Energetic approach to gradient plasticity

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We formulate a problem of the evolution of elasto-plastic materials subjected to external loads in the framework of large deformations and multiplicative plasticity. Our model includes gradients of the plastic strain and of hardening variables. We prove the existence of the so-called energetic solution. The stored energy density function is assumed to be quasiconvex in the elastic strain which makes our results applicable to relaxed models of shape memory materials, for instance.

1 Introduction, notation

The elastic-plastic behavior of crystalline materials poses a challenge for mathematical analysis on the microscopic, the mesoscopic, and the macroscopic scales. Here, we study a rate-independent model arising in the crystal plasticity. A common and successful approach to the analysis of crystalline materials is via energy minimization; see e.g. Ortiz & Repetto [31]. This is manifested for elastic crystals, even for those with the potential of undergoing phase transitions. The applicability of variational methods has been broadened to include rate-independent evolution. Typically, these models are characterized by energy minimization of a functional including macroscopic quantities such as the macroscopic deformation gradient as well as a dissipation functional. In order to introduce a physically relevant scale to our problem we assume, following earlier works of Dillon & Kratochvíl [8], Gurtin and Gurtin & Anand [15, 16], Mainik & Mielke [21] and others that our energy functional depends also on the gradient of the plastic tensor. The gradient term models non-local effects caused by short-range interactions among dislocations. It is not clear, however, which function of the gradient should be used. We refer to Kratochvíl & Sedláček [18], to Bakó & Groma [1], and to Zaiser et al. [35] for attempts to derive it from statistical physics revealing thus complexity of the problem, for more details. A related approach to non-local models in damage and plasticity was undertaken in Bažant & Jirásek [3], see also [7, 9, 10, 12, 22].

In this paper, we formulate the so-called *energetic solution* due to Mielke et al. [29] to our problem. This concept of solution is based on two requirements. First, as a consequence of the conservation law for linear momentum, all work put into the system by external forces or boundary conditions is spent on increasing the stored energy or it is dissipated. Secondly, the formulation must satisfy the second law of thermodynamics, which has in the present mechanical framework the form of a dissipation inequality. The last requirement enters the framework as the assumption of the existence of a nonnegative convex potential of dissipative forces. As a consequence the imposed deformation evolves in such a way that the sum of stored and dissipated energies is always minimized. The main advantage of this approach is that it allows us to exploit theory of the modern calculus of variations and suggests a numerical approach to this problem.

To expose the essence of the mathematical structure of the energetic approach we first analyze a proto-model called here a material with internal variables. It freely follows the exposition of Francfort & Mielke [11] and we recall it here to motivate the notion of the energetic solution. In the second step, the framework is applied to elasto-plastic materials by specifications of some internal variables. One of the main results is that the described energetic approach can be identified with crystal plasticity with strain gradients in the version formulated by Gurtin [15]. Gurtin's model is formulated in the

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mathematical language of differential equations. From the point of view of numerical solution of a boundary value problem of crystal plasticity the energetic formulation is more convenient.

Our results are closely related to [21] where the authors proved the existence of energetic solutions to strain gradient plasticity with polyconvex energy density allowing even for $+\infty$; see [2] and to Giacomini & Lussardi [13] where the linear-elastoplasticity framework is considered. Here we allow for finite quasiconvex stored energy density and large deformations. This is motivated by relaxation theory in the calculus of variation where the effective macroscopic energy density is quasiconvexification of the microscopic one. Thus, our results may be applied to plasticity of materials with developing microstructures as in shape memory alloys, [4], or [26] for instance. We refer an interested reader to [19] for a model describing cyclic plasticity in these materials. Another related paper is Carstensen et al. [5] where the authors use the energetic approach to plasticity without strain gradients.

In what follows, $\Omega \subset \mathbb{R}^n$, is an open bounded domain, $L^\beta(\Omega; \mathbb{R}^n)$, $1 \leq \beta < +\infty$ denotes the usual Lebesgue space of mappings $\Omega \to \mathbb{R}^n$ whose modulus is integrable with the power β and $L^\infty(\Omega; \mathbb{R}^n)$ is the space of measurable and essentially bounded mappings $\Omega \to \mathbb{R}^n$. Further, $W^{1,\beta}(\Omega; \mathbb{R}^n)$ standardly represents the space of mappings which live in $L^\beta(\Omega; \mathbb{R}^n)$ and their gradients belong to $L^\beta(\Omega; \mathbb{R}^{n \times n})$. Finally, $W_0^{1,\beta}(\Omega; \mathbb{R}^n)$ is a subspace of $W^{1,\beta}(\Omega; \mathbb{R}^n)$ of maps with the zero trace on $\partial\Omega$. The weak convergence in $L^\beta(\Omega; \mathbb{R}^n)$ is defined as follows: $y_k \to y$ weakly in $L^\beta(\Omega; \mathbb{R}^n)$ if $\int_\Omega y_k(x) \cdot \varphi(x) \, dx \to \int_\Omega y(x) \cdot \varphi(x) \, dx$ for all $\varphi \in L^{\beta'}(\Omega; \mathbb{R}^n)$ where $\beta' = \beta/(\beta - 1)$ if $1 < \beta < +\infty$, $\beta' = 1$ if $\beta = +\infty$ and $\beta' = +\infty$ for $\beta = 1$. Weak convergence of mappings and gradients in L^β then defines the weak convergence in $W^{1,\beta}(\Omega; \mathbb{R}^n)$. Finally, $C(\Omega)$ or $C(\mathbb{R}^{n \times n})$ denotes function spaces of functions continuous on Ω or $\mathbb{R}^{n \times n}$, respectively, and $C^1(\Omega)$ denotes the spaces of continuously differentiable functions.

If $f : \mathbb{R}^n \to \mathbb{R}$ is convex but possibly nonsmooth we define its subdifferential at a point $x_0 \in \mathbb{R}^n$ as the set of all $v \in \mathbb{R}^n$ such that $f(x) \geq f(x_0) + v \cdot (x - x_0)$ for all $x \in \mathbb{R}^n$. The subdifferential of f will be denoted $\partial^{\text{sub}} f$ and its elements will be called subgradients of f at x_0.

2   Materials with internal variables

Consider a material whose elastic properties depend on internal variables $z \in Z \subset \mathbb{R}^m$. The stored energy density is then $\mathcal{W} = \mathcal{W}(F_e, z)$, where $F_e \in \mathbb{R}^{n \times n}$ is the elastic strain. We are interested in the *rate-independent* evolution of the material. To this end, we assume the existence of a nonnegative convex potential $\delta = \delta(\dot{z})$ of dissipative forces, where $\dot{z}$ denotes the time derivative of z. In order to ensure rate-independence, δ must be positively one-homogeneous, i.e., $\delta(\alpha\dot{z}) = \alpha\delta(\dot{z})$ for all $\alpha > 0$. Finally, we define for $z \in Z$ a thermodynamic force

$$Q := -\frac{\partial}{\partial z} \mathcal{W}(F_e, z) . \tag{1}$$

The evolution rule is introduced in the form

$$Q(t) \in \partial^{\text{sub}}\delta(\dot{z}(t)), \tag{2}$$

where $\partial^{\text{sub}}\delta$ is the subdifferential of δ. Hence, there is $\omega(t) \in \partial^{\text{sub}}\delta(\dot{z}(t))$ such that $Q(t) = \omega(t)$. Maximal monotonicity of the subdifferential implies that for all $\theta \in \partial^{\text{sub}}\delta(\xi)$ we have

$$\langle \omega(t) - \theta, \dot{z}(t) - \xi \rangle \geq 0 . \tag{3}$$

Remark 2.1. In particular, taking $\xi = 0$ and realizing that the one-homogeneity of δ yields $\delta(\dot{z}) = \langle \omega, \dot{z} \rangle$ for all $\omega \in \partial^{\text{sub}}\delta(\dot{z})$ we get

$$\delta(\dot{z}(t)) = \langle \omega(t), \dot{z}(t) \rangle = \langle Q(t), \dot{z}(t) \rangle \geq \langle \theta, \dot{z}(t) \rangle \rangle \tag{4}$$

for all $\theta \in \partial^{\text{sub}}\delta(0)$. Inequality (4) expresses the so-called *maximum dissipation principle* (see e.g. Hill [17] or Simo [33]) which says that thermodynamic forces "available" in the so-called *elastic domain* $\partial^{\text{sub}}\delta(0)$ are not strong enough to overcome frictional forces.

In what follows, $\Omega \subset \mathbb{R}^n$, is a bounded Lipschitz domain representing the so-called reference configuration, ν is the outer unit normal to $\partial\Omega$, and $\partial\Omega \supset \Gamma_0, \Gamma_1$ which are disjoint. The elastic deformation will be denoted $y : \Omega \to \mathbb{R}^n$. The evolution of the system will be controlled by external forces. Let $f(t) : \Omega \to \mathbb{R}^n$ be the (volume) density of external body forces and $g(t) : \Gamma_1 \subset \partial\Omega \to \mathbb{R}^n$ be the (surface) density of surface forces. The equilibrium equations governing mechanical behavior of the system are:

$$-\text{div}\left(\frac{\partial}{\partial \nabla y} \mathcal{W}(\nabla y(t), z(t))\right) = f(t) \text{ in } \Omega , \tag{5}$$

$$y(t, x) = y_0(x) \text{ on } \Gamma_0 \,, \tag{6}$$

$$\frac{\partial}{\partial \nabla y} \mathcal{W}(\nabla y(t), z(t)) \nu(x) = g(t, x) \text{ on } \Gamma_1 \,. \tag{7}$$

The full system characterizing the proto-model consists of (5)-(7) supplemented by (2):

$$-\frac{\partial}{\partial z} \mathcal{W}(\nabla y(t), z(t)) \in \partial \delta(\dot{z}(t)) \,, \ z(0) = z_0 \,, \ z \in Z \,, \tag{8}$$

where $z_0 \in Z$ is an initial condition for the internal variable.

In order to regularize our problem we may add the gradient of the internal variable , i.e., for $\omega \geq 1$ and $\varepsilon > 0$ put

$$\mathcal{W}(\nabla y, z) + \frac{\varepsilon}{\omega} |\nabla z|^\omega \,. \tag{}$$

The evolution rule changes to

$$\varepsilon \mathrm{div}(|\nabla z(t)|^{\omega-2} \nabla z(t)) - \frac{\partial}{\partial z} \mathcal{W}(\nabla y(t), z(t)) \in \partial \delta(\dot{z}(t)) \,, \tag{9}$$

$$z(0) = z_0 \,, z \in Z \,,$$

so we have the thermodynamic force

$$Q(t) := \varepsilon \mathrm{div}(|\nabla z(t)|^{\omega-2} \nabla z(t)) - \frac{\partial}{\partial z(t)} \mathcal{W}(\nabla y(t), z(t)) \,. \tag{10}$$

The potential energy of our system can be written ($\epsilon := \varepsilon/\omega$)

$$\mathcal{I}(t, y(t), z(t)) := \int_\Omega \mathcal{W}(\nabla y(t), z(t)) \, \mathrm{d}x + \epsilon \int_\Omega |\nabla z(t)|^\omega \, \mathrm{d}x - L(t, y(t)) \,, \tag{11}$$

where the work done by external forces is

$$L(t, y(t)) := \int_\Omega f(t) \cdot y(t) \, \mathrm{d}x + \int_{\Gamma_1} g(t) \cdot y(t) \, \mathrm{d}S \tag{12}$$

and the following energy balance is satisfied

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{I}(t, y(t), z(t)) = \dot{L}(t, y(t)) - \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{Diss}(z; [0, t]) \,, \tag{13}$$

where

$$\mathrm{Diss}(z; [0, t]) := \int_0^t \int_\Omega \delta(\dot{z}(s)) \, \mathrm{d}x \mathrm{d}s \,. \tag{}$$

Hence, the integration with respect to time gives

$$\mathcal{I}(t, y(t), z(t)) + \mathrm{Diss}(z; [0, t]) = \mathcal{I}(0, y(0), z(0)) + \int_0^t \dot{L}(s, y(s)) \, \mathrm{d}s \,. \tag{}$$

We can also consider a more general form of δ which can also depend on (x, z), i.e. $\delta := \delta(x, z, \dot{z})$.

Typically, however, we do not have enough smoothness in the internal variable to compute the time derivative on the right-hand side of (13).

Following Mielke [23] we define a dissipation distance between two values of internal variables $z_0, z_1 \in Z$ as

$$D(x, z_0, z_1) := \inf_z \left\{ \int_0^1 \delta(x, z(s), \dot{z}(s)) \, \mathrm{d}s; \ z(0) = z_0 \,, \ z(1) = z_1 \right\} \,, \tag{14}$$

where $z \in C^1([0,1]; Z)$, and set

$$\mathcal{D}(z_1, z_2) = \int_\Omega D(x, z_1(x), z_2(x)) \, \mathrm{d}x \, , \tag{15}$$

where $z_1, z_2 \in \mathbb{Z} := \{z : \Omega \to \mathbb{R}^M; \ z(x) \in Z \text{ a.e. in } \Omega\}$. We assume that $\mathbb{Z}$ is equipped with strong and weak topologies which define notions of convergence used below.

Following [11, 21] we impose the following assumptions on $\mathcal{D}$: (i) Weak lower semicontinuity:

$$\mathcal{D}(z, \tilde{z}) \le \liminf_{k \to \infty} \mathcal{D}(z_k, \tilde{z}_k) \, , \tag{16}$$

whenever $z_k \rightharpoonup z$ and $\tilde{z}_k \rightharpoonup \tilde{z}$.

(ii) Positivity: If $\{z_k\} \subset \mathbb{Z}$ is bounded and $\min\{\mathcal{D}(z_k, z), \mathcal{D}(z, z_k)\} \to 0$ then

$$z_k \rightharpoonup z \, . \tag{17}$$

2.1 Energetic solution

Suppose that we look for the time evolution of $y(t) \in \mathbb{Y} \subset \{y : \Omega \to \mathbb{R}^n\}$ and $z(t) \in \mathbb{Z}$ during the time interval $[0, T]$. The following two properties are the key ingredients of the so-called energetic solution due to Mielke and Theil [28, 29].

(i) Stability inequality:

$\forall t \in [0, T] \, , \ \tilde{z} \in \mathbb{Z} \, , y \in \mathbb{Y}:$

$$\mathcal{I}(t, y(t), z(t)) \le \mathcal{I}(t, \tilde{y}, \tilde{z}) + \mathcal{D}(z(t), \tilde{z}) \tag{18}$$

(ii) Energy balance: $\forall \, 0 \le t \le T$

$$\mathcal{I}(t, y(t), z(t)) + \mathrm{Var}(\mathcal{D}, z; [0, t]) = \mathcal{I}(s, y(0), z(0)) + \int_0^t \dot{L}(\xi, y(\xi)) \, \mathrm{d}\xi \, , \tag{19}$$

where

$$\mathrm{Var}(\mathcal{D}, z; [s, t]) := \sup\left\{ \sum_{i=1}^N \mathcal{D}(z(t_i), z(t_{i-1})); \ \{t_i\} \text{ partition of } [s, t] \right\} . \tag{}$$

Definition 2.2. The mapping $t \mapsto (y(t), z(t)) \in \mathbb{Y} \times \mathbb{Z}$ is an energetic solution to the problem $(\mathcal{I}, \delta, L)$ if the stability inequality and the energy balance are satisfied.

Remark 2.3. Notice that the stability inequality (i) can be written in the form $\forall t \in [0, T] \, , \tilde{z} \in \mathbb{Z}, \tilde{y} \in \mathbb{Y}:$

$$\mathcal{I}(t, y(t), z(t)) + \mathcal{D}(z(t), z(t)) \le \mathcal{I}(t, \tilde{y}, \tilde{z}) + \mathcal{D}(z(t), \tilde{z}) \, , \tag{}$$

i.e., that $y(t), z(t)$ always minimizes $(\tilde{y}, \tilde{z}) \mapsto \mathcal{I}(t, \tilde{y}, \tilde{z}) + \mathcal{D}(z(t), \tilde{z})$. It means that the equilibrium configurations are characterized by energetic minima. Contrary to elasticity theory the minimized energy is not only the overall elastic one described by $\mathcal{I}$ but the dissipated energy is added.

It is convenient to put $\mathbb{Q} := \mathbb{Y} \times \mathbb{Z}$ and to set $q := (y, z)$. Moreover, we define the set of stable states at time t as

$$\mathcal{S}(t) := \{q \in \mathbb{Q} : \forall \tilde{q} \in \mathbb{Q} : \ \mathcal{I}(t, q) \le \mathcal{I}(t, \tilde{q}) + \mathcal{D}(q, \tilde{q})\} \tag{20}$$

and

$$\mathcal{S}_{[0,T]} := \cup_{t \in [0,T]} \{t\} \times \mathcal{S}(t) \, . \tag{21}$$

Moreover, a sequence $\{(t_k, q_k)\}_{k \in \mathbb{N}}$ is called *stable* if $q_k \in \mathcal{S}(t_k)$.

3 Applications to elasto-plasticity

Now we apply the energetic approach to an elasto-plastic problem.

3.1 Problem statement

In what follows $y : \Omega \to \mathbb{R}^n$ will be a deformation of a body $\Omega \subset \mathbb{R}^n$ (in a fixed reference configuration) with the deformation gradient $F = \nabla y$. In particular, y covers both *elastic*, as well as *plastic* deformation. We define the multiplicative split, $F = F_{\mathrm{e}} F_{\mathrm{p}}$, into an elastic part F_{e} and an irreversible plastic part F_{p} which belongs to $\mathrm{SL}(n) := \{A \in \mathbb{R}^{n \times n}; \det A = 1\}$. The so-called plastic strain F_{p} and the vector $p \in \mathbb{R}^m$ of hardening variables are internal variables influencing elasticity. In other words, $z(x) = (F_{\mathrm{p}}(x), p(x)) \in \mathrm{SL}(n) \times \mathbb{R}^m$ for almost all $x \in \Omega$.

The energy functional $\mathcal{I}$ takes the form

$$\mathcal{I}(t, y(t), z(t)) := \int_\Omega \mathcal{W}(x, \nabla y F_{\mathrm{p}}^{-1}, F_{\mathrm{p}}, \nabla F_{\mathrm{p}}, p, \nabla p)\, \mathrm{d}x - L(t, y(t)) \tag{22}$$

with L given by (12).

In order to ease the notation we omit the dependence of $\mathcal{W}$ on x, however, all the theory developed in this paper may include nonhomogeneous $\mathcal{W}$, too.

In what follows, we suppose that

$$y \in \mathbb{Y} := \{y \in W^{1,d}(\Omega; \mathbb{R}^n); \ y = y_0 \text{ on } \Gamma_0\},$$

where $\Gamma_0 \subset \partial\Omega$ with a positive surface measure. Moreover, we suppose that $\Gamma_0 \cap \Gamma_1 = \emptyset$. Further

$$\mathbb{Z} := \{(F_p, p) \in W^{1,\beta}(\Omega; \mathbb{R}^{n \times n}) \times W^{1,\omega}(\Omega; \mathbb{R}^m) : F_p(x) \in \mathrm{SL}(n) \text{ for a.e. } x \in \Omega\}.$$

As $q = (y, z)$ it will be advantageous and will make no confusion to write $\mathcal{D}$ as dependent on q, i.e.,

$$\mathcal{D}(q_1, q_2) := \mathcal{D}(z_1, z_2)$$

if $q_1 = (y_1, z_1)$ and $q_2 = (y_2, z_2)$. Similarly, we may write $\mathcal{I}$ in terms of $q = (y, z)$ as

$$\mathcal{I}(t, q(t)) = \int_\Omega \mathcal{W}(x, \nabla y F_{\mathrm{p}}^{-1}, F_{\mathrm{p}}, \nabla F_{\mathrm{p}}, p, \nabla p)\, \mathrm{d}x - L(t, q(t)),$$

where, obviously, $L(t, q(t) := L(t, y(t))$.

In this situation, $Q = (Q_1, Q_2)$ are *conjugate plastic stress* and *conjugate hardening forces*, respectively,

$$Q_1 = \mathrm{div}\left(\frac{\partial \mathcal{W}(\nabla y F_{\mathrm{p}}^{-1}, F_{\mathrm{p}}, \nabla F_{\mathrm{p}}, p, \nabla p)}{\partial \nabla F_{\mathrm{p}}}\right) - \frac{\partial \mathcal{W}(\nabla y F_{\mathrm{p}}^{-1}, F_{\mathrm{p}}, \nabla F_{\mathrm{p}}, p, \nabla p)}{\partial F_{\mathrm{p}}} \tag{23}$$

and

$$Q_2 = \mathrm{div}\left(\frac{\partial \mathcal{W}(\nabla y F_{\mathrm{p}}^{-1}, F_{\mathrm{p}}, \nabla F_{\mathrm{p}}, p, \nabla p)}{\partial \nabla p}\right) - \frac{\partial \mathcal{W}(\nabla y F_{\mathrm{p}}^{-1}, F_{\mathrm{p}}, \nabla F_{\mathrm{p}}, p, \nabla p)}{\partial p}. \tag{24}$$

The elastic domain is defined as

$$\mathcal{Q}(x, z) = \partial_z^{\mathrm{sub}} \delta(x, z, 0). \tag{25}$$

Remark 3.1. The principle of maximal dissipation asserts that

$$Q_1 : \dot{F}_p + Q_2 \cdot \dot{p} \tag{26}$$

is maximal if $\dot{F}_p$ and $\dot{p}$ are kept fixed and $(Q_1, Q_2) \in \mathcal{Q}(x, z)$. This means that for all $(A, B) \in \mathcal{Q}(x, z)$.

$$Q_1 : \dot{F}_p + Q_2 \cdot \dot{p} \geq A : \dot{F}_p + B \cdot \dot{p}. \tag{27}$$

Finally, we include two examples covered by our approach.

Example 3.2. (Simple shear carried by single slip) Consider a single slip system defined by two orthonormal vectors $a, b \in \mathbb{R}^3$ such that a is the glide direction and b is the slip-plane normal. Further suppose that we have a particular case of the so-called *separable* material where

$$\mathcal{W}(x, F_{\mathrm{e}}, z) = \mathcal{W}_1 + (F_{\mathrm{e}}) + \frac{\varepsilon}{2} |\nabla F_{\mathrm{p}}|^2 \,,$$

where $F_{\mathrm{p}}(t) = \mathbb{I} + \gamma(t) a \otimes b$, where γ is the plastic slip. The slip system is generally not fixed in the reference configuration. The slip-plane normal $\tilde{b}$ in the reference configuration has the form $\tilde{b} = (F_{\mathrm{p}})^\top b$. However, in this special case we have that $\tilde{b} = b$, so that the slip-plane normal is kept constant during the process.

Due to the special case of F_{p} we may identify $z := (\gamma, p)$ because F_{p} depends only on γ.

Choose the dissipation metric

$$\delta(z, \dot{z}) = \delta(\gamma, p, \dot{\gamma}, \dot{p}) \,,$$

$$\delta(\gamma, p, \dot{\gamma}, \dot{p}) = \begin{cases} p|\dot{\gamma}| & \text{if} \quad \dot{p} \geq H|\dot{\gamma}| \,, \\ +\infty & \text{otherwise,} \end{cases}$$

where H is the so-called hardening function.

The evolution rule reads:

$$\varepsilon \Delta \gamma \in \partial^{\mathrm{sub}}(p|\dot{\gamma}|) \,.$$

The elastic domain $\partial^{\mathrm{sub}}\delta(\gamma, p, 0, 0) = [-p, p]$ if $\dot{p} \geq H|\dot{\gamma}|$ and $(-\infty, \infty)$ otherwise. The boundary of the elastic domain $\pm p - \varepsilon \Delta \gamma = 0$ defines the yield surface. Thus, the energetic approach recovers Gurtin's calculations on shear bands in single-slip, see [15].

The dissipation quasi distance is

$$\mathcal{D}(\gamma_1, p_1, \gamma_2, p_2) = \begin{cases} p_2|\gamma_2 - \gamma_1| & \text{if} \quad p_2 - p_1 \geq H|\gamma_2 - \gamma_1| \,, \\ +\infty & \text{otherwise.} \end{cases}$$

Example 3.3. (Multi slip) Suppose that we have a multi slip system described by glide directions $\{a_i\}$ and slip-plane normals $\{b_i\}$, where $a_i \cdot b_i = 0$ and $|a_i| = |b_i| = 1$ for all $1 \leq i \leq N$.

Following [15] we define for $\gamma = (\gamma_i)_{i=1}^N$ and $p = (p_i)_{i=1}^N$

$$\delta(\gamma, p, \dot{\gamma}, \dot{p}) = \begin{cases} \sum_{i=1}^N p_i|\dot{\gamma}_i| & \text{if} \quad \dot{p}_i \geq \sum_{i=1}^N H_{ij}|\dot{\gamma}_j| \,, \\ +\infty & \text{otherwise,} \end{cases}$$

where $H = (H_{ij})$ is a hardening matrix which may generally depend on p.

In order to set up a mathematical formulation of our problem we will need the following definitions and results on weak lower semicontinuity of integral functionals.

3.2   Quasiconvex functions

Let $\Omega \subset \mathbb{R}^n$ be a bounded regular domain. We say that a function $v : \mathbb{R}^{n \times n} \to \mathbb{R}$ is quasiconvex if for any $s_0 \in \mathbb{R}^{n \times n}$ and any $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^n)$,

$$v(s_0)|\Omega| \leq \int_\Omega v(s_0 + \nabla\varphi(x)) \, \mathrm{d}x \,. \tag{28}$$

The notion of quasiconvexity which generalizes the usual convexity is important because of the following result, see e.g. Dacorogna [6].

Lemma 3.4. *Let* $v : \mathbb{R}^{n \times n} \to \mathbb{R}$ *be quasiconvex and such that for all* $s \in \mathbb{R}^{n \times n}$ $0 \leq v \leq C(1 + |\cdot|^\beta)$. *Then the functional* $I : W^{1,\beta}(\Omega; \mathbb{R}^n) \to \mathbb{R}$ *given by*

$$I(y) = \int_\Omega v(\nabla y(x)) \, \mathrm{d}x$$

is weakly sequentially lower semicontinuous. This means that if $y_k \to y$ *weakly in* $W^{1,\beta}(\Omega; \mathbb{R}^n)$ *then* $I(y) \leq \liminf_{k \to \infty} I(y_k)$.

In fact, we need a slightly more general lower semicontinuity result than that given in Lemma 3.4.

Lemma 3.5. *Let $\alpha^{-1} + \beta^{-1} \leq d^{-1} < n^{-1} < 1$. Let $\{A_k\} \subset L^{\beta/(n-1)}(\Omega; \mathbb{R}^{n \times n})$ converge strongly to A, and $y_k \to y$ weakly in $W^{1,d}(\Omega; \mathbb{R}^n)$, let $\{\nabla y_k A_k\}_{k \in \mathbb{N}}$ be bounded in $L^{\alpha}(\Omega; \mathbb{R}^{n \times n})$, and let for almost all $x \in \Omega$ $0 \leq v(x, \cdot) \leq C(1 + | \cdot |^{\alpha})$ be quasiconvex and $v \in L^{\infty}(\Omega; C(\mathbb{R}^{n \times n}))$. Then $I : W^{1,d}(\Omega; \mathbb{R}^n) \times L^{\beta/(n-1)}(\Omega; \mathbb{R}^{n \times n}) \to \mathbb{R}$ given by*

$$I(y, A) = \int_{\Omega} v(x, \nabla y(x) A(x)) \, \mathrm{d}x$$

satisfies $I(y, A) \leq \liminf_{k \to \infty} I(y_k, A_k)$, i.e., I is sequentially (weakly,strongly)-lower semicontinuous in $W^{1,d}(\Omega; \mathbb{R}^n) \times L^{\beta/(n-1)}(\Omega; \mathbb{R}^{n \times n})$.

P r o o f. We give a proof based on Young measures, see e.g. [32] for details on the subject. There is a subsequence (not relabeled) of $\{\nabla y_k, A_k\}_{k \in \mathbb{N}}$ generating an L^{γ}-Young measure, $\gamma := \min(d, \beta/(n-1))$, $\nu \otimes \delta_A$ where $\nu = \{\nu_x\}_{x \in \Omega}$ is the $W^{1,d}$-gradient Young measure generated by $\{\nabla y_k\}$. Moreover, as the α-th moment of $\nu \otimes \delta_A$ is finite, i.e.,

$$\int_{\Omega} \int_{\mathbb{R}^{n \times n}} |sA(x)|^{\alpha} \nu_x(\mathrm{d}s) \, \mathrm{d}x < +\infty \,,$$

which means that $\nu \otimes \delta$ is an L^{α}-Young measure. We have

$$\lim_{k \to \infty} \int_{\Omega} v(x, \nabla y_k(x) A_k(x)) \, \mathrm{d}x = \int_{\Omega} \int_{\mathbb{R}^{n \times n}} v(x, sA(x)) \nu_x(\mathrm{d}s) \, \mathrm{d}x \,. \tag{29}$$

Thus, $s \mapsto v(x, sA(x))$ is quasiconvex and by ([32]) for a.a. $x \in \Omega$

$$\int_{\mathbb{R}^{n \times n}} v(x, sA(x)) \nu_x(\mathrm{d}s) \geq v \left(x, \int_{\mathbb{R}^{n \times n}} sA(x) \nu_x(\mathrm{d}s) \right) = v(x, \nabla y(x) A(x)) \,.$$

Integrating this inequality over Ω and putting it into (29) gives the statement. □

3.3 Assumptions on problem data

As in Gurtin [15] we will consider so-called *separable materials*, i.e., materials where the elasto-plastic energy density has the form

$$\mathcal{W}(x, F_e, F_p, \nabla F_p, p, \nabla p) := \mathcal{W}_1(x, F_e) + \mathcal{W}_2(x, F_p, \nabla F_p, p, \nabla p) \,. \tag{30}$$

We start with assumptions on $\mathcal{W}$:

(i) $\mathcal{W}_1, \mathcal{W}_2 \geq 0$ are measurable in $x \in \Omega$ and continuous in their other arguments.

(ii) Suppose that there are two constants $C, c > 0$ so that the following assumptions hold for constants $C, c, c_1 > 0$, $\alpha, \beta > n$, $\omega > n$, and for almost all $x \in \Omega$:

$$C(1 + |A|^{\alpha} + |F_p|^{\beta} + |G|^{\beta} + |p|^{\omega} + |\pi|^{\omega}) \geq \mathcal{W}(x, A, F_p, G, p, \pi)$$

$$\geq c(|A|^{\alpha} + |F_p|^{\beta} + |G|^{\beta} + |p|^{\omega} + |\pi|^{\omega}) - c_1 \,, \tag{31}$$

where $| \cdot |$ denotes the Euclidean norm;

(iii) $\mathcal{W}(x, \cdot, F_p, G, p, \pi)$ is quasiconvex for almost all $x \in \Omega$ and all $(F_p, G, p, \pi) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n \times n} \times \mathbb{R}^m \times \mathbb{R}^{n \times m}$;

(iv) $\mathcal{W}(x, A, F_p, \cdot, p, \cdot)$ is convex for almost all $x \in \Omega$ and all $(A, F_p, p) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^m$.

We recall the following assumptions on $\mathcal{D}$:

(i) Lower semicontinuity:

$$\mathcal{D}(z, \tilde{z}) \leq \liminf_{k \to \infty} \mathcal{D}(z_k, \tilde{z}_k) \,, \tag{32}$$

whenever $z_k \rightharpoonup z$ and $\tilde{z}_k \rightharpoonup \tilde{z}$.

(ii) Positivity: If $\{z_k\} \subset Z$ is bounded and $\min\{\mathcal{D}(z_k, z), \mathcal{D}(z, z_k)\} \to 0$ then $z_k \rightharpoonup z$.

In order to prove the existence of a solution to (37) we must impose some data qualifications. In what follows, we assume that

$$f \in C^1 \left([0, T]; L^{d^*}(\Omega; \mathbb{R}^n) \right) \,, \tag{33}$$

$$g \in C^1\left([0,T]; L^{d^{\#}}\left(\Gamma_1; \mathbb{R}^n\right)\right) , \tag{34}$$

where $d^* \geq nd/(n-d)$ if $1 \leq d < n$ or $d^* \geq 1$ otherwise. Similarly, we suppose that $d^{\#} \geq (nd-d)/(n-d)$ if $d < n$ or $d^{\#} \geq 1$ otherwise.

3.4   Auxiliary results

Proposition 3.6. *Let $\mathcal{I}$ be weakly sequentially lower semicontinuous and let (33) and (34) hold. Let for all $(t_*, q_*) \in [0,T] \times \mathbb{Q}$ and all stable sequences $\{(t_k, q_k)\}_{k \in \mathbb{N}}$ such that $w\text{-}\lim_{k\to\infty}(t_k, q_k) = (t_*, q_*)$ it holds that for all $\tilde{q} \in \mathbb{Q}$ there is $\{\tilde{q}_k\} \subset \mathbb{Q}$ that*

$$\limsup_{k\to\infty}(\mathcal{I}(t_k, \tilde{q}_k) + \mathcal{D}(q_k, \tilde{q}_k)) \leq \mathcal{I}(t_*, \tilde{q}) + \mathcal{D}(q_*, \tilde{q}) . \tag{35}$$

Then $\mathcal{I}$ is weakly continuous along stable sequences and $q_* \in \mathcal{S}(t_*)$, i.e. q_* is in the stable set at t_*; cf. (20).

P r o o f.   We follow the proof of Prop. 4.2 in [21]. Take $\tilde{q} = q_*$ in (35) and notice that (35) holds with $\tilde{q}_k = q_k$, $k \in \mathbb{N}$. Then we get

$$\limsup_{k\to\infty}\mathcal{I}(t_k, q_k) \leq \limsup_{k\to\infty}((\mathcal{I}(t_k, \tilde{q}_k) + \mathcal{D}(q_k, \tilde{q}_k)) \leq \mathcal{I}(t_*, \tilde{q}) + \mathcal{D}(q_*, \tilde{q}) = \mathcal{I}(t_*, q_*) .$$

We have further

$$\lim_{k\to\infty}|\mathcal{I}(t_k, q_k) - \mathcal{I}(t_*, q_k)| = \lim_{k\to\infty}|L(t_k, q_k) - L(t_*, q_k)| = 0 ,$$

due to the assumptions (33) and (34) on f and g, respectively.

Since $\mathcal{I}$ is weakly lower semicontinuous we have

$$\liminf_{k\to\infty}\mathcal{I}(t_k, q_k) = \lim_{k\to\infty}(\mathcal{I}(t_k, q_k) - \mathcal{I}(t_*, q_k)) + \liminf_{k\to\infty}\mathcal{I}(t_*, q_k) \geq \mathcal{I}(t_*, q_*) .$$

This together with (35) gives weak continuity of $\mathcal{I}(t_k, q_k) \to \mathcal{I}(t_*, q_*)$. Finally, we have

$$\mathcal{I}(t_*, q_*) = \lim_{k\to\infty}\mathcal{I}(t_k, q_k) \leq \liminf_{k\to\infty}(\mathcal{I}(t_k, \tilde{q}_k) + \mathcal{D}(q_k, \tilde{q}_k)) \leq \mathcal{I}(t_*, \tilde{q}) + \mathcal{D}(q_*, \tilde{q}) .$$

The arbitrariness of $\tilde{q} \in \mathbb{Q}$ shows the stability of q_*. □

The key point is, however, to ensure validity of (35). If $\mathcal{D} : \mathbb{Q} \times \mathbb{Q} \to [0, +\infty)$, i.e., no irreversibility constraint is imposed on plastic processes, then it is sufficient if D from (14) satisfies

$$D(x, z_1, z_2) \leq c(x) + C(|F_{p_1}|^{\beta^*-\epsilon} + |F_{p_2}|^{\beta^*-\epsilon} + |p_1|^{\omega^*-\epsilon} + |p_2|^{\omega^*-\epsilon}) , \tag{36}$$

where $\epsilon > 0$ is small enough and $\beta^* := n\beta/(n-\beta)$ if $n > \beta$ and $\beta^* < +\infty$ if $\beta \geq n$. Similarly, $\omega^* := n\omega/(n-\omega)$ if $n > \omega$ and $\omega^* < +\infty$ if $\omega \geq n$. Then the compact embedding ensures continuity of $\mathcal{D}$.

If $\mathcal{D} : \mathbb{Q} \times \mathbb{Q} \to [0, +\infty]$ we must be more careful. Following [21] we impose the following *sufficient* conditions on D from (14):

(A) $D(x, \cdot, \cdot) : D(x) \to [0, +\infty)$ is continuous, $D(x) := \{(z_1, z_2); D(x, z_1, z_2) < +\infty\}$,

(B) For every $R > 0$ there is $K > 0$ such that for almost all $x \in \Omega$ $D(x, z_1, z_2) < K$ if $z_1, z_2 \in D(x)$ and $|z_1|, |z_2| < R$, and

(C) There is $v^* \in \mathbb{R}^M$ such that for all $\alpha, R > 0$ there is $\rho > 0$ such that for almost every $x \in \Omega$ and every z_0, z_1, z_2:

$$|z - z_0| < \rho \text{ and } (z_0, z_1) \in D(x) \text{ implies } (z, z_1 + (0, \alpha v^*)) \in D(x) .$$

Proposition 3.7. *Let $\beta, \omega > n$. Let D satisfy (A)–(C). Then (35) holds.*

P r o o f.   We follow [21]. If $\mathcal{D}(q_*, \tilde{q}) = +\infty$ in (35) the proof is finished. So, we assume that

$$\mathcal{D}(q_*, \tilde{q}) \in \mathbb{R} .$$

If $q_j \rightharpoonup q_*$ we have due to the compact embedding that

$$\rho_k := \|F_{\mathrm{p}k} - F_{\mathrm{p}*}\|_{C(\bar{\Omega};\mathbb{R}^{n \times n})} + \|p_k - p_*\|_{C(\bar{\Omega};\mathbb{R}^{n \times n})} \to 0 .$$

Thus, $|z_k| + |z_*| + |\tilde{z}| < R$ if k is large enough. We define $\tilde{z}_k := (\tilde{F}_{\mathrm{p}}, \tilde{p} + \alpha_k v^*)$ where $\alpha_k \to 0$ and relates to ρ_k as in (C). Then $(z_k, \tilde{z}_k) \in D(x)$ in Ω and we have $|z_k|, |\tilde{z}_k| < R$. Continuity of D gives the pointwise convergence of $D(x, z_k, \tilde{z}_k) \to D(z, z_*, \tilde{z})$ pointwise. The condition (B) together with the Lebesgue dominated convergence theorem implies that $\mathcal{D}(q_j, \tilde{q}_j) \to \mathcal{D}(q_*, \tilde{q})$. Assumptions (33), (34) imply that (35) is fulfilled with equality. □

3.5 Incremental problems

Next, we define the following sequence of incremental problems. We consider a stable initial condition $q_\tau^0 := q^0 \in \mathbb{Q}$.

Let us take $\tau > 0$, a time step, chosen in the way that $N = T/\tau \in \mathbb{N}$. For $1 \le k \le N$, $t_k := k\tau$, find $q_\tau^k \in \mathbb{Q}$ such that q_τ^k solves

$$\left. \begin{array}{l} \text{minimize} \quad \mathcal{I}(t_k, q) + \mathcal{D}(q_\tau^{k-1}, q) \\ \text{subject to} \quad q_\tau^k \in \mathbb{Q} \, . \end{array} \right\} \tag{37}$$

Proposition 3.8. *Let $\alpha^{-1} + \beta^{-1} \le d^{-1} < n^{-1}$ and $\omega > n$. Let the assumption on $\mathcal{W}$ and $\mathcal{D}$ be satisfied. Let further (33) and (34) be satisfied. Then the problem (37) has a solution for all $k = 1, \ldots, T/\tau$.*

P r o o f. First, notice that $F_{\mathrm{p}}^{-1} = (\mathrm{cof}F)^\top$, where "cof" stands for the cofactor matrix. Suppose that $q_\tau^{k-1} \in \mathbb{Q}$ is known and that $\{q_j\} \subset \mathbb{Q}$ is a minimizing sequence for $q \mapsto \mathcal{I}(t_k, q) + \mathcal{D}(q_\tau^{k-1}, q)$. We use Young's and Hölder's inequalities as in [21] to obtain the following pointwise inequality for any member of the minimizing sequence (the index j is omitted for simplicity)

$$|FF_{\mathrm{p}}^{-1}| \ge \frac{|F|}{|F_{\mathrm{p}}|} \ge r\theta^{r/(r-1)}|F|^{1/r} - (r-1)\theta|F_{\mathrm{p}}|^{1/(r-1)} \tag{38}$$

valid for all $r > 1$ and all $\theta > 0$.

Taking into account that $F_{\mathrm{e}} = FF_{\mathrm{p}}^{-1} \in L^\alpha(\Omega; \mathbb{R}^{n \times n})$, $F_{\mathrm{p}} \in L^\beta(\Omega; \mathbb{R}^{n \times n})$, and $F \in L^d(\Omega; \mathbb{R}^{n \times n})$ together with Hölder's inequality we get for $r := \alpha/d > 1$ and $\frac{1}{b} := \frac{1}{d} - \frac{1}{\alpha} \ge \frac{1}{\beta}$

$$\|FF_{\mathrm{p}}^{-1}\|_{L^\alpha(\Omega; \mathbb{R}^{n \times n})}^\alpha \ge \frac{\|F\|_{L^d(\Omega; \mathbb{R}^{n \times n})}^\alpha}{\|F_{\mathrm{p}}\|_{L^\beta(\Omega; \mathbb{R}^{n \times n})}^\alpha} \tag{39}$$

$$\ge r\theta^{r/(r-1)}\|F\|_{L^d(\Omega; \mathbb{R}^{n \times n})}^d - (r-1)\theta\|F_{\mathrm{p}}\|_{L^b(\Omega; \mathbb{R}^{n \times n})}^b \, .$$

Using this inequality for θ small enough in the lower bound (31) of $\mathcal{W}$ integrated over Ω implies together with the Poincaré inequality a uniform bound on $\|y_j\|_{W^{1,d}(\Omega; \mathbb{R}^n)}$ for all $j \in \mathbb{N}$. Having this bound we get uniform bounds on $\|z_j\|_{W^{1,\beta}(\Omega; \mathbb{R}^{n \times n}) \times W^{1,\omega}(\Omega; \mathbb{R}^m)}$ due to (31). Then Lemma 3.5 shows the lower semicontinuity of the $\mathcal{W}_{\mathrm{e}}$ part of the stored energy along the minimizing sequence and the convexity of $\mathcal{W}_{\mathrm{p}}$ in the gradient terms implies the weak lower semicontinuity of the minimized functional. The proof is then finished by the direct method of the calculus of variations. $\square$

We denote q_τ a piecewise constant interpolation of $q_\tau^k := (y_\tau^k, z_\tau^k)$, i.e., $q_\tau(t) = q_\tau^k$ if $t \in [k\tau, (k+1)\tau)$ and $k = 1, \ldots, T/\tau - 1$. Finally, $q_\tau(T) = q_\tau^N$. Analogously, $L_\tau(t, q_\tau) = L(k\tau, q_\tau)$ is a piecewise constant interpolation of L and $\mathcal{I}_\tau(t, q_\tau) = \mathcal{I}(t, q_\tau)$ is a piecewise constant interpolation of $\mathcal{I}$.

Proposition 3.9. *Let (33) and (34) be satisfied. Then the problem (37) has a solution $q_\tau(t)$ which is stable, i.e., for all $t \in [0, T]$ and for every $q \in \mathbb{Q}$,*

$$\mathcal{I}_\tau(t, q_\tau(t)) \le \mathcal{I}_\tau(t, q) + \mathcal{D}(q_\tau(t), q) \, . \tag{40}$$

Moreover, for all $t_1 \le t_2$ from the set $\{k\tau\}_{k=0}^N$, the following discrete energy inequalities hold if one extends the definition of $q_\tau(t)$ by setting $q_\tau(t) := q^0$ if $t < 0$.

$$-\int_{t_1}^{t_2} \dot{L}(t, q_\tau(t-\tau)) \, \mathrm{d}t \le \mathcal{I}(t_2, q_\tau(t_2)) + \mathrm{Var}(\mathcal{D}, q_\tau; [t_1, t_2]) - \mathcal{I}(t_1, q_\tau(t_1))$$

$$\le -\int_{t_1}^{t_2} \dot{L}(t, q_\tau(t)) \, \mathrm{d}t. \tag{41}$$

P r o o f. The existence of a solution to (37) was proved in the Proposition 3.8.

The stability estimate (40) follows from the minimizing property of q_τ^k and the properties of $\mathcal{D}$. Indeed, by minimality of q_τ^k,

$$\mathcal{I}\left(k\tau, q_\tau^k\right) + \mathcal{D}\left(q_\tau^{k-1}, q_\tau^k\right) \le \mathcal{I}\left(k\tau, q\right) + \mathcal{D}\left(q_\tau^{k-1}, q\right), \tag{42}$$

from which we infer that

$$\mathcal{I}\left(k\tau, q_\tau^k\right) \le \mathcal{I}\left(k\tau, q\right) + \mathcal{D}\left(q_\tau^{k-1}, q\right) - \mathcal{D}\left(q_\tau^{k-1}, q_\tau^k\right).$$

However, the structure of the metric implies that

$$\mathcal{D}\left(q_\tau^{k-1}, q\right) - \mathcal{D}\left(q_\tau^{k-1}, q_\tau^k\right) \le \mathcal{D}\left(q_\tau^k, q\right),$$

from which (40) follows.

Next, we demonstrate the validity of the energy inequality (41), following the arguments in Mielke et al. [29]. For this part, let us test the stability of q_τ^{k-1} with $q := q_\tau^k$. This gives

$$\mathcal{I}\left((k-1)\tau, q_\tau^{k-1}\right) \le \mathcal{I}\left((k-1)\tau, q_\tau^k\right) + \mathcal{D}\left(q_\tau^{k-1}, q_\tau^k,\right) \tag{43}$$

$$= \mathcal{I}\left(k\tau, q_\tau^k\right) + L\left(k\tau, q_\tau^k\right) - L\left((k-1)\tau, q_\tau^k\right) + \mathcal{D}\left(q_\tau^{k-1}, q_\tau^k\right).$$

Suppose that $0 \le k_1 \le k_2 \le N$ and that $t_1 = k_1\tau$ and $t_2 = k_2\tau$. A summation of (43) over $k = k_1 + 1, \ldots, k_2$ yields

$$\sum_{k=k_1+1}^{k_2} \left[L\left((k-1)\tau, q_\tau^k\right) - L\left(k\tau, q_\tau^k\right) \right] \le \mathcal{I}\left(k_2\tau, q_\tau^{k_2}\right) - \mathcal{I}\left(k_1\tau, q_\tau^{k_1}\right) \tag{44}$$

$$+ \sum_{k=k_1+1}^{k_2} \mathcal{D}\left(q_\tau^{k-1}, q_\tau^k\right).$$

We rewrite this inequality in terms of q_τ to see that it is the first inequality in (41),

$$-\int_{t_1}^{t_2} \dot{L}\left(t, q_\tau(t-\tau)\right) \, \mathrm{d}t \le \mathcal{I}\left(k_2\tau, q_\tau^{k_2}\right) - \mathcal{I}\left(k_1\tau, q_\tau^{k_1}\right) + \sum_{k=k_1+1}^{k_2} \mathcal{D}\left(q_\tau^{k-1}, q_\tau^k\right)$$

$$= \mathcal{I}\left(k_2\tau, q_\tau^{k_2}\right) - \mathcal{I}\left(k_1\tau, q_\tau^{k_1}\right) + \mathrm{Var}\left(\mathcal{D}, q_\tau; [t_1, t_2]\right)$$

(the explicit form of $\mathrm{Var}\left(\mathcal{D}, q_\tau; [t_1, t_2]\right)$ holds since we consider a step function). To prove the validity of the second inequality in (41), we rely on the minimality of q_τ^k, when compared with q_τ^{k-1} in (42). That is,

$$\mathcal{I}\left(k\tau, q_\tau^k\right) + \mathcal{D}\left(q_\tau^{k-1}, q_\tau^k\right) \le \mathcal{I}\left(k\tau, q_\tau^{k-1}\right) = \mathcal{I}\left((k-1)\tau, q_\tau^{k-1}\right) + L\left((k-1)\tau, q_\tau^{k-1}\right) - L\left(k\tau, q_\tau^{k-1}\right).$$

Similarly as in the previous argument, a summation over $k = k_1 + 1, \ldots, k_2$ is employed to find that

$$\mathcal{I}\left(k_2\tau, q_\tau^{k_2}\right) - \mathcal{I}\left(k_1\tau, q_\tau^{k_1}\right) + \sum_{k=k_1+1}^{k_2} \mathcal{D}\left(q_\tau^{k-1}, q_\tau^k\right)$$

$$\le \sum_{k=k_1+1}^{k_2} \left[L\left((k-1)\tau, q_\tau^{k-1}\right) - L\left(k\tau, q_\tau^{k-1}\right) \right],$$

so that

$$\mathcal{I}\left(k_2\tau, q_\tau^{k_2}\right) - \mathcal{I}\left(k_1\tau, q_\tau^{k_1}\right) + \mathrm{Var}\left(\mathcal{D}, q_\tau; [t_1, t_2]\right) \le -\int_{t_1}^{t_2} \dot{L}\left(t, q_\tau(t)\right) \, \mathrm{d}t,$$

which is the second inequality in (41). $\square$

The next proposition gives the *a priori* bounds needed to pass to the limit as the step size goes to zero.

Proposition 3.10. *Let (33) and (34) be satisfied. Then there is $\kappa \in \mathbb{R}$ such that*

$$\|y_\tau\|_{L^\infty\left(0,T;W^{1,d}(\Omega;\mathbb{R}^n)\right)} \le \kappa, \tag{45}$$

$$\mathrm{Var}(\mathcal{D}, q_\tau; [0, T])) \le \kappa, \tag{46}$$

for $\hat{\mathcal{I}}_\tau(t) := \mathcal{I}_\tau(t, q_\tau(t))$,

$$\|\hat{\mathcal{I}}_\tau\|_{BV(0,T)} \leq \kappa. \tag{47}$$

$$\|z_\tau\|_{L^\infty\left(0,T;W^{1,\alpha}(\Omega;\mathbb{R}^{n\times n})\times W^{1,\beta}(\Omega;\mathbb{R}^m)\right)} \leq \kappa, \tag{48}$$

P r o o f.   Set for $q = (y, F_{\mathrm{p}}, p) \in \mathbb{Q}$.

$$V(q) = \int_\Omega \mathcal{W}(x, \nabla y(x) F_{\mathrm{p}}^{-1}, F_{\mathrm{p}}(x), \nabla F_{\mathrm{p}}, p(x), \nabla p(x))\, \mathrm{d}x\ .$$

The growth conditions on W imply that

$$\|y\|_{W^{1,d}(\Omega;\mathbb{R}^n)}^d + \|F_p\|_{W^{1,\beta}(\Omega;\mathbb{R}^{n\times n})}^\beta + \|p\|_{W^{1,\omega}(\Omega;\mathbb{R}^m)}^\omega \leq V(q)\ . \tag{49}$$

Using this inequality for $q := q_\tau^{k_2}$ and the energy inequality for $k_1 = 0$ we get

$$V(q_\tau^{k_2}) - L(k_2\tau, q_\tau^{k_2}) - V(q_\tau^0) + L(0, q_\tau^0) \leq \sum_{k=1}^{k_2} \left[L\left((k-1)\tau, q_\tau^{k-1}\right) - L\left(k\tau, q_\tau^{k-1}\right) \right]\ .$$

Hence,

$$V(q_\tau^{k_2}) \leq \sum_{k=1}^{k_2} \left[L\left((k-1)\tau, q_\tau^{k-1}\right) - L\left(k\tau, q_\tau^{k-1}\right) \right] + L(k_2\tau, q_\tau^{k_2})\ . \tag{50}$$

So, denoting $Y_\tau := \max_{1 \leq \ell \leq T/\tau} \|y_\tau^\ell\|_{W^{1,d}(\Omega;\mathbb{R}^n)}^d$ we have

$$Y_\tau \leq \sum_{k=1}^{k_2} \left[L\left((k-1)\tau, q_\tau^{k-1}\right) - L\left(k\tau, q_\tau^{k-1}\right) \right] + C\ . \tag{51}$$

This gives the bound (45). This estimate together with (50) gives us the estimates (46)–(48). $\qquad\square$

The following lemma is proved in [20]. Let us first denote $\mathbb{X} := L^\beta(\Omega;\mathbb{R}^{n\times n}) \times L^\omega(\Omega;\mathbb{R}^m)$. Notice that if (16) and (17) hold for $\mathcal{D}$ and $\mathbb{Z}$ then they hold in $\mathbb{X}$ with the strong convergence in $\mathbb{X}$.

Lemma 3.11. *Let $\mathcal{D} : \mathbb{X} \times \mathbb{X} \to [0, +\infty]$. Let $\mathcal{K}$ be a compact subset of $\mathbb{X}$. Then for every sequence $\{z_k\}_{k\in\mathbb{N}}$, $z_k :$ $[0,T] \to \mathcal{K}$ for which $\sup_{k\in\mathbb{N}} \mathrm{Var}(\mathcal{D}_{\mathrm{p}}, z_k; [0,T]) < C$, $C > 0$, there exists a subsequence (not relabeled), a function $z : [0,T] \to \mathcal{K}$, and a function $\Delta : [0,T] \to [0,C]$ such that:*
(i) $\mathrm{Var}(\mathcal{D}, z_k; [0,t]) \to \Delta(t)$ for all $t \in [0,T]$,
(ii) $z_k \to z$ for all $t \in [0,T]$, and
(iii) $\mathrm{Var}(\mathcal{D}_{\mathrm{p}}, z; [t_0, t_1]) \leq \lim_{t\to t_1+} \Delta(t) - \lim_{t\to t_0-} \Delta(t)$ for all $0 \leq t_0 < t_1 \leq T$.

Finally, we proved the existence of an energetic solution.

Theorem 3.12. *Let $\alpha^{-1} + \beta^{-1} \leq d^{-1} < n^{-1} < 1$ and $\omega > n$. Let $q^0 \in \mathbb{Q}$ be a stable initial condition. Let the assumptions on $\mathcal{W}$, $\mathcal{D}$, f and g from Section (3.3) hold. Let further (36) or (A), (B), (C) hold. Then there is a process $q : [0,T] \to \mathbb{Q}$ with $q(t) = (y(t), z(t))$ such that q is an energetic solution according to Definition 2.2. The following limit passages are also valid:*
(i) for a t-dependent (not relabeled) subsequence w-$\lim_{\tau\to 0} y_\tau(t) = y(t)$ in $W^{1,d}(\Omega;\mathbb{R}^n)$ for all $t \in [0,T]$,
(ii) for a (not relabeled) subsequence $\lim_{\tau\to 0} z_\tau(t) = z(t)$ in $\mathbb{X}$ for all $t \in [0,T]$,
(iii) for a (not relabeled) subsequence $\lim_{\tau\to 0} \mathcal{I}_\tau(t, q_\tau) = \mathcal{I}(t, q(t))$ for all $t \in [0,T]$, and
(iv) for a (not relabeled) subsequence $\lim_{\tau\to 0} \mathrm{Var}(\mathcal{D}, q_\tau; [0,t]) = \mathrm{Var}(\mathcal{D}, q; [0,t])$ for all $t \in [0,T]$.

P r o o f.   The proof is divided into two steps and follows [11].

Step 1: The points (i), (ii), and (iii) follow from the a-priori estimates in Proposition 3.10 and Lemma 3.11. Notice also, $\theta_\tau(t) := \frac{\partial L}{\partial t}(t, q_\tau)$ is bounded in $L^\infty(0,T)$, so that there is a weak* limit of a subsequence (not relabeled) called θ. Following [24] we set $\theta_{\mathrm{i}}(t) := \liminf_{\tau\to 0} \theta_\tau(t)$. By the Fatou's lemma $\theta_{\mathrm{i}} \leq \theta$. Altogether, it implies the existence of the limit $q(t) = (y(t), z(t))$. We immediately get that $q \in \mathbb{Q}$.

Put $S(t,\tau) := \min_{k\in\mathbb{N}\cup\{0\}}\{k\tau;\ k\tau \geq t\}$. Then $\lim_{\tau\to 0} S(t,\tau) = t$ and by the definition $q_\tau(t) := q_\tau(S(t,\tau)) \in \mathcal{S}(S(t,\tau))$. Moreover, by our assumptions on $\mathcal{D}$ we know that (35) holds. Therefore $q(t) \in \mathcal{S}(t)$, i.e., the limit is stable by Proposition 3.6. Proposition 3.6) also implies (iii) .

Step 2: We have $q_\tau(t) = q_\tau(k\tau)$ if $0 \leq t - k\tau \leq \tau$. Hence, using (41) we get for some $C, C_1 > 0$

$$\mathcal{I}(t, q_\tau(t)) + \mathrm{Var}(\mathcal{D}, q_\tau; [0, t]) \leq \mathcal{I}(k\tau, q_\tau(k\tau)) + \mathrm{Var}(\mathcal{D}, q_\tau; [0, k\tau]) + C\tau$$

$$\leq \mathcal{I}(0, q_\tau(0)) - \int_0^{k\tau} \dot{L}(s, q_\tau(s))\, \mathrm{d}s + C\tau$$

$$\leq \mathcal{I}(0, q_\tau(0)) - \int_0^t \dot{L}(s, q_\tau(s))\, \mathrm{d}s + C_1\tau\;.$$

Further, using Lemma 3.11 (i) and weak lower semicontinuity of the variation we get for $\tau \to 0$

$$\mathcal{I}(t, q(t)) + \Delta(t) + \mathrm{Var}(\mathcal{D}, q; [0, t]) \leq \mathcal{I}(0, q(0)) - \int_0^t \theta(s)\, \mathrm{d}s\;.$$

Let us denote $\theta_\mathrm{i}(s) := \liminf_{\tau \to 0} \dot{L}(s, q_\tau(s))$

As $\Delta(t) \geq \mathrm{Var}(\mathcal{D}, q; [0, t])$ and by the Fatou's lemma $\int_0^t \theta(s)\, \mathrm{d}s \geq \int_0^t \theta_\mathrm{i}(s)\, \mathrm{d}s$ for a.a. $t \in [0, T]$ we get

$$\mathcal{I}(t, q(t)) + \mathrm{Var}(\mathcal{D}, q; [0, t]) \leq \mathcal{I}(0, q(0)) - \int_0^t \theta_\mathrm{i}(s)\, \mathrm{d}s\;.$$

Moreover, we have due to Step 2 we get that $\theta_\mathrm{i}(s) = \dot{L}(s, q(s))$. Altogether we get the upper energy estimate

$$\mathcal{I}(t, q(t)) + \mathrm{Var}(\mathcal{D}, q; [0, t]) \leq \mathcal{I}(0, q(0)) - \int_0^t \dot{L}(s, q(s))\, \mathrm{d}s\;. \tag{52}$$

In order to get the lower estimate we exploit the fact that $q(t)$ is stable for all $t \in [0, T]$. Take a (possibly non-uniform) partition of a time interval $[t_1, t_2] \subset [0, T]$ such that $t_1 = \vartheta_0 < \vartheta_1 < \vartheta_2 < \vartheta_K = t_2$ such that $\max_i(\vartheta_i - \vartheta_{i-1}) =: \vartheta \to 0$ as $K \to \infty$ and we are going to test the stability of $q(\vartheta_{k-1})$ with $q(\vartheta_k)$, $k = k_1 + 1, \ldots, k_2$. Analogously as in (44) we get

$$\sum_{k=1}^K \left[L\left((\vartheta_{k-1}, q(\vartheta_k)) - L\left(\vartheta_k, q(\vartheta_k)\right) \right] \leq \mathcal{I}\left(t_2, q(t_2) - \mathcal{I}\left(t_1, q(t_1)\right) \tag{53}$$

$$+ \sum_{k=+1}^K \mathcal{D}\left(q(\vartheta_{k-1}), q(\vartheta_k)\right)\;.$$

Hence,

$$\sum_{k=1}^K - \int_{\vartheta_{k-1}}^{\vartheta_k} \dot{L}(s, q(\vartheta_k))\, \mathrm{d}s \leq \mathcal{I}\left(t_2, q(t_2) - \mathcal{I}\left(t_1, q(t_1)\right) + \mathrm{Var}(\mathcal{D}, q; [t_1, t_2])\;. \tag{54}$$

Finally, we realize that

$$\sum_{k=1}^K \int_{\vartheta_{k-1}}^{\vartheta_k} \dot{L}(s, q(k\tau))\, \mathrm{d}s = \sum_{k=+1}^K \dot{L}(\vartheta_k, q(\vartheta_k))(\vartheta_k - \vartheta_{k-1})$$

$$+ \sum_{k=1}^K \int_{\vartheta_{k-1}}^{\vartheta_k} \left(\dot{L}(s, q(\vartheta_k)) - \dot{L}(\vartheta_k, q(\vartheta_k)) \right)\, \mathrm{d}s\;. \tag{55}$$

The second term on the right-hand side of (55) tends to zero as $\vartheta \to 0$ because the time derivative of external forces is uniformly continuous in time by (33) and (34). The first term on the right-hand side converges to $\int_{t_1}^{t_2} \dot{L}(s, q(s))\, \mathrm{d}s$ by Dal Maso et al. [7], Lemma 4.12.

The upper and lower estimates give us the energy balance

$$\mathcal{I}(t, q(t)) + \mathrm{Var}(\mathcal{D}, q; [0, t]) = \mathcal{I}(0, q(0)) - \int_0^t \dot{L}(s, q(s))\, \mathrm{d}s\;. \tag{56}$$

Moreover,

$$\mathcal{I}(0, q(0)) + \int_0^t \theta(s)\,\mathrm{d}s \le \mathcal{I}(t, q(t)) + \mathrm{Var}(\mathcal{D}, q; [0, t])) \le \mathcal{I}(t, q(t)) + \Delta(t)$$

$$\le \mathcal{I}(0, q(0)) + \int_0^t \theta_\mathrm{i}(s)\,\mathrm{d}s \le \mathcal{I}(0, q(0)) + \int_0^t \theta(s)\,\mathrm{d}s \,. \tag{57}$$

Thus, all inequalities in (57) are equalities and we get that (iv) holds. □

4 Conclusions

This paper provides a mathematical framework for a general plasticity theory where the *full* gradient of the plastic strain is included; cf. [15]. On the other hand, the structure of incremental problems (37) in our approach motivates numerical methods used to approximate energetic solutions. Namely, after suitable approximation of y and $z = (F_\mathrm{p}, p)$ by piecewise affine elements, for instance, we must numerically solve a sequence of global minimization problems. We hope to address this problem in a future paper. We refer to [27] for a general numerical approach to rate-independent problems.

Author, please clarify missing parts of the references; the black squares (■■) should be replaced by the respective element (please give, give, if available, the complete names of contributors); abbreviated journal title, volume number, pages and year for journal citations; book and/or series title, editors, publishing house, place of publication and year for book citations)

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