is reflexive.*

Proof. If $\mathcal{M}$ is hyperreflexive then $\mathcal{M}$ is clearly reflexive. Conversely, let $\mathcal{M}$ be reflexive. Let $\mathcal{M}_1=\mathcal{M}\cap F(X,Y)$ and let $\mathcal{M}_2$ be any subspace of $\mathcal{M}$ such that $\mathcal{M}=\mathcal{M}_1\oplus\mathcal{M}_2$. Choose a basis $A_1,\dots,A_k$ of $\mathcal{M}_1$ and a basis $B_1,\dots,B_l$ of $\mathcal{M}_2$.

Let $M=\bigcap_{i=1}^k \ker A_i$. Then $\operatorname{codim} M<\infty$. By the previous result for the operators $B_i|_M$, there is a constant $d_1>0$ such that

(7)

$$\operatorname{dist}\big(T|_M, \operatorname{span}\{B_1|_M,\dots,B_l|_M\}\big) \leqslant d_1\cdot\alpha\big(T|_M, \operatorname{span}\{B_1|_M,\dots,B_l|_M\}\big).$$

Let $P\in B(X)$ be a projection onto M and $F=\ker P$. Let $F'=\operatorname{span}\{Sf : S\in\mathcal{M}, f\in F\}$. Clearly $\dim F'<\infty$. By Lemma 2.2, there is a subspace $M'\subset Y$ such that $\operatorname{codim} M'<\infty$ and $\|f'+m'\|\geqslant$

$\frac{1}{3}\max\{\|f'\|,\|m'\|\}$ for all $f' \in F'$ and $m' \in M'$. Set $M'' = M \cap \bigcap_{i=1}^{l} B_i^{-1}M'$. Clearly $\operatorname{codim} M'' < \infty$.

Let $u \in M''$ be a "separating vector" for the operators $B_i|_{M''}$, i.e., $\|u\| = 1$ and there is a constant $d_2 > 0$ such that $\left\|\sum_{i=1}^{l} \gamma_i B_i u\right\| \geqslant d_2 \max|\gamma_i|$ for all $\gamma_1,\dots,\gamma_l \in \mathbb{C}$. Such a vector exists by Theorem 2.3 and Lemma 2.1(9).

It follows from [11, Corollary 2.8] that since $\mathcal{M}$ is reflexive, $\mathcal{M}_1$ is also reflexive. For the sake of completeness we include the proof of this here. Since $\mathcal{M}_1 \subset \mathcal{M}$, we have $\operatorname{ref}\mathcal{M}_1 \subset \operatorname{ref}\mathcal{M}$. By reflexivity of $\mathcal{M}$, we have $\operatorname{ref}\mathcal{M}_1 \subset \mathcal{M}$. To show the reflexivity of $\mathcal{M}_1 = \mathcal{M}\cap F(X,Y)$, it is enough to show that $\operatorname{ref}\mathcal{M}_1 \subset F(X,Y)$. Let $B \in \operatorname{ref}\mathcal{M}_1$. Then, for all $u \in X$, $Bu \in \operatorname{span}\{A_i x : i = 1,\dots,k, x \in X\}$. Hence $\operatorname{rank} B < \infty$ and $B \in F(X,Y)$.

Now, for $i = 1,\dots,k$ consider the operators $\tilde{A}_i \colon F \to \operatorname{span}\{A_1x,\dots, A_kx : x \in F\}$ induced by A_i. Since the operators $A_1,\dots,A_k$ are equal to zero on M, it is easy to see that $\tilde{\mathcal{M}}_1 = \operatorname{span}\{\tilde{A}_1,\dots,\tilde{A}_k\}$ is reflexive. As it was observed in the introduction, $\tilde{\mathcal{M}}_1$ is hyperreflexive. Thus there exists a constant $d_3 > 0$ with the following property: if $\varepsilon > 0$ and $T\colon F \to \operatorname{span}\{A_1x,\dots,A_kx : x \in F\}$ satisfies $\operatorname{dist}(Tx, \operatorname{span}\{A_1x,\dots,A_kx\}) \leqslant \varepsilon$ for all $x \in F$, $\|x\| = 1$ then there exist numbers $\gamma_1,\dots,\gamma_k \in \mathbb{C}$ such that $\left\|T - \sum_{i=1}^{n} \gamma_i A_i|_F\right\| \leqslant d_3\varepsilon$.

We show now that $\mathcal{M}$ is hyperreflexive. Let $\varepsilon > 0$, $T \in B(X,Y)$ and let

$$\operatorname{dist}(Tx, \mathcal{M}x) \leqslant \varepsilon$$

for all $x \in X$, $\|x\| = 1$. By (7) there exist numbers $\beta_1,\dots\beta_l \in \mathbb{C}$ such that

$$\left\|T|_M - \sum_{j=1}^{l} \beta_j B_j|_M\right\| \leqslant d_1\,\varepsilon. \tag{8}$$

Set $S = T - \sum_{j=1}^{l} \beta_j B_j$. Thus $\|S|_M\| \leqslant d_1\varepsilon$ and S satisfies $\operatorname{dist}(Sx, \mathcal{M}x) \leqslant \varepsilon$ for all $x \in X$, $\|x\| = 1$.

Let $x \in F$, $\|x\| = 1$. Then there are numbers $\lambda_1,\dots,\lambda_k,\mu_1,\dots,\mu_l \in \mathbb{C}$ such that

$$\left\|Sx - \sum_{i=1}^{k} \lambda_i A_i x - \sum_{j=1}^{l} \mu_j B_j x\right\| \leqslant \varepsilon.$$

Similarly, there are numbers $\lambda_1',\dots,\lambda_k',\mu_1',\dots,\mu_l'\in\mathbb{C}$ such that

$$\Big\|S(x+u)-\sum_{i=1}^k\lambda_i'A_i(x+u)-\sum_{j=1}^l\mu_j'B_j(x+u)\Big\|\leqslant\varepsilon\|x+u\|\leqslant 2\varepsilon.$$

By subtracting we have

$$\Big\|Su+\sum_{i=1}^k(\lambda_i-\lambda_i')A_ix-\sum_{j=1}^l(\mu_j-\mu_j')B_jx-\sum_{j=1}^l\mu_j'B_ju\Big\|\leqslant 3\varepsilon,$$

since $A_i\,u=0$ for all i. By the definitions of M'' and F' and by (8), we have

$$\begin{aligned}\Big\|\sum_{j=1}^l\mu_j'B_ju\Big\|\leqslant 3\Big\|\sum_{i=1}^k(\lambda_i-\lambda_i')A_ix+\sum_{j=1}^l(\mu_j-\mu_j')B_jx-\sum_{j=1}^l\mu_j'B_ju\Big\|\\ \leqslant 3(3\varepsilon+\|Su\|)\leqslant 3\varepsilon(3+d_1).\end{aligned}$$

Since $\Big\|\sum_{j=1}^l\mu_j'B_ju\Big\|\geqslant d_2\max|\mu_j'|$, we have $\max|\mu_j'|\leqslant 3\varepsilon\frac{d_1+3}{d_2}$. Thus we have

$$\begin{aligned}\Big\|Sx-\sum_{i=1}^n\lambda_i'A_ix\Big\|&\leqslant\|Sx-S(x+u)\|\\ &+\Big\|S(x+u)-\sum_{i=1}^n\lambda_i'A_ix-\sum_{j=1}^l\mu_j'B_j(x+u)\Big\|+\Big\|\sum_{j=1}^l\mu_j'B_j(x+u)\Big\|\\ &\leqslant\|Su\|+2\varepsilon+\sum_{j=1}^l|\mu_j'|\cdot\|B_j\|\cdot 2\leqslant d_4\,\varepsilon,\end{aligned}$$

where $d_4=d_1+2+\frac{3d_1+9}{d_2}\cdot 2\sum_{j=1}^l\|B_j\|$. Thus there exist numbers $\gamma_1,\dots,\gamma_k\in\mathbb{C}$ such that $\Big\|S|_F-\sum_{i=1}^k\gamma_iA_i|_F\Big\|\leqslant d_3\,d_4\,\varepsilon$.

Let $f \in F$, $m \in M$ and $\|f+m\| = 1$. Then $\|m\| = \|P(f+m)\| \leqslant \|P\|$ and $\|f\| \leqslant \|f+m\| + \|m\| \leqslant 1 + \|P\|$. Since $A|_M = 0$, we have

$$\Big\|T(f+m) - \sum_{i=1}^{k} \gamma_i A_i(f+m) - \sum_{j=1}^{l} \beta_j B_j(f+m)\Big\|$$

$$\Big\|S(f+m) - \sum_{i=1}^{k} \gamma_i A_i f\Big\| \leqslant \|Sm\| + \Big\|Sf - \sum_{i=1}^{k} \gamma_i A_i f\Big\| \leqslant d_1\,\varepsilon\,\|m\| + d_3\,d_4\,\varepsilon\,\|f\|.$$

Thus $\Big\|T - \sum\limits_{i=1}^{k} \gamma_i A_i - \sum\limits_{j=1}^{l} \beta_j B_j\Big\| \leqslant \varepsilon\big(d_1\,\|P\| + d_3\,d_4\,(\|P\|+1)\big)$, and so $\mathcal{M}$ is hyperreflexive. □

3. Examples and Corollaries

The example from [9], mentioned in the introduction shows also that there is no constant in the condition (2) for the hyperreflexivity of a finite-dimensional subspace depending only on the dimension of the subspace. Bellow we give another example of this kind.

Example 3.1. Let $H = \mathbb{C}^3$ with the Hilbert norm. For $\varepsilon > 0$ consider the operators $A_{1,\varepsilon} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \oplus [\varepsilon]$ and $A_{2,\varepsilon} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus [0]$. Let $\mathcal{M}_\varepsilon = \operatorname{span}\{A_{1,\varepsilon}, A_{2,\varepsilon}\}$. Clearly $\dim \mathcal{M}_\varepsilon = 2$. It is easy to verify that $\mathcal{M}_\varepsilon$ is reflexive for all ε.

Let $T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \oplus [0]$. For $\beta, \gamma \in \mathbb{C}$ we have

$$\|\beta A_{1,\varepsilon} + \gamma A_{2,\varepsilon} - T\| = \left\| \begin{bmatrix} \beta - 1 & \gamma \\ 0 & \beta \end{bmatrix} \oplus [\varepsilon\beta] \right\| \geqslant \max\{|\beta - 1|, |\beta|\} \geqslant \tfrac{1}{2}.$$

Thus $\operatorname{dist}(T, \mathcal{M}_\varepsilon) \geqslant \frac{1}{2}$ for all $\varepsilon > 0$.

Let $x = \begin{bmatrix} a \\ b \end{bmatrix} \oplus [c] \in H$, $\|x\| = 1$. If $b \neq 0$ then $\operatorname{dist}(Tx, \mathcal{M}_\varepsilon x) \leq \|ab^{-1}A_{2,\varepsilon}x - Tx\| = 0$. If $b = 0$ then $\operatorname{dist}(Tx, \mathcal{M}_\varepsilon x) \leq \|A_{1,\varepsilon}x - Tx\| = \left\| \begin{bmatrix} 0 \\ 0 \end{bmatrix} \oplus [\varepsilon c] \right\| \leq \varepsilon$. Thus $\alpha(T, \mathcal{M}_\varepsilon) \leqslant \varepsilon$ and there is no constant $C > 0$ such that

$$\operatorname{dist}(T, \mathcal{M}_\varepsilon) \leqslant C \cdot \alpha(T, \mathcal{M}_\varepsilon)$$

for all $\varepsilon > 0$.

Now we consider finite dimensional subspaces of $B(H)$, where H is a Hilbert space. It is well known that $B(H)$ is the dual of the trace class operators. If $\mathcal{M}$ is a w^*-closed subspace of $B(H)$, in particular if $\dim\mathcal{M} < \infty$, then $\mathcal{M}$ is reflexive if and only if $\mathcal{M}_\perp \cap F_1(H)$ is total in $\mathcal{M}_\perp$ (see for example [2]). According to [2], a subspace $\mathcal{M}$ is called k-reflexive if $\mathcal{M}_\perp \cap F_k(H)$ is total in $\mathcal{M}_\perp$. In [12] it was shown that each n dimensional subspace is $[\sqrt{2n}]$–reflexive ($[\cdot]$ denotes the integer part). For any subspace $\mathcal{M} \subset B(H)$ and $T \in B(H)$, as it was suggested in [8], we can consider

$$\alpha_k(T,\mathcal{M}) = \sup\bigl\{|\langle T,t\rangle| : t \in \mathcal{M}_\perp,\ \|t\| \leqslant 1,\ \operatorname{rank} t \leqslant k\bigr\}$$

(compare with (3)). As in [8] we can call the subspace $\mathcal{M} \subset B(H)$ *k-hyperreflexive* if there is a constant C such that

$$\operatorname{dist}(T,\mathcal{M}) \leqslant C\alpha_k(T,\mathcal{M})$$

for each operator $T \in B(H)$. We will show the following

Corollary 3.2. *Let $\mathcal{M} \subset B(H)$ and* $\dim\mathcal{M} = n$. *Then $\mathcal{M}$ is $[\sqrt{2n}]$-hyperreflexive.*

Proof. Let $k = [\sqrt{2n}]$. By $\mathcal{M}^{(k)}$ we denote the k-th amplification of $\mathcal{M}$

$$\mathcal{M}^{(k)} = \bigl\{\underbrace{S \oplus \cdots \oplus S}_{k} :\ S \in \mathcal{M}\bigr\} \subset B\left(H^{(k)}\right),$$

where $H^{(k)}$ is the direct sum of k-copies of H, $H^{(k)} = \underbrace{H \oplus \cdots \oplus H}_{k}$.

Since $\dim\mathcal{M} = n$, $\mathcal{M}$ is k-reflexive by [12, Theorem 12]. By [2], $\mathcal{M}^{(k)}$ is reflexive. Since $\dim\mathcal{M}^{(k)} = n$, it is also hyperreflexive. Hence [8, Theorem 3.5] implies that $\mathcal{M}$ is k-hyperreflexive. □

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