Duality in the space of regulated functions and the play operator

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Abstract

As a new weak convergence concept motivated originally by singular perturbation problems, we investigate the *wbo*-convergence in the space of regulated functions with values in a Hilbert space. Using a generalization of the Tvrdý Representation Theorem for bounded linear functionals on the space of regulated functions, we establish the relationship between the weak and the *wbo*-convergences. The classical play operator from the mathematical theory of hysteresis is used as an important tool in the analysis.

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Introduction

We consider a real Hilbert space X endowed with a scalar product $\langle \cdot, \cdot \rangle$ and norm $|x| := \langle x, x \rangle^{1/2}$ for $x \in X$. We mainly work with *regulated functions* (cf. [1]) with values in X, that is, functions of real variable which at each point of their domain of definition admit both finite one-sided limits, see Definition 1.1 below. The space of regulated functions $[a, b] \to X$ will be denoted by $G(a, b\,; X)$ according to [20].

Regulated functions play an important role for instance in applications to differential equations with singular right-hand sides, see [18], or to the Skorokhod problem, see [5]. Our aim here is to use the so-called *play operator* from the mathematical theory of hysteresis as presented in [3, 4, 10, 11, 21] for investigating a specific notion of weak convergence in $G(a, b\,; X)$ which we call "weak bounded oscillation" or *wbo*-convergence and which arises naturally in singular perturbation problems, see [12]. Namely, we say that a sequence $\{f_n\}$ in $G(a, b\,; X)$ *wbo*-converges to f if $f_n(t)$ converge weakly to $f(t)$ for every $t \in [a, b]$ and, in addition, it has uniformly bounded oscillation in a sense made precise below in Definition 2.1. We will prove, using special properties of the play operator, that a subset of $G(a, b\,; X)$ has uniformly bounded oscillation if and only if it has uniformly bounded ε-variation. Fraňková has introduced in [7] the latter property and proved, as a generalization of the Helly Selection Principle, that every bounded sequence in $G(a, b\,; X)$ with uniformly bounded ε-variation contains a subsequence which converges pointwise at each point $t \in [a, b]$ to a regulated function provided $\dim X < \infty$. This result has been extended to the general Hilbert case in [13]. The main goal of the present

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paper is to investigate the relationship between wbo-convergent sequences and duality issues in $G(a,b\,;\,X)$. We prove in Theorem 2.6, as a generalization of representation theorems from [20], that the dual $G(a,b\,;\,X)'$ to $G(a,b\,;\,X)$ is isomorphic to an explicit subspace of $BV(a,b\,;\,X)\times BV_0(a,b\,;\,X)$ via the Young integral, where $BV(a,b\,;\,X)$ is the space of functions of bounded variation and $BV_0(a,b\,;\,X)$ is its subspace consisting of all functions from $BV(a,b\,;\,X)$ which vanish almost everywhere. As a direct consequence, we obtain that the duals to both spaces $G_L(a,b\,;\,X)$ and $G_R(a,b\,;\,X)$ of left- and right-continuous regulated functions, respectively, are isometrically isomorphic to $BV(a,b\,;\,X)$. The wbo-convergence is neither weaker nor stronger than the natural weak convergence in $G(a,b\,;\,X)$, it has nevertheless interesting properties with respect to the duality $G \leftrightarrow BV$ which we describe in some detail in Theorems 2.5 and 2.7 – 2.8.

The structure of the paper is the following. In Section 1 we give a brief overview of the theory of regulated functions, of the Young integral in the form of [8, 16], and of the play operator as solution operator of a rate-independent variational inequality (which is actually a special case of the Skorokhod problem). The main results are stated in Section 2, and Section 3 is devoted to proofs.

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1 Preliminaries

This paper is based in a substantial way on results of [13]. In order to fix the notation, we list without proof in this section those of them which will be needed in the sequel.

1.1 Regulated functions and total variation

Consider a nondegenerate closed interval $[a,b]\subset\mathbb{R}$, and denote by $\mathcal{D}_{a,b}$ the set of all partitions of the form

$$d \;=\; \{t_0,\dots,t_m\}\,,\quad a=t_0<t_1<\ldots<t_m=b\,.$$

For a given function $g:[a,b]\to X$ and a given partition $d\in\mathcal{D}_{a,b}$ we define the *variation* $\mathcal{V}_d(g)$ of g on d by the formula

$$\mathcal{V}_d(g) \;:=\; \sum_{j=1}^{m} |g(t_j)-g(t_{j-1})|$$

and the *total variation* $\mathrm{Var}_{[a,b]}\, g$ of g by

$$\operatorname*{Var}_{[a,b]} g \;:=\; \sup\{\mathcal{V}_d(g)\,;\, d\in\mathcal{D}_{a,b}\}\,.$$

In a standard way (cf. [2]) we denote the set of functions of bounded variation by

$$BV(a,b\,;\,X) \;:=\; \{g:[a,b]\to X\,;\, \operatorname*{Var}_{[a,b]} g<\infty\}\,. \tag{1.1}$$

Let us further introduce the set $S(a,b\,;\,X)$ of all *step functions* of the form

$$w(t) \;:=\; \sum_{k=0}^{m}\hat{c}_k\,\chi_{\{t_k\}}(t) \,+\, \sum_{k=1}^{m} c_k\,\chi_{]t_{k-1},t_k[}(t)\,,\quad t\in[a,b]\,, \tag{1.2}$$

where $d = \{t_0, \ldots, t_m\} \in \mathcal{D}_{a,b}$ is a given partition, χ_A for $A \subset [a,b]$ is the characteristic function of the set A, and $\hat{c}_0, \ldots, \hat{c}_m, c_1, \ldots, c_m$ are given elements from X.

It is well-known (see e. g. the Appendix of [2]) that every function of bounded variation with values in a Banach space admits one-sided limits at each point of its domain of definition. Following [1], we separate this property from the notion of total variation and introduce the following definition.

Definition 1.1 *We say that a function $f : [a,b] \to X$ is* regulated *if for every $t \in [a,b]$ there exist both one-sided limits $f(t+), f(t-) \in X$ with the convention $f(a-) = f(a)$, $f(b+) = f(b)$.*

According to [20], we denote by $G(a,b\,;\,X)$ the set of all regulated functions $f : [a,b] \to X$.

For a given function $g \in G(a,b\,;\,X)$ and a given partition $d \in \mathcal{D}_{a,b}$ we define the *essential variation* $\overline{\mathcal{V}}_d(g)$ of g on d by the formula

$$\overline{\mathcal{V}}_d(g) \;:=\; \sum_{j=1}^{m} |g(t_j-) - g(t_{j-1}+)| \,+\, \sum_{j=0}^{m} |g(t_j+) - g(t_j-)|$$

and the *total essential variation* $\overline{\mathrm{Var}}_{[a,b]}\, g$ of g by

$$\overline{\mathrm{Var}}_{[a,b]}\, g \;:=\; \sup\{\overline{\mathcal{V}}_d(g)\,;\, d \in \mathcal{D}_{a,b}\}\,.$$

We denote the space of functions of *essentially bounded variation* by

$$\overline{BV}(a,b\,;\,X) \;:=\; \{g : [a,b] \to X\,;\, \overline{\mathrm{Var}}_{[a,b]}\, g < \infty\}\,. \tag{1.3}$$

The terminology has been taken from [6], although we restrict ourselves a priori to regulated functions which makes the analysis easier. This however means here in particular that $\mathcal{V}_d(g)$ is defined for *every* function $g : [a,b] \to X$, but $\overline{\mathcal{V}}_d(g)$ only for a regulated function g.

We summarize some easy basic properties of the above spaces in Lemma 1.2 below the proof of which is left to the reader.

Lemma 1.2

(i) *Every regulated function is bounded.*

(ii) *For every $g \in G(a,b\,;\,X)$ we have $\overline{\mathrm{Var}}_{[a,b]}\, g \le \mathrm{Var}_{[a,b]}\, g$.*

(iii) *The sets $S(a,b\,;\,X)$, $BV(a,b\,;\,X)$, $\overline{BV}(a,b\,;\,X)$, $G(a,b\,;\,X)$ are vector spaces satisfying the inclusions*

$$S(a,b\,;\,X) \subset BV(a,b\,;\,X) \subset \overline{BV}(a,b\,;\,X) \subset G(a,b\,;\,X)\,. \tag{1.4}$$

(iv) *Let $G_L(a,b\,;\,X)$ be the subset of left-continuous functions in $G(a,b\,;\,X)$. Then $\overline{\mathrm{Var}}_{[a,b]}\, g = \mathrm{Var}_{[a,b]}\, g$ for every $g \in G_L(a,b\,;\,X)$, and $G_L(a,b\,;\,X) \cap \overline{BV}(a,b\,;\,X) \subset BV(a,b\,;\,X)$.*

Similarly as in Lemma 1.2, we denote by $G_R(a,b\,;\,X)$ the space of right-continuous regulated functions on $[a,b]$. We further set $BV_L(a,b\,;\,X) = BV(a,b\,;\,X) \cap G_L(a,b\,;\,X)$, $BV_R(a,b\,;\,X) = BV(a,b\,;\,X) \cap G_R(a,b\,;\,X)$, and $S_L(a,b\,;\,X) = S(a,b\,;\,X) \cap G_L(a,b\,;\,X)$, $S_R(a,b\,;\,X) = S(a,b\,;\,X) \cap G_R(a,b\,;\,X)$. Let us introduce in $G(a,b\,;\,X)$ a system of seminorms

$$\|f\|_{[s,t]} \;:=\; \sup\{|f(\tau)|\,;\,\tau \in [s,t]\} \tag{1.5}$$

for any subinterval $[s,t] \subset [a,b]$. Indeed, $\|\cdot\|_{[a,b]}$ is a norm. With this norm, $G(a,b\,;\,X)$ becomes a Banach space, and the injections in (1.4) are dense. Let us note that the space $C(a,b\,;\,X)$ of *continuous functions* $f : [a,b] \to X$ is a closed subspace of $G(a,b\,;\,X)$ with respect to the norm $\|\cdot\|_{[a,b]}$.

1.2 The Young integral

We say that a partition $\hat{d}$ is a *refinement* of $d \in \mathcal{D}_{a,b}$ and write $\hat{d} \succ d$ if $\hat{d} \in \mathcal{D}_{a,b}$ and $d \subset \hat{d}$.

Let $d = \{t_0,\ldots,t_m\} \in \mathcal{D}_{a,b}$ be a partition. We denote by $\mathcal{B}(d)$ the set of special refinements D of d (the so-called *P-partitions*, see [17]) of the form

$$D \;=\; \{t_0,\tau_1,t_1,\ldots,t_{m-1},\tau_m,t_m\}\,,\;\; a = t_0 < \tau_1 < t_1 < \tau_2 < t_2 < \ldots t_{m-1} < \tau_m < t_m = b\,. \tag{1.6}$$

For given functions $f : [a,b] \to X$, $g \in G(a,b\,;\,X)$ and partitions $d \in \mathcal{D}_{a,b}$, $D \in \mathcal{B}(d)$ of the form (1.6) we define the integral sum $S_D(f\,\Delta g)$ by the formula

$$S_D(f\,\Delta g) \;=\; \sum_{j=1}^{m} \langle f(\tau_j), g(t_j-) - g(t_{j-1}+)\rangle \;+\; \sum_{j=0}^{m} \langle f(t_j), g(t_j+) - g(t_j-)\rangle\;, \tag{1.7}$$

again with the convention $g(a-) = g(a)$, $g(b+) = g(b)$.

Definition 1.3 *We say that $J \in \mathbb{R}$ is the* Young integral *over $[a,b]$ of f with respect to g and denote*

$$J \;=\; \int_a^b \langle f(t), dg(t)\rangle\;, \tag{1.8}$$

if for every $\varepsilon > 0$ there exists $d_\varepsilon \in \mathcal{D}_{a,b}$ such that for every $d \succ d_\varepsilon$ and $D \in \mathcal{B}(d)$ we have

$$|J \;-\; S_D(f\,\Delta g)| \;\le\; \varepsilon\,. \tag{1.9}$$

It is an easy exercise to check that if the value J in Definition 1.3 exists, then it is uniquely determined. In what follows, whenever we write $\int_a^b \langle f(t), dg(t)\rangle \;=\; J$, we interpret it as *'the function f is Young integrable with respect to g in $[a,b]$ and the integral equals to J.'*

For the sake of completeness, we list the following properties of the Young integral which are common for most integrals; detailed proofs can be found e. g. in [13].

Proposition 1.4

(i) *Let $\int_a^b\langle f_1(t), dg(t)\rangle$, $\int_a^b\langle f_2(t), dg(t)\rangle$ exist. Then we have*

$$\int_a^b \langle (f_1+f_2)(t), dg(t)\rangle \;=\; \int_a^b \langle f_1(t), dg(t)\rangle + \int_a^b \langle f_2(t), dg(t)\rangle\,. \tag{1.10}$$

(ii) *Let* $\int_a^b \langle f(t), dg_1(t)\rangle$, $\int_a^b \langle f(t), dg_2(t)\rangle$ *exist. Then we have*

$$\int_a^b \langle f(t), d(g_1+g_2)(t)\rangle \;=\; \int_a^b \langle f(t), dg_1(t)\rangle + \int_a^b \langle f(t), dg_2(t)\rangle\,. \tag{1.11}$$

(iii) *Let* $\int_a^b \langle f(t), dg(t)\rangle$ *exist. Then for every constant* $\lambda\in\mathbb{R}$ *we have*

$$\int_a^b \langle \lambda f(t), dg(t)\rangle \;=\; \int_a^b \langle f(t), d(\lambda g)(t)\rangle \;=\; \lambda\int_a^b \langle f(t), dg(t)\rangle\,. \tag{1.12}$$

Proposition 1.5 *Let* $f:[a,b]\to X$, $g\in G(a,b\,;\,X)$ *be given functions and let* $[r,s]\subset[a,b]$ *be a nondegenerate interval.*

(i) *Assume that* $\int_a^b \langle f(t), dg(t)\rangle$ *exists. Then* $\int_r^s \langle f(t), dg(t)\rangle$ *exists.*

(ii) *Assume that* $\int_r^s \langle f(t), dg(t)\rangle$ *exists. Then we have*

$$\int_a^b \Big\langle \Big(f\,\chi_{]r,s[}\Big)(t), dg(t)\Big\rangle \;=\; \int_r^s \langle f(t), dg(t)\rangle - \langle f(r), g(r+)-g(r)\rangle - \langle f(s), g(s)-g(s-)\rangle\,. \tag{1.13}$$

(iii) *Assume that* $\int_a^s \langle f(t), dg(t)\rangle$, $\int_s^b \langle f(t), dg(t)\rangle$ *exist for some* $s\in\,]a,b[$. *Then we have*

$$\int_a^b \langle f(t), dg(t)\rangle \;=\; \int_a^s \langle f(t), dg(t)\rangle + \int_s^b \langle f(t), dg(t)\rangle\,. \tag{1.14}$$

In order to preserve the consistency of (1.14) also in the limit cases $s=a$ and $s=b$, we set

$$\int_s^s \langle f(t), dg(t)\rangle \;=\; 0 \qquad \forall s\in[a,b]\,,\ \forall\, f,g:[a,b]\to X\,. \tag{1.15}$$

Remark 1.6 Proposition 1.5 needs some comment. Here and in the sequel, whenever we integrate functions f,g defined in $[a,b]$ over an interval $[r,s]\subset[a,b]$, we implicitly consider their restrictions $f|_{[r,s]}, g|_{[r,s]}$. In particular, we have e. g. $f|_{[r,s]}(s+)=f(s)$, $f|_{[r,s]}(r-)=f(r)$, similarly as in Definition 1.1.

The following explicit formulas will be needed in the sequel.

Proposition 1.7 *For every* $f:[a,b]\to X$, $g\in G(a,b\,;\,X)$, $a\le r\le b$ *and* $v\in X$ *we have*

(i) $\displaystyle\int_a^b \Big\langle v\,\chi_{\{r\}}(t), dg(t)\Big\rangle \;=\; \langle v, g(r+)-g(r-)\rangle$,

(ii) $\displaystyle\int_a^b \Big\langle f(t), d\Big(v\,\chi_{\{r\}}\Big)(t)\Big\rangle \;=\; \begin{cases} 0 & \text{if } r\in\,]a,b[\ , \\ -\langle f(a), v\rangle & \text{if } r=a\,, \\ \langle f(b), v\rangle & \text{if } r=b\,, \end{cases}$

(iii) $\displaystyle\int_a^b \Big\langle v\,\chi_{]r,s[}(t), dg(t)\Big\rangle \;=\; \langle v, g(s-)-g(r+)\rangle \qquad \forall\, s\in\,]r,b]$,

(iv) $\displaystyle\int_a^b \Big\langle f(t), d\Big(v\,\chi_{]r,s[}\Big)(t)\Big\rangle \;=\; \langle f(r)-f(s), v\rangle \qquad \forall\, s\in\,]r,b]\;.$

Theorem 1.8 *If $f\in G(a,b\,;\,X)$ and $g\in \overline{BV}(a,b\,;\,X)$, then $\int_a^b \langle f(t), dg(t)\rangle$ exists and satisfies the estimate*

$$\left|\int_a^b \langle f(t), dg(t)\rangle\right| \;\le\; \|f\|_{[a,b]}\, \overline{\operatorname*{Var}_{[a,b]}}\, g\,. \tag{1.16}$$

Moreover, for every $g\in\overline{BV}(a,b\,;\,X)$ we have

$$\overline{\operatorname*{Var}_{[a,b]}}\, g \;=\; \sup\left\{\int_a^b \langle f(t), dg(t)\rangle \;;\; f\in S(a,b\,;\,B_1(0))\right\}, \tag{1.17}$$

where $B_r(x_0)$ for $r>0$ and $x_0\in X$ denotes the ball $\{x\in X\,;\,|x-x_0|\le r\}$ centered at x_0 with radius r.

The assertion of Theorem 1.8 is contained in [13, Corollary 3.10 and Theorem 3.11]. We will also need the following statement as a slight modification of the second part of [13, Theorem 3.11].

Theorem 1.9 *If $f\in BV(a,b\,;\,X)$ and $g\in G(a,b\,;\,X)$, then $\int_a^b \langle f(t), dg(t)\rangle$ exists and satisfies the estimates*

$$\left|\langle f(a), g(a)\rangle + \int_a^b \langle f(t), dg(t)\rangle\right| \;\le\; \Big(|f(b)| + \operatorname*{Var}_{[a,b]} f\Big)\, \|g\|_{[a,b]}\,, \tag{1.18}$$

$$\left|\langle f(b), g(b)\rangle - \int_a^b \langle f(t), dg(t)\rangle\right| \;\le\; \Big(|f(a)| + \operatorname*{Var}_{[a,b]} f\Big)\, \|g\|_{[a,b]}\,. \tag{1.19}$$

Moreover, for every $f\in BV(a,b\,;\,X)$ we have

$$|f(b)| + \operatorname*{Var}_{[a,b]} f \;=\; \sup\left\{\langle f(a), g(a)\rangle + \int_a^b \langle f(t), dg(t)\rangle \;;\; g\in S_R(a,b\,;\,B_1(0))\right\}, \tag{1.20}$$

$$|f(a)| + \operatorname*{Var}_{[a,b]} f \;=\; \sup\left\{\langle f(b), g(b)\rangle - \int_a^b \langle f(t), dg(t)\rangle \;;\; g\in S_L(a,b\,;\,B_1(0))\right\}. \tag{1.21}$$

Proof of Theorem 1.9. For the existence of $\int_a^b \langle f(t), dg(t)\rangle$ we refer to [13, Corollary 3.10]. Inequalities (1.18) – (1.19) follow easily if we consider an arbitrary partition D of the form (1.6), and rewrite (1.7) as

$$\begin{aligned} S_D(f\,\Delta g) \;&=\; \langle f(b), g(b)\rangle - \langle f(a), g(a)\rangle \\ &\quad + \sum_{j=1}^{m} \langle f(\tau_j) - f(t_j), g(t_j-)\rangle - \sum_{j=0}^{m-1} \langle f(\tau_{j+1}) - f(t_j), g(t_j+)\rangle\,. \end{aligned}$$

To prove (1.20) – (1.21), we fix $\varepsilon>0$ and find a partition $d=\{t_0,\dots,t_m\}\in\mathcal{D}_{a,b}$ such that

$$\sum_{j=1}^{m} |f(t_j) - f(t_{j-1})| \;\ge\; \operatorname*{Var}_{[a,b]} f \;-\; \varepsilon\,. \tag{1.22}$$

Let $\sigma : X \to X$ be the function

$$\sigma(x) = x/|x| \quad \text{for} \ x \neq 0\,, \quad \sigma(0) = 0\,,$$

and for $t \in [a,b]$ put

$$g(t) \ = \ \sigma(f(b))\, \chi_{\{b\}}(t) - \sum_{j=1}^{m} \sigma(f(t_j) - f(t_{j-1}))\, \chi_{[t_{j-1},t_j[}(t)\,. \tag{1.23}$$

We then infer from (1.22), Proposition 1.7 (ii), (iv) and Proposition 1.4 (ii) that

$$\langle f(a), g(a)\rangle + \int_a^b \langle f(t), dg(t)\rangle \ = \ |f(b)| + \sum_{j=1}^{m} |f(t_j) - f(t_{j-1}| \ \geq \ |f(b)| + \operatorname*{Var}_{[a,b]} f - \varepsilon\,, \tag{1.24}$$

and (1.20) follows from (1.18) and (1.24). The proof of (1.21) is analogous. ∎

As an easy consequence of Theorems 1.8, 1.9 and of the dense injection $BV(a,b\,;X) \subset G(a,b\,;X)$, we have the following convergence result.

Proposition 1.10 *Consider* $f, f_n \in G(a,b\,;X)$, $g, g_n \in \overline{BV}(a,b\,;X)$, $n \in \mathbb{N}$ *such that*

$$\lim_{n\to\infty} \|f - f_n\|_{[a,b]} = 0\,, \quad \lim_{n\to\infty} \|g - g_n\|_{[a,b]} = 0\,, \quad \sup_{n\in\mathbb{N}} \overline{\operatorname*{Var}_{[a,b]}}\, g_n \ = \ C \ < \ \infty\,.$$

Then

$$\int_a^b \langle f(t), dg(t)\rangle \ = \ \lim_{n\to\infty} \int_a^b \langle f_n(t), dg_n(t)\rangle\,. \tag{1.25}$$

To conclude this section, we recall two integration-by-parts formulas.

Theorem 1.11 *For every* $f \in G(a,b\,;X)$, $g \in BV(a,b\,;X)$ *we have*

$$\begin{aligned} \int_a^b \langle f(t), dg(t)\rangle + \int_a^b \langle g(t), df(t)\rangle \ &= \ \langle f(b), g(b)\rangle - \langle f(a), g(a)\rangle \\ &+ \sum_{t\in[a,b]} \Big(\langle f(t) - f(t-)\,,\, g(t) - g(t-)\rangle - \langle f(t+) - f(t)\,,\, g(t+) - g(t)\rangle \Big)\,. \end{aligned} \tag{1.26}$$

Corollary 1.12 *For every* $g \in \overline{BV}(a,b\,;X)$ *we have*

$$\int_a^b \langle g(t+), dg(t)\rangle \ = \ \frac{1}{2}\left(|g(b)|^2 - |g(a)|^2\right) + \frac{1}{2} \sum_{t\in[a,b]} |g(t+) - g(t-)|^2\,. \tag{1.27}$$

1.3 The play operator

We fix a convex closed (not necessarily bounded) set $Z \subset X$ with $0 \in \operatorname{Int} Z$, and a constant $r > 0$ such that

$$B_r(0) \subset Z\,. \tag{1.28}$$

For a given initial condition $x_0 \in Z$ and an input $u \in G_L(a,b\,;X)$, we look for a solution $\xi \in BV_L(a,b\,;X)$ to the problem

$$
\begin{aligned}
&\textbf{(}\boldsymbol{\mathcal{P}}\textbf{)} \quad \text{(i)} \quad u(t) - \xi(t) \in Z \quad \forall t \in [a,b]\,, \\
&\qquad\ \ \text{(ii)} \quad u(a) - \xi(a) = x_0\,, \\
&\qquad\ \ \text{(iii)} \quad \int_a^b \langle u(\tau+) - \xi(\tau+) - y(\tau), d\xi(\tau)\rangle \ \geq\ 0 \qquad \forall y \in G(a,b;Z)\,.
\end{aligned}
$$

Problem **($\boldsymbol{\mathcal{P}}$)** has been considered in [13] in a more general setting. We cite without proof the following result as a special case of [13, Theorem 2.4] (up to the obvious variable transformation $[0,T] \to [a,b]$).

Theorem 1.13 *For every $(x_0, u) \in Z \times G_L(a,b\,;\,X)$ there exists a unique solution $\xi \in BV_L(a,b\,;\,X)$ to Problem* **($\boldsymbol{\mathcal{P}}$)**, *and the inequality*

$$
|\xi(t+) - \xi(t)| \ \leq\ |u(t+) - u(t)| \tag{1.29}
$$

holds for every $t \in [a,b]$. Moreover, if $\{(x_0^n, u_n)\,;\, n \in \mathbb{N}\}$ is a sequence in $Z \times G_L(a,b\,;\,X)$ such that $|x_0^n - x_0| \to 0$, $\|u_n - u\|_{[a,b]} \to 0$ as $n \to \infty$, and if ξ_n, ξ are the solutions corresponding to the data $(x_0^n, u_n), (x_0, u)$, respectively, then $\|\xi_n - \xi\|_{[a,b]} \to 0$, and there exists a constant $C > 0$ independent of n such that $\operatorname{Var}_{[a,b]} \xi_n \leq C$.

The solution mapping

$$
\mathfrak{p}_Z \,:\, Z \times G_L(a,b\,;\,X) \to BV_L(a,b\,;\,X) \,:\, (x_0, u) \mapsto \xi\,, \tag{1.30}
$$

is called the *play operator*. As a consequence of (1.29), we see in particular that $\xi := \mathfrak{p}_Z[x_0, u]$ is continuous if u is continuous, hence it is a natural extension of the classical play operator which has many applications in hysteresis modelling, in particular in plasticity, piezoelectricity, ferromagnetism etc., see [3, 11, 14, 21] and the references therein.

The following useful statement has been proved in [13, Lemma 4.2].

Proposition 1.14 *Let $(x_0, u) \in Z \times G_L(a,b\,;\,X)$ be given and let $\xi = \mathfrak{p}_Z[x_0, u]$. Then for every $a \leq s < t \leq b$ we have*

$$
\int_s^t \langle u(\tau+) - \xi(\tau+) - y(\tau), d\xi(\tau)\rangle \ \geq\ 0 \quad \forall y \in G(s,t;Z)\,. \tag{1.31}
$$

2 Main results

We start by introducing some definitions. By $\mathbb{R}^+$ we denote the open interval $]0, \infty[$.

Definition 2.1 *Let $U \subset G(a,b\,;\,X)$ be an arbitrary set, and let $\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$ be an arbitrary function. Then U is said to have*

(i) uniformly bounded ε-variation *if there exists a non-increasing function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$
\forall \varepsilon > 0 \ \ \forall u \in U \ \ \exists \psi \in BV(a,b\,;\,X) : \quad \|u - \psi\|_{[a,b]} \leq \varepsilon\,, \quad \operatorname*{Var}_{[a,b]} \psi \leq L(\varepsilon)\,. \tag{2.1}
$$

(ii) uniformly bounded oscillation *if*

- *there exists a constant* $R > 0$ *such that*

$$|u(t) - u(s)| \leq R \qquad \forall u \in U \quad \forall s, t \in [a, b], \tag{2.2}$$

- *there exists a non-increasing function* $N : \mathbb{R}^+ \to \mathbb{R}^+$ *such that for every* $r > 0$ *and every system* $\{]a_k, b_k[\,;\, k = 1, \ldots, m\}$ *of pairwise disjoint intervals* $]a_k, b_k[\subset [a, b]$ *the implication*

$$\Big(|u(b_k) - u(a_k)| \geq r \quad \forall k = 1, \ldots, m\Big) \quad \Rightarrow \quad m \leq N(r) \tag{2.3}$$

holds for every $u \in U$.

(iii) uniformly bounded Γ-variation *if for every partition* $a = t_0 < t_1 < \ldots < t_m = b$ *we have*

$$\sum_{j=1}^{m} \Gamma(|u(t_j) - u(t_{j-1})|) \leq 1 \qquad \forall u \in U.$$

We now state the first main result of this paper.

Theorem 2.2 *Let* $U \subset G(a, b\,; X)$ *be a given set. Then the following three conditions are equivalent.*

(i) U *has uniformly bounded* ε*-variation.*

(ii) U *has uniformly bounded oscillation.*

(iii) *There exists a non-decreasing function* $\Gamma : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\Gamma(+\infty) = +\infty$*, and* U *has uniformly bounded* Γ*-variation.*

The equivalence (i) $\Leftrightarrow$ (iii) restricted to continuous inputs along with quantitative estimates has been established in [19, Theorem 17] in the power law case $\Gamma(r) = Cr^p$ with $C > 0$ and $p \geq 1$. The concept of uniformly bounded ε-variation has been introduced in [7, Definition 3.3] with the intention to extend the Helly Selection Principle to regulated functions. For our purposes, we state this result in the following form.

Proposition 2.3 *Let* $\{f_n \,;\, n \in \mathbb{N}\}$ *be a bounded sequence of functions from* $G(a, b\,; X)$ *which has uniformly bounded* ε*-variation. Then there exist* $f \in G(a, b\,; X)$ *and a subsequence* $\{f_{n_k}\}$ *of* $\{f_n\}$ *such that* $f_{n_k}(t)$ *converge weakly to* $f(t)$ *as* $k \to \infty$ *for every* $t \in [a, b]$.

The original reference goes back to [7, Theorem 3.8] in the case $\dim X < \infty$. The extension to a general Hilbert space X has been done in [13, Theorem 5.2].

Definition 2.4 *A sequence* $\{f_n\}$ *in* $G(a, b\,; X)$ *is said to wbo*-converge *to a function* $f \in G(a, b\,; X)$*, if the set* $\{f_n \,;\, n \in \mathbb{N}\}$ *has uniformly bounded oscillation and* $f_n(t) \to f(t)$ *as* $n \to \infty$ *weakly in* X *for every* $t \in [a, b]$.

The following equivalence statement reflects the "weak" character of the wbo-convergence.

Theorem 2.5 *Let $f \in G(a,b\,;X)$ be given, and let $\{f_n\,;\, n \in \mathbb{N}\}$ be a bounded sequence in $G(a,b\,;X)$ such that $f_n(t) \to f(t)$ weakly in X for every $t \in [a,b]$. Then the following two conditions are equivalent.*

(i) *f_n have uniformly bounded oscillation;*

(ii) *For every sequence $\{g_n\}$ in $\overline{BV}(a,b\,;X)$ such that $\|g_n - g\|_{[a,b]} \to 0$ as $n \to \infty$ and $\overline{\mathrm{Var}}_{[a,b]}\, g_n \le C$ independently of n we have*

$$\int_a^b \langle f_n(t), dg_n(t)\rangle \to \int_a^b \langle f(t), dg(t)\rangle\ .$$

Representation formulas for bounded linear functionals on $G_L(a,b\,;X)$ using the Kurzweil integral have been derived for the first time in [20]. We prove in the next section the following extension to the whole space $G(a,b\,;X)$. To this end, we denote by $BV_0(a,b\,;X)$ the closed subspace of $BV(a,b\,;X)$ consisting of all functions $\mu \in BV(a,b\,;X)$ which vanish everywhere except on a countable set.

Theorem 2.6 *Let $P : G(a,b\,;X) \to \mathbb{R}$ be a bounded linear functional. Then there exist uniquely determined functions $\mu \in BV_0(a,b\,;X)$ and $f,\hat f \in BV(a,b\,;X)$ such that $\mu(b) = f(b)$, $\mu(a) = \hat f(a)$, and for every $g \in G(a,b\,;X)$ we have*

$$P(g) \;=\; \langle f(a), g(a)\rangle + \int_a^b \langle f(t), dg(t)\rangle - \sum_{t\in[a,b]} \langle \mu(t), g(t+) - g(t)\rangle\ , \tag{2.4}$$

$$P(g) \;=\; \left\langle \hat f(b), g(b)\right\rangle - \int_a^b \left\langle \hat f(t), dg(t)\right\rangle + \sum_{t\in[a,b]} \langle \mu(t), g(t) - g(t-)\rangle\ . \tag{2.5}$$

Moreover, the norm

$$\|P\| \;=\; \sup\{P(g)\,;\, g \in G(a,b\,;X)\,,\ \|g\|_{[a,b]} \le 1\} \tag{2.6}$$

of P satisfies the estimates

$$\|P\| \;\le\; |\mu(a)| + |f(b)| + \mathrm{Var}_{[a,b]}\, f + \mathrm{Var}_{[a,b]}\, \mu \;\le 3\,\|P\|\,, \tag{2.7}$$

$$\|P\| \;\le\; |\hat f(a)| + |\mu(b)| + \mathrm{Var}_{[a,b]}\, \hat f + \mathrm{Var}_{[a,b]}\, \mu \;\le 3\,\|P\|\,. \tag{2.8}$$

In particular, the dual $G(a,b\,;X)'$ to $G(a,b\,;X)$ is isomorphic to each of the two spaces $V_b = \{(f,\mu) \in BV(a,b\,;X) \times BV_0(a,b\,;X)\,;\, f(b) = \mu(b)\}$ and $V_a = \{(\hat f,\mu) \in BV(a,b\,;X) \times BV_0(a,b\,;X)\,;\, \hat f(a) = \mu(a)\}$.

Note that in (2.4) – (2.5), we sum over a countable set, and $\sum_{t\in[a,b]} |\mu(t)| \le \mathrm{Var}_{[a,b]}\, \mu < \infty$, hence the formula is meaningful. Furthermore, identities (2.4) – (2.5) imply that arbitrary functionals $P_R \in G_R(a,b\,;X)'$, $P_L \in G_L(a,b\,;X)'$ admit representations of the form

$$P_R(g) \;=\; \langle f(a), g(a)\rangle + \int_a^b \langle f(t), dg(t)\rangle \qquad \text{for} \quad g \in G_R(a,b\,;X)\,, \tag{2.9}$$

$$P_L(g) \;=\; \left\langle \hat f(b), g(b)\right\rangle - \int_a^b \left\langle \hat f(t), dg(t)\right\rangle \qquad \text{for} \quad g \in G_L(a,b\,;X) \tag{2.10}$$

with functions $f, \hat{f} \in BV(a,b\,;\,X)$. We will see in the proof of Theorem 2.6 that $f, \hat{f}$ are also uniquely determined. Moreover, by virtue of Theorem 1.9, we have

$$\|P_R\| \;=\; |f(b)| + \operatorname*{Var}_{[a,b]} f\,, \qquad \|P_L\| \;=\; |\hat{f}(a)| + \operatorname*{Var}_{[a,b]} \hat{f}\,, \tag{2.11}$$

hence both $G_R(a,b\,;\,X)'$, $G_L(a,b\,;\,X)'$ are isometrically isomorphic to $BV(a,b\,;\,X)$. This is a slight improvement with respect to Theorem 3.8 of [20], where bounded linear functionals P_L on $G_L(a,b\,;\,\mathbb{R})$ (with an obvious extension to X-valued functions) are represented in the form

$$P_L(g) \;=\; \langle c, g(a)\rangle + \int_a^b \langle p(t), dg(t)\rangle \tag{2.12}$$

with a vector $c \in X$ and a function $p \in BV(a,b\,;\,X)$. We thus "save" one component, although (2.12) is indeed (note that g is left-continuous!) equivalent to (2.10) with

$$\hat{f}(t) \;=\; c - p(t)\ \chi_{[a,b[}(t)\,.$$

Let us recall also the Riesz Representation Theorem (see [9, Chapter 4, §6] for real-valued functions) which states that every bounded linear functional P_C on $C(a,b\,;\,X)$ can be represented in a unique way as

$$P_C(g) \;=\; \int_a^b \langle g(t), df^*(t)\rangle \qquad \forall g \in C(a,b\,;\,X)\,, \qquad \|P_C\| \;=\; \operatorname*{Var}_{[a,b]} f^* \tag{2.13}$$

with some $f^* \in BV(a,b\,;\,X)$ such that $f^*(0) = 0$ and $f^*(t) = (f^*(t+) + f^*(t-))/2$ for all $t \in]a,b[$. Note that this also follows from (2.10) if we put $f_*(t) = \hat{f}(t) - \hat{f}(a)\,\chi_{\{a\}}(t)$, $f^*(t) = (f_*(t+)+f_*(t-))/2$ for $t \in]a,b[$, $f^*(t) = f_*(t)$ for $t = a, b$, and integrate by parts using Theorem 1.11. A natural question is whether a functional P on $G_L(a,b\,;\,X)$ or $G_R(a,b\,;\,X)$ can also be represented in the form

$$P(g) \;=\; \langle f(b), g(b)\rangle - \int_a^b \langle g(t), df(t)\rangle\,, \tag{2.14}$$

$$P(g) \;=\; \left\langle \hat{f}(a), g(a)\right\rangle + \int_a^b \left\langle g(t), d\hat{f}(t)\right\rangle \tag{2.15}$$

analogous to (2.13) with functions $f, \hat{f} \in \overline{BV}(a,b\,;\,X)$. The answer is in general negative. It suffices to consider the functional $P_R(g) = \langle g(s-), v\rangle$ for some fixed $s \in]a,b[$ and $0 \neq v \in X$, which satisfies (2.4) with $\mu \equiv 0$ and $f = v\,\chi_{[a,s[}$. On the other hand, the sequence $\{g_n\}$, $g_n(t) = v\,\chi_{[s_n,s[}(t)$, $s_n \nearrow s$ has uniformly bounded oscillation and $g_n(t) \to 0$ for every $t \in [a,b]$, hence $\int_a^b \langle g_n(t), df(t)\rangle \to 0$ for every $f \in \overline{BV}(a,b\,;\,X)$ according to Theorem 2.5, but $P_R(g_n) = |v|^2$ for every $n \in \mathbb{N}$ in contradiction with (2.14), (2.15). The same argument applies to the functional $P_L(g) = \langle g(s+), v\rangle$, with $g_n(t) = v\,\chi_{]s,s_n]}(t)$ and $s_n \searrow s$.

In the above example, the functions g_n *wbo*-converge to 0, but do not converge weakly in the usual sense. On the other hand, the functions

$$\hat{g}_n \;=\; \sum_{k=2^{n-1}+1}^{2^n-1} v\ \chi_{[a+(1/k)-2^{-2n},\, a+(1/k)[} \tag{2.16}$$

for $n \geq n_0$ sufficiently large with a fixed vector $0 \neq v \in X$ provide an example of a sequence in $G_R(a,b\,;\,X)$ which does not have uniformly bounded oscillation, but $\hat{g}_n(t) \to 0$ as $n \to \infty$

for every $t \in [a,b]$ and $P_R(\hat{g}_n) \to 0$ for every $P_R \in G_R(a,b\,;\,X)'$. Indeed, for every functional P_R of the form (2.9) we have

$$|P_R(\hat{g}_n)| \;\le\; |v| \sum_{k=2^{n-1}+1}^{2^n-1} \left| f\left(a+\frac{1}{k}\right) - f\left(a+\frac{1}{k}-2^{-2n}\right)\right| \;\le\; |v| \operatorname*{Var}_{[a+2^{-n},\, a+2^{1-n}]} f\,. \tag{2.17}$$

This yields that

$$\sum_{n=n_0}^{\infty} |P_R(\hat{g}_n)| \;\le\; |v| \operatorname*{Var}_{[a,b]} f \;<\; \infty\,, \tag{2.18}$$

hence $P_R(\hat{g}_n) \to 0$ as $n \to \infty$. We see that there is no direct implication between the weak and the *wbo*-convergence in $G_R(a,b\,;\,X)$ or $G_L(a,b\,;\,X)$.

We conclude this section by two statements illustrating in a different way the duality between $G(a,b\,;\,X)$ and $BV(a,b\,;\,X)$ with respect to convergences in Theorem 2.5.

Theorem 2.7 *Let $P : G(a,b\,;\,X) \to \mathbb{R}$ be a bounded linear functional. Then the following two conditions are equivalent.*

(i) *$P(g_n) \to P(g)$ for every wbo-convergent sequence $g_n \to g$ in $G(a,b\,;\,X)$;*

(ii) *There exist $f, \hat{f} \in BV(a,b\,;\,X)$ such that for every $g \in G(a,b\,;\,X)$ the identities (2.14), (2.15) hold.*

Theorem 2.8 *Let $Q_L : BV_L(a,b\,;\,X) \to \mathbb{R}$ be a bounded linear functional. Then the following two conditions are equivalent.*

(i) *$Q_L(g_n) \to Q_L(g)$ for every sequence $\{g_n\}$ in $BV_L(a,b\,;\,X)$ such that $\|g_n - g\|_{[a,b]} \to 0$ as $n \to \infty$ and $\operatorname{Var}_{[a,b]} g_n \le C$ independently of n;*

(ii) *There exists $\hat{f} \in G(a,b\,;\,X)$ such that*

$$Q_L(g) \;=\; \left\langle \hat{f}(b), g(b)\right\rangle - \int_a^b \left\langle \hat{f}(t), dg(t)\right\rangle \qquad \forall g \in BV_L(a,b\,;\,X)\,. \tag{2.19}$$

The next section is devoted to proofs of the results listed above.

3 Proofs of main results

Before passing to the proof of Theorem 2.2, we need the following auxiliary result which uses the concept of play operator defined in (1.30). In its proof we develop an idea which goes back to A. Vladimirov, see [10, Section 19] or [19, Theorem 11], cf. also [15, Chapter 2]. and [13, Lemma 4.4].

Lemma 3.1 *Let $Z \subset X$ be a convex closed set with $B_r(0) \subset Z$ for some $r > 0$, and let $K > 0$ be given. Let $\mathfrak{p}_Z : Z \times G_L(a,b\,;\,X) \to BV_L(a,b\,;\,X)$ be the play operator with characteristic Z. Let $U \subset G_L(a,b\,;\,X)$ be a set with uniformly bounded oscillation. Then there exists $C > 0$ such that*

$$\operatorname*{Var}_{[a,b]} \mathfrak{p}_Z[x^0, u] \;\le\; C \tag{3.1}$$

for every $x^0 \in Z \cap B_K(0)$ and every $u \in U$.

Proof of Lemma 3.1 . Let $x^0 \in Z \cap B_K(0)$ and $u \in U$ be arbitrary. We define the set

$$S \; = \; \{t \in]a,b]\,;\, |u(t+) - u(t)| > r\}\,. \tag{3.2}$$

Then $S = \{s_i\}_{i=1}^p$ with

$$a < s_1 < \ldots < s_p < b\,, \quad 0 \le p \le N(r) \tag{3.3}$$

by Definition 2.1 (ii). We put $s_0 = a$, $s_{p+1} = b$, and for $i = 1, \ldots, p+1$ define recursively the sequences

$$t_0^i \;=\; s_{i-1}\,, \tag{3.4}$$

$$t_k^i \;=\; \sup\{t \in]t_{k-1}^i, b]\,;\, |u(\tau) - u(t_{k-1}^i+)| \le r/2 \;\; \forall \tau \in]t_{k-1}^i, t[\,\} \tag{3.5}$$

as long as $t_{k-1}^i < s_i$. It follows from (3.5) that $t_k^i \le s_i$. Indeed, assuming $t_k^i > s_i$ would imply $|u(s_i) - u(t_{k-1}^i+)| \le r/2$, $|u(s_i+) - u(t_{k-1}^i+)| \le r/2$, hence $|u(s_i+) - u(s_i)| \le r$ in contradiction with (3.2).

Let us consider now a fixed $i \in \{1, \ldots, p+1\}$, and assume that for some $q \in \mathbb{N}$ we have $s_{i-1} = t_0^i < t_1^i < \ldots < t_q^i < s_i$. For each $k = 1, \ldots, q$ we have by (3.5) that $|u(t_k^i+) - u(t_{k-1}^i+)| \ge r/2$. We can thus find points $a_k^i < t_k^i \le b_k^i$, $a_k^i \ge b_{k-1}^i$, such that

$$|u(b_k^i) - u(a_k^i)| \ge r/4 \qquad \text{for} \;\; k = 1, \ldots, q\,. \tag{3.6}$$

The uniformly bounded oscillation hypothesis then implies that $q \le N(r/4)$. For each $i = 1, \ldots, p+1$ there exists therefore $q_i \in \mathbb{N}$ such that

$$s_{i-1} = t_0^i < t_1^i < \ldots < t_{q_i}^i = s_i\,, \qquad q_i \le q^* := N(r/4) + 1\,. \tag{3.7}$$

Consider now a fixed $k \in \{1, \ldots, q_i\}$. By Lemma 1.14 we have for every $y \in G(a,b\,;\,Z)$, and $\varepsilon \in]0, (t_k^i - t_{k-1}^i)/2[$ that

$$\int_{t_{k-1}^i+\varepsilon}^{t_k^i-\varepsilon} \langle u(\tau+) - \xi(\tau+) - y(\tau), d\xi(\tau)\rangle \;\ge\; 0\,. \tag{3.8}$$

In particular, we may choose an arbitrary $w \in S(0,T\,;\,B_1(0))$ and put in (3.8)

$$y(\tau) \;=\; \Big(u(\tau+) - u(t_{k-1}+) + \frac{r}{2}\, w(\tau)\Big)\; \chi_{]t_{k-1}^i, t_k^i[}\,(\tau)$$

Indeed, then $\|y\|_{[a,b]} \le r$ according to (3.5), hence $y(\tau) \in Z$ for all $\tau \in [a,b]$. Then (3.8) yields

$$\frac{r}{2} \int_{t_{k-1}^i+\varepsilon}^{t_k^i-\varepsilon} \langle w(\tau), d\xi(\tau)\rangle \;\le\; \int_{t_{k-1}^i+\varepsilon}^{t_k^i-\varepsilon} \Big\langle u(t_{k-1}^i+) - \xi(\tau+), d\xi(\tau)\Big\rangle$$

and from Theorem 1.8 and Corollary 1.12 it follows that

$$r \overline{\operatorname*{Var}_{[t_{k-1}^i+\varepsilon, t_k^i-\varepsilon]}} \xi \;\le\; |u(t_{k-1}^i+) - \xi(t_{k-1}^i+\varepsilon)|^2 - |u(t_{k-1}^i+) - \xi(t_k^i-\varepsilon)|^2\,.$$

Letting $\varepsilon \to 0$ (note that ξ is left-continuous), we obtain on the one hand that

$$|u(t_{k-1}^i+) - \xi(t_{k-1}^i+)| \;\ge\; |u(t_{k-1}^i+) - \xi(t_k^i)|\,, \tag{3.9}$$

and, on the other hand,

$$r \operatorname*{Var}_{[t^i_{k-1},t^i_k]} \xi \;\le\; r\,|\xi(t^i_{k-1}+) - \xi(t^i_{k-1})| + |u(t^i_{k-1}+) - \xi(t^i_{k-1}+)|^2 \tag{3.10}$$

for every $i = 1,\dots,p+1$ and $k = 1,\dots,q_i$.

We now denote $x^i_k = u(t^i_k) - \xi(t^i_k) \in Z$, $x^{i+}_k = u(t^i_k+) - \xi(t^i_k+) \in Z$ whenever it makes sense. Then it follows from (3.9)

$$|x^i_k| \;\le\; |x^{i+}_{k-1}| + |u(t^i_k) - u(t^i_{k-1}+)| \;\le\; |x^{i+}_{k-1}| + R \qquad \text{for } k = 1,\dots,q_i\,. \tag{3.11}$$

On the other hand, inequality (1.29) yields

$$|x^{i+}_{k-1}| \;\le\; |x^i_{k-1}| + 2\,|u(t^i_{k-1}+) - u(t^i_{k-1})| \;\le\; |x^i_{k-1}| + 2R \qquad \text{for } k = 1,\dots,q_i\,. \tag{3.12}$$

Summing up the above inequalities over k, we obtain that $|x^i_k| \le |x^i_0| + 3Rq^*$ for all $i = 1,\dots,p+1$ and $k = 1,\dots,q_i$, hence $|x^{i+1}_0| \;=\; |x^i_{q_i}| \;\le\; |x^i_0| + 3Rq^*$ for $i = 1,\dots,p$, and

$$|x^i_k| \;\le\; |x^0| + 3(p+1)Rq^* \le K + 3(p+1)Rq^*\,, \tag{3.13}$$

$$|x^{i+}_{k-1}| \;\le\; K + 2R + 3(p+1)Rq^* \tag{3.14}$$

for all $i = 1,\dots,p+1$ and $k = 1,\dots,q_i$. From (1.29), (3.10) and (3.14) it further follows that

$$\operatorname*{Var}_{[t^i_{k-1},t^i_k]} \xi \;\le\; |u(t^i_{k-1}+) - u(t^i_{k-1})| + \frac{1}{r}|x^{i+}_{k-1}|^2 \;\le\; R + \frac{1}{r}(K + 2R + 3(p+1)Rq^*)^2 \tag{3.15}$$

independently of k and i. Summing up the above inequality over k and i we thus obtain

$$\operatorname*{Var}_{[a,b]} \xi \;\le\; (1+p)q^* \left(R + \frac{1}{r}(K + 2R + 3(p+1)Rq^*)^2\right) \tag{3.16}$$

which together with (3.3) and (3.7) completes the proof. ■

Proof of Theorem 2.2.

(i) $\Rightarrow$ (ii) Let U have uniformly bounded ε-variation, and let $]a_k, b_k[\,,\ k = 1,\dots,m$ be pairwise disjoint subintervals of $[a,b]$ such that for some $u \in U$ and $r > 0$ we have $|u(b_k) - u(a_k)| \ge r$. For $\varepsilon = r/4$ we find $\psi_\varepsilon \in BV(a,b\,;\,X)$ such that $\|u - \psi_\varepsilon\|_{[a,b]} \le r/4$, $\operatorname{Var}_{[a,b]} \psi_\varepsilon \le L(r/4)$. Then

$$L(r/4) \;\ge\; \sum_{k=1}^m |\psi_\varepsilon(b_k) - \psi_\varepsilon(a_k)| \;\ge\; \sum_{k=1}^m (|u(b_k) - u(a_k)| - r/2) \;\ge\; \frac{mr}{2},$$

hence $m \le 2/rL(r/4)$. We now repeat the argument with $\varepsilon = 1$, and for each $a \le s < t \le b$ we obtain

$$|u(t) - u(s)| \;\le\; 2 + |\psi_1(t) - \psi_1(s)| \;\le\; 2 + \operatorname*{Var}_{[a,b]} \psi_1 \;\le\; 2 + L(1)\,,$$

and the assertion is proved.

(ii) $\Rightarrow$ (i) Assume that U has uniformly bounded oscillation, and consider an arbitrary $\varepsilon > 0$. For $t \in [a,b]$ and $u \in U$ put $u_-(t) := u(t-)$, $\xi(t) := \mathfrak{p}_Z[0, u_-](t)$ with $Z = B_{\varepsilon/2}(0)$. The set

$U_- = \{u_- \,;\, u \in U\}$ has uniformly bounded oscillation, and by Lemma 3.1 there exists $C_\varepsilon > 0$ independent of u such that

$$\operatorname*{Var}_{[a,b]} \xi \le C_\varepsilon\,, \qquad \|u_- - \xi\|_{[a,b]} \le \varepsilon/2\,. \tag{3.17}$$

For $u \in U$ we introduce the set

$$S(u) \;=\; \{t \in [a,b]\,;\, u(t) \ne u_-(t)\}\,, \quad S_\varepsilon(u) \;=\; \{t \in [a,b]\,;\, |u(t) - u_-(t)| \ge \varepsilon/2\}\,, \tag{3.18}$$

and define the function

$$\psi(t) \;:=\; \xi(t) + (u(t) - u_-(t))\, \chi_{S_\varepsilon(u)}(t)\,. \tag{3.19}$$

By hypothesis of uniformly bounded oscillation, the number of elements $\# S_\varepsilon(u)$ of $S_\varepsilon(u)$ is bounded above by a constant independent of u, say,

$$\# S_\varepsilon(u) \;\le\; N(\varepsilon/4)\,, \tag{3.20}$$

hence

$$\operatorname*{Var}_{[a,b]} \psi \;\le\; \operatorname*{Var}_{[a,b]} \xi + 2 \sum_{t \in S_\varepsilon(u)} |u(t) - u_-(t)| \;\le\; C_\varepsilon + 2RN(\varepsilon/4)\,. \tag{3.21}$$

We moreover have

$$|u(t) - \psi(t)| \;=\; \begin{cases} |u_-(t) - \xi(t)| \le \varepsilon/2 & \text{for } \ t \in [a,b] \setminus (S(u) \setminus S_\varepsilon(u))\,, \\ |u(t) - u_-(t) + u_-(t) - \xi(t)| \le \varepsilon & \text{for } \ t \in S(u) \setminus S_\varepsilon(u)\,, \end{cases} \tag{3.22}$$

hence U has uniformly bounded ε-variation, and the implication follows.

(ii) $\Rightarrow$ **(iii)** Let U have uniformly bounded oscillation with a function $N(r)$ and bound R. We define a function Γ by the formula

$$\Gamma(r) \;=\; \begin{cases} \frac{1}{2^k N(R/k)} & \text{for } \ r \in \left]\frac{R}{k}, \frac{R}{k-1}\right]\,, \quad k = 2, 3, \dots\,, \\ \Gamma(R) + r - R & \text{for } \ r > R\,. \end{cases} \tag{3.23}$$

Let $u \in U$ and $a = t_0 < \ldots < t_m = b$ be arbitrary, and let A_k be the sets

$$A_k \;=\; \left\{ j \in \{1, \dots, m\}\,;\, |u(t_j) - u(t_{j-1})| \in \left]\frac{R}{k}, \frac{R}{k-1}\right] \right\} \ \text{ for } \ k = 2, 3, \dots \tag{3.24}$$

The number of elements of A_k does not exceed $N(R/k)$, and we have

$$\sum_{j=1}^{m} \Gamma(|u(t_j) - u(t_{j-1})|) \;=\; \sum_{k=2}^{\infty} \sum_{j \in A_k} \Gamma(|u(t_j) - u(t_{j-1})|) \;\le\; \sum_{k=2}^{\infty} N\left(\frac{R}{k}\right) \Gamma\left(\frac{R}{k-1}\right) \;\le\; 1\,, \tag{3.25}$$

which we wanted to prove.

(iii) $\Rightarrow$ **(ii)** This part is easy. Let $]a_k, b_k[\,$, $k = 1, \dots, m$ be pairwise disjoint subintervals of $[a,b]$ such that for some $u \in U$ and $r > 0$ we have $|u(b_k) - u(a_k)| \ge r$. Then

$$1 \;\ge\; \sum_{k=1}^{m} \Gamma(|u(b_k) - u(a_k)|) \;\ge\; m\, \Gamma(r)\,, \tag{3.26}$$

hence $m \le 1/\Gamma(r)$. Furthermore, for each $a \le s < t \le b$ we have $\Gamma(|u(t) - u(s)|) \le 1$, hence $|u(t) - u(s)| \le \sup\{r \ge 0\,;\, \Gamma(r) \le 1\}$, and the proof is complete. ■

Proof of Theorem 2.5.

The implication (i) $\Rightarrow$ (ii) has been established in Proposition 5.4 of [13]. We only prove here the converse. Assume that $\int_a^b \langle f_n(t), dg_n(t)\rangle \to \int_a^b \langle f(t), dg(t)\rangle$ for every sequence $\{g_n\}$ in $\overline{BV}(a,b\,;\,X)$ such that $\|g_n - g\|_{[a,b]} \to 0$ as $n \to \infty$ and $\overline{\mathrm{Var}}_{[a,b]}\, g_n \le C$ independently of n, and that $f_n(t) \to f(t)$ weakly in X for every $t \in [a,b]$. To check that the sequence $\{f_n\}$ has uniformly bounded oscillation, we proceed by contradiction. Consider a sequence of pairwise disjoint intervals $]a_k^n, b_k^n[$, $k = 1,\dots,m(n)$ in $[a,b]$ such that for some $r > 0$ we have $|f_n(b_k^n) - f_n(a_k^n)| \ge r$ for all n and k, and assume that $\limsup_{n\to\infty} m(n) = \infty$. We find $v_k^n \in B_1(0)$ such that $\langle v_k^n, f_n(b_k^n) - f_n(a_k^n)\rangle = |f_n(b_k^n) - f_n(a_k^n)|$ for all n and k, and set

$$g_n(t) \;=\; \frac{1}{m(n)} \sum_{k=1}^{m(n)} v_k^n\ \chi_{\,]a_k^n, b_k^n[}\,(t)$$

for $n \in \mathbb{N}$ and $t \in [a,b]$. Then $\mathrm{Var}_{[a,b]}\, g_n \le 2$, there exists a subsequence of $\{g_n\}$ which converges uniformly to 0, and

$$\int_a^b \langle f_n(t), dg_n(t)\rangle \;=\; \frac{1}{m(n)} \sum_{k=1}^{m(n)} |f_n(b_k^n) - f_n(a_k^n)| \;\ge\; r$$

which is a contradiction. ■

Proof of Theorem 2.6.

It suffices to prove (2.4); (2.5) is then obtained by putting

$$\hat f(t) = f(a) - f(t) + \mu(t)\,. \tag{3.27}$$

Then $\hat f(b) = f(a)$, $\hat f(a) = \mu(a)$. As an easy consequence of Propositions 1.7 (i) and 1.10, we have for every $g \in G(a,b\,;\,X)$ the identity

$$\int_a^b \langle \mu(t), dg(t)\rangle \;=\; \sum_{t\in[a,b]} \langle \mu(t), g(t+) - g(t-)\rangle\ , \tag{3.28}$$

and (2.5) follows.

In the proof of (2.4) we proceed analogously as in the Riesz Representation Theorem in [9]. For each fixed $t \in [a,b]$, the mappings $v \mapsto P(v\,\chi_{[t,b]})$, $v \mapsto P(v\,\chi_{\{t\}})$ are bounded linear functionals on X. There exist therefore elements denoted by $f(t)$, $\mu(t)$, respectively, such that

$$\langle f(t), v\rangle \;=\; P(v\,\chi_{[t,b]})\,, \quad \langle \mu(t), v\rangle \;=\; P(v\,\chi_{\{t\}}) \qquad \forall v \in X\,. \tag{3.29}$$

For an arbitrary partition $a = t_0 < t_1 < \ldots < t_m = b$ and an arbitrary sequence $c_0, c_1, \dots, c_m \in B_1(0)$ we have

$$\left|\sum_{j=1}^m \langle f(t_j) - f(t_{j-1}), c_j\rangle\right| \;=\; \left|P\left(\sum_{j=1}^m c_j \chi_{[t_{j-1},t_j[}\right)\right| \;\le\; \|P\|\,, \tag{3.30}$$

$$\left|\sum_{j=0}^m \langle \mu(t_j), c_j\rangle\right| \;=\; \left|P\left(\sum_{j=0}^m c_j \chi_{\{t_j\}}\right)\right| \;\le\; \|P\|\,, \tag{3.31}$$

hence $f \in BV(a,b\,;X)$, $\mu \in BV(a,b\,;X)$, $\mu(t) \neq 0$ in at most countably many points. Furthermore, for every function $g \in S(a,b\,;X)$ of the form

$$g = \sum_{j=0}^{m} \hat{c}_j\, \chi_{\{t_j\}} + \sum_{j=1}^{m} c_j\, \chi_{]t_{j-1},t_j[} \tag{3.32}$$

with any choice of the partition $a = t_0 < t_1 < \ldots < t_m = b$ and of the sequences $\hat{c}_0, \hat{c}_1, \ldots, \hat{c}_m$, $c_1, \ldots, c_m \in X$ we have $P(c_j\, \chi_{]t_{j-1},t_j[}) = \langle f(t_{j-1}) - f(t_j) - \mu(t_{j-1}), c_j \rangle$, hence

$$P(g) \;=\; \langle \mu(b), \hat{c}_m \rangle + \sum_{j=0}^{m-1} \langle \mu(t_j), \hat{c}_j - c_{j+1} \rangle - \sum_{j=1}^{m} \langle f(t_j) - f(t_{j-1}), c_j \rangle\,. \tag{3.33}$$

On the other hand, Proposition 1.7 yields

$$\int_a^b \langle f(t), dg(t) \rangle \;=\; \langle f(b), \hat{c}_m \rangle - \langle f(a), \hat{c}_0 \rangle - \sum_{j=1}^{m} \langle f(t_j) - f(t_{j-1}), c_j \rangle\,, \tag{3.34}$$

hence (note that $f(b) = \mu(b)$)

$$P(g) \;=\; \langle f(a), \hat{c}_0 \rangle + \int_a^b \langle f(t), dg(t) \rangle + \sum_{j=0}^{m-1} \langle \mu(t_j), \hat{c}_j - c_{j+1} \rangle \tag{3.35}$$

which is nothing but formula (2.4) for every $g \in S(a,b\,;X)$ of the form (3.32). Every function $g \in G(a,b\,;X)$ can be approximated by a uniformly convergent sequence of functions from $S(a,b\,;X)$, hence, by Theorem 1.9, identity (2.4) holds for every $g \in G(a,b\,;X)$.

The uniqueness of f, $\hat{f}$, μ is easy. Assume for instance that

$$\langle f(a), g(a) \rangle + \int_a^b \langle f(t), dg(t) \rangle - \sum_{t\in[a,b]} \langle \mu(t), g(t+) - g(t) \rangle \;=\; 0 \qquad \forall g \in G(a,b\,;X)\,. \tag{3.36}$$

Choosing in (3.36) $g(t) = g_s(t) := \mu(s)\, \chi_{\{s\}}(t)$ for $s \in [a,b[$ and $t \in [a,b]$ we obtain from Proposition 1.7 that $\mu(s) = 0$ for all $s \in [a,b[$. Similarly, the choice $g(t) = \tilde{g}_s(t) := f(s)\, \chi_{[s,b]}(t)$ for $s \in [a,b]$ and $t \in [a,b]$ yields that $f(s) = 0$ for all $s \in [a,b]$, hence also $\mu(b) = f(b) = 0$. The uniqueness of $\hat{f}$ can be obtained similarly. Note that the functions $\tilde{g}_s$ are right-continuous. The same argument therefore confirms the uniqueness in the representation formulas (2.9), (2.10).

We now pass to the proof of (2.7). From Theorem 1.9 it immediately follows that

$$\|P\| \;\le\; |f(b)| + \operatorname*{Var}_{[a,b]} f + 2 \sum_{t\in[a,b[} |\mu(t)| \;\le\; |\mu(a)| + |f(b)| + \operatorname*{Var}_{[a,b]} f + \operatorname*{Var}_{[a,b]} \mu\,. \tag{3.37}$$

Conversely, for every $\varepsilon > 0$ we use Theorem 1.9 and find $g_\varepsilon \in G(a,b\,;X)$, $\|g_\varepsilon\|_{[a,b]} \le 1$, such that

$$\langle f(a), g_\varepsilon(a) \rangle + \int_a^b \langle f(t), dg_\varepsilon(t) \rangle \;>\; |f(b)| + \operatorname*{Var}_{[a,b]} f - \varepsilon\,. \tag{3.38}$$

We construct a finite set $D_\varepsilon \subset [a,b]$, $D_\varepsilon = \{t_1^\varepsilon, \ldots, t_{m(\varepsilon)}^\varepsilon\}$, such that $a \in D_\varepsilon$ if $\mu(a) \neq 0$, $b \in D_\varepsilon$ if $\mu(b) \neq 0$, $\mu(t_i^\varepsilon) \neq 0$ for $i = 1, \ldots, m(\varepsilon)$ and

$$\sum_{t\in[a,b]\setminus D_\varepsilon} |\mu(t)| \;<\; \frac{\varepsilon}{2}\,. \tag{3.39}$$

Let λ_ε be the function

$$\lambda_\varepsilon(t) \; = \; \sum_{i=1}^{m(\varepsilon)} \left(g_\varepsilon(t_i^\varepsilon +) - g_\varepsilon(t_i^\varepsilon) + 2\frac{\mu(t_i^\varepsilon)}{|\mu(t_i^\varepsilon)|}\right) \chi_{\{t_i^\varepsilon\}}(t) \qquad \text{for } \; t \in [a,b]\,, \tag{3.40}$$

and set

$$g(t) \; = \; g_\varepsilon(t) + \lambda_\varepsilon(t) \qquad \text{for } \; t \in [a,b]\,. \tag{3.41}$$

Proposition 1.7 yields (note that $\langle f(b), \lambda_\varepsilon(b)\rangle = 2|\mu(b)| = 2|f(b)|$)

$$\langle f(a), g(a)\rangle + \int_a^b \langle f(t), dg(t)\rangle \; = \; \langle f(a), g_\varepsilon(a)\rangle + \int_a^b \langle f(t), dg_\varepsilon(t)\rangle + 2|f(b)|\,. \tag{3.42}$$

Furthermore, for $i = 1, \ldots, m(\varepsilon)$, $t_i^\varepsilon \neq b$ we have

$$\langle \mu(t_i^\varepsilon), g(t_i^\varepsilon +) - g(t_i^\varepsilon)\rangle \; = \; \langle \mu(t_i^\varepsilon), g_\varepsilon(t_i^\varepsilon +) - g_\varepsilon(t_i^\varepsilon) - \lambda_\varepsilon(t_i^\varepsilon)\rangle \; = \; -2|\mu(t_i^\varepsilon)|\,, \tag{3.43}$$

hence, by virtue of (3.39) – (3.40),

$$-\sum_{t\in D_\varepsilon} \langle \mu(t), g(t+) - g(t)\rangle \; = \; 2\sum_{t\in D_\varepsilon\setminus\{b\}} |\mu(t)| \; \geq \; |\mu(a)| + \operatorname*{Var}_{[a,b]} \mu - |\mu(b)| - \varepsilon\,, \tag{3.44}$$

$$\left| \sum_{t\in[a,b]\setminus D_\varepsilon} \langle \mu(t), g(t+) - g(t)\rangle \right| \; \leq \; 2\sum_{t\in[a,b[\,\setminus D_\varepsilon} |\mu(t)| \; \leq \; \varepsilon\,. \tag{3.45}$$

From (3.38), (3.42), and (3.44) – (3.45) it follows that

$$\begin{aligned} P(g) \; &= \; \langle f(a), g(a)\rangle + \int_a^b \langle f(t), dg(t)\rangle - \sum_{t\in[a,b]} \langle \mu(t), g(t+) - g(t)\rangle \\ &\geq \; |\mu(a)| + |f(b)| + \operatorname*{Var}_{[a,b]} f + \operatorname*{Var}_{[a,b]} \mu - 3\varepsilon\,, \end{aligned} \tag{3.46}$$

hence

$$|\mu(a)| + |f(b)| + \operatorname*{Var}_{[a,b]} f + \operatorname*{Var}_{[a,b]} \mu \; \leq \; \|P\|\,\|g\|_{[a,b]} + 3\varepsilon \; \leq \; 3\,(\|P\| + \varepsilon)\,. \tag{3.47}$$

Since ε is arbitrary, we obtain (2.7) from (3.37) and (3.47). The proof of (2.8) is analogous. ∎

Proof of Theorem 2.7.

The implication (ii) $\Rightarrow$ (i) follows from Theorem 2.5 with $g_n \equiv g$. To prove the converse, we define the functions f, $\hat{f}$, and μ as in (3.29), (3.27). For every $v \in X$ and every $t \in [a,b[$ we have by hypothesis $P(v\,\chi_{]t,b]}) = \lim_{s\to t+} P(v\,\chi_{[s,b]})$, hence $f(t) = f(t+) + \mu(t)$, and similarly for $t \in]a,b]$ we have $P(v\,\chi_{[t,b]}) = \lim_{s\to t-} P(v\,\chi_{[s,b]})$, hence $f(t) = f(t-)$. We now invoke the integration-by-parts formula (1.26) which in combination with (2.4) yields (2.14). Using the fact that $\mu(t+) = 0$ for $t \in [a,b[$, $\mu(t-) = 0$ for $t \in]a,b]$, we obtain from (3.27) that $\hat{f}(t) = \hat{f}(t-) + \mu(t)$, $\hat{f}(t) = \hat{f}(t+)$ for every $t \in [a,b]$, and argue as above to obtain (2.15). ∎

Proof of Theorem 2.8.

The implication (ii) $\Rightarrow$ (i) follows again from Theorem 2.5 with $f_n \equiv \hat{f}$. To prove the converse, we argue as in the proof of Theorem 2.6 and define the function $\hat{f} : [0,T] \to X$ by the formula

$$\left\langle \hat{f}(t), v \right\rangle \;=\; Q_L(v\, \chi_{[a,t]}) \quad \forall v \in X \quad \forall t \in [a,b]\,. \tag{3.48}$$

We check by contradiction that $\hat{f} \in G(a,b\,;\,X)$. Assuming that there exist $t \in]a,b]$, $\delta > 0$, and sequences $s_j \nearrow t$, $t_j \nearrow t$, $s_j < t_j < s_{j+1} < t_{j+1}$ for all $j \in \mathbb{N}$, such that $|\hat{f}(t_j) - \hat{f}(s_j)| \geq \delta$, it suffices to choose $v_j \in X$ in such a way that

$$\left\langle \hat{f}(t_j) - \hat{f}(s_j), v_j \right\rangle \;=\; |\hat{f}(t_j) - \hat{f}(s_j)|\,, \quad |v_j| = 1 \qquad \forall j \in \mathbb{N}\,,$$

and put

$$g_n \;=\; 2^{-n} \sum_{j=2^{n-1}+1}^{2^n} v_j\, \chi_{\,]s_j,t_j]} \qquad \text{for } \; n \in \mathbb{N}\,. \tag{3.49}$$

Then $\|g_n\|_{[a,b]} \to 0$, $\operatorname{Var}_{[a,b]} g_n = 1$ for $n \in \mathbb{N}$, and

$$Q_L(g_n) \;=\; 2^{-n} \sum_{j=2^{n-1}+1}^{2^n} \left\langle v_j, \hat{f}(t_j) - \hat{f}(s_j) \right\rangle \;\geq\; \frac{\delta}{2}$$

which is a contradiction. This implies that $\hat{f}(t-)$ exists for all $t \in]a,b]$. We analogously prove that $\hat{f}(t+)$ exists for all $t \in [a,b[$ by considering sequences $s_j \searrow t$, $t_j \searrow t$, $s_j > t_j > s_{j+1} > t_{j+1}$ for all $j \in \mathbb{N}$, hence $\hat{f} \in G(a,b\,;\,X)$.

For each function g of the form

$$g \;=\; c_0\, \chi_{\{a\}} + \sum_{j=1}^{m} c_j\, \chi_{\,]t_{j-1},t_j]}\,, \quad a = t_0 < t_1 < \ldots < t_m = b \tag{3.50}$$

we have by Proposition 1.7 that

$$Q_L(g) \;=\; \left\langle \hat{f}(a), c_0 \right\rangle + \sum_{j=1}^{m} \left\langle \hat{f}(t_j) - \hat{f}(t_{j-1}), c_j \right\rangle \;=\; \left\langle \hat{f}(b), g(b) \right\rangle - \int_a^b \left\langle \hat{f}(t), dg(t) \right\rangle. \tag{3.51}$$

Each function $g \in BV_L(a,b\,;\,X)$ can be uniformly approximated by step functions g_n of the form (3.50) and such that $\operatorname{Var}_{[a,b]} g_n \leq \operatorname{Var}_{[a,b]} g$, cf. e.g. [13, Proposition 1.3]. Passing to the limit as $n \to \infty$ and using Proposition 1.10 we obtain the assertion. ■

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