Nonlinear Analysis 71 (2009) 1213–1226

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

On the co-derivative of normal cone mappings to inequality systems☆

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ARTICLE INFO

Article history:
Received 22 July 2008
Accepted 6 November 2008

MSC:
90C30
49J53

Keywords:
Mordukhovich coderivative
Normal cone mapping
Calmness
Inequality constraints

ABSTRACT

The paper deals with co-derivative formulae for normal cone mappings to smooth inequality systems. Both the regular (Linear Independence Constraint Qualification satisfied) and nonregular (Mangasarian–Fromovitz Constraint Qualification satisfied) cases are considered. A major part of the results relies on general transformation formulae previously obtained by Mordukhovich and Outrata. This allows one to derive exact formulae for general smooth, regular and polyhedral, possibly nonregular systems. In the nonregular, nonpolyhedral case a generalized transformation formula by Mordukhovich and Outrata applies, however, a major difficulty consists in checking a calmness condition of a certain multivalued mapping. The paper provides a translation of this condition in terms of much easier to verify constraint qualifications. The final section is devoted to the situation where the calmness condition is violated. A series of examples illustrates the use and comparison of the presented formulae.

1. Introduction

The Mordukhovich co-derivative has become an important tool for the characterization of stability and optimality in variational analysis. We refer to the basic monograph [1] for definitions, properties, calculus rules and applications of this object. When dealing with generalized equations or variational inequalities, the multivalued mappings for which the co-derivative is to be calculated are typically given by normal cones N_Ω to certain closed sets Ω. For complementarity problems, e.g., $\Omega = \mathbb{R}^n_+$, an explicit, ready-to-use formula for the co-derivative $D^*N_{\mathbb{R}^n_+}$ is available. However, in many applications, Ω is often more complicated than just $\mathbb{R}^n_+$. For example, Ω may be a general polyhedron or a set described by a finite number of smooth inequalities. In such cases (see [2,1]), given that certain constraint qualifications hold true, the existence of convenient calculus rules for the co-derivative allow similar formulae to be obtained as well. If Ω is defined via a smooth inequality system satisfying the Linear Independence Constraint Qualification (LICQ), then the co-derivative D^*N_Ω can be led back to the well-known formula for $D^*N_{\mathbb{R}^n_+}$ with the addition of a second order term and a linear transformation. In the nonregular case, i.e., LICQ is violated, a slightly more complicated transformation formula (involving a union over nonuniquely defined multipliers) can be applied provided that the Mangasarian–Fromovitz Constraint Qualification (MFCQ) holds as well as the additional requirement that a certain associated multifunction is calm (see [3]). In general, this transformation formula holds true as an inclusion only, thus leaving a gap between the precise expression for the co-derivative and the one comfortably calculated from the formula. Closing this gap amounts to calculating the co-derivative

☆ This work was supported by the DFG Research Center MATHEON "Mathematics for key technologies" in Berlin and by the Grant Agency of the Czech Academy of Sciences (Grant No. IAA 1030405).

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doi:10.1016/j.na.2008.11.089

'from scratch'. Important examples where precise formulae for the co-derivative could be obtained in the nonregular case are general polyhedra where LICQ may be violated (see [4]) and the second order cone, which also admits no description satisfying LICQ (see [5]).

The aim of this paper is to provide explicit, readily-applicable co-derivative formulae for normal cone mappings to possibly nonregular inequality systems. The first part reviews some precise expressions for the co-derivative in the regular and nonregular polyhedral setting. Similar to the reduction approach by Bonnans and Shapiro ([6], p. 240), it is illustrated how nonregularity of the given set Ω can be shifted in certain special situations to the simpler image set, which might happen to be polyhedral, thus allowing one to apply the previously mentioned co-derivative formula for certain nonregular, nonpolyhedral sets Ω as well.

A second part of the paper deals with the transformation formula for the nonregular systems mentioned above. The formula is first used to derive an alternative explicit expression for the co-derivative in case of Ω being polyhedral (possibly nonregular). Some examples are then furnished with the intent of contrasting the formula's ease-of-use versus its possible lack of precision when compared to that which one obtains by applying the precise formula.

In the polyhedral case, the mentioned calmness condition required for the application of the transformation formula happens to be automatically satisfied. However, for nonlinear inequality systems this is no longer true and one is therefore required to verify calmness.

The original calmness condition is formulated for a multifunction of complicated structure involving primal and dual variables. A major part of the paper is therefore devoted to a reformulation of this condition as a constraint qualification, i.e., in terms of primal variables only. More precisely, by associating the respective equality system with the original inequality system describing Ω, one then checks calmness of this equality system along with that for all its subsystems. A reasonable constraint qualification ensuring this property is derived for the special case that the number of binding inequalities exceeds the spacial dimension.

The final section of the paper addresses the most complicated situation: where MFCQ holds true but the mentioned calmness condition is violated. Partial, albeit less precise, upper estimates for the coderivative are then discussed for this situation.

2. Some concepts and tools of variational analysis

We start with the definitions of the main objects in our investigation. For a closed set $\Lambda \subseteq \mathbb{R}^n$ and a point $\bar{x} \in \Lambda$, the *Fréchet normal cone* to Λ at $\bar{x} \in \Lambda$ is defined by

$$\hat{N}_\Lambda(\bar{x}) := \{x^* \in \mathbb{R}^n \mid \left\langle x^*, x - \bar{x}\right\rangle \leq o\left(\|x - \bar{x}\|\right) \forall x \in \Lambda\}.$$

The *Mordukhovich normal cone* to Λ at $\bar{x} \in \Lambda$ results from the Fréchet normal cone in the following way:

$$N_\Lambda(\bar{x}) := \operatorname*{Limsup}_{x \to \bar{x}, x \in \Lambda} \hat{N}_\Lambda(x).$$

The 'Limsup' in the definition above is the upper limit of sets in the sense of Kuratowski-Painlevé, cf. [7].

For a multifunction $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$, consider a point of its graph: $(x, y) \in \operatorname{gph} \Phi$. The Mordukhovich normal cone induces the following co-derivative $D^*\Phi\,(x, y) : \mathbb{R}^p \rightrightarrows \mathbb{R}^n$ of Φ at (x, y):

$$D^*\Phi\,(x, y)\left(y^*\right) = \{x^* \in \mathbb{R}^n \mid \left(x^*, -y^*\right) \in N_{\operatorname{gph}\Phi}\,(x, y)\} \quad \forall y^* \in \mathbb{R}^p.$$

Similarly, the more elementary Fréchet normal cone induces the Fréchet co-derivative

$$\hat{D}^*\Phi\,(x, y)\left(y^*\right) = \{x^* \in \mathbb{R}^n \mid \left(x^*, -y^*\right) \in \hat{N}_{\operatorname{gph}\Phi}\,(x, y)\} \quad \forall y^* \in \mathbb{R}^p.$$

The relation between both concepts is given by

$$D^*\Phi\,(\bar{x}, \bar{y})\left(\bar{y}^*\right) = \operatorname*{Limsup}_{\substack{(x,y)\to(\bar{x},\bar{y}) \\ y\in\Phi(x) \\ y^*\to\bar{y}^*}} \hat{D}^*\Phi\,(x, y)\left(y^*\right).$$

Moreover, the co-derivative enjoys the following robustness property:

$$D^*\Phi\,(\bar{x}, \bar{y})\left(\bar{y}^*\right) = \operatorname*{Limsup}_{\substack{(x,y)\to(\bar{x},\bar{y}) \\ y\in\Phi(x) \\ y^*\to\bar{y}^*}} D^*\Phi\,(x, y)\left(y^*\right).$$

A multifunction $Z : Y \rightrightarrows X$ between metric spaces is said to be calm at a point $(\bar{y}, \bar{x})$ belonging to its graph, if there exist $L, \varepsilon > 0$, such that

$$d(x, Z(\bar{y})) \leq L d(y, \bar{y}) \quad \forall x \in Z(y) \cap \mathbb{B}\,(\bar{x}, \varepsilon)\, \forall y \in \mathbb{B}\,(\bar{y}, \varepsilon)\,.$$

Here 'd' and '$\mathbb{B}$' refer to the distances and balls with corresponding radii in the respective metric space. For the special multifunction $Z : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^p$, defined by

$$Z\,(\alpha, \beta) := \{x \in \mathbb{R}^p | G_1(x) = \alpha, G_2(x) \leq \beta\},$$

where $G_1 : \mathbb{R}^p \to \mathbb{R}^n$ and $G_1 : \mathbb{R}^p \to \mathbb{R}^m$ are continuous mappings, it is easy to see that calmness of Z at $(0, 0, \bar{x})$ for some $\bar{x}$ satisfying $G_1(\bar{x}) = 0$ and $G_2(\bar{x}) = 0$ is equivalent with the existence of $L, \varepsilon > 0$, such that

$$d(x, Z(0,0)) \le L\left(\sum_i |G_{1i}(x)| + \sum_i [G_{2i}(x)]_+\right) \quad \forall x \in \mathbb{B}(\bar{x}, \varepsilon). \tag{1}$$

Here, $[y]_+ := \max\{y, 0\}$.

3. On the co-derivative of normal cone mappings

3.1. Regular constraint systems

The following theorem recalls a basic transformation formula for co-derivatives that was established first as an inclusion by Mordukhovich and Outrata in [2] (Theorem 3.4) and later as an equality in [1] (Theorem 1.127):

Theorem 3.1. *Let $C = F^{-1}(P)$, where $F : \mathbb{R}^n \to \mathbb{R}^m$ is twice continuously differentiable and $P \subseteq \mathbb{R}^m$ is some closed subset. Consider points $\bar{x} \in C$ and $\bar{v} \in N_C(\bar{x})$. If the Jacobian $\nabla F(\bar{x})$ is surjective, then*

$$D^*N_C(\bar{x}, \bar{v})(v^*) = \left(\sum_{i=1}^m \bar{\lambda}_i \nabla^2 F_i(\bar{x})\right) v^* + \nabla^T F(\bar{x}) D^*N_P\left(F(\bar{x}), \bar{\lambda}\right)\left(\nabla F(\bar{x}) v^*\right). \tag{2}$$

Here, the F_i are the components of F and $\bar{\lambda}$ is the unique solution of the equation $\nabla^T F(\bar{x})\bar{\lambda} = \bar{v}$, i.e.,

$$\bar{\lambda} = \left(\nabla F(\bar{x})\nabla^T F(\bar{x})\right)^{-1} \nabla F(\bar{x})\bar{v}.$$

The value of transformation formula (2) relies on the fact that, starting with the co-derivative for normal cone mappings to simple objects (such as an orthant), one may pass to nonlinearly transformed constraint systems (such as differentiable inequalities). So, for instance, if

$$C = \{x \in \mathbb{R}^n \mid F_i(x) \le 0 \ (i = 1, \ldots, m)\},$$

where the F_i are twice continuously differentiable and $\bar{x} \in C$ satisfies the *Linear Independence Constraint Qualification*, then, putting

$$F := (F_1, \ldots, F_m)^T \quad P := \mathbb{R}^m_-,$$

one may calculate D^*N_C from $D^*N_{\mathbb{R}^m_-}$ via Theorem 3.1. To do so, one may access the following representation (see, e.g., [8,3,4, (Cor. 3.5)]) for any $(x, v) \in \operatorname{gr} N_{\mathbb{R}^m_-}$:

$$D^*N_{\mathbb{R}^m_-}(x, v)(v^*) = \begin{cases} \emptyset & \text{if } \exists i : v_i v_i^* \neq 0 \\ \{x^* | x_i^* = 0 \ \forall i \in I_1, x_i^* \ge 0 \ \forall i \in I_2\} & \text{else} \end{cases} \tag{3}$$

where

$$I_1 := \{i | x_i < 0\} \cup \{i | v_i = 0, v_i^* < 0\}, \qquad I_2 := \{i | x_i = 0, v_i = 0, v_i^* > 0\}.$$

Formula (2) is of use even in the linear case:

Corollary 3.1. *Let $C := \{x \in \mathbb{R}^n \mid Ax \le b\}$, where $b \in \mathbb{R}^m$ and A is some matrix of order (m, n) having rank m. Then, for $\bar{x} \in C$ and $\bar{v} \in N_C(\bar{x})$, it holds that*

$$D^*N_C(\bar{x}, \bar{v})(v^*) = \begin{cases} \emptyset & \text{if } \exists i : \bar{\lambda}_i \left\langle a_i, v^*\right\rangle \neq 0 \\ \{A^T x^* | x_i^* = 0 \ \forall i \in \tilde{I}_1, x_i^* \ge 0 \ \forall i \in \tilde{I}_2\} & \text{else} \end{cases}$$

where

$$\tilde{I}_1 := \{i | \langle a_i, \bar{x}\rangle < b_i\} \cup \{i | \bar{\lambda}_i = 0, \left\langle a_i, v^*\right\rangle < 0\}, \qquad \tilde{I}_2 := \{i | \langle a_i, \bar{x}\rangle = b_i, \bar{\lambda}_i = 0, \left\langle a_i, v^*\right\rangle > 0\},$$

$\bar{\lambda} = \left(AA^T\right)^{-1} A\bar{v}$ and the a_i refer to the rows of A.

Proof. Putting $F(x) := Ax - b$, (2) yields that

$$D^*N_C(\bar{x}, \bar{v})(v^*) = A^T D^*N_{\mathbb{R}^m_-}\left(A\bar{x} - b, \bar{\lambda}\right)\left(Av^*\right).$$

Now, the result follows from (3). □

The full-rank assumption in the corollary can in fact be localized, thus allowing the formula to be applied to any regular polyhedra defined by possibly many inequalities. Of course in this case, the matrix A must then replaced by the submatrix corresponding to active inequalities.

3.2. Nonregular constraint systems — polyhedral image sets

Corollary 3.1 does not, however, apply to nonregular polyhedra. For example, one cannot derive a co-derivative formula for the polyhedral set $x_3 \geq \max\{|x_1|, |x_2|\}$. Nevertheless, by using the well-known representation of polyhedral normal cone mappings from Dontchev and Rockafellar ([8], proof of Theorem 2), one may derive an explicit co-derivative formula for arbitrary polyhedra

$$C := \{x \in \mathbb{R}^n \mid Ax \leq b\},$$

where $b \in \mathbb{R}^m$ and A is some matrix of order (m, n). To this aim, denote by a_i the rows of A and consider arbitrary $\bar{x} \in C$ and $\bar{v} \in N_C(\bar{x})$. Then, $\bar{v} = A^T\lambda$ for some $\lambda \in \mathbb{R}^m_+$. Define, the index sets $I := \{i|\langle a_i, \bar{x}\rangle = b_i\}$ and $J := \{j|\lambda_j > 0\}$, then it is easily noted that $J \subseteq I$. Finally, with each $I' \subseteq I$ associate its characteristic index set $\chi(I')$ consisting of those indices $j \in I$ such that for all $h \in \mathbb{R}^n$ the following implication holds true:

$$\langle a_i, h\rangle \leq 0 \quad (i \in I \setminus I'), \qquad \langle a_i, h\rangle = 0 \quad (i \in I') \Longrightarrow \langle a_j, h\rangle = 0.$$

Clearly, $I' \subseteq \chi(I') \subseteq I$, $\chi(I') \subseteq \chi(I'')$ for $I' \subseteq I''$ and $I' = \chi(I')$ if the submatrix $\{a_i\}_{i\in I}$ has full rank. Given this setting, the following relations hold true ([4], Prop. 3.2 and Cor. 3.4):

Theorem 3.2. *With the notation introduced above, one has that*

$$D^*N_C(\bar{x}, \bar{v})(v^*) = \left\{x^* \,\middle|\, (x^*, -v^*) \in \bigcup_{J\subseteq I_1\subseteq I_2\subseteq I} P_{I_1,I_2} \times Q_{I_1,I_2}\right\}, \tag{4}$$

where

$$P_{I_1,I_2} = \text{con}\,\{a_i|i \in \chi(I_2) \setminus I_1\} + \text{span}\,\{a_i|i \in I_1\}$$
$$Q_{I_1,I_2} = \{h \in \mathbb{R}^n|\,\langle a_i, h\rangle = 0\,(i \in I_1)\,, \langle a_i, h\rangle \leq 0\,(i \in \chi(I_2) \setminus I_1)\}$$

and 'con' and 'span' refer to the convex conic and linear hulls, respectively.

A more convenient expression avoiding the union above can be used in the following upper estimation:

$$D^*N_C(\bar{x}, \bar{v})(v^*) \subseteq \text{con}\,\{a_i|i \in \chi\left(I^a(v^*) \cup I^b(v^*)\right) \setminus I^a(v^*)\} + \text{span}\,\{a_i|i \in I^a(v^*)\} \tag{5}$$

if $\langle a_i, v^*\rangle = 0$ *for all* $i \in J$ *and* $\langle a_i, v^*\rangle \geq 0$ *for all* $i \in \chi(J) \setminus J$*, whereas* $D^*N_C(\bar{x}, \bar{v})(v^*) = \emptyset$ *otherwise. Here,*

$$I^a(v^*) := \{i \in I|\langle a_i, v^*\rangle = 0\}, \quad I^b(v^*) := \{i \in I|\langle a_i, v^*\rangle > 0\}.$$

Corollary 3.2. $D^*N_C(\bar{x}, \bar{v})(0) = \text{span}\,\{a_i|i \in I\}$.

Proof. Since $0 \in Q_{I_1,I_2}$ for any index sets I_1, I_2, it follows from (4) that

$$D^*N_C(\bar{x}, \bar{v})(0) = \bigcup_{J\subseteq I_1\subseteq I_2\subseteq I} P_{I_1,I_2} = P_{I,I}.$$

Here, the last equality relies on the fact that $P_{I_1,I_2} \subseteq P_{I_3,I_4}$, whenever $I_1 \subseteq I_3$ and $I_2 \subseteq I_4$. □

To illustrate these characterizations, consider the following two examples:

Example 3.1. Let $C := Ax \leq 0$, where

$$A := \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Put $\bar{x} := 0$ and $\bar{v} := a_1 + a_2 = (0, 0, 2)$. Then, $I = \{1, 2, 3, 4\}$, $J = \{1, 2\}$ and $\chi(J) = I$. Referring to (5), the condition '$\langle a_i, v^*\rangle = 0$ for all $i \in J$ and $\langle a_i, v^*\rangle \geq 0$ for all $i \in \chi(J) \setminus J$' reduces to '$v^* = 0$'. Moreover, $I^a(0) = I$ and $I^b(0) = \emptyset$. Consequently, $D^*N_C(\bar{x}, \bar{v})(v^*) = \emptyset$ for $v^* \neq 0$. On the other hand, by Corollary 3.2,

$$D^*N_C(\bar{x}, \bar{v})(0) = \text{span}\,\{a_i|i \in I\} = \text{Im}\,A^T = \mathbb{R}^3.$$

Example 3.2. In the previous example, put $\bar{x} := 0$ and $\bar{v} := a_1 + a_3 = (-1, -1, 2)$. Then, $I = \{1, 2, 3, 4\}$, $J = \{1, 3\}$ and $\chi(J) = J$. Now, the condition '$\langle a_i, v^*\rangle = 0$ for all $i \in J$ and $\langle a_i, v^*\rangle \geq 0$ for all $i \in \chi(J) \setminus J$' reduces to $v_1^* = v_2^* = v_3^*$. Consequently, $D^*N_C(\bar{x}, \bar{v})(v^*) = \emptyset$ if this last identity is violated. If it holds true, then $D^*N_C(\bar{x}, \bar{v})(0) = \mathbb{R}^3$ by Corollary 3.2 and

$$D^*N_C(\bar{x}, \bar{v})(t, t, t) \subseteq \begin{cases} \text{con}\,\{a_2, a_4\} + \text{span}\,\{a_1, a_3\} & \text{if } t > 0 \\ \text{span}\,\{a_1, a_3\} & \text{if } t < 0. \end{cases}$$

This follows easily from (5), from the definition of A and from the already stated identity $\chi(\{1, 3\}) = \{1, 3\}$.

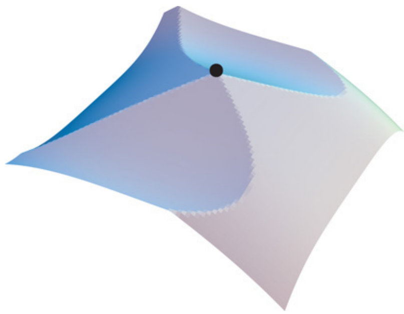

Fig. 1. Illustration of the boundary of the constraint set in Example 3.3.

We combine the previous results for general linear and regular nonlinear constraint systems in order to calculate the co-derivative in a special nonregular, nonlinear setting. We assume that

$$C := \{x \in \mathbb{R}^n \mid AF(x) \le b\}, \tag{6}$$

where $F : \mathbb{R}^n \to \mathbb{R}^s$ is twice continuously differentiable, $b \in \mathbb{R}^m$ and A is some matrix of order (m, s). Suppose also that $\nabla F(\bar{x})$ is surjective. Note that in order to calculate D^*N_C, we cannot invoke Theorem 3.1 because surjectivity of $\nabla(AF)(\bar{x})$ may be violated. Nevertheless, we may rewrite the constraint set as

$$C := F^{-1}(P), \quad P := \{y \in \mathbb{R}^s \mid Ay \le b\} \tag{7}$$

and then apply Theorem 3.1, recalling that we are able to calculate D^*N_P via Theorem 3.2. We illustrate this fact in the following example:

Example 3.3. Let

$$C := \{(x_1, x_2, x_3) \mid x_3 \le -\|(x_1 + x_1^3 + x_2^4, x_1^3 + x_2 - x_2^3)\|_\infty\}.$$

Evidently, C can be equivalently represented by the nonlinear inequality system

$$\begin{aligned} -x_1 - x_1^3 - x_2^4 + x_3 &\le 0 \\ x_1 + x_1^3 + x_2^4 + x_3 &\le 0 \\ -x_1^3 - x_2 + x_2^3 + x_3 &\le 0 \\ x_1^3 + x_2 - x_2^3 + x_3 &\le 0. \end{aligned}$$

Fig. 1 illustrates the boundary of this constraint set. At $\bar{x} = 0 \in C$, all inequalities are active, so their gradients cannot be linearly independent (the nonregularity can also be recognized from Fig. 1, where the graph exhibits four creases meeting at $\bar{x}$). This prevents an application of Theorem 3.1. However, we may write C in the form (7), where $b = 0$, A is as in Example 3.1 and

$$F(x) = (x_1 + x_1^3 + x_2^4, x_1^3 + x_2 - x_2^3, x_3)^T.$$

Evidently, $\nabla F(0) = I_3$ is surjective. As a normal vector $\bar{v} \in N_C(0)$ choose for example $\bar{v} = (-1, -1, 2)$. Because of $\nabla^2 F_i(0) = 0$ for $i = 1, 2, 3$, Theorem 3.1 provides the formula

$$D^*N_C(0, \bar{v})(v^*) = D^*N_P(0, \bar{v})\left(v^*\right).$$

Hence, we may use for $D^*N_C(0, \bar{v})$ exactly the same estimates as obtained in Example 3.2.

3.3. Nonregular constraint systems — the use of calmness

In [3] (Th. 3.1), it was shown how the assumption of calmness for a certain multifunction allows one to weaken the surjectivity condition in a result like Theorem 3.1 to a condition that, in the setting of Theorem 3.1, amounts to the *Mangasarian–Fromovitz Constraint Qualification* (MFCQ). Specifying those ideas to our setting, one gets the following generalization of Theorem 3.1:

Theorem 3.3. *Consider the set $C = \{x \in \mathbb{R}^n \mid F_i(x) \le 0 \ (i = 1, \dots, m)\}$, where $F : \mathbb{R}^n \to \mathbb{R}^m$ is twice continuously differentiable. Fix some $\bar{x} \in C$ and $\bar{v} \in N_C(\bar{x})$ such that, without loss of generality, $F(\bar{x}) = 0$ and suppose that the following two constraint qualifications are fulfilled:*

1. *The rows of $\{\nabla F(\bar{x})\}$ are positively linearly independent (i.e., MFCQ is satisfied at $\bar{x}$)*

2. *The multifunction*

$$M(\vartheta) := \{(x, \lambda) \mid (F(x), \lambda) + \vartheta \in \operatorname{Gr} N_{\mathbb{R}^m_-}\}$$

is calm at $\left(0, \bar{x}, \bar{\lambda}\right)$ *for all* $\bar{\lambda} \geq 0$ *with* $\nabla^T F(\bar{x})\bar{\lambda} = \bar{v}$.

Then,

$$D^* N_C(\bar{x}, \bar{v})(v^*) \subseteq \bigcup_{\bar{\lambda} \geq 0, \nabla^T F(\bar{x})\bar{\lambda} = \bar{v}} \left\{ \left(\sum_{i=1}^m \bar{\lambda}_i \nabla^2 F_i(\bar{x}) \right) v^* + \nabla^T F(\bar{x}) D^* N_{\mathbb{R}^m_-}\left(0, \bar{\lambda}\right)\left(\nabla F(\bar{x})\, v^*\right) \right\}.$$

As a first application of Theorem 3.3, we recover an alternative estimate of (4) and (5) in terms of dual (multipliers) rather than primal (characteristic index sets) objects.

Corollary 3.3. *Let*

$$C := \{x \in \mathbb{R}^n \mid Ax \leq b\},$$

where $b \in \mathbb{R}^m$ *and* A *is some matrix of order* (m, n). *Fix some* $\bar{x} \in C$ *and* $\bar{v} \in N_C(\bar{x})$ *with* $A\bar{x} = b$. *If there exists some* $\xi \in \mathbb{R}^n$ *such that* $A\xi < 0$ *(component-wise), then, with the notation of Theorem 3.2,*

$$D^* N_C(\bar{x}, \bar{v})(v^*) \subseteq \operatorname{con}\{a_i | \langle a_i, v^* \rangle > 0\} + \operatorname{span}\{a_i | \langle a_i, v^* \rangle = 0\},$$

whenever there exists some $\bar{\lambda} \geq 0$ *such that* $A^T\bar{\lambda} = \bar{v}$ *and* $\bar{\lambda}_i \langle a_i, v^* \rangle = 0$ *for all* $i = 1, \ldots, m$. *Otherwise,* $D^* N_C(\bar{x}, \bar{v})(v^*) = \emptyset$.

Proof. In the setting of Theorem 3.3, put $F(x) := Ax - b$. Observe, that the existence of $\xi \in \mathbb{R}^n$ such that $A\xi < 0$ implies via Gordan's Theorem the first constraint qualification of the Theorem. Moreover, the multifunction considered in the second constraint qualification happens to be polyhedral, so it is calm by Robinson's well-known upper Lipschitz result for polyhedral multifunctions. Hence, one may conclude that

$$D^* N_C(\bar{x}, \bar{v})(v^*) \subseteq \bigcup_{\bar{\lambda} \geq 0, A^T\bar{\lambda} = \bar{v}} A^T D^* N_{\mathbb{R}^m_-}\left(0, \bar{\lambda}\right)\left(Av^*\right). \tag{8}$$

From (3) we derive that the union on the right-hand side takes place only if $\bar{\lambda}_i \langle a_i, v^* \rangle = 0$ for all $i = 1, \ldots, m$, in which case

$$\begin{aligned} A^T D^* N_{\mathbb{R}^m_-}\left(0, \bar{\lambda}\right)\left(Av^*\right) &= \{A^T x^* | x_i^* = 0\ \forall i : \bar{\lambda}_i = 0, \left\langle a_i, v^* \right\rangle < 0; x_i^* \geq 0\ \forall i : \bar{\lambda}_i = 0, \langle a_i, v^* \rangle > 0\} \\ &= \operatorname{con}\{a_i | \langle a_i, v^* \rangle > 0\} + \operatorname{span}\{a_i | \langle a_i, v^* \rangle = 0\}. \end{aligned}$$

This yields the assertion of the corollary. □

The last corollary provides a more handy formula for calculating the coderivative of normal cone mappings to polyhedra when compared to Theorem 3.2, where characteristic index sets need to be calculated, on the other hand, it may be less precise than the latter in certain circumstances. This shall be illustrated by revisiting Examples 3.1 and 3.2:

Example 3.4 (Example 3.2 *Revisited*). With the data from Example 3.2, the only $\bar{\lambda} \geq 0$ with $A^T\bar{\lambda} = \bar{v}$ is $\bar{\lambda} = (1, 0, 1, 0)$. Hence, by Corollary 3.3,

$$D^* N_C(\bar{x}, \bar{v})(v^*) \neq \varnothing \iff \langle a_1, v^* \rangle = \langle a_3, v^* \rangle = 0 \iff v_1^* = v_2^* = v_3^*.$$

Moreover,

$$D^* N_C(\bar{x}, \bar{v})(t, t, t) \subseteq \begin{cases} \operatorname{span}\{a_1, a_2, a_3, a_4\} = \mathbb{R}^3 & \text{if } t = 0 \\ \operatorname{con}\{a_2, a_4\} + \operatorname{span}\{a_1, a_3\} & \text{if } t > 0 \\ \operatorname{span}\{a_1, a_3\} & \text{if } t < 0. \end{cases}$$

Thus, we completely recover the results of Example 3.2 obtained via Theorem 3.2.

Example 3.5 (Example 3.1 *Revisited*). With the data from Example 3.1, there are three possibilities for $\bar{\lambda} \geq 0$ with $A^T\bar{\lambda} = \bar{v}$: $\bar{\lambda} = (0, 0, 1, 1)$, $\bar{\lambda} = (1, 1, 0, 0)$ and $\bar{\lambda} = (r, r, s, s)$ for $r, s > 0$ and $r + s = 1$. Now, Corollary 3.3 implies

$$D^* N_C(\bar{x}, \bar{v})(v^*) \subseteq \begin{cases} \operatorname{span}\{a_1, a_2, a_3, a_4\} = \mathbb{R}^3 & \text{if } v^* = 0 \\ \operatorname{con}\{a_1\} + \operatorname{span}\{a_3, a_4\} & \text{if } v_2^* = v_3^* = 0,\ v_1^* < 0 \\ \operatorname{con}\{a_2\} + \operatorname{span}\{a_3, a_4\} & \text{if } v_2^* = v_3^* = 0,\ v_1^* > 0 \\ \operatorname{con}\{a_3\} + \operatorname{span}\{a_1, a_2\} & \text{if } v_1^* = v_3^* = 0,\ v_2^* < 0 \\ \operatorname{con}\{a_4\} + \operatorname{span}\{a_1, a_2\} & \text{if } v_1^* = v_3^* = 0,\ v_2^* < 0 \\ \emptyset & \text{else.} \end{cases}$$

In contrast to this result, the application of Theorem 3.2 in Example 3.1 has shown that $D^* N_C(\bar{x}, \bar{v})(v^*) = \emptyset$ whenever $v^* \neq 0$. In other words, the formula of Corollary 3.3 creates some additional artificial expressions in the coderivative formula.

We now turn to an application of Theorem 3.3 in a nonlinear setting. The crucial calmness condition required there has been investigated in [9] (see Th. 2, Th. 6, Ex. 6). In general, the conditions for calmness used there may be difficult to verify. Therefore, we provide a different characterization here, where the calmness property needs to be verified only for certain constraint systems defined as subsystems of the original inequality constraints in the space of x-variables. For the definition of calmness used in the following, we refer to Section 2.

Proposition 3.1. *If for all* $\emptyset \neq I \subseteq \{1, \ldots, m\}$ *the multifunctions*

$$H_I(\alpha) = \{x \mid F_i(x) = \alpha_i (i \in I), F_i(x) \leq 0 (i \in I^c)\}$$

are calm at $(0, \bar{x})$*, then the multifunction* M *introduced in Theorem 3.3 is calm at* $\left(0, \bar{x}, \bar{\lambda}\right)$ *for any* $\bar{\lambda}$ *specified there.*

Proof. Throughout this proof we use the 1-norm of vectors. Note first, that for $I = \emptyset$, H_I is trivially calm as a constant multifunction. Hence, this special case can be excluded from the assumption. Next, observe that, by $F(\bar{x}) = 0$, one has indeed $(0, \bar{x}) \in \operatorname{gr} H_I$ for all $I \subseteq \{1, \ldots, m\}$. The calmness assumption means that for any $I \subseteq \{1, \ldots, m\}$, there exist constants $\delta_I, \varepsilon_I, L_I > 0$ such that

$$d(x, H_I(0)) \leq L_I \|\alpha\| \quad \forall x \in \mathbb{B}_{\delta_I}(\bar{x}) \cap H_I(\alpha)\ \forall \alpha : \alpha_i \in (-\varepsilon_I, \varepsilon_I)\ (i \in I).$$

Putting

$$\delta := \min_{I \subseteq \{1,\ldots,m\}} \delta_I, \qquad \varepsilon := \min_{I \subseteq \{1,\ldots,m\}} \varepsilon_I, \qquad L := \max_{I \subseteq \{1,\ldots,m\}} L_I,$$

one obtains that $\delta, \varepsilon, L > 0$ and

$$d(x, H_I(0)) \leq L \|\alpha\| \\ \forall x \in \mathbb{B}_{\delta}(\bar{x}) \cap H_I(\alpha) \quad \forall \alpha : \alpha_i \in (-\varepsilon, \varepsilon)\ (i \in I), \forall I \subseteq \{1, \ldots, m\}. \tag{9}$$

Due to $F(\bar{x}) = 0$, we may further shrink $\delta > 0$ such that

$$|F_i(x)| \leq \varepsilon \quad \forall x \in \mathbb{B}_{\delta}(\bar{x}) \quad \forall i \in \{1, \ldots, m\}. \tag{10}$$

Now, consider any $\bar{\lambda} \geq 0$. Then, $\bar{\lambda} \in N_{\mathbb{R}^m_-}(0)$ and so $\left(\bar{x}, \bar{\lambda}\right) \in M(0)$. We show that

$$d((x, \lambda), M(0)) \leq (L + 1) \|\vartheta\| \\ \forall (x, \lambda) \in M(\vartheta) \cap \left(\mathbb{B}_{\delta}(\bar{x}) \times \mathbb{R}^m\right) \quad \forall \vartheta = (\vartheta_1, \vartheta_2) \in \mathbb{B}_{\varepsilon}(0) \times \mathbb{R}^m. \tag{11}$$

This would prove the asserted calmness of M at $\left(0, \bar{x}, \bar{\lambda}\right)$. To this aim, choose arbitrary $\vartheta = (\vartheta_1, \vartheta_2) \in \mathbb{B}_{\varepsilon}(0) \times \mathbb{R}^m$ and $(x, \lambda) \in M(\vartheta) \cap (\mathbb{B}_{\varepsilon}(\bar{x}) \times \mathbb{R}^m)$. Note first that $(x, \lambda) \in M(\vartheta)$ amounts to $\lambda + \vartheta_2 \in N_{\mathbb{R}^m_-}(F(x) + \vartheta_1)$. Accordingly,

$$F(x) + \vartheta_1 \leq 0, \quad \lambda + \vartheta_2 \geq 0, \quad (\lambda_i + \vartheta_{2i})(F_i(x) + \vartheta_{1i}) = 0 \quad \forall i \in \{1, \ldots, m\}. \tag{12}$$

For the fixed x, define

$$I_x := \{i \in \{1, \ldots, m\} | F_i(x) + \vartheta_{1i} = 0 \text{ or } F_i(x) \geq 0\}.$$

Choose $\tilde{x} \in H_{I_x}(0)$ such that $\|x - \tilde{x}\| = d(x, H_{I_x}(0))$. Note that by definition of I_x, $F_i(x) < 0$ for all $i \in (I_x)^c$. Consequently, $x \in \mathbb{B}_{\varepsilon}(\bar{x}) \cap H_{I_x}(\alpha)$ for α defined by

$$\alpha_i := F_i(x) \quad (i \in I_x).$$

Since also (10) ensures that $\alpha_i \in (-\varepsilon, \varepsilon)$ for all $i \in I_x$, we may apply (9) to derive that

$$d(x, H_{I_x}(0)) \leq L \|\alpha\| = L \sum_{i \in I_x} |F_i(x)|.$$

Now, if $i \in I_x$ is such that $F_i(x) + \vartheta_{1i} = 0$, then $|F_i(x)| = |\vartheta_{1i}|$. Otherwise, by (12), $F_i(x) + \vartheta_{1i} < 0$ and, by definition of I_x, $F_i(x) \geq 0$. This implies $|F_i(x)| \leq |\vartheta_{1i}|$. In any case we may conclude that

$$\|x - \tilde{x}\| = d(x, H_{I_x}(0)) \leq L \|\vartheta_1\|.$$

Next, define $\tilde{\lambda} \in \mathbb{R}^m$ by $\tilde{\lambda}_i := \lambda_i + \vartheta_{2i}$ if $i \in I_x$ and $\tilde{\lambda}_i := 0$ if $i \in (I_x)^c$. Then, $\tilde{\lambda} \geq 0$ by (12). Moreover, $\tilde{x} \in H_{I_x}(0)$ entails that $F_i(\tilde{x}) = 0$ if $i \in I_x$ and $F_i(\tilde{x}) \leq 0$ if $i \in (I_x)^c$. In particular, $\tilde{\lambda}_i F_i(\tilde{x}) = 0$ for all $i \in \{1, \ldots, m\}$. This means that $\tilde{\lambda} \in N_{\mathbb{R}^m_-}\left(F(\tilde{x})\right)$ and, hence, $(\tilde{x}, \tilde{\lambda}) \in M(0)$. Finally, observe that, for $i \in (I_x)^c$, one has $F_i(x) + \vartheta_{1i} < 0$ and, thus, by (12), $\lambda_i = -\vartheta_{2i}$. This proves that $\tilde{\lambda} - \lambda = \vartheta_2$. Consequently,

$$\begin{aligned} d((x, \lambda), M(0)) &\leq \|(x, \lambda) - (\tilde{x}, \tilde{\lambda})\| = \|x - \tilde{x}\| + \|\lambda - \tilde{\lambda}\| \\ &\leq L\|\vartheta_1\| + \|\vartheta_2\| \leq (L + 1) \|\vartheta\| \end{aligned}$$

which shows (11). □

We refer the reader to [10] for methods to check calmness of constraint systems like those given by the multifunctions H_I in the previous proposition. The next proposition shows how to get rid of inequalities for the verification of calmness in the previous proposition. More precisely, calmness must only be checked for all equality subsystems. This proposition, which yields a slightly stronger result than needed, requires a technical lemma, the proof of which is shifted to the Appendix (Lemma 3.1).

Proposition 3.2. *If for all $I \subseteq \{1, \ldots, m\}$ the multifunctions*

$$\tilde{H}_I(\alpha) := \{x | F_i(x) = \alpha_i \, (i \in I)\}$$

are calm at $(0, \bar{x})$, then the multifunctions

$$\bar{H}_I(\alpha) = \left\{x | F_i(x) = \alpha_i \, (i \in I), \quad F_i(x) \le \alpha_i \left(i \in I^c\right)\right\}$$

are also calm at $(0, \bar{x})$ for all $I \subseteq \{1, \ldots, m\}$. In particular, the multifunctions $H_I(\alpha)$ introduced in Proposition 3.1 are calm at $(0, \bar{x})$ for all $I \subseteq \{1, \ldots, m\}$.

Proof. We proceed by induction over the number m of components of F. Consider first the case $m = 1$. We either have $I = \emptyset$ or $I = \{1\}$. In the second case, one has $\bar{H}_I = \tilde{H}_I$ due to $m = 1$, hence calmness of $\bar{H}_I$ follows from that of $\tilde{H}_I$. In the first case, we apply Lemma 3.1 proved in the Appendix. Referring to the notation of this lemma, we put $I^* = \emptyset$ and check the two assumptions made there. As the only set $I \subseteq \{1\}$ with $I \neq I^*$ is given by $I = \{1\}$ and then, as before, $\bar{H}_I = \tilde{H}_I$, calmness of $\bar{H}_I$ follows from that of $\tilde{H}_I$. This shows the first assumption of Lemma 3.1 to hold true. Concerning the second assumption, one has $i' = 1$ and, hence, M reduces to the trivial constant multifunction $M(\alpha, \beta) \equiv \mathbb{R}^n$ which is calm. On the other hand, the second multifunction introduced there reduces to $\bar{M} = \tilde{H}_I$, hence calmness of $\bar{M}$ follows from that of $\tilde{H}_I$. As a consequence, Lemma 3.1 yields calmness of $\bar{H}_{I^*} = \bar{H}_\emptyset$. Summarizing, the assertion of our proposition follows for the case $m = 1$. Next assume that the Proposition holds true for all $m \le k$ and consider the case $m = k + 1$. By assumption, the $\tilde{H}_I$ are calm at $(0, \bar{x})$ for all $I \subseteq \{1, \ldots, k + 1\}$. In particular, the multifunction $\bar{M}$ considered in the second assumption of Lemma 3.1 and corresponding to the case $\#I = 1$ is calm at $(0, \bar{x})$. Moreover, the induction hypothesis ensures that also the multifunctions

$$\{x | F_i(x) = \alpha_i \, (i \in I), F_i(x) \le \alpha_i \, (i \in J \diagdown I)\} \tag{13}$$

are calm at $(0, \bar{x})$ for all subsets $I \subseteq J$ and all $J \subseteq \{1, \ldots, k + 1\}$ with $\#J = k$. Since the multifunction M considered in the second assumption of Lemma 3.1 is of type (13) with $J = \{1, \ldots, k + 1\} \diagdown \left\{i'\right\}$, it follows that M is calm at $(0, 0, \bar{x})$. Summarizing, the second assumption of Lemma 3.1 is always satisfied no matter how the index set $I^* \subseteq \{1, \ldots, k + 1\}$ is chosen in the Lemma. Therefore, it is enough to check the first assumption for its application.

Now, choose an arbitrary $I^* \subseteq \{1, \ldots, k + 1\}$. We have to show that $\bar{H}_{I^*}$ is calm at $(0, \bar{x})$. If $I^* = \{1, \ldots, k + 1\}$, then $\bar{H}_{I^*} = \tilde{H}_{I^*}$ and calmness of $\bar{H}_{I^*}$ follows from that of $\tilde{H}_{I^*}$. If $\#I^* = k$, then the only choice for the index set I considered in the first assumption of Lemma 3.1 is $I = \{1, \ldots, k + 1\}$. According to what we have shown just before, $\bar{H}_I$ is calm, so we have shown that the $\bar{H}_{I^*}$ are calm at $(0, \bar{x})$ whenever $\#I^* \ge k$. Passing to the case $\#I^* = k - 1$ and recalling that the index set I considered in the first assumption of Lemma 3.1 is always strictly larger than I^*, one derives calmness of $\bar{H}_I$ on the basis of what we have shown before due to $\#I > \#I^* = k - 1$ which amounts to $\#I \ge k$. So, the first assumption of Lemma 3.1 is satisfied again and we derive calmness of $\bar{H}_{I^*}$ whenever $\#I^* \ge k - 1$. Proceeding this way until $\#I^* = 0$, we get the desired calmness at $(0, \bar{x})$ for all subsets $I^* \subseteq \{1, \ldots, k + 1\}$.

That the calmness of the $\bar{H}_I$ implies the calmness of the corresponding H_I introduced in Proposition 3.1, is an immediate consequence of the calmness definition and of the evident relations $\bar{H}_I(\alpha, 0) = H_I(\alpha)$. □

We emphasize that without considering subsystems, a result analogous to Proposition 3.2 cannot be obtained for a single constraint system. For instance, for $F(x) := (x^2, x)$ one has that the equality system $F_1(x) = \alpha_1, F_2(x) = \alpha_2$ is calm at $(0, 0)$, whereas the inequality system $F_1(x) \le \alpha_1, F_2(x) \le \alpha_2$ is not. The reason is that subsystems need not inherit calmness (for instance, the equality subsystem $F_1(x) = \alpha_1$ fails to be calm at $(0, 0)$).

We may combine Theorem 3.3, Propositions 3.1 and 3.2 to get an assumption which completely relies on constraint systems induced by F and thus can be considered to be a CQ (weaker than surjectivity) for the mapping F.

Theorem 3.4. *In the setting of Theorem 3.3 assume that*

1. *MFCQ is satisfied at $\bar{x}$;*
2. *all perturbed equality subsystems*

$$\{x \mid F_i(x) = \alpha_i(i \in I)\} \quad I \subseteq \{1, \ldots, m\}$$

are calm at $(0, \bar{x})$.

Then, the coderivative formula of Theorem 3.3 holds true.

Remark 3.1. If we consider the couple of constraint qualifications imposed in Theorem 3.4 as a single one and give it the name CQ*, then the following holds true for the inequality system $F(x) \le 0$:

$$\text{LICQ} \Longrightarrow \text{CQ}^* \Longrightarrow \text{MFCQ},$$

where MFCQ and LICQ refer to the Mangasarian–Fromovitz and Linear Independence constraint qualifications, respectively, with the latter being the same as the surjectivity condition imposed in Theorem 3.1. Indeed, the second implication being evident, suppose that $F(x) \leq 0$ satisfies LICQ at $\bar{x}$. Then, all gradients $\{\nabla F_i(\bar{x})\}_{i=1,\ldots,m}$ – and trivially all subsets of gradients – are linearly independent. But linear independence of a set of gradients implies the Aubin property and, hence, calmness for the corresponding set of equations. Consequently, CQ^* follows from LICQ. Summarizing, CQ^* is something in between LICQ and MFCQ and it seems that it is closely related to the constant rank constraint qualification CRCQ (see [11]).

At the end of this section, we provide a useful and easy to check constraint qualification ensuring condition 2. in Theorem 3.4.

Proposition 3.3. *Assume that at $\bar{x}$ the following full rank constraint qualification is satisfied:*

$$rank\ \{\nabla F_i\,(\bar{x})\}_{i\in I} = \min\{n, \#I\} \quad \forall I \subseteq \{1,\ldots,m\}\,. \tag{14}$$

Then, the multifunctions $\tilde{H}_I$ introduced in Proposition 3.2 are calm at $(0,\bar{x})$ for all $I \subseteq \{1,\ldots,m\}$.

Proof. Choose an arbitrary $I \subseteq \{1,\ldots,m\}$. Consider first the case that $\#I \leq n$. Then, by (14), the set of gradients $\{\nabla F_i\,(\bar{x})\}_{i\in I}$ is linearly independent. Consequently, $\tilde{H}_I$ is calm at $(0,\bar{x})$. Now, if $\#I > n$, then select an arbitrary $J \subseteq I$ with $\#J = n$. By (14), the set of gradients $\{\nabla F_i\,(\bar{x})\}_{i\in J}$ is linearly independent, hence $\tilde{H}_J(0) = \{\bar{x}\}$ by the inverse function theorem. Since $F\,(\bar{x}) = 0$ and $\tilde{H}_I(0) \subseteq \tilde{H}_J(0)$, it follows that $\tilde{H}_I(0) = \tilde{H}_J(0)$. Moreover, according to what has been mentioned before, $\tilde{H}_J$ is calm at $(0,\bar{x})$. Consequently, there are constants $L, \varepsilon > 0$ such that

$$d(x,\tilde{H}_J(0)) \leq L\,\|\tilde{\alpha}\| \quad \forall x \in \tilde{H}_J(\tilde{\alpha}) \cap \mathbb{B}_\varepsilon\,(\bar{x}) \quad \forall \tilde{\alpha} \in \mathbb{B}_\varepsilon\,(0)\,.$$

From here it follows with $\tilde{H}_I(\alpha) \subseteq \tilde{H}_J(\tilde{\alpha})$, where $\tilde{\alpha}$ is the subvector of α according to the index set $J \subseteq I$, that

$$d(x,\tilde{H}_I(0)) = d(x,\tilde{H}_J(0)) \leq L\,\|\tilde{\alpha}\| \leq L\,\|\alpha\| \quad \forall x \in \tilde{H}_I(\alpha) \cap \mathbb{B}_\varepsilon\,(\bar{x})\ \forall \alpha \in \mathbb{B}_\varepsilon\,(0)\,.$$

This, however, amounts to calmness of $\tilde{H}_I$ at $(0,\bar{x})$. □

As an illustration, we revisit Example 3.3. Because this example is nonlinear, we cannot take for granted that assumption 2. of Theorem 3.3 will be automatically satisfied as we could in Examples 3.4 and 3.5. On the other hand, the four constraint gradients in this example, though linearly dependent in $\mathbb{R}^3$ satisfy the full rank constraint qualification (14). Indeed, any of the 4 triples that can be selected from the original set of gradients happens to be a linearly independent set. Therefore, the second assumption of Theorem 3.4 is satisfied by virtue of Proposition 3.3. Since the Mangasarian–Fromovitz Constraint Qualification is easily seen to be fulfilled at $\bar{x}$, we may apply the co-derivative formula of Theorem 3.3. Doing so would yield the same result as in the linearized examples discussed before.

3.4. Beyond calmness

Before we address the issue of computing the co-derivative in the event that the calmness condition is violated, we provide an instructive counterexample to Theorem 3.3 showing that the provided formula does not hold when the calmness condition is dropped.

Example 3.6. Let $F\,(x_1,x_2) := (-x_2, \varphi(x_1) - x_2)$, where $\varphi(t) := t^5 \sin(1/t)$ for $t \neq 0$ and $\varphi(0) := 0$. Since φ is twice continuously differentiable, so is F and one has

$$\nabla F_1\,(x_1,x_2) = (0,-1)\,, \qquad \nabla F_2\,(x_1,x_2) = \left(\varphi'(x_1), -1\right). \tag{15}$$

We choose $\bar{x} := (0,0)$. Then, $F\,(\bar{x}) = (0,0)$ and, taking into account that $\varphi'(0) = 0$, it holds that

$$\nabla F_1\,(\bar{x}) = \nabla F_2\,(\bar{x}) = (0,-1)\,. \tag{16}$$

This means that both gradients are positively linearly independent (i.e., MFCQ is satisfied). Of course they are linearly dependent, thus preventing us from applying (2). Summarizing, all assumptions of Theorem 3.3 are fulfilled except calmness (this could be easily checked directly, but we shall see it as a consequence of the conclusion of that theorem being violated). We choose $v^* := 0$ and $\bar{v} := \nabla F_1\,(\bar{x}) + \nabla F_2\,(\bar{x})$ (implying that $\bar{v} \in N_C\,(\bar{x})$ in view of MFCQ). Formally applying Theorem 3.3 would yield the inclusion

$$D^*N_C\,(\bar{x},\bar{v})\,(0) \subseteq \bigcup_{\bar{\lambda}\geq 0,\nabla^T F(\bar{x})=\bar{v}} \nabla^T F\,(\bar{x})\, D^*N_{\mathbb{R}^2_-}\left(0,\bar{\lambda}\right)(0).$$

Given the fact that $D^*N_{\mathbb{R}^2_-}\left(0,\bar{\lambda}\right)(0) = \mathbb{R}^2$, regardless of the value of $\bar{\lambda} \geq 0$ (see (3)), and taking into account (16), we end up at the inclusion

$$D^*N_C\,(\bar{x},\bar{v})\,(0) \subseteq \{0\} \times \mathbb{R}. \tag{17}$$

To see that this is wrong, consider the sequences

$$x^k := (1/(k\pi), 0), \qquad v_k := \nabla F_1\left(x^k\right) + \nabla F_2\left(x^k\right).$$

Then taking into account that $\varphi(x_1^k) = 0$ and, thus, $F\left(x^k\right) = (0, 0)$, it follows that

$$x^k \to \bar{x}, \quad x^k \in C, \qquad v_k \to \bar{v}, \quad v_k \in N_C\left(x^k\right).$$

Here, the last relation relies on the fact that MFCQ is an open property and and pertains to hold at x^k close to $\bar{x}$. Furthermore, we observe that $\varphi'(x_1^k) \neq 0$, which, as a consequence of (15), implies $\nabla F\left(x^k\right)$ is surjective; in fact, $\nabla F\left(x^k\right)$ is even a regular matrix. This allows to apply (2) at $\left(x^k, v_k\right)$:

$$D^* N_C\left(x^k, v_k\right)(0) = \nabla^T F\left(x^k\right) D^* N_{\mathbb{R}^2_-}(0, (1, 1))(0) = \nabla^T F\left(x^k\right) \mathbb{R}^2 = \mathbb{R}^2,$$

where the last equality follows from the fact that, rank $\nabla^T F\left(x^k\right) = \operatorname{rank} \nabla F\left(x^k\right) = 2$. Exploiting the robustness property of the co-derivative, we let $k \to \infty$ and thus derive

$$\mathbb{R}^2 \supseteq D^* N_C(\bar{x}, \bar{v})(0) \supseteq \operatorname*{Limsup}_{k\to\infty} D^* N_C\left(x^k, v_k\right)(0) = \mathbb{R}^2,$$

therefore $D^* N_C(\bar{x}, \bar{v})(0) = \mathbb{R}^2$, contradicting (17).

We see then that by dropping the calmness condition, one can no longer expect the formula of Theorem 3.3 to hold true. Nevertheless, the formula may serve as a part of calculating the co-derivative in a more elementary (according to its basic definition) aggregation process. More precisely, by introducing the set

$$C^* := \{x \in \operatorname{bd} C \mid \nabla F(x) \text{ is surjective}\},$$

we have for any $\bar{v}^* \in \mathbb{R}^n$ (see Section 2),

$$\begin{aligned} D^* N_C(\bar{x}, \bar{v})(\bar{v}^*) &= \operatorname*{Limsup}_{x\to\bar{x}, v\to\bar{v}, x\in C, v\in N_C(x), v^*\to\bar{v}^*} \hat{D}^* N_C(x, v)(\bar{v}^*) \\ &= \operatorname*{Limsup}_{x\to\bar{x}, v\to\bar{v}, x\in \operatorname{int} C, v\in N_C(x), v^*\to\bar{v}^*} \hat{D}^* N_C(x, v)(\bar{v}^*) \cup \underbrace{\left\{ \operatorname*{Limsup}_{x\to\bar{x}, x\in C\setminus C^*, v\to\bar{v}, v\in N_C(x), v^*\to\bar{v}^*} \hat{D}^* N_C(x, v)(\bar{v}^*) \right\}}_{P(\bar{v}^*)} \\ &\quad \cup \left\{ \operatorname*{Limsup}_{x\to\bar{x}, x\in C^*, v\to\bar{v}, v\in N_C(x), v^*\to\bar{v}^*} \hat{D}^* N_C(x, v)(\bar{v}^*) \right\} \\ &= \{0\} \cup P(\bar{v}^*) \cup \underbrace{\left\{ \operatorname*{Limsup}_{x\to\bar{x}, x\in C^*, v\to\bar{v}, v\in N_C(x), v^*\to\bar{v}^*} D^* N_C(x, v)(\bar{v}^*) \right\}}_{R(\bar{v}^*)}. \end{aligned}$$

Here, the first term is trivial and follows easily from the definition of the Fréchet coderivative evaluated in the interior of the feasible set, whereas the last equality results from the robustness (outer semicontinuity) of the co-derivative. In this way, we have subdivided the computation of the co-derivative into a **p**athological part $P(\bar{v}^*)$, where Fréchet coderivatives have to be calculated and aggregated in an elementary way, and a **r**egular part $R(\bar{v}^*)$, where we may exploit formula (2) in the aggregation process. It is important to observe that the pathological part is small in the following sense ([12], Th. 2.1): If the MFCQ is satisfied everywhere in the feasible set C, then the subset of points around which the feasible set may be locally described by a regular constraint system (i.e., with surjective Jacobian $\nabla F(x)$) is open and dense in the boundary of C. In the following, we derive some upper estimates of the regular part.

Proposition 3.4. *Let $C = \{x \in \mathbb{R}^n \mid F(x) \leq 0\}$, where $F \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^m)$. Consider some $\bar{x} \in C$ with $F(\bar{x}) = 0$ and $\bar{v} \in N_C(\bar{x})$. Assume that MFCQ is satisfied at $\bar{x}$. Then, for all $\bar{v}^* \in \mathbb{R}^n$,*

$$R\left(\bar{v}^*\right) \subseteq \bigcup_{\bar{\lambda}\geq 0, \nabla F(\bar{x})\bar{\lambda}=\bar{v}} \left\{ \left(\sum_{i=1}^m \bar{\lambda}_i \nabla^2 F_i(\bar{x}) \right) \bar{v}^* + \operatorname*{Limsup}_{x\to\bar{x}, x\in C^*} \left[\nabla^T F(x) D^* N_{\mathbb{R}^m_-}\left(0, \bar{\lambda}\right) \left(\nabla F(\bar{x}) \bar{v}^*\right) \right] \right\}.$$

Proof. Let $\bar{v}^* \in \mathbb{R}^n$ and $x^* \in R(\bar{v}^*)$ be arbitrarily given. By definition, there are sequences

$$x_k \to \bar{x}, \quad x_k \in C^*, \quad v_k \to \bar{v}, \quad v_k \in N_C(x_k), \quad v_k^* \to \bar{v}^*, \quad x_k^* \to x^*, \quad x_k^* \in D^* N_C(x_k, v_k)\left(v_k^*\right).$$

By the definition of C^*, $\nabla F(x_k)$ is surjective. Then, applying the transformation formula (2), we obtain

$$x_k^* \in \left(\sum_{i=1}^m \lambda_i^k \nabla^2 F_i(x_k)\right) v_k^* + \nabla^T F(x_k) D^* N_{\mathbb{R}^m_-}\left(F(x_k), \lambda^k\right)\left(\nabla F(x_k) v_k^*\right),$$

where $\lambda^k \geq 0$ is the unique solution of the equation $\nabla F(x_k)\lambda = v_k$. Therefore, we may write

$$x_k^* = \left(\sum_{i=1}^m \lambda_i^k \nabla^2 F_i(x_k)\right) v_k^* + w_k^*$$

for some

$$w_k^* \in \nabla^T F(x_k) D^* N_{\mathbb{R}^m_-}\left(F(x_k), \lambda^k\right)\left(\nabla F(x_k) v_k^*\right). \tag{18}$$

MFCQ being satisfied at $\bar{x}$ guarantees boundedness of the sequence $\left\{\lambda^k\right\}$ (see, e.g., Prop. 4.43 in [6]). Therefore, upon passing to a subsequence, which we do not relabel, we may assume that $\lambda^k \to \mu$ for some $\mu \geq 0$. It follows that $w_k^* \to w^*$ for

$$w^* := x^* - \left(\sum_{i=1}^m \mu_i \nabla^2 F_i(\bar{x})\right) \bar{v}^*. \tag{19}$$

Moreover, $\nabla F(x_k)\lambda^k = v_k$ entails that

$$\nabla F(\bar{x})\mu = \bar{v}. \tag{20}$$

We claim that, for k large enough,

$$D^* N_{\mathbb{R}^m_-}\left(F(x_k), \lambda^k\right)\left(\nabla F(x_k) v_k^*\right) \subseteq D^* N_{\mathbb{R}^m_-}(0, \mu)\left(\nabla F(\bar{x}) \bar{v}^*\right). \tag{21}$$

According to (3), we may write

$$D^* N_{\mathbb{R}^m_-}\left(F(x_k), \lambda^k\right)\left(\nabla F(x_k) v_k^*\right) = A_1^k \times \cdots \times A_m^k,$$

where

$$A_i^k = \begin{cases} \varnothing & \text{if } \lambda_i^k \nabla F_i(x_k) v_k^* \neq 0 \\ \{0\} & \text{if } \lambda_i^k = 0 \text{ and } \nabla F_i(x_k) v_k^* < 0 \\ \mathbb{R}_+ & \text{if } \lambda_i^k = 0 \text{ and } \nabla F_i(x_k) v_k^* > 0 \\ \mathbb{R} & \text{if } \nabla F_i(x_k) v_k^* = 0. \end{cases}$$

Similarly,

$$D^* N_{\mathbb{R}^m_-}(0, \mu)\left(\nabla F(\bar{x}) \bar{v}^*\right) = A_1 \times \cdots \times A_m,$$

where

$$A_i = \begin{cases} \varnothing & \text{if } \mu_i \nabla F_i(\bar{x}) \bar{v}^* \neq 0 \\ \{0\} & \text{if } \mu_i = 0 \text{ and } \nabla F_i(\bar{x}) \bar{v}^* < 0 \\ \mathbb{R}_+ & \text{if } \mu_i = 0 \text{ and } \nabla F_i(\bar{x}) \bar{v}^* > 0 \\ \mathbb{R} & \text{if } \nabla F_i(\bar{x}) \bar{v}^* = 0. \end{cases}$$

In order to verify (21) it is enough to show that $A_i^k \subseteq A_i$ for all large enough $k \in \mathbb{N}$ and all $i \in \{1, \ldots, m\}$. Let i be an arbitrary such index. We proceed by case distinction. If $\nabla F_i(\bar{x}) \bar{v}^* = 0$, then $A_i = \mathbb{R}$ and $A_i^k \subseteq A_i$ holds trivially. If $\nabla F_i(\bar{x}) \bar{v}^* < 0$, then $\nabla F_i(x_k) v_k^* < 0$ for all large enough k. In particular, $A_i^k \subseteq \{0\}$. If, additionally, $\mu_i = 0$, then $A_i = \{0\}$ showing again the inclusion to hold. Otherwise, if $\mu_i > 0$, then $\lambda_i^k > 0$ for all large enough and, so, $A_i^k = \varnothing$ which implies the inclusion a third time. A similar reasoning applies to the remaining case of $\nabla F_i(\bar{x}) \bar{v}^* > 0$ upon replacing $\{0\}$ by $\mathbb{R}_+$. This finally proves (21).

Multiplying (21) from the left by $\nabla^T F(x_k)$, we may infer from (18) that

$$w_k^* \in \nabla^T F(x_k) D^* N_{\mathbb{R}^m_-}(0, \mu)\left(\nabla F(\bar{x}) \bar{v}^*\right).$$

In other words,

$$w^* \in \underset{x \to \bar{x}, x \in C^*}{\text{Limsup}} \left[\nabla^T F(x) D^* N_{\mathbb{R}^m_-}(0, \mu)\left(\nabla F(\bar{x}) \bar{v}^*\right)\right]$$

which entails via (19) and (20) that

$$x^* \in \left(\sum_{i=1}^m \mu_i \nabla^2 F_i(\bar{x})\right)\bar{v}^* + \operatorname*{Limsup}_{x\to\bar{x},x\in C^*}\left[\nabla^T F(x) D^* N_{\mathbb{R}^m_-}(0,\mu)\left(\nabla F(\bar{x})\bar{v}^*\right)\right]$$
$$\subseteq \bigcup_{\bar{\lambda}\geq 0,\nabla F(\bar{x})\bar{\lambda}=\bar{v}}\left\{\left(\sum_{i=1}^m \bar{\lambda}_i \nabla^2 F_i(\bar{x})\right)\bar{v}^* + \operatorname*{Limsup}_{x\to\bar{x},x\in C^*}\left[\nabla^T F(x) D^* N_{\mathbb{R}^m_-}\left(0,\bar{\lambda}\right)\left(\nabla F(\bar{x})\bar{v}^*\right)\right]\right\}. \quad \square$$

If the mapping

$$x \mapsto \nabla^T F(x) D^* N_{\mathbb{R}^m_-}\left(0,\bar{\lambda}\right)\left(\nabla F(\bar{x})\bar{v}^*\right)$$

happens to be outer semicontinuous, then one could replace the 'Limsup' in the formula of the last Proposition by the limiting expression

$$\nabla^T F(\bar{x}) D^* N_{\mathbb{R}^m_-}\left(0,\bar{\lambda}\right)\left(\nabla F(\bar{x})\bar{v}^*\right)$$

and would be back to the formula of Theorem 3.3. Unfortunately, this outer semicontinuity does not hold true in general (without the calmness condition of Theorem 3.3), such that the 'Limsup' may become strictly larger (see Example 3.6). Nevertheless, we may verify outer semicontinuity in a special situation:

Proposition 3.5. *In the setting of Proposition 3.4, suppose that*

$$\nabla F_i(\bar{x})\bar{v}^* \neq 0 \quad i = 1,\ldots,m.$$

Then, for any $\bar{\lambda} \geq 0$ with $\nabla F(\bar{x})\bar{\lambda} = \bar{v}$, one has that

$$\operatorname*{Limsup}_{x\to\bar{x},x\in C^*}\left[\nabla^T F(x) D^* N_{\mathbb{R}^m_-}\left(0,\bar{\lambda}\right)\left(\nabla F(\bar{x})\bar{v}^*\right)\right] \subseteq \nabla^T F(\bar{x}) D^* N_{\mathbb{R}^m_-}\left(0,\bar{\lambda}\right)\left(\nabla F(\bar{x})\bar{v}^*\right).$$

Proof. Consider an arbitrary w^* in the left-hand side of the asserted inclusion. Accordingly, there are sequences $x_k \to \bar{x}$ and $w_k^* \to w^*$ such that $x_k \in C^*$ and $w_k^* = \nabla^T F(x_k)\alpha^k$ for certain

$$\alpha^k \in D^* N_{\mathbb{R}^m_-}\left(0,\bar{\lambda}\right)\left(\nabla F(\bar{x})\bar{v}^*\right).$$

Recalling the representation of the set $D^* N_{\mathbb{R}^m_-}\left(0,\bar{\lambda}\right)(\nabla F(\bar{x})\bar{v}^*)$ provided in the proof of Proposition 3.4 (with $\bar{\lambda}$ replaced by μ), we see that $\alpha^k \in \mathbb{R}^m_+$. Now an argument already used in the proof of Proposition 3.4 (based on MFCQ being satisfied at $\bar{x}$) allows to derive boundedness of the sequence $\{\alpha^k\}$. Hence, $\alpha^{k_l} \to \alpha$ for some subsequence and some $\alpha \geq 0$. It follows that $w^* = \nabla^T F(\bar{x})\alpha$. Since $\alpha \in D^* N_{\mathbb{R}^m_-}\left(0,\bar{\lambda}\right)(\nabla F(\bar{x})\bar{v}^*)$ by closedness of this latter set, we are done. $\square$

Corollary 3.4. *In the setting of Proposition 3.4 and under the additional assumption of Proposition 3.5, it holds true that*

$$R\left(\bar{v}^*\right) = \varnothing \quad \text{if } \bar{v} \neq 0$$

and

$$R\left(\bar{v}^*\right) \subseteq \operatorname{con}\left\{\nabla^T F_i(\bar{x}) \,|\, i \in I\right\} \quad \text{if } \bar{v} = 0,$$

where 'con' denotes the convex conic hull and $I := \{i | \nabla F_i(\bar{x})\bar{v}^* > 0\}$.

Proof. If $\bar{v} \neq 0$, then in the representation $\nabla F(\bar{x})\bar{\lambda} = \bar{v}$ there must be at least one i such that $\bar{\lambda}_i > 0$. Since, by assumption in Proposition 3.5 one also has that $\nabla F_i(\bar{x})\bar{v}^* \neq 0$, it follows from (3) that $D^* N_{\mathbb{R}^m_-}\left(0,\bar{\lambda}\right)(\nabla F(\bar{x})\bar{v}^*) = \varnothing$. As a consequence of Propositions 3.4 and 3.5, $R(\bar{v}^*) = \varnothing$. If, in contrast, $\bar{v} = 0$, then also $\bar{\lambda} = 0$ (by MFCQ being satisfied at $\bar{x}$), hence (see (3) again),

$$D^* N_{\mathbb{R}^m_-}\left(0,\bar{\lambda}\right)\left(\nabla F(\bar{x})\bar{v}^*\right) = A_1 \times \cdots \times A_m,$$

where either $A_i = \{0\}$ (if $\nabla F_i(\bar{x})\bar{v}^* < 0$) or $A_i = \mathbb{R}_+$ (if $\nabla F_i(\bar{x})\bar{v}^* > 0$). This proves the second inclusion along with Propositions 3.4 and 3.5. $\square$

Unfortunately, it seems to be difficult to find similar meaningful upper estimates for $R(\bar{v}^*)$ if the assumption of Proposition 3.5 is violated. Similarly, it remains an open question if there is a general characterization of the previously mentioned contribution $P(\bar{v}^*)$ along the pathological set $C \setminus C^*$.

Appendix

Lemma 3.1. *Fix an arbitrary $I^* \subseteq \{1,\ldots,m\}$. Referring back to the multifunctions $\bar{H}_I$ introduced in Proposition 3.2, assume that*

1. *For all $I \neq I^*$ with $I^* \subseteq I \subseteq \{1,\ldots,m\}$ the $\bar{H}_I$ are calm at $(0,\bar{x})$.*
2. *For some $i' \in I \setminus I^*$ the multifunctions*

$$M(\alpha,\beta) := \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{cases} F_i(x) = \alpha_i \left(i \in I^*\right), \\ F_j(x) \le \beta_j \left(j \in \{1,\ldots,m\} \setminus \left(I^* \cup \left\{i'\right\}\right)\right) \end{cases} \right\},$$

$$\bar{M}(t) := \left\{x \in \mathbb{R}^n | F_{i'}(x) = t\right\}$$

are calm at $(0,0,\bar{x})$ and $(0,\bar{x})$, respectively.

Then, $\bar{H}_{I^*}$ is calm at $(0,\bar{x})$.

Proof. Assume that $\bar{H}_{I^*}$ fails to be calm at $(0,\bar{x})$. Then, by (1), there is a sequence $x_k \to \bar{x}$ such that

$$d(x_k, \bar{H}_{I^*}(0)) > k\left(\sum_{i\in I^*} |F_i(x_k)| + \sum_{j\in\{1,\ldots,m\}\setminus I^*} \left[F_j(x_k)\right]_+\right). \tag{22}$$

Suppose there is some index $j' \in \{1,\ldots,m\} \setminus I^*$ and some subsequence x_{k_l} with $F_{j'}\left(x_{k_l}\right) \ge 0$. Put $I' := I^* \cup \left\{j'\right\}$. Due to $\bar{H}_{I'}(0) \subseteq \bar{H}_{I^*}(0)$ and to $x_{k_l} \in \bar{H}_{I'}\left(F\left(x_{k_l}\right)\right)$ one would arrive from (22) at

$$d(x_{k_l}, \bar{H}_{I'}(0)) > k_l\left(\sum_{i\in I'} \left|F_i\left(x_{k_l}\right)\right| + \sum_{j\in\{1,\ldots,m\}\setminus I'} \left[F_j\left(x_{k_l}\right)\right]_+\right),$$

a contradiction with assumption 1. Hence, there is some k_0 such that

$$F_j(x_k) < 0 \quad \forall k \ge k_0 \quad \forall j \in \{1,\ldots,m\} \setminus I^*. \tag{23}$$

Together with (22), this implies that

$$d(x_k, \bar{H}_{I^*}(0)) > k \sum_{i\in I^*} |F_i(x_k)|. \tag{24}$$

We claim the existence of some $\rho > 0$ and $k_1 \ge k_0$ such that

$$\sum_{i\in I^*} |F_i(x_k)| > \rho\,|F_{i'}(x_k)| \quad \forall k \ge k_1, \tag{25}$$

where i' refers to assumption 2. Indeed, otherwise there was a subsequence x_{k_l} such that

$$\sum_{i\in I^*} \left|F_i\left(x_{k_l}\right)\right| \le l^{-1}\left|F_{i'}\left(x_{k_l}\right)\right|.$$

In the following, we lead this relation to a contradiction. Now, justified by $\bar{x} \in \bar{M}(0) \neq \emptyset$, where $\bar{M}$ is defined in assumption 2, we may select for any l some $y_l \in \bar{M}(0)$ such that

$$d(x_{k_l}, \bar{M}(0)) = \left\|x_{k_l} - y_l\right\|.$$

The assumed calmness at $(0,\bar{x})$ of $\bar{M}$ entails the existence of some $L_1 > 0$ such that

$$d(x_{k_l}, \bar{M}(0)) \le L_1\left|F_{i'}\left(x_{k_l}\right)\right| \to_l 0$$

which in turn implies that $y_l \to \bar{x}$. Consequently, for all large enough l,

$$\left|F_{i'}\left(x_{k_l}\right)\right| = \left|F_{i'}\left(x_{k_l}\right) - F_{i'}(y_l)\right| \le L_2\left\|x_{k_l} - y_l\right\|$$

where L_2 is some Lipschitz modulus of $F_{i'}$ near $\bar{x}$. Now, referring to the multifunction M defined in assumption 2, we observe by virtue of (23) that, for all large enough l, $x_{k_l} \in M\left(\alpha^{(l)}, 0\right)$, where $\alpha_i^{(l)} := F_i\left(x_{k_l}\right)$ for $i \in I^*$. Now, the assumed calmness at $(0,\bar{x})$ of M leads to

$$\begin{aligned} d(x_{k_l}, M(0,0)) &\le L_3\left\|\alpha^{(l)}\right\| = L_3\sum_{i\in I^*}\left|F_i\left(x_{k_l}\right)\right| \le l^{-1}L_3\left|F_{i'}\left(x_{k_l}\right)\right| \\ &\le l^{-1}L_3L_2\left\|x_{k_l} - y_l\right\| = l^{-1}L_3L_2\,d(x_{k_l}, \bar{M}(0)), \end{aligned}$$

for all large enough l. If also $l > L_3 L_2$, then

$$d(x_{k_l}, M(0,0)) < d(x_{k_l}, \bar{M}(0)). \tag{26}$$

Now, justified by $\bar{x} \in M(0,0) \neq \emptyset$, we may select $z_l \in M(0,0)$ such that

$$d(x_{k_l}, M(0,0)) = \left\| x_{k_l} - z_l \right\| \quad \forall l.$$

It follows from (26) that $z_l \notin \bar{M}(0)$, whence $F_{i'}(z_l) \neq 0$. Recalling that $F_{i'}\left(x_{k_l}\right) < 0$ for large enough l (see (23)), one would find in case of $F_{i'}(z_l) > 0$ some z' on the line segment $\left[x_{k_l}, z_l\right]$ with $F_{i'}\left(z'\right) = 0$ and $\left\| x_{k_l} - z' \right\| < \left\| x_{k_l} - z_l \right\|$ yielding a contradiction with (26) due to $z' \in \bar{M}(0)$. Therefore, $F_{i'}(z_l) < 0$ and, hence, one may invoke the definition of M to infer from $z_l \in M(0,0)$ that $z_l \in \bar{H}_{I^*}(0)$ for large enough l. Now, (23) and (24) provide, for large enough l that

$$k_l \left(\sum_{i \in I^*} |F_i(x_k)| + \sum_{j \in \{1,\ldots,m\} \setminus (I^* \cup \{i'\})} \left[F_j\left(x_{k_l}\right)\right]_+ \right) = k_l \sum_{i \in I^*} \left| F_i\left(x_{k_l}\right) \right| < d(x_{k_l}, \bar{H}_{I^*}(0)) \leq \left\| x_{k_l} - z_l \right\| = d(x_{k_l}, M(0,0)),$$

a contradiction with the assumed calmness at $(0, 0, \bar{x})$ of M. This contradiction proves the desired relation (25). Using this, we may continue (24) as

$$\begin{aligned} d(x_k, \bar{H}_{I^*}(0)) &> k \left(\frac{1}{\rho+1} \sum_{i \in I^*} |F_i(x_k)| + \frac{\rho}{\rho+1} \sum_{i \in I^*} |F_i(x_k)| \right) \\ &> k \frac{\rho}{\rho+1} \left(\sum_{i \in I^* \cup \{i'\}} |F_i(x_k)| \right) \\ &= k \frac{\rho}{\rho+1} \left(\sum_{i \in I^* \cup \{i'\}} |F_i(x_k)| + \sum_{j \in \{1,\ldots,m\} \setminus (I^* \cup \{i'\})} \left[F_j(x_k)\right]_+ \right) \\ &\quad \forall k \geq k_1, \end{aligned}$$

where in the last relation, we exploited again (23). Put $I' := I^* \cup \left\{i'\right\}$. Due to $\bar{H}_{I'}(0) \subseteq \bar{H}_{I^*}(0)$ we end up at the relation

$$d(x_k, \bar{H}_{I'}(0)) > k \frac{\rho}{\rho+1} \left(\sum_{i \in I'} |F_i(x_k)| + \sum_{j \in \{1,\ldots,m\} \setminus I'} \left[F_j(x_k)\right]_+ \right) \quad \forall k \geq k_1.$$

This, however, is in contradiction with the assumed calmness at $(0, \bar{x})$ of $\bar{H}_{I'}$ (see assumption 1.) Hence, we have finally led to a contradiction our initial assumption that $\bar{H}_{I^*}$ fails to be calm at $(0, \bar{x})$. □

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