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RESEARCH REPORT

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Filtering with mixed continuous and discrete states: special case

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Abstract
The paper deals with state estimation, organized in the entry-wise way. The main purpose of the presented research work is to make a step towards the joint filtering of states of a mixed (both of continuous and discrete-valued nature) type. The paper describes a general probabilistic solution of the entry-wise filtering, which provides the estimates of entries of a state vector, updated entry-wise. The proposed solution is based on factorization of a state-space model via application of the chain rule. The paper proposes the solutions with specific models. In the case of continuous variables described by linear Gaussian models the entry-wise updating of the state estimate is reached with the help of decomposition of precision matrices. The proposed factorized version of Kalman filter covers the state estimation in this case. The paper considers a special form of the state vector with a discrete-valued entry placed at the end. Its estimation is based on Bayesian filtering applied to discrete distribution. Practical experiments demonstrate the filtering with mixed-type states from the urban traffic control area, which is a main application of the research.

Keywords: state estimation; entry-wise filtering; state-space model; mixed-type (continuous and discrete) data

Chapter 1

INTRODUCTION

The paper is devoted to the problem of state estimation with involved data of a mixed (both of continuous and discrete-valued nature) type. The estimation of continuous Gaussian state variables is thoroughly worked out with the help of the famous Kalman filter [1]. Gaussian state-space models have shown themselves well in such an application area as traffic control systems (as well as many others) [2, 3, 4]. However, some of the state variables are of a discrete-valued nature. The estimation with data models of both types is highly desirable, but the needed support is rather less developed [5]. Incorporation of the discrete state entries and the joint filtering of the mixed-type state (currently as a special case) is the task, addressed in the paper.

Handling with the mixed-type data is known to be a hard problem addressed within completely different context of logistic regression. Analysis of a series of published works in this field [6, 7, 8, 9, 10] (a detailed overview can be found in [11]) shown, that despite many existing approaches, the mixed-type state estimation still calls for a reliable solution. The present paper proposes the solution of this problem in the form of the entry-wise organized filtering. Specialization of the solution to linear Gaussian state-space model (in the case of continuous data) is proposed as a factorized version of Kalman filter. Majority of research found in the field of factorized filtering [12, 13] are primarily directed at reduction of computational complexity via a lesser rank of a covariance matrix. Exploitation of matrix factorization with the aim of the entry-wise updating of the state estimate under Bayesian methodology [14] was proposed in [15] with a reduced form of the state matrix. Later the work [16] proposed the probabilistic solution of the factorized Bayesian prediction and filtering, based on applying the chain rule. However, both algorithms were rather hard for implementation and restricted in application. Subsequent papers [17, 18] continued this line by a series of experiments with the factorized Kalman filter, based on $L'DL$ decomposition of covariance matrices. Nevertheless, the results in [17, 18] were not too optimistic from a position of the entry-wise updating of the state estimate. Recently, the paper [19] proposed the novel relatively successful algorithm with simultaneous data and time updating of the state estimate, which used, however, a non-standard system model, taken as the joint probability density function (pdf). The present paper uses the experience of the last mentioned works. The proposed solution is based on the modified form of the filtering from [19], but with exploitation of the "classical-approach" state-space model and LDL'-factorized precision matrices. It improves the algorithm and gets rid of most inconsistencies and inaccuracies. The proposed entry-wise filtering allows a presence of a discrete state entry, currently placed at the end of the state vector. This special case of the mixed-type state estimation is covered by the paper.

The outline of the paper is as follows. Basic facts about the models used, Bayesian filtering and the chain rule are provided in Section 2. The main part of the paper is organized so that Section 3 relates to the entry-wise filtering, assuming a continuous nature of variables, while Sections 4-5 considers the mixed one. Section 3 provides the general solution and proposes the factorized Kalman filtering with simultaneous data and time updating. Section 4 is devoted to involvement of the discrete state and the filtering with both continuous and discrete variables. Section 5 provides the illustrative experiments with data from traffic control area.

Chapter 2

PRELIMINARIES

The probabilistic description of the system, a state of which has to be estimated, is provided by a state-space model in the following form.

2.1 State-space model

The *observation model*, specified by the conditional probability density function (pdf)

$$f(y_t|u_t, x_t), \tag{2.1}$$

relates the system output y_t to the system input u_t and the unobserved system state x_t at discrete time moments $t \in t^* \equiv \{0, \ldots, \mathring{t}\}$, where $\mathring{t}$ is the cardinality of the set t^* and $\equiv$ means equivalence.
The *state evolution model*

$$f(x_{t+1}|u_{t+1}, x_t), \tag{2.2}$$

describes the evolution of the system state x_t.

The estimation of the finite-dimensional system state calls for application of Bayesian filtering.

2.2 Bayesian filtering

Bayesian filtering, estimating the system state, includes the following coupled formulas.
Data updating

$$\begin{aligned} f\left(x_t\middle|d^t\right) &= \frac{f(y_t|u_t, x_t)\, f\left(x_t\middle|u_t, d^{t-1}\right)}{\int f(y_t|u_t, x_t)\, f\left(x_t\middle|u_t, d^{t-1}\right)\, dx_t}, \\ &\propto f(y_t|u_t, x_t)\, f\left(x_t\middle|u_t, d^{t-1}\right), \end{aligned} \tag{2.3}$$

($\propto$ means proportionality) incorporates the experience contained in the data d^t, where $d^t = (d_0, \ldots, d_{\mathring{t}})$ and $d_t \equiv (y_t, u_t)$.
Time updating

$$f\left(x_{t+1}\middle|u_{t+1}, d^t\right) = \int f(x_{t+1}|u_{t+1}, x_t)\, f\left(x_t\middle|d^t\right)\, dx_t, \tag{2.4}$$

fulfills the state prediction.

The filtering does not depend on the control strategy $\{f(u_t|d^{t-1})\}_{t\in t^*}$ but on the generated inputs only. The prior pdf $f(x_0|u_0)$, which expresses the subjective prior knowledge on the state x_0, starts the recursions.

2.3 Model factorization by the chain rule

Application of the chain rule [20] to the models (2.1-2.2) enables to represent them as the product of pdfs of the individual entries of the output and the state vectors respectively.

$$f\left(y_t | u_t, x_t\right) = \prod_{j=1}^{\mathring{y}} f\left(y_{j;t} | y_{j+1:\mathring{y};t}, u_t, x_{1:\mathring{x};t}\right), \tag{2.5}$$

$$f\left(x_{t+1} | u_{t+1}, x_t\right) = \prod_{i=1}^{\mathring{x}} f\left(x_{i;t+1} | x_{i+1:\mathring{x};t+1}, u_{t+1}, x_{1:\mathring{x};t}\right), \tag{2.6}$$

where $\mathring{y}$ and $\mathring{x}$ denote number of entries of column vectors y_t and x_t respectively, $j = \{1, \ldots, \mathring{y}\}$, $i = \{1, \ldots, \mathring{x}\}$. The input u_t is assumed to be factorized whenever it would be necessary from computational point of view, nevertheless modeling of input entries is out of interest in the present paper. A notation in the form $x_{i+1:\mathring{x};t}$ in (2.5-2.6) denotes a sequence of the vector entries from $(i+1)$ to $\mathring{x}$, i.e. $\{x_{i+1;t}, x_{i+2;t}, \ldots, x_{\mathring{x};t}\}$, which is empty, when $(i+1) > \mathring{x}$. The filtering of the individual state entries is the main task addressed in the paper.

Chapter 3

FILTERING WITH SIMULTANEOUS DATA AND TIME UPDATING

Let's assume in the present section, that the state variable is of continuous type and that Gaussian state-space model could be used with available Gaussian prior on x_0 and Gaussian observations. In this case the application of Bayesian filtering (2.3-2.4) provides Kalman filter. The required state estimate should be obtained as the product of pdfs, corresponding to the i-th state entries, i.e.

$$f\left(x_{t+1} \middle| u_{t+1}, d^t\right) = \prod_{i=1}^{\mathring{x}} f\left(x_{i;t+1} \middle| x_{i+1:\mathring{x};t+1}, u_{t+1}, d^t\right). \tag{3.1}$$

However, traditional execution of data updating and time updating (2.3-2.4) spoils the form (3.1) due to the following reasons. In the data updating (2.3) the observation model $f\left(y_t \middle| u_t, x_t\right)$ after factorization (2.5) always remains with the output, conditioned on all entries of the vector x_t, while the prior pdf $f\left(x_t \middle| u_t, d^{t-1}\right)$ with the help of the chain rule takes the form $\prod_{i=1}^{\mathring{x}} f\left(x_{i;t} \middle| x_{i+1:\mathring{x};t}, u_t, d^{t-1}\right)$. It hampers correct entry-wise calculation of the data updating. In case of multi-output Gaussian observation model it may cause superfluous decompositions and assumptions about the triangular form of the matrix, corresponding to the state (more detailed explanation is available in Subsection 3.1). After several attempts [16, 21] it was clear, that, concerning the entry-wise updating, the standard case (2.3-2.4) is restricted by the single-output observation model. The solution, which the paper proposes, is to organize the filtering so that the calculation can be fulfilled at one integration step. It helps to get rid of the state entries, which should be integrated out, simultaneously in all pdfs, involved in the filtering.

General probabilistic formulation of the entry-wise filtering is based on simultaneous performance of the data and time updating steps (2.3-2.4). An advocated formula of such updating is as follows.

$$f\left(x_{t+1} \middle| u_{t+1}, d^t\right) \propto \int f\left(x_{t+1} \middle| u_{t+1}, x_t\right) \left\{ \underbrace{f\left(y_t \middle| u_t, x_t\right) f\left(x_t \middle| u_t, d^{t-1}\right)}_{\propto f\left(x_t \middle| d^t\right)} \right\} dx_t, \tag{3.2}$$

which is obtained by trivial substitution of the state estimate updated by measurements (2.3) into the time updating (2.4). Substitution of the factorized forms of the models (2.5-2.6) in (3.2) and decomposition of the prior distribution according to the chain rule provide the following form of (3.2).

$$\begin{aligned}
\prod_{i=1}^{\mathring{x}} f\left(x_{i;t+1} \middle| x_{i+1:\mathring{x};t+1}, u_{t+1}, d^t\right) \propto & \int \prod_{i=1}^{\mathring{x}} f\left(x_{i;t+1} \middle| x_{i+1:\mathring{x};t+1}, u_{t+1}, x_{1:\mathring{x};t}\right), \\
& \times \prod_{j=1}^{\mathring{y}} f\left(y_{j;t} \middle| y_{j+1:\mathring{y};t}, u_t, x_{1:\mathring{x};t}\right), \\
& \times \prod_{i=1}^{\mathring{x}} f\left(x_{i;t} \middle| x_{i+1:\mathring{x};t}, u_t, d^{t-1}\right) dx_t,
\end{aligned} \tag{3.3}$$

where the presence of variable x_t assumes integration over all the entries of the respective vector. The state estimate in (3.3) presents the product of the updated pdfs $f\left(x_{i;t+1}|x_{i+1:\mathring{x};t+1}, u_{t+1}, d^t\right)$. Each of them corresponds to the i-th entry of the state vector.

3.1 Entry-wise filtering with Gaussian state-space model

Linear Gaussian models (2.1-2.2), used for demonstrating of the entry-wise filtering, are as follows.

$$\begin{aligned} &\textit{observation model} & y_t &= Cx_t + Hu_t + v_t, & (3.4)\\ &\textit{state evolution model} & x_{t+1} &= Ax_t + Bu_{t+1} + \omega_t, & (3.5)\end{aligned}$$

where C, H, A and B are the known matrices of appropriate dimensions; v_t is a measurement (Gaussian) noise with zero mean and known covariance matrix R_v; ω_t is a process (Gaussian) noise with zero mean and known covariance matrix R_w.

Application of the relation (3.3) to the models (3.4-3.5) leads to a factorized version of Kalman filter. The preserving of the factorized form of the state estimate is reached via LDL' decomposition [14] of the precision (i.e. inverse covariance) matrices. Such the decomposition supposes L to be a lower triangular matrix with unit diagonal, D to be a diagonal one and $'$ denoting transposition. This kind of matrix decomposition is used throughout the paper.

The factorization of models (3.4-3.5) can be clearly demonstrated via exploitation of quadratic forms, contained inside the exponents of multivariate Gaussian distributions, corresponding to (3.4-3.5). Firstly, one should factorize the observation model (3.4). The measurement noise covariance matrix R_v is inverted into a precision matrix and decomposed so that

$$R_v^{-1} = L_v D_v L_v'. \tag{3.6}$$

The resulted factorized quadratic form, corresponding to Gaussian model (3.4), is as follows.

$$[L_v' y_t - \underbrace{L_v' H u_t}_{\rho_t} - \underbrace{L_v' C}_{\mathcal{A}} x_t]' D_v [L_v' y_t - \rho_t - \mathcal{A} x_t]. \tag{3.7}$$

Due to expression (3.7), Gaussian distribution of the j-th entry of the output vector can be written in the form

$$\mathcal{N}_{y_{j;t}}\left(\rho_{j;t} - \sum_{k=j+1}^{\mathring{y}} L_{v;kj} y_{k;t} + \sum_{l=1}^{\mathring{x}} \mathcal{A}_{jl} x_{l;t}, \frac{1}{D_{v;jj}}\right), \tag{3.8}$$

where $L_{v;kj}$, $\mathcal{A}_{jl}$ and $D_{v;jj}$ are the elements of matrices L_v, $\mathcal{A}$ and D_v respectively. Factorization of the state evolution model (3.5) is made in the similar way. The process noise covariance matrix R_w is inverted into the precision matrix and decomposed so that

$$R_w^{-1} = L_w D_w L_w'. \tag{3.9}$$

The factorized Gaussian quadratic form, corresponding to the model (3.5), becomes now

$$[L_w' x_{t+1} - \underbrace{L_w' B u_{t+1}}_{z_{t+1}} - \underbrace{L_w' A}_{\Xi} x_t]' D_w [L_w' x_{t+1} - z_{t+1} - \Xi x_t], \tag{3.10}$$

and provides Gaussian distribution of the i-th state entry as follows.

$$\mathcal{N}_{x_{i;t+1}}\left(z_{i;t+1} - \sum_{k=i+1}^{\mathring{x}} L_{w;ki} x_{k;t+1} + \sum_{l=1}^{\mathring{x}} \Xi_{il} x_{l;t}, \frac{1}{D_{w;ii}}\right), \tag{3.11}$$

where $L_{w;ki}$, Ξ_{il} and $D_{w;ii}$ are the elements of matrices L_w, Ξ and D_w respectively.

Gaussian prior distribution to be incorporated in (3.3) is chosen with mean μ_0 and covariance matrix P_0, usually provided by experts from the application domain. The factorization of the prior distribution is obtained via decomposition of the initial precision matrix with $t = 0$

$$P_t^{-1} = L_{p|t} D_{p|t} L_{p|t}', \tag{3.12}$$

which provides the following quadratic form, corresponding to Gaussian distribution of the initial state x_t with $t = 0$.

$$[L'_{p|t}x_t - \mu_t^f]' D_{p|t} [L'_{p|t}x_t - \mu_t^f] \text{ with } \mu_t^f = L'_{p|t}\mu_t. \quad (3.13)$$

It allows to present Gaussian distributions of the initial state entries as

$$\mathcal{N}_{x_{i;t}} \left(\mu_{i;t}^f - \sum_{k=i+1}^{\mathring{x}} L_{p|t;ki} x_{k;t}, \frac{1}{D_{p|t;ii}} \right), \quad (3.14)$$

where $L_{p|t;ki}$ and $D_{p|t;ii}$ are the elements of matrices $L_{p|t}$ and $D_{p|t}$ respectively.

Now one can substitute Gaussian distributions with quadratic forms (3.7), (3.10) and (3.13) in the simultaneous data and time updating (3.2). After this substitution the function to be integrated is as follows.

$$\begin{aligned} & \int \exp\left\{ -\frac{1}{2} \left[\underbrace{L'_w x_{t+1} - z_{t+1}}_{\beta_1} - \Xi x_t \right]' D_w \left[L'_w x_{t+1} - z_{t+1} - \Xi x_t \right] \right\}, \quad (3.15) \\ \times \; & \exp\left\{ -\frac{1}{2} \left[\underbrace{L'_v y_t - \rho_t}_{\beta_2} - \mathcal{A} x_t \right]' D_v \left[L'_v y_t - \rho_t - \mathcal{A} x_t \right] \right\}, \\ \times \; & \exp\left\{ -\frac{1}{2} \left[\underbrace{\mu_t^f}_{\beta_3} - L'_{p|t} x_t \right]' D_{p|t} \left[\mu_t^f - L'_{p|t} x_t \right] \right\} dx_t, \end{aligned}$$

with additional notations $\beta = [\beta_1;\ \beta_2;\ \beta_3]$, where β_i is a column vector. After completion of squares [20] for x_t in (3.15) and subsequent integration of non-normalized Gaussian pdf [16], the variable x_t is being integrated out. The resulted expression is proportional to $\exp\left\{-\frac{1}{2}\lambda\right\}$, with the following remainder λ, obtained after integration.

$$\lambda = \beta' \left(\Omega_t - \Omega_t [\Xi;\ \mathcal{A};\ L'_{p|t}] \Gamma_t^{-1} [\Xi;\ \mathcal{A};\ L'_{p|t}]' \Omega_t \right) \beta, \quad (3.16)$$

where

$$\begin{aligned} \Omega_t &= \text{diag}\left[D_w,\ D_v,\ D_{p|t} \right], \quad (3.17) \\ \Gamma_t &= [\Xi;\ \mathcal{A};\ L'_{p|t}]' \Omega_t [\Xi;\ \mathcal{A};\ L'_{p|t}]. \quad (3.18) \end{aligned}$$

With the help of algebraic rearrangement of the remainder (3.16) using completion of squares for x_{t+1}, one obtains the following Gaussian quadratic form for the factorized state

$$\begin{aligned} & \left[L'_w x_{t+1} - z_{t+1} - \tilde{D}_t^{-1} (D_w \Xi \Gamma_t^{-1} (\mathcal{A}' D_v (L'_v y_t - \rho_t) + L_{p|t} D_{p|t} \mu_t^f)) \right]' \tilde{D}_t, \quad (3.19) \\ \times \; & \left[L'_w x_{t+1} - z_{t+1} - \tilde{D}_t^{-1} (D_w \Xi \Gamma_t^{-1} (\mathcal{A}' D_v (L'_v y_t - \rho_t) + L_{p|t} D_{p|t} \mu_t^f)) \right], \end{aligned}$$

where

$$\tilde{D}_t = D_w - D_w \Xi \Gamma_t^{-1} \Xi' D_w. \quad (3.20)$$

The matrix $\tilde{D}_t$, obtained in (3.20) is decomposed so that

$$\tilde{D}_t = L_{u|t} D_{u|t} L'_{u|t}. \quad (3.21)$$

The decomposition (3.21) and factorization of (3.19) (i.e. multiplication of quadratic form by respective triangular matrix $L_{u|t}$) enables to obtain the following result.

$$\left[\underbrace{L'_{u|t} L'_w}_{L'_{p|t+1}} x_{t+1} - \mu_{t+1}^f \right]' \underbrace{D_{u|t}}_{D_{p|t+1}} \left[L'_{p|t+1} x_{t+1} - \mu_{t+1}^f \right], \quad (3.22)$$

where

$$\mu^f_{t+1} \quad = \quad L'_{u|t}\left(z_{t+1} + \tilde{D}_t^{-1}(D_w \Xi \Gamma_t^{-1}(\mathcal{A}' D_v (L'_v y_t - \rho_t) + L_{p|t} D_{p|t} \mu^f_t))\right). \tag{3.23}$$

The obtained result (3.22) preserves the form of the prior distribution (3.13) and expresses the simultaneous data and time updating of the decomposed matrices $L'_{p|t}$ and the factorized mean value μ^f_t. Finally, the resulting estimate of the i-th state entry keeps the factorized form (3.14).

$$\mathcal{N}_{x_{i;t+1}}\left(\mu^f_{i;t+1} - \sum_{k=i+1}^{\mathring{x}} L_{p|t+1;ki} x_{k;t+1}, \frac{1}{D_{p|t+1;ii}}\right). \tag{3.24}$$

The obtained form (3.24) reflects realization of the entry-wise updating in (3.3), where the individual pdf $f\left(x_{i;t+1}|\, x_{i+1:\mathring{x};t+1}, u_{t+1}, d^t\right)$ is corresponding to the i-th state entry distribution in (3.24). Thereby, the proposed algorithm enables the modeling of the state entries by the rows of respective vectors (matrices), operating with the entries mean values, starting from the last one (i.e for $i = \mathring{x}$).

3.2 Verification of entry-wise Kalman filtering

Correct performance of the proposed updating of the state estimate in the factorized form is verified by unfactorization of the obtained results (3.22-3.23) and their comparison with results of the “classical” well-elaborated Kalman filter [1, 20], and vice versa (i.e. factorization of the state estimates, provided by the Kalman filter). The Kalman filter, which computationally coincides with the solution of Bayesian filtering (2.3-2.4) applied to the models (3.4-3.5), provides the state estimate with mean vector μ_{t+1} and covariance matrix P_{t+1} with the help of the following grouped equations.

$$\begin{aligned}
\textit{data updating} \quad K_t &= P_t C'(C P_t C' + R_v)^{-1}, && (3.25)\\
\mu_t &= \mu_t + K_t(y_t - C\mu_t - H u_t), && (3.26)\\
P_t &= P_t - P_t C'(C P_t C' + R_v)^{-1} C P_t && (3.27)\\
\textit{time updating} \quad \mu_{t+1} &= A\mu_t + B u_t, && (3.28)\\
P_{t+1} &= A P_t A' + R_w. && (3.29)
\end{aligned}$$

The transformation of the results (3.22-3.23) into the non-factorized form $\mathcal{N}_{x_{t+1}}(\mu_{t+1}, P_{t+1})$ to be compared with (3.25-3.29) is fulfilled as follows.

$$\begin{aligned}
(L'_{p|t+1})^{-1}\mu^f_{t+1} &= \mu_{t+1}, && (3.30)\\
(L_{p|t+1} D_{p|t+1} L'_{p|t+1})^{-1} &= P_{t+1}. && (3.31)
\end{aligned}$$

Obviously, the state estimate $\mathcal{N}_{x_{t+1}}(\mu_{t+1}, P_{t+1})$, obtained in (3.28-3.29), can be transformed to the factorized form by LDL'-decomposition of the precision matrix and multiplication of the mean value by the transposed triangular matrix.

$$\begin{aligned}
P^{-1}_{t+1} &= LDL', && (3.32)\\
L'\mu_{t+1} &= \mu^f_{t+1}, && (3.33)
\end{aligned}$$

which executes the double-check of the proposed factorization. The variances of the individual state entries are verified as inverse elements of diagonal matrix D.

Comparison of both implementations in the unified form successfully provides identical results.

Chapter 4

DISCRETE STATE INCORPORATION AND MIXED STATE ESTIMATION

Let's see how the entry-wise filtering (3.3) could be applied to the mixed-type state. Currently a special case of the state vector with mixed-type entries is considered, where a discrete entry is placed in the end of the vector. This solution nevertheless is planned to be extended later.

Let's consider the state vector $[x_{1;t}, x_{2;t}, \ldots, x_{\mathring{x}-1;t}, x_{\mathring{x};t}] \equiv [x_{1:\mathring{x}-1;t}, \tilde{x}_t]$. The state entries

$$x_{1;t}, x_{2;t}, \ldots, x_{\mathring{x}-1;t}$$

of continuous type are supposed to be estimated by the factorized Kalman filter. The last entry $x_{\mathring{x};t}$, denoted by $\tilde{x}_t$, is a discrete scalar one with a set of possible discrete values $\tilde{x}^* \equiv \{\mathbf{0}, \mathbf{1}\}$. Due to its position in the end of the state vector, the discrete entry can be considered individually from other foregoing state entries. It means, that it should be estimated by a suitable filter for subsequent exploitation of its mean value in (3.3).

One can be now focused on the estimation of the involved discrete state $\tilde{x}_t$. To facilitate calculations, the output vector $[y_{1;t}, y_{2;t}, \ldots, y_{\mathring{y}-1;t}, y_{\mathring{y};t}]$ should be transformed into the similar form $[y_{1:\mathring{y}-1;t}, \tilde{y}_t]$. The last entry $y_{\mathring{y};t} \equiv \tilde{y}_t$ is supposed to be a discrete (eventually discretized via thresholds, delivered by experts from the application domain) scalar one with a set of possible discrete values $\{\mathbf{0}, \mathbf{1}\}$. The input u_t is exploited in calculations via a threshold θ_u, given by the experts (which assumes two cases: $u_t \geq \theta_u$ and $u_t < \theta_u$).

The models (2.1-2.2) take the following forms with the involved discrete (discretized) variables.

$$\textit{observation model} \qquad f\left(\tilde{y}_t | u_t, x_{1:\mathring{x}-1;t}, \tilde{x}_t\right), \tag{4.1}$$

$$\textit{state evolution model} \qquad f\left(\tilde{x}_{t+1} | u_{t+1}, x_{1:\mathring{x}-1;t}, \tilde{x}_t\right). \tag{4.2}$$

In general, an assumption about omitting of the states $x_{1:\mathring{x}-1;t}$ in (4.1-4.2) is made. The forms of these probabilistic models strongly depend on the application field. In the considered area (see Section 5) such the assumption seems to be not necessary. The model (4.1) is supposed to be described by the discrete Bernoulli distribution, shown in Table 4.1, where p with respective indices denotes a probability (assumed to be known) of taking the possible values of the output, conditioned on values of the state and input. The distribution from Table 4.1 can be also written in the product form with the help of Kronecker delta,

Table 4.1: Discrete distribution of the output

	$\tilde{y}_t = \mathbf{0}$	$\tilde{y}_t = \mathbf{1}$
$\tilde{x}_t = \mathbf{0}, u_t \geq \theta_u$	$p_{\mathbf{0}\vert\mathbf{01}}$	$p_{\mathbf{1}\vert\mathbf{01}}$
$\tilde{x}_t = \mathbf{0}, u_t < \theta_u$	$p_{\mathbf{0}\vert\mathbf{00}}$	$p_{\mathbf{1}\vert\mathbf{00}}$
$\tilde{x}_t = \mathbf{1}, u_t \geq \theta_u$	$p_{\mathbf{0}\vert\mathbf{11}}$	$p_{\mathbf{1}\vert\mathbf{11}}$
$\tilde{x}_t = \mathbf{1}, u_t < \theta_u$	$p_{\mathbf{0}\vert\mathbf{10}}$	$p_{\mathbf{1}\vert\mathbf{10}}$

Table 4.2: Discrete distribution of the state

	$\tilde{x}_{t+1} = \mathbf{0}$	$\tilde{x}_{t+1} = \mathbf{1}$
$\tilde{x}_t = \mathbf{0}, u_{t+1} \geq \theta_u$	$\tilde{p}_{\mathbf{0}\vert\mathbf{01}}$	$\tilde{p}_{\mathbf{1}\vert\mathbf{01}}$
$\tilde{x}_t = \mathbf{0}, u_{t+1} < \theta_u$	$\tilde{p}_{\mathbf{0}\vert\mathbf{00}}$	$\tilde{p}_{\mathbf{1}\vert\mathbf{00}}$
$\tilde{x}_t = \mathbf{1}, u_{t+1} \geq \theta_u$	$\tilde{p}_{\mathbf{0}\vert\mathbf{11}}$	$\tilde{p}_{\mathbf{1}\vert\mathbf{11}}$
$\tilde{x}_t = \mathbf{1}, u_{t+1} < \theta_u$	$\tilde{p}_{\mathbf{0}\vert\mathbf{10}}$	$\tilde{p}_{\mathbf{1}\vert\mathbf{10}}$

Table 4.3: Prior discrete distribution

$\tilde{x}_t = \mathbf{0}$	$\tilde{x}_t = \mathbf{1}$
$p_{\mathbf{0}(t)}$	$p_{\mathbf{1}(t)}$

which expresses a choice of an occurred situation from the possible ones.

$$f\left(\tilde{y}_t \middle| u_t, \tilde{x}_t\right) = \prod_{\tilde{x}_t \in \{\mathbf{0},\mathbf{1}\}} \prod_{u_t \in u^*} p_{\mathbf{0}|\tilde{x}_t u_t}^{\delta(\tilde{y}_t,\mathbf{0})} p_{\mathbf{1}|\tilde{x}_t u_t}^{\delta(\tilde{y}_t,\mathbf{1})}. \tag{4.3}$$

Similarly, the state evolution model (4.2) is related to Bernoulli distribution, provided in Table 4.2, where $\tilde{p}$ denotes a known probability of taking the possible values of the state, conditioned on its previous values and on the input. The product form of the distribution from Table 4.2 is as follows.

$$f\left(\tilde{x}_{t+1} \middle| u_{t+1}, \tilde{x}_t\right) = \prod_{\tilde{x}_t \in \{\mathbf{0},\mathbf{1}\}} \prod_{u_{t+1} \in u^*} \tilde{p}_{\mathbf{0}|\tilde{x}_t u_{t+1}}^{\delta(\tilde{x}_{t+1},\mathbf{0})} \tilde{p}_{\mathbf{1}|\tilde{x}_t u_{t+1}}^{\delta(\tilde{x}_{t+1},\mathbf{1})}. \tag{4.4}$$

The prior distribution of the discrete state entry $\tilde{x}_t$, shown in Table 4.3, is also chosen as the Bernoulli one. Its product form can be written as

$$f\left(\tilde{x}_t \middle| u_t, d^{t-1}\right) = p_{\mathbf{0}(t)}^{\delta(\tilde{x}_t,\mathbf{0})} \left(1 - p_{\mathbf{0}(t)}\right)^{\delta(\tilde{x}_t,\mathbf{1})} = \prod_{\mathbf{k} \in \tilde{x}^*} p_{\mathbf{k}(t)}^{\delta(\tilde{x}_t,\mathbf{k})}, \tag{4.5}$$

where $\sum_{\mathbf{k} \in \tilde{x}^*} p_{\mathbf{k}(t)} = 1,\ p_{\mathbf{k}(t)} > 0\ \forall\ \mathbf{k}$.

The estimation of the discrete state entry is proposed as the direct application of Bayesian filtering (2.3-2.4) to the models (4.1-4.2) and, respectively, (4.3-4.4). According to the mentioned models, the relation (2.3) for Bernoulli distributions (4.3) and (4.5) takes the following form, providing the updating of the state estimate by the measurements.

$$\begin{aligned} f\left(\tilde{x}_t \middle| d^t\right) &= \frac{f\left(\tilde{y}_t \middle| u_t, \tilde{x}_t\right) f\left(\tilde{x}_t \middle| u_t, d^{t-1}\right)}{\int f\left(\tilde{y}_t \middle| u_t, \tilde{x}_t\right) f\left(\tilde{x}_t \middle| u_t, d^{t-1}\right) d\tilde{x}_t}, \\ &= \frac{\prod_{\tilde{x}_t \in \{\mathbf{0},\mathbf{1}\}} \prod_{u_t \in u^*} p_{\mathbf{0}|\tilde{x}_t u_t}^{\delta(\tilde{y}_t,\mathbf{0})} p_{\mathbf{1}|\tilde{x}_t u_t}^{\delta(\tilde{y}_t,\mathbf{1})} p_{\mathbf{0}(t)}^{\delta(\tilde{x}_t,\mathbf{0})} \left(1 - p_{\mathbf{0}(t)}\right)^{\delta(\tilde{x}_t,\mathbf{1})}}{\sum_{\tilde{x}_t \in \{\mathbf{0},\mathbf{1}\}} \prod_{\tilde{x}_t \in \{\mathbf{0},\mathbf{1}\}} \prod_{u_t \in u^*} p_{\mathbf{0}|\tilde{x}_t u_t}^{\delta(\tilde{y}_t,\mathbf{0})} p_{\mathbf{1}|\tilde{x}_t u_t}^{\delta(\tilde{y}_t,\mathbf{1})} p_{\mathbf{0}(t)}^{\delta(\tilde{x}_t,\mathbf{0})} \left(1 - p_{\mathbf{0}(t)}\right)^{\delta(\tilde{x}_t,\mathbf{1})}}, \end{aligned} \tag{4.6}$$

where the integration is replaced by the regular summation and which is evolved via substitution of the probabilities from Table 4.1 into (4.6), according to the actual values of the output and the input. The resulting distribution is as follows.

$$f\left(\tilde{x}_t \middle| d^t\right) = p_{\dagger\mathbf{0}(t)}^{\delta(\tilde{x}_t,\mathbf{0})} p_{\dagger\mathbf{1}(t)}^{\delta(\tilde{x}_t,\mathbf{1})}, \tag{4.7}$$

where $p_{\dagger\mathbf{0}(t)}$ and $p_{\dagger\mathbf{1}(t)}$ are obtained with the help of normalization of the products of the corresponding probabilities from Table 4.1 and prior probabilities $p_{\mathbf{0}(t)}$ and $\left(1 - p_{\mathbf{0}(t)}\right)$ respectively.

The time updating (2.4) for Bernoulli distribution (4.4) and according to the intermediate result (4.7) takes the following form.

$$f\left(\tilde{x}_{t+1} \middle| u_{t+1}, d^t\right) = \int f\left(\tilde{x}_{t+1} \middle| u_{t+1}, \tilde{x}_t\right) f\left(\tilde{x}_t \middle| d^t\right) d\tilde{x}_t, \tag{4.8}$$

$$= \sum_{\tilde{x}_t \in \{\mathbf{0},\mathbf{1}\}} \prod_{\tilde{x}_t \in \{\mathbf{0},\mathbf{1}\}} \prod_{u_{t+1} \in u^*} \tilde{p}_{\mathbf{0}|\tilde{x}_t u_{t+1}}^{\delta(\tilde{x}_{t+1},\mathbf{0})} \tilde{p}_{\mathbf{1}|\tilde{x}_t u_{t+1}}^{\delta(\tilde{x}_{t+1},\mathbf{1})} p_{\dagger\mathbf{0}(t)}^{\delta(\tilde{x}_t,\mathbf{0})} p_{\dagger\mathbf{1}(t)}^{\delta(\tilde{x}_t,\mathbf{1})}. \tag{4.9}$$

The resulted updated state estimate of the discrete entry is obtained as follows.

$$f\left(\tilde{x}_{t+1}\middle| u_{t+1}, d^t\right) = p_{\mathbf{0}(t+1)}^{\delta(\tilde{x}_{t+1},\mathbf{0})}(1-p_{\mathbf{0}(t+1)})^{\delta(\tilde{x}_{t+1},\mathbf{1})}, \tag{4.10}$$

with the updated probability

$$p_{\mathbf{0}(t+1)} = \sum_{\tilde{x}_t\in\{\mathbf{0},\mathbf{1}\}} \prod_{\tilde{x}_t\in\{\mathbf{0},\mathbf{1}\}} \prod_{u_{t+1}\in u^*} \tilde{p}_{\mathbf{0}|\tilde{x}_t u_{t+1}}^{\delta(\tilde{x}_{t+1},\mathbf{0})} p_{\dagger\mathbf{0}(t)}^{\delta(\tilde{x}_t,\mathbf{0})} p_{\dagger\mathbf{1}(t)}^{\delta(\tilde{x}_t,\mathbf{1})}, \tag{4.11}$$

$$= \tilde{p}_{\mathbf{0}|\mathbf{0}u_{t+1}} p_{\dagger\mathbf{0}(t)} + \tilde{p}_{\mathbf{0}|\mathbf{1}u_{t+1}} p_{\dagger\mathbf{1}(t)}, \tag{4.12}$$

where (4.12) is calculated according to the known values of the input and substitution of the corresponding probabilities from Table 4.2. The probability of value **1** is

$$p_{\mathbf{1}(t+1)} = (1-p_{\mathbf{0}(t+1)}), \text{ which can be directly calculated as} \tag{4.13}$$

$$= \sum_{\tilde{x}_t\in\{\mathbf{0},\mathbf{1}\}} \prod_{\tilde{x}_t\in\{\mathbf{0},\mathbf{1}\}} \prod_{u_{t+1}\in u^*} \tilde{p}_{\mathbf{1}|\tilde{x}_t u_{t+1}}^{\delta(\tilde{x}_{t+1},\mathbf{1})} p_{\dagger\mathbf{0}(t)}^{\delta(\tilde{x}_t,\mathbf{0})} p_{\dagger\mathbf{1}(t)}^{\delta(\tilde{x}_t,\mathbf{1})}, \tag{4.14}$$

$$= \tilde{p}_{\mathbf{1}|\mathbf{0}u_{t+1}} p_{\dagger\mathbf{0}(t)} + \tilde{p}_{\mathbf{1}|\mathbf{1}u_{t+1}} p_{\dagger\mathbf{1}(t)}, \tag{4.15}$$

where, similarly, (4.15) is obtained according to the input values and substitution of the probabilities from Table 4.2. The obtained distribution (4.10) is taken as the prior one for the next step of the discrete state estimation (4.6) with actual available measurements.

For the sake of simplicity, the filtering related to the scalar discrete entries is described. In general, the involved discrete state can be a vector. In this case the state dimension can be reduced via specific denoting, which leads the solution to the proposed scalar-entry one. It means, that for the discrete state $\tilde{x}_t \equiv [\tilde{x}_{1;t}, \ldots, \tilde{x}_{\mathbf{s};t}]'$ with finite number $\mathbf{s}$ and $\mathbf{k} = \{1, \ldots, \mathbf{s}\}$, each entry $\tilde{x}_{\mathbf{k};t} \in \tilde{x}^*$ with its possible values is treated as an individual possible value of a new scalar state $\mathcal{X}_t$. However, the present paper is focused on the considered case.

After estimation of the discrete state entry its mean value can be involved in the consequent entry-wise filtering (3.3) along with Gaussian entries. The mean value is calculated as a sum of possible values of the entry multiplied by the updated probabilities, i.e.

$$\sum_{\mathbf{k}\in\tilde{x}^*} \prod_{\tilde{x}_t\in\tilde{x}^*} \tilde{x}_{t+1} p_{\mathbf{k}(t+1)} \equiv \mu^f_{\mathring{x};t+1}, \tag{4.16}$$

which means, that for the considered case of the state vector and starting at $i = \{\mathring{x}, \mathring{x}-1, \ldots, 1\}$, the relation (3.3) exploits $\mu^f_{\mathring{x};t}$ for its last $\mathring{x}$-th pdf. The rest of the pdfs for $i = \{\mathring{x}-1, \ldots, 1\}$ are evolved according to (3.24) with the help of the factorized Kalman filter, proposed in Subsection 3.1.

Chapter 5

EXPERIMENTS

In urban traffic control, which is a target application domain of the research, the state variables of the continuous nature are used for modeling of a length of a car queue on an intersection. The length of the queue expresses a state of the transport network most adequately, but it is not directly observable and has to be estimated. Various discrete-valued state variables could be involved into the estimation (signal lights, level of service, visibility, road surface etc). The paper presents an experiment of the filtering with the mixed-type state, where the discrete entry indicates the queue existence on the intersection lane.

The simulated system, used for demonstration of the proposed filtering, is provided by the traffic microsimulator AIMSUN [22]. The system represents the intersection with four arms, each with one input and one output lane. Each lane is equipped by a measuring detector. The input detectors are placed about 100 meters before the stop line at the input lane of the intersection arm, and the output detectors – at the output lane. The detectors measure the following quantities: intensity, expressing a number of the cars, passing through an intersection lane per hour $[c/h]$, and occupancy, reflecting a proportion of a time period of activating the detector by cars [%]. The scope of the paper does not allow to describe the specific features of the traffic control in details, but the physical interpretation of the traffic system model can be explained. According to [2], Gaussian state-space model (3.4-3.5) is specialized to the traffic control area in the following way.

$$
\begin{aligned}
\textit{observation model} \qquad y_t &= C_t x_t + H_t u_t + v_t, && (5.1)\\
\textit{state evolution model} \qquad x_{t+1} &= A_t x_t + B_t u_{t+1} + F_t + \omega_t. && (5.2)
\end{aligned}
$$

The system output y_t in (5.1) relates to the column vector Y_t of output intensities, provided by the output detectors of the intersection lanes, i.e. $Y_t = [y_{1;t}, \ldots, y_{n;t}]'$, $n = 4$ is a number of lanes (identical to the number of arms for the given system). The state x_t in models (5.1-5.2) expresses the length of the car queue at the intersection lanes in cars $[c]$. One car is supposed to have about 6 meters. The queue length is not directly observed and has to be estimated. The state vector x_t relates to $\xi_t = [\xi_{1;t}, \ldots, \xi_{4;t}]'$, where $\xi_{i;t}$ is a queue length to be estimated. According to [2, 3, 4], the general idea of the car queue length evolution lies in the statement, that the queue length at the i-th intersection lane is equal to the previous queue plus arrived cars minus departed cars. In general, it can be expressed in the following way.

$$
\xi_{i;t+1} = \delta_{i;t}\xi_{i;t} - \underbrace{[\delta_{i;t}S_i + (1-\delta_{i;t})I_{i;t}]u_t}_{\textit{departed cars}} + \underbrace{I_{i;t},}_{\textit{arrived}} \; , i = \{1, \ldots, n\}, \qquad (5.3)
$$

where S_i is the known saturated flow of the i-th lane (the maximal number of cars, which can pass through the lane per hour in the case of the green light) with the following values in cars in proportion to the time period, equal to ninety seconds: $S_1 = 27$, $S_2 = 20.5$, $S_3 = 23$, $S_4 = 27$. $I_{i;t}$ is the input intensity, and u_t is the time of the green light in seconds $[s]$ (proportional to the time period of sampling) with the values $u_t = [0.5 \;\; 0.4]'$. $\delta_{i;t}$ is a discrete queue indicator so that $\delta_{i;t} = 1$, if the queue exists, and $\delta_{i;t} = 0$ otherwise. The paper proposes to estimate it as the discrete state $\tilde{x}_t$, which means, that a probability of the queue existence, obtained according to (4.16), is used in (5.3).

As the discrete output to be exploited in the estimation of the queue indicator according to (4.6) and (4.8), one can use the discretized value $\tilde{y}_{i;t} \in \{0, 1\}$ of the input occupancy $O^I_{i;t}$. The value $\tilde{y}_{i;t}$ expresses either the high ($\tilde{y}_{i;t} = 1$) occupancy of the input detector or the low one ($\tilde{y}_{i;t} = 0$) on the i-th intersection lane, discretized via the average value θ_{y_i} chosen by the experts. According to Table 4.1, the following probabilities of taking the value $\tilde{y}_{i;t} = 1$ are to be used in relation (4.6): $p_{\mathbf{1}|\mathbf{01}} = 0.98$,

$p_{\mathbf{1|00}} = 0.8$, $p_{\mathbf{1|11}} = 0.11$, $p_{\mathbf{1|10}} = 0.01$ (the probabilities of the opposite value are taken as one minus the corresponding probability). The threshold $\theta_u = 0.5$ is given by the traffic experts.

The probabilities of the queue existence for the value $\delta_{i;t+1} = 1$ to be used for the discrete state evolution, according to the model (4.2) and Table 4.2, are as follows: $\tilde{p}_{\mathbf{1|01}} = 0.57$, $\tilde{p}_{\mathbf{1|00}} = 0.54$, $\tilde{p}_{\mathbf{1|11}} = 0.02$, $\tilde{p}_{\mathbf{1|10}} = 0.01$. The prior probability, corresponding to Table 4.3, expresses the prior knowledge about the queue existence on the i-th intersection lane. Besides the knowledge of traffic experts about problematic traffic regions, the prior probability of the queue existence primarily depends on the time of the day. The daily course of the traffic has a dynamically changing character, starting at very low driving activities on the roads in the night and having the main peak-hours in the morning and the late afternoon time. In the case of the estimation, made for the *daily* course, the usual practice in the traffic control area is to start the filter about 4 a.m. It defines the low prior value as $p_{\mathbf{1}(t)} = 0.1$ to be incorporated in (4.6). The queue indicator $\delta_{i;t}$ is identified with its mean value, calculated according to (4.16), and substituted in models (5.1-5.2), dealing with the length of the queue. The time-varying matrices C_t and H_t for the considered simulated system are composed as follows [2, 3, 4].

$$C_t = \begin{bmatrix} 0 & \alpha_{21}(1-\delta_{2,t}) & \alpha_{31}(1-\delta_{3,t}) & \alpha_{41}(1-\delta_{4,t}) \\ \alpha_{12}(1-\delta_{1,t}) & 0 & \alpha_{32}(1-\delta_{3,t}) & \alpha_{42}(1-\delta_{4,t}) \\ \alpha_{13}(1-\delta_{1,t}) & \alpha_{23}(1-\delta_{2,t}) & 0 & \alpha_{43}(1-\delta_{4,t}) \\ \alpha_{14}(1-\delta_{1,t}) & \alpha_{24}(1-\delta_{2,t}) & \alpha_{34}(1-\delta_{3,t}) & 0 \end{bmatrix}, H_t = \begin{bmatrix} H1_t & 0 \\ 0 & H2_t \\ H3_t & 0 \\ 0 & H4_t \end{bmatrix}, \quad (5.4)$$

with
$$H1_t = \sum_{k=2}^{m=4} \alpha_{k1}((1-\delta_{k,t})I_{k,t} + \delta_{k,t}S_k), \quad (5.5)$$

$$H2_t = \alpha_{12}((1-\delta_{1,t})I_{1,t} + \delta_{1,t}S_1) + \sum_{k=3}^{m=4} \alpha_{k2}((1-\delta_{k,t})I_{k,t} + \delta_{k,t}S_k), \quad (5.6)$$

$$H3_t = \sum_{k=1}^{m=2} \alpha_{k3}((1-\delta_{k,t})I_{k,t} + \delta_{k,t}S_k) + \alpha_{43}((1-\delta_{4,t})I_{4,t} + \delta_{4,t}S_4), \quad (5.7)$$

$$H4_t = \sum_{k=1}^{m=3} \alpha_{k4}((1-\delta_{k,t})I_{k,t} + \delta_{k,t}S_k), \quad (5.8)$$

where α_{ij} is the known (constant) parameter of the turn rate, reflecting the ratio of cars going from the i-th arm to the j-th arm, $j \neq i$, in percent [%]. The provided values of this parameter are $\alpha_{12} = 0.3$, $\alpha_{13} = 0.5$, $\alpha_{14} = 0.2$, $\alpha_{21} = 0.3$, $\alpha_{23} = 0.2$, $\alpha_{24} = 0.5$, $\alpha_{31} = 0.5$, $\alpha_{32} = 0.2$, $\alpha_{34} = 0.3$, $\alpha_{41} = 0.2$, $\alpha_{42} = 0.5$, $\alpha_{43} = 0.3$. For the state evolution model (5.2) the matrices A_t, B_t and F_t are as follows.

$$A_t = \begin{bmatrix} \delta_{1,t} & 0 & 0 & 0 \\ 0 & \delta_{2,t} & 0 & 0 \\ 0 & 0 & \delta_{3,t} & 0 \\ 0 & 0 & 0 & \delta_{4,t} \end{bmatrix}, \ F_t = \begin{bmatrix} I_{1,t} \\ I_{2,t} \\ I_{3,t} \\ I_{4,t} \end{bmatrix}, \quad (5.9)$$

$$B_t = \begin{bmatrix} -(\delta_{1,t}S_1 + (1-\delta_{1,t})I_{1,t}) & 0 \\ 0 & -(\delta_{2,t}S_2 + (1-\delta_{2,t})I_{2,t}) \\ -(\delta_{3,t}S_3 + (1-\delta_{3,t})I_{3,t}) & 0 \\ 0 & -(\delta_{4,t}S_4 + (1-\delta_{4,t})I_{4,t}) \end{bmatrix}. \quad (5.10)$$

The noises v_t and ω_t are defined according to (3.4-3.5) with the covariance matrices R_v and R_w respectively. The covariances are computed as a mean of squares of differences between the state (or output) value and its conditional mean. The mean is substituted by the samples of the daily (or for the corresponding time of a day) course of the state (or output), which is constructed as a spline approximation of several last periodic courses (e.g. courses during the workdays of a week). The resulted covariance matrices, used for the experimental part of the work, are respectively

$$R_v = \begin{bmatrix} 4.0757 & 0.2023 & 0.2860 & 0.0148 \\ 0.2023 & 4.9410 & 0.4509 & 0.0505 \\ 0.2860 & 0.4509 & 4.3145 & -0.1486 \\ 0.0148 & 0.0505 & -0.1486 & 4.2407 \end{bmatrix}. \quad (5.11)$$

and

$$R_w = \begin{bmatrix} 1.7898 & 0.2446 & -0.0387 & 0.0166 \\ 0.2446 & 1.2599 & 0.0263 & -0.0091 \\ -0.0387 & 0.0263 & 1.5738 & 0.0200 \\ 0.0166 & -0.0091 & 0.0200 & 1.3482 \end{bmatrix}. \quad (5.12)$$

The estimation of the queue length is made for the daily course of the traffic, starting the factorized Kalman filter about 4 a.m. with the zero-mean initial states. It is naturally caused by the low night intensities. The initial covariance matrix, chosen by the experts is as follows.

$$P_0 = \begin{bmatrix} 2.8836 & 0.0789 & 0.2260 & -0.0002 \\ 0.0789 & 2.9479 & 0.1090 & 0.0979 \\ 0.2260 & 0.1090 & 1.7431 & -0.0514 \\ -0.0002 & 0.0979 & -0.0514 & 2.5355 \end{bmatrix}. \tag{5.13}$$

The simulated traffic system, constructed in the described way, has been used for experiments. The filtering of the mixed-type states, proposed in Sections 3-4, has been applied to the estimation of the continuous queue lengths and the discrete queue existence indicators. The simulated data for the experiments were identified with the real measurements. The data set was available for 960 time periods, which corresponds to the 24-hours course of the traffic. Fig. 5.1 shows the results of the filtering of the car queue length, compared with the simulated one, and the estimated queue indicator at the intersection input lane 3 of the considered traffic system (the estimation results for the rest of the lanes are very similar). For better viewing, Fig. 5.1 demonstrates a shorter traffic course for 800 time periods, i.e. from 4 a.m up to the midnight. The queue existence is adequately indicated for the morning and the late afternoon peak-hours of the daily traffic. The estimated and simulated car queue lengths are in an adequate correspondence. The obtained variances of the estimated queue lengths for all the input lanes, evolved as

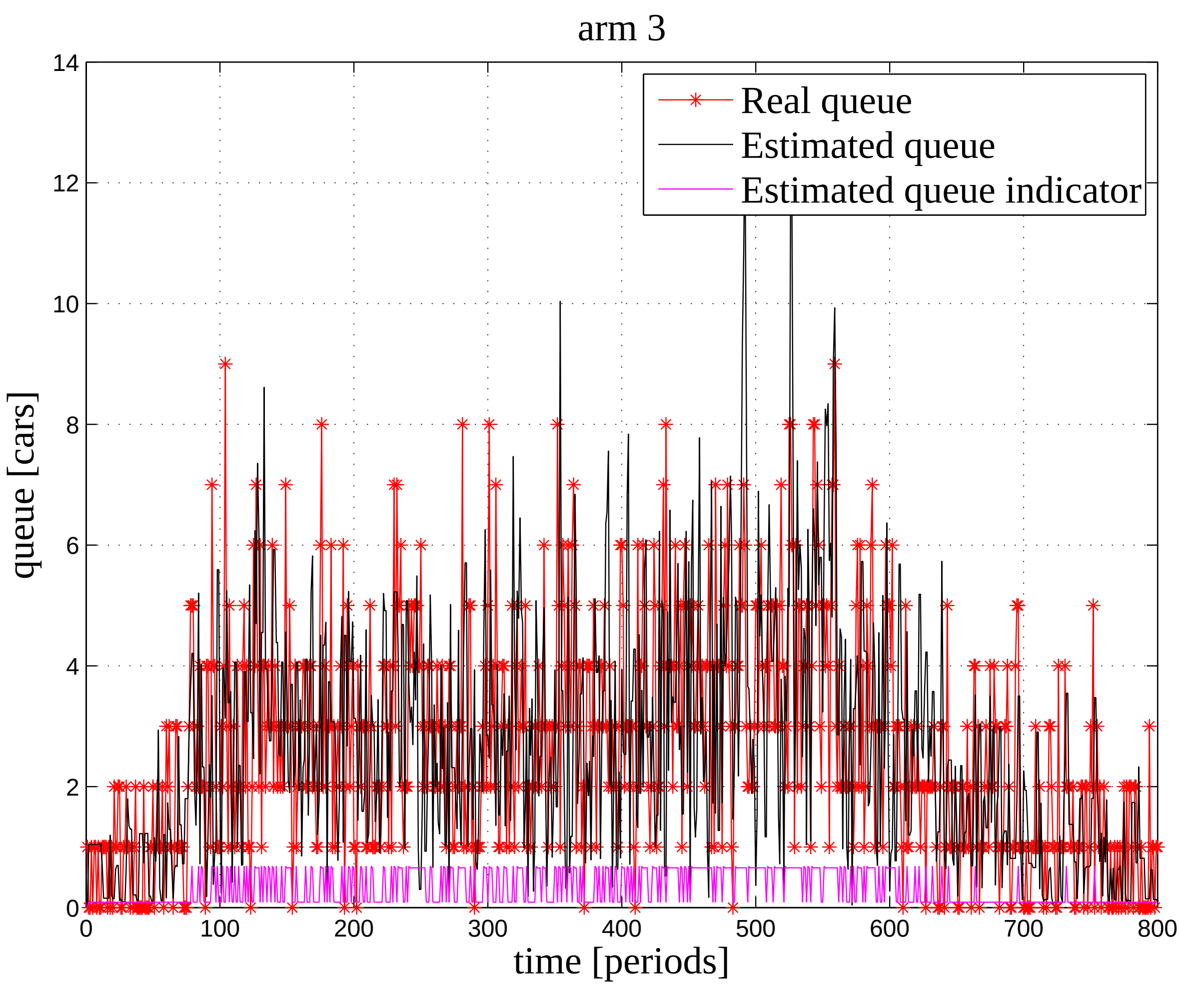

Figure 5.1: Mixed-type state estimation at intersection lane 3

the inverse elements of the matrix $D_{p|t+1}$ according to (3.24), are respectively as follows: 3.4257, 1.3872, 1.5996, 1.7332. Fig. 5.2 plots the mean value of the queue indicator, obtained according to (4.16), at the intersection input lane 3 against its values, calculated directly from the simulations, i.e. $\delta_{i;t} = 0$, if $(\xi_{i;t} + I_{i;t}u_t) < S_{i;t}u_t$ and $\delta_{i;t} = 1$, if $(\xi_{i;t} + I_{i;t}u_t) \geq S_{i;t}u_t$ [4].

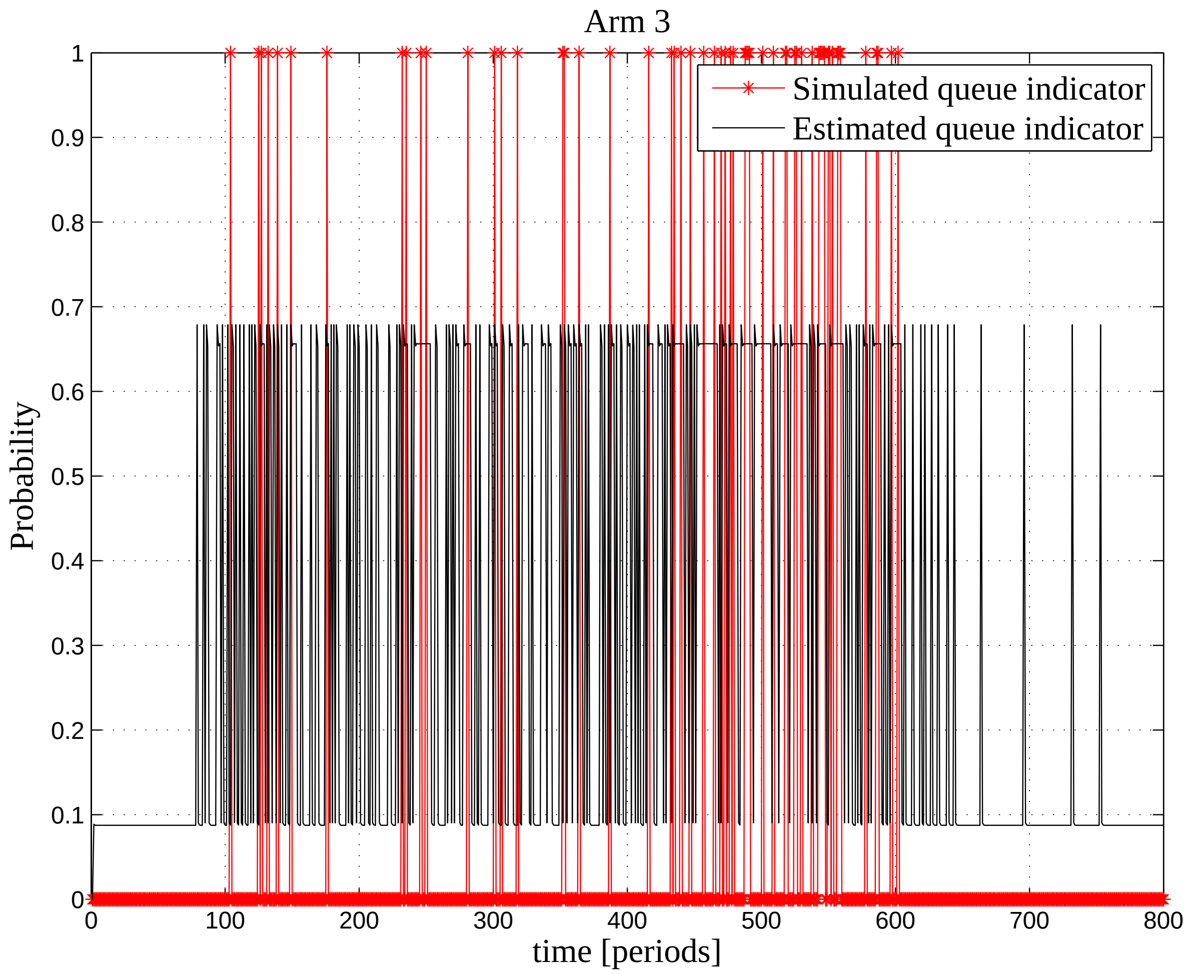

Figure 5.2: Discrete queue indicator estimation at intersection lane 3

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