Projective Moment Invariants

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Abstract—The paper is devoted to the moment invariants with respect to projective transform. It has been a common belief that such invariants do not exist. We show that projective moment invariants exist in a form of infinite series containing moments with positive as well as negative indices.

Index Terms—Projective transform, moment invariants, object recognition.

1 Introduction

MOMENT invariants have become a powerful tool for recognizing objects regardless of their particular position, orientation, viewing angle, and gray-level variations. There is a well-elaborated theory on rotation moment invariants [1], [2], [3], [4], [5], [6], [7], affine moment invariants [8], [9], [10], and even on invariants to color changes and blurring [11], [12], [13], [14], [15]. In practice, however, we often face object deformations described by *projective transform*. Projective transform is an exact model of photographing a planar scene by a pin-hole camera whose optical axis is not perpendicular to the scene. For small objects and large camera-to-scene distance, the perspective effect is negligible. Projective transform then can be approximated by affine transform and affine moment invariants are sufficient for object description and recognition. In many computer vision tasks, such as in mobile robot navigation and 3D scene analysis, this assumption is not fulfilled and the necessity of having projective invariants arises.

Major difficulties with projective moment invariants originate from the fact that the projective transform is not linear, its Jacobian is a function of spatial coordinates and it does not preserve the center of gravity of the object. The theory of algebraic invariants, which was successfully applied to derive affine moment invariants, as well as the theory of complex and orthogonal moments, which were used to find rotation invariants, cannot be exploited in the case of projective invariants.

Moments are not the only possible theoretical tools to construct projective invariants. Invariants based on local shape properties have been described—differential invariants of object boundary [16], [17], [18] and various invariants defined by means of salient points [19], [20], [21], [22], [23]. Since they have different nature and different usage than global moment invariants, we will not consider them further.

This is, to our knowledge, the first paper demonstrating the existence of projective moment invariants. We show they have a form of infinite series which may contain also moments with negative indices.

2 Basic Terms

First, let us introduce a few basic terms:

Definition 1. *By* image function *(or* image*) we understand any real function $f(x,y)$ having a finite integral and a bounded support S such that $x > 0, y > 0$ for all points of S.*

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Definition 2. Geometric moment *(or just* moment *for short)* m_{pq} *of order* $(p+q)$ *of image* $f(x,y)$ *is defined as*

$$m_{pq} = \int_S \int x^p y^q f(x,y) \mathrm{d}x \mathrm{d}y. \tag{1}$$

Traditionally, only $p, q \geq 0$ are considered, but, in this paper, we allow the indices p, q to be arbitrary integers. This is correct if $f(x,y)$ satisfies Definition 1. Thus, our moments are projections not only onto a set of polynomials but also onto a set of rational functions.

Definition 3. Projective transform *is a transformation of spatial coordinates* (x,y) *into new coordinate system* (x', y') *defined by the equations*

$$\begin{aligned} x' &= \frac{a_0 + a_1 x + a_2 y}{1 + c_1 x + c_2 y}, \\ y' &= \frac{b_0 + b_1 x + b_2 y}{1 + c_1 x + c_2 y}. \end{aligned} \tag{2}$$

The coefficients c_1, c_2 cause nonlinear perspective distortion of the image. If they both equal zero, the projective transform (2) becomes linear.

Definition 4. *By* Projective moment invariant *we understand any function F of geometric moments such that $F(f(x,y)) = F(f(x',y'))$ for any image f and arbitrary projective transformation (2).*

3 Finite Projective Invariants

Only few papers on projective moment invariants have been published so far and all of them considered only invariants composed of a finite number of moments.

Voss and Susse [24] have proven that, if $a_0 = a_2 = b_0 = b_1 = 0$ (Voss called such particular case "rein transform"), then there exist moment-like invariants of the form

$$R_{pq} = \int_S \int \frac{x^p y^q}{p^a(x,y)} f(x,y) \mathrm{d}x \mathrm{d}y, \tag{3}$$

where $p(x,y)$ is a homogeneous polynomial of order $n > 0$ of the form

$$p(x,y) = \sum_{k+j=n} p_{kj} x^k y^j,$$

k and j are arbitrary integers, p_{kj} are coefficients, and $a = (p+q+3)/n$. Unfortunately, this special case of invariants cannot be generalized to the projective transform (2).

Van Gool et al. [25] proved the nonexistence of finite projective invariants using the Lie group theory. We present here another proof which is done in a more readable manner without the group theory tools.

Let us decompose the projective transform (2) into eight one-parametric transformations.

Horizontal and vertical translations:

$$(a) \quad \begin{aligned} x' &= x + \alpha \\ y' &= y \end{aligned} \qquad (b) \quad \begin{aligned} x' &= x \\ y' &= y + \beta, \end{aligned}$$

scaling and stretching:

$$(c) \quad \begin{aligned} x' &= \omega x \\ y' &= \omega y \end{aligned} \qquad (d) \quad \begin{aligned} x' &= r x \\ y' &= \tfrac{1}{r} y, \end{aligned}$$

horizontal and vertical skewing:

$$(e) \quad \begin{aligned} x' &= x + t_1 y \\ y' &= y \end{aligned} \qquad (f) \quad \begin{aligned} x' &= x \\ y' &= t_2 x + y, \end{aligned}$$

and horizontal and vertical pure projections:

$$(g)\quad \begin{aligned} x' &= \tfrac{x}{1+c_1 x} \\ y' &= \tfrac{y}{1+c_1 x} \end{aligned} \qquad (h)\quad \begin{aligned} x' &= \tfrac{x}{1+c_2 y} \\ y' &= \tfrac{y}{1+c_2 y}. \end{aligned}$$

Any projective invariant F must be invariant to all elementary transformations $(a)-(h)$. Each transform imposes special constraints on F. Particularly, from the invariance to horizontal pure projection (g), it follows that the derivative of F with respect to parameter c_1 must be zero (assuming all derivatives exist):

$$\frac{\mathrm{d}F}{\mathrm{d}c_1} = \sum_p \sum_q \frac{\partial F}{\partial m_{pq}} \frac{\mathrm{d}m_{pq}}{\mathrm{d}c_1} = 0. \tag{4}$$

The derivatives of moments can be expressed as

$$\begin{aligned} \frac{\mathrm{d}m_{pq}}{\mathrm{d}c_1} &= \frac{\mathrm{d}}{\mathrm{d}c_1} \int_S \int \frac{x^p y^q}{(1+c_1 x)^{p+q}} \frac{1}{|1+c_1 x|^3} f(x,y)\mathrm{d}x\mathrm{d}y \\ &= \int_S \int (-p-q-3) \frac{x^{p+1} y^q}{(1+c_1 x)^{p+q+1}} \frac{1}{|1+c_1 x|^3} f(x,y)\mathrm{d}x\mathrm{d}y \\ &= -(p+q+3) m_{p+1,q}. \end{aligned} \tag{5}$$

Thus, (4) becomes

$$-\sum_p \sum_q (p+q+3) m_{p+1,q} \frac{\partial F}{\partial m_{pq}} = 0. \tag{6}$$

The constraint (6) must be fulfilled for any image. Assuming that F contains only a finite number of moments, we denote their maximum order as r. However, (6) contains moments up to the order $r+1$. If (6) would be satisfied for some image, we always could construct another image with identical moments up to the order r and different moments of the order $r+1$ such that (6) would not be satisfied for this new image. Thus, finite projective moment invariants cannot exist.

This result can be extended to prove the nonexistence of invariants that would have a form of infinite series with each term equal to a finite product of moments of nonnegative indices.

4 Infinite Projective Invariants

The only possible form of projective moment invariants is an infinite series of products of moments with both positive and negative indices.

To derive the invariants, we use the *triangle method*. Let $V_1, V_2, V_3;\ V_i = (x_i, y_i)$, be three arbitrary noncollinear points in the image. Let $A(1,2,3)$ be the area of the triangle with vertices V_1, V_2, V_3;

$$A(1,2,3) = \frac{1}{2} \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}. \tag{7}$$

When the horizontal pure projection (g) is applied, then the area becomes

$$A'(1,2,3) = \frac{A(1,2,3)}{(1+c_1 x_1)(1+c_1 x_2)(1+c_1 x_3)}. \tag{8}$$

Similarly, after the vertical pure projection (h), we get

$$A'(1,2,3) = \frac{A(1,2,3)}{(1+c_2 y_1)(1+c_2 y_2)(1+c_2 y_3)} \tag{9}$$

and, after the scaling (c), it becomes

$$A'(1,2,3) = \omega^2 A(1,2,3). \tag{10}$$

Under the other elementary transformations, the triangle area is invariant.

To obtain a projective invariant, we take a proper power of $A(1,2,3)$ multiplied by intensity values in the vertices and integrate it over all possible points of the image. Such a functional is invariant to projective transform only if the Jacobian, appearing in the integrand due to substitution, is eliminated. Jacobian J of horizontal pure projection is

$$J = \frac{1}{(1+c_1 x)^3}. \tag{11}$$

To eliminate it, the power of $A(1,2,3)$ must equal minus three (as will be demonstrated in Section 4.1). To derive more invariants, we consider more than one triangle, take product P of appropriate powers of their areas, and integrate it as in the previous case. The elimination of the Jacobian is possible only if the cumulative power of each vertex in P equals minus three. Below, we show explicitly the derivation of simple invariants.

4.1 Case 1: Three Vertices

The simplest invariant can be constructed by considering only one triangle and setting $P = A^{-3}(1,2,3)$. The corresponding invariant is then defined as

$$\begin{aligned} I_1 = \int_S & A^{-3}(1,2,3) f(x_1,y_1) f(x_2,y_2) f(x_3,y_3) \\ & \mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}x_3\mathrm{d}y_3. \end{aligned} \tag{12}$$

Under horizontal pure projection, I_1 is transformed as

$$\begin{aligned} I_1' = \int_S & A^{-3}(1,2,3)(1+c_1 x_1)^3 (1+c_1 x_2)^3 (1+c_1 x_3)^3 \\ & \frac{1}{|1+c_1 x_1|^3} \frac{1}{|1+c_1 x_2|^3} \frac{1}{|1+c_1 x_3|^3} f(x_1,y_1) f(x_2,y_2) f(x_3,y_3) \\ & \mathrm{d}x_1\mathrm{d}y_1\mathrm{d}x_2\mathrm{d}y_2\mathrm{d}x_3\mathrm{d}y_3 = \mathrm{sign}(J) \cdot I_1. \end{aligned} \tag{13}$$

Nonsingular projective transform divides the image plane into two parts by a straight line. In one part, its Jacobian is positive, while, in the other part, it is negative. When the whole image is in the part with positive Jacobian (which is always true in a real-world imaging), then I_1 is an absolute invariant with respect to horizontal pure projection. The proofs for vertical pure projection and scaling are straightforward; the invariance to the other elementary transformations is trivial because their Jacobians equal one and $A(1,2,3)$ itself is invariant to them.

To express I_1 by means of moments, we expand $A^{-3}(1,2,3)$ into power series of monomials $x_i y_j$ and integrate it term-wise. As a result of this process, we obtain a series of moment products, where each product consists of three moments of both negative and positive indices:

$$\begin{aligned} I_1 = 2^3 \sum_{i,j,k=0}^{\infty} & \sum_{\substack{i_1+i_2+i_3+i_4+i_5=i \\ i_1,i_2,i_3,i_4,i_5 \geq 0}} \frac{i!(-1)^{i_2+i_4}}{i_1!i_2!i_3!i_4!i_5!} \\ & \sum_{\substack{j_1+j_2+j_3+j_4+j_5=j \\ j_1,j_2,j_3,j_4,j_5 \geq 0}} \frac{j!(-1)^{j_2+j_4}}{j_1!j_2!j_3!j_4!j_5!} \\ & \sum_{\substack{k_1+k_2+k_3+k_4+k_5=k \\ k_1,k_2,k_3,k_4,k_5 \geq 0}} \frac{k!(-1)^{k_2+k_4}}{k_1!k_2!k_3!k_4!k_5!} \\ & m_{-1-i+i_5+k_1+k_2+j_3+j_4,\,-1-k+k_3+i_1+i_4+j_2+j_5} \\ & m_{-1-j+j_5+i_1+i_2+k_3+k_4,\,-1-i+i_3+j_1+j_4+k_2+k_5} \\ & m_{-1-k+k_5+j_1+j_2+i_3+i_4,\,-1-j+j_3+i_2+i_5+k_1+k_4}. \end{aligned} \tag{14}$$

4.2 Case 2: Four Vertices

In case of four vertices, we define

$$P = A^{-1}(1,2,3) A^{-1}(2,1,4) A^{-1}(3,4,1) A^{-1}(4,3,2)$$

and, consequently, the respective projective invariant

$$I_2 = \int_S P \cdot f(x_1, y_1) f(x_2, y_2) f(x_3, y_3) f(x_4, y_4) \qquad \mathrm{d}x_1 \mathrm{d}y_1 \mathrm{d}x_2 \mathrm{d}y_2 \mathrm{d}x_3 \mathrm{d}y_3 \mathrm{d}x_4 \mathrm{d}y_4. \tag{15}$$

Similarly to the previous case, we expand $A^{-1}(1,2,3)A^{-1}(2,1,4)$ $A^{-1}(3,4,1)A^{-1}(4,3,2)$ into a power series and get the moment-related expression

$$\begin{aligned} I_2 = 2^4 \sum_{i,j,k,\ell=0}^{\infty} & \sum_{\substack{i_1+i_2+i_3+i_4+i_5=i \\ i_1,i_2,i_3,i_4,i_5 \geq 0}} \frac{i!(-1)^{i_2+i_4}}{i_1!i_2!i_3!i_4!i_5!} \\ & \sum_{\substack{j_1+j_2+j_3+j_4+j_5=j \\ j_1,j_2,j_3,j_4,j_5 \geq 0}} \frac{j!(-1)^{j_2+j_4}}{j_1!j_2!j_3!j_4!j_5!} \\ & \sum_{\substack{k_1+k_2+k_3+k_4+k_5=k \\ k_1,k_2,k_3,k_4,k_5 \geq 0}} \frac{k!(-1)^{k_2+k_4}}{k_1!k_2!k_3!k_4!k_5!} \\ & \sum_{\substack{\ell_1+\ell_2+\ell_3+\ell_4+\ell_5=\ell \\ \ell_1,\ell_2,\ell_3,\ell_4,\ell_5 \geq 0}} \frac{\ell!(-1)^{\ell_2+\ell_4}}{\ell_1!\ell_2!\ell_3!\ell_4!\ell_5!} \\ & m_{-1-i+i_5+j_1+j_2+k_3+k_4,-1-j+j_3+i_1+i_4+k_2+k_5} \\ & m_{-1-j+j_5+i_1+i_2+\ell_3+\ell_4,-1-i+i_3+j_1+j_4+\ell_2+\ell_5} \\ & m_{-1-k+k_5+i_3+i_4+\ell_1+\ell_2,-1-\ell+\ell_3+i_2+i_5+k_1+k_4} \\ & m_{-1-\ell+\ell_5+j_3+j_4+k_1+k_2,-1-k+k_3+j_2+j_5+\ell_1+\ell_4}. \end{aligned} \tag{16}$$

4.3 Case 3: Five and More Vertices

We can continue the above process with five and more vertices. P is no longer unique; for instance, for five vertices, each product of the type

$$A^{k_1}(1,2,3)A^{k_2}(1,2,4)A^{k_3}(2,1,5)A^{k_4}(3,1,4)A^{k_5}(3,5,1)$$
$$A^{k_6}(4,5,1)A^{k_7}(2,4,3)A^{k_8}(5,3,2)A^{k_9}(5,4,2)A^{k_{10}}(4,3,5),$$

where

$$\begin{aligned} k_1 + k_2 + k_3 + k_4 + k_5 + k_6 &= -3 \\ k_1 + k_2 + k_3 + k_7 + k_8 + k_9 &= -3 \\ k_1 + k_4 + k_5 + k_7 + k_8 + k_{10} &= -3 \\ k_2 + k_4 + k_5 + k_7 + k_9 + k_{10} &= -3 \\ k_3 + k_5 + k_6 + k_8 + k_9 + k_{10} &= -3 \end{aligned}$$

can be formally used to derive one new projective invariant. However, such new invariants might be dependent on the previous invariants.

5 Numerical Experiment

In this section, we demonstrate the performance of the invariants on real gray-level images that have undergone computer-simulated perspective projections.

In Fig. 1, we can see Lisa image deformed by nine projective transformations (2). The nonlinearity of these transformations, which can be measured by the value of $c = |c_1| + |c_2|$, varied from zero to $4 \cdot 10^{-3}$. Evaluating several first terms only, we approximated the values of the invariants (14), (16) of all distorted images (more precisely, only indices i, j, k less than six were used in the case of (14) and i, j, k, ℓ less than five in the case of (16)). To measure the invariance under distortions, we calculated relative errors of both invariants. Relative error $e(I)$ of invariant I is defined as

$$e(I) = |\frac{I - m(I)}{m(I)}|,$$

where $m(I)$ is the mean value of I over all deformations. The results are summarized in Table 1. Contrary to theoretical expectation, the values of both invariants slightly varied depending on the particular deformation. This was caused mainly by two factors. First, only a finite number of terms was used to evaluate the infinite series in the definitions of the invariants. Second, discrete projective transforms include image resampling, which violates our assumption of "exact" projective distortion of the image and, consequently, violates the invariance property.

Fig. 1. The Lisa image deformed by projective transformations.

6 Conclusion

In this paper, we have shown that, contrary to common belief, projective moment invariants do exist. We have proven they have a form of infinite series containing moments with positive as well as negative indices and we have derived two of them explicitly.

A very important observation is that our invariants are not affected by a fundamental limitation, which was discovered by Åström [26] and which many popular projective invariants suffer from. Åström proved that all closed curves in a plane are projectively equivalent (with some ε-tolerance) and, consequently, any projective invariant defined on the space of closed curves must be either constant or discontinuous. In other words, he showed that object recognition by invariants calculated only from the boundary is practically impossible because they either have no discrimination power or are extremely unstable. Our moment invariants are calculated from the whole object, including its inner structure, colors, etc. Since the Åström's theorem on projective equivalence does not hold for 2D objects (images) including their interiors, it has no impact on the moment invariants.

However, our moment invariants suffer from another limitation. Being calculated from the whole object, the moments are very sensitive to partial occlusion. This is why moment invariants can hardly be used in occluded object recognition. These properties predestinate potential applications of our invariants—we envisage their usage, for instance, in classification of projectively deformed objects and in matching/registration of images of planar scenes obtained from different viewing angles, which is often required in computer vision and remote sensing.

TABLE 1
Relative Errors of the Invariants of the Lisa Image

$c[10^{-4}]$	0	5	10	15	20	25	30	35	40
$e(I_1)$	1.34	0.33	0.34	0.47	0.02	0.15	0.21	0.11	0.64
$e(I_2)$	0.53	0.41	0.50	0.49	0.02	0.02	0.43	0.61	0.22

Although the proof of existence is a significant contribution to moment theory, many questions still remain open. The invariants are formally defined by infinite series but there is no guarantee that all of them converge.

A very important issue is the independence of the invariants because dependent invariants do not contribute to discrimination power at all. However, a general method to identify convergent invariants and to discard dependent invariants has not been found yet. Detailed investigation of these issues will be the subject of our future research.

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