Charles University in Prague
Faculty of Mathematics and Physics
Department of Probability and Mathematical Statistics
and
Academy of Sciences of the Czech Republic
Institute of Information Theory and Automation

Doctoral Thesis

Mgr. Bc. Michal Červinka

Hierachical Structures in Equilibrium Problems

Supervisor:
Doc. Ing. Jiří V. Outrata, DrSc.

Prague, May 2008

Branch M5 – Econometrics and Operations Research

Karlova Univerzita v Praze
Matematicko-fyzikální fakulta
Katedra pravděpodobnosti a matematické statistiky
a
Akademie věd České Republiky
Ústav teorie informace a automatizace, v.v.i

Disertační práce

Mgr. Bc. Michal Červinka

Hierarchické struktury v ekvilibriálních úlohách

Školitel:
Doc. Ing. Jiří V. Outrata, DrSc.

Praha, květen 2008

Obor M5 – Ekonometrie a operační výzkum

Preface

This doctoral thesis would not be written if not for my supervisor, Jiří Outrata. When I entered the Ph.D. program at the Charles University, I had very little if any knowledge of modern variational analysis and nonlinear optimization. Over the years, my supervisor was patiently answering my questions, explained me the concepts of modern optimization theory, each time pointed me to relevant books and papers. He carefully read my working papers and preprints uncountably many times, always improving the texts. His critical reviews, insights and ideas lead to the creation of this thesis. I am deeply indebt to him for providing me with this opportunity, for his guidance and constant patience.

Also, if not for him, I would not have the opportunity to discuss my results with other researches in my field. I am grateful to Boris S. Mordukhovich for interesting discussions and topics which lead to a joint work with him and Jiří Outrata, resulting in Sections 4.1, 4.3 and 5.3.

I would like to express my gratitude to Daniel Ralph, for allowing me to spend two unforgettable months at the Judge Business School, University of Cambridge during the summer 2006. I thank him for nearly everyday discussions despite his busy schedule, which helped me to get deeper understanding of stationarity concepts discussed in Section 2.3. Also, he proposed an idea to generalize the homotopy method in such a way that it could be applied to a special type of EPEC. The results we achieved together during my stay in Cambridge form a basis of Section 5.2.

I also thank to Jong-Shi Pang and Alejandro Jofré for suggesting the generalization of concept of solutions to mixed strategies, which lead me to results in Section 3.3.

The Institute of Information Theory and Automation of the Academy of Sciences of the Czech Republic has been a great working environment. I would like thank for the financial support of The Ryoichi Sasakawa Young Leaders Fellowship Fund, which allowed me to visit Daniel Ralph in Cambridge. The work on results presented in this thesis was also supported by the Grant Agency of the Charles University under grant GAUK 7645/2007 and by the Grant Agency of the Academy of Sciences of the Czech Republic under grant IAA 1030405.

A few other people deserve special mention: Michal Kočvara, for help in creating figures, Václav Kratochvíl, for his great help in creating computer codes in Matlab, my students, for cheerful hours every week; and my former and current colleagues, classmates and friends, too many to be listed here.

Last but surely not the least, this thesis would not be possible without the support

of my parents, my sister and her family. Thank you for always being there for me. This thesis is dedicated to you.

Michal Červinka
Prague, May 2008

Contents

Abbreviations

EPCC	equilibrium problem with complementarity constraints
EPEC	equilibrium problem with equilibrium constraints
GLICQ	generalized linear independence constraint qualification
GMFCQ	generalized Mangasarian-Fromowitz constraint qualification
ISO	independent system operator
KKT	Karush-Kuhn-Tucker
LICQ	linear independence constraint qualification
MCP	mixed complementarity problem
MFCQ	Mangasarian-Fromowitz constraint qualification
MOPCC	multiobjective problem with complementarity constraints
MOPEC	multiobjective problem with equilibrium constraints
MPCC	mathematical program with complementarity constraints
MPEC	mathematical program with equilibrium constraints
NCP	nonlinear complementarity problem
NLP	nonlinear program
OD	origin-destination
SOPEC	set-valued optimization problem with equilibrium constraints
SRC	strong regularity condition
TNLP	tightened nonlinear program

Notation

Spaces and Orthants

$\mathbb{R}$	the real numbers
$\mathbb{R}_-$	the left half line
$\mathbb{R}_+$	the right half line
$\mathbb{R}^n$	the n-dimensional real vector space
$\mathbb{R}^n_-$	the nonpositive orthant in $\mathbb{R}^n$
$\mathbb{R}^n_+$	the nonnegative orthant in $\mathbb{R}^n$

Sets

$\emptyset$	empty set
$\{x\}$	the set consisting of the vector x
$\{x\}^\perp$	the orthogonal complement of vector x
(a,b)	an open interval in $\mathbb{R}$
$[a,b]$	a closed interval in $\mathbb{R}$
conv $\mathcal{S}$	convex hull of the set $\mathcal{S}$
cone $\mathcal{S}$	conic hull of the set $\mathcal{S}$
cl $\mathcal{S}$	closure of the set $\mathcal{S}$
int $\mathcal{S}$	interior of the set $\mathcal{S}$
rint $\mathcal{S}$	relative interior of the set $\mathcal{S}$
bdry $\mathcal{S}$	boundary of the set $\mathcal{S}$
$\mathcal{S}_1 \subset \mathcal{S}_2$	$\mathcal{S}_1$ is a subset of $\mathcal{S}_2$
$\lvert I\rvert$	cardinality of a finite set I
$\mathcal{P}(I)$	the set of all subsets of a finite set I
$\mathcal{S}_1 \times \mathcal{S}_2$	Carthesian product of sets $\mathcal{S}_1$ and $\mathcal{S}_2$
$\bigtimes_{i=1}^n \mathcal{S}_i$	Carthesian product of sets $\mathcal{S}_i, i = 1, \ldots, n$
$\arg\min_{x\in\Omega} f(x)$	the set of points where the minimum of the real-valued function f on the set Ω is attained
$\arg\max_{x\in\Omega} f(x)$	the set of points where the maximum of the real-valued function f on the set Ω is attained
$\mathbb{B}$	the closed unit ball
$\mathbb{B}(x)$	the closed unit ball around x

Cones

$T(x;\Omega)$	the Bouligand-Severi contingent cone to Ω at x
$T_C(x;\Omega)$	the Clarke tangent cone to Ω at x
$N(x;\Omega)$	the limiting normal cone to Ω at x
$N_C(x;\Omega)$	the Clarke normal cone to Ω at x
$\hat{N}(x;\Omega)$	the Fréchet normal cone to Ω at x
$K(x,y;\Omega)$	the critical cone of Ω with respect to x and $x-y$
K^*	the polar cone to K
K^-	the negative polar cone to K

Vectors

$x \in \mathbb{R}^n$	column vector in $\mathbb{R}^n$
$x^\top$	transpose of vector x
(x, y)	column vector $(x^\top, y^\top)^\top$
x_i	ith component of vector x
x_{-i}	the vector in $\mathbb{R}^{n-1}$ consisting of components $x_j, j \neq i$
x_I	the vector in $\mathbb{R}^{\lvert I \rvert}$ consisting of components $x_i, i \in I$
x^{-i}	the vector $(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^m)$ with $x^j \in \mathbb{R}^n, j = 1, \ldots, m$
$x \geq y$	componentwise comparison $x_i \geq y_i, i = 1, \ldots, n$
$x > y$	componentwise strict comparison $x_i > y_i, i = 1, \ldots, n$
$\langle x, y \rangle := x^\top y$	the standard inner product of vectors in $\mathbb{R}^n$
$\lvert\lvert x \rvert\rvert$	the Euclidean norm of a vector $x \in \mathbb{R}^n$
$\min\{x, y\}$	the vector whose ith component is min $\{x_i, y_i\}$
$x \perp y$	orthogonality of vectors x and y in $\mathbb{R}^n$

Functions and Multifunction

$f : \mathbb{R}^n \to \mathbb{R}^m$	a function that maps $\mathbb{R}^n$ to $\mathbb{R}^m$
$f_i : \mathbb{R}^n \to \mathbb{R}$	the ith component function of f
$F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$	a multifunction that maps $\mathbb{R}^n$ to subsets of $\mathbb{R}^m$
epi f	the epigraph of function f
epi F	the generalized epigraph of multifunction F
$\mathcal{E}_F$	the epigraphical multifunction of multifunction F
Dom F	the domain of multifunction F
Gph F	the graph of multifunction F
Ker F	the kernel of operator F
$\nabla f(x)$	the Jacobian of $f : \mathbb{R}^n \to \mathbb{R}^m$ (the gradient of $f : \mathbb{R}^n \to \mathbb{R}$)
$\nabla_x f(x)$	the partial Jacobian of $f : \mathbb{R}^n \to \mathbb{R}^m$ (the partial gradient of $f : \mathbb{R}^n \to \mathbb{R}$) with respect to x
$\nabla f_I(x)$	the submatrix of the $m \times n$ matrix $\nabla f(x)$ with rows indexed by $i \in I \subset \{1, \ldots, m\}$
$\nabla f_{I,J}(x)$	the submatrix of the $m \times n$ matrix $\nabla f(x)$ with rows indexed by $i \in I \subset \{1, \ldots, m\}$ and columns by $j \in J \subset \{1, \ldots, n\}$
$\partial f(x)$	limiting subdifferential of f at x
$\bar{\partial} f(x)$	generalized Jacobian (Clarke subdifferential) of f at x
$\partial_K F(x, y)$	limiting subdifferential of multifunction F at $(x, y) \in$ epi F with respect to a cone K
$\partial_K^\infty F(x, y)$	singular subdifferential of multifunction F at $(x, y) \in$ epi F with respect to a cone K
$D^* F(x, y)$	coderivative of a multifunction F at $(x, y) \in$ Gph F
$\Pi(x; \Omega)$	the Euclidean projector of x onto the closure of Ω
dist $(x; \Omega)$	Euclidean distance between x and Ω
$F \circ G$	composition of mappings F and G

Matrices

E	the identity matrix of appropriate order
$A^\top$	transpose of a matrix A
A_j	the jth row of a matrix A
A_I	the submatrix of a matrix A with rows $A_j, j \in J$
$A_I^\top$	transpose of the submatrix of a matrix A with rows $A_j, j \in J$
A_{x^i}	the submatrix of a matrix A with rows of A which correspond to components of vector x^i in the product $Ax, x = (x^1, \ldots, x^n)$
Q_{x^i,x^i}	the square submatrix of a square matrix Q with rows and columns of Q which correspond to components of vector x^i in the product $Qx, x = (x^1, \ldots, x^n)$
$\det A$	determinant of a matrix A
$\mathrm{Adj}\ A$	adjunct matrix of a matrix A
A^{-1}	inverse matrix of a matrix A
$\mathrm{diag}\,(A^1, \ldots, A^n)$	block diagonal matrix with the ith block equal to matrix A^i

Sequences

$\{x^{(k)}\}$	a sequence in $\mathbb{R}^n$
$x \to \bar{x}$	x converges to $\bar{x}$
$x \xrightarrow{\Omega} \bar{x}$	x converges to $\bar{x}$ with $x \in \Omega$
$x \searrow \bar{x}$	x converges to $\bar{x}$ with $x > \bar{x}$
liminf	lower limit for real numbers
limsup	upper limit for real numbers
Lim inf	lower/inner limit for multifunctions
Lim sup	upper/outer limit for multifunctions

Oligopolistic market problem

$x^i \in \mathbb{R}$	production of the ith leader
$y^j \in \mathbb{R}$	production of the jth follower
$\omega \subset \mathbb{R}^n$	the set of geometric constraints of leaders
T	overall production quantity on the market
p	inverse demand function/market price
φ^i	objective function of the ith leader
f^j	objective function of the $j$the follower
c^i	cost function of the ith producer

Forward-spot market model

x	production vector
s	spot sales vector
f	forward position vector
p	inverse demand function/spot price
c_i	cost function of the ith producer

Deregulated electricity market model

L	set of links
q_i	injection/withdrawal at node i
C_{ij}	transmission limit on the link ij
$\phi_{ij,k}$	contribution of injection/withdrawal at node k to the link ij
p_i	price at node i

Traffic equilibrium problem

G	transportation network
$\mathcal{N}$	set of nodes
$\mathcal{A}$	set of arcs
W	set of OD pairs in G
R_w	set of all paths connecting OD pair $w \in W$
R	set of all routes
F_r	flow on route $r \in R$
v_a	flow on arc $a \in \mathcal{A}$
Δ	incidence matrix with elements δ_{ar}
C_r	costs of using route $r \in R$
D_w	traffic demand between OD pair $w \in W$
μ_w	minimum travel costs between OD pair $w \in W$
y_a	capacity on arc $a \in \mathcal{A}$
α	the value of time
t_a	travel time on arc $a \in \mathcal{A}$
ηI_a	costs of firm providing arc $a \in \mathcal{A}$

Chapter 1

Introduction

In past century, the study of conflicting situation, a collision of interest, received a considerable scientific interest. Although some game-theoretical results can be traced to the 18th century, the first rigorous results were developed in the 1920s by Borel and von Neumann. The establishment of game theory as a scientific field is usually related to the publication of [50] in 1944. Since then, a great variety of scientific disciplines, like economics, biology, sociology and politics, become interested in study of conflicting situations.

An individual facing a decision takes into account different outcomes. However, he or she may not be the only decision-making person and the resulting outcome often depends on multi-person decision. In this case, optimality is not a well defined concept and instead, we speak of *equilibria*.

There is a great variety of different equilibrium concepts. Among the two widely used belongs a solution to a noncooperative game, where, roughly speaking, each player can not improve his or her outcome by altering his or her decision unilaterally. This concept, named *Nash equilibrium* concept, was introduced in the early 1950s in [34]. A different situation arises when cooperation is present. We then speak of a *Pareto optimal solution* when there is no other joint decision such that the performance of at least one player can be improved without degrading the performance of the others.

Probably the first study of a hierarchical model of conflicting situations is due to Stackelberg [51]. Nowadays, a *Stackelberg* (or sometimes termed also single-leader-follower) *game* is used to model an economic situation when on the market the dominant firm (e.g., due to some temporal advantage), called the market *leader* (or upper-level player), maximizes its profits under the assumption that all other firms present on the market, called *followers* (or lower-level players), play a noncooperative strategy. Mathematically, this situation is modeled via *bilevel optimization problems* (namely when only one follower is present on the market) and *mathematical programs with equilibrium constraints* (MPECs). The MPEC class of optimization problems was introduced in 1970s motivated by other applications to mechanics and network design. In past decade MPECs received an extensive interest of mathematicians. Following the progress in computational power of computers, there is now a wide range of algorithmic approaches to MPECs. We refer the reader to monographs [25], [39] and [30, Chapter 5] on MPECs, and [12] on bilevel programming.

Our main interest in this thesis, however, is focused on the conflicting situations leading to problems which in a sense lie in between Nash and Stackelberg games, to the so-called *multi-leader-follower games*. This situation occurs, as the name suggests, when more than one player is in a dominant position and hence has to take into account not just the reaction of players on the lower level but also of the remaining leaders.

Concerning the behavior of the leaders, one can again distinguish two situations: the decision making of the leaders forms a Nash equilibrium on the upper level, or all leaders cooperate in order to achieve an upper-level Pareto optimal strategy. To express mathematically the former situation one can use the novel paradigm of *equilibrium problems with equilibrium constraints* (EPECs). This class of hierarchical decision making models was probably directly addressed for the first time in [47]. The latter situation leads to a different class of hierarchical problems, nowadays called *multiobjective problems with equilibrium constraints* (MOPECs).

The aim is, of course, to find (local) solutions to the mentioned problems. For this purpose, various stationarity concepts have been introduced. To verify that a given point is stationary is in general easier then to check that it is a local solution. However, for a local solution to be stationary, certain *constraint qualification* must hold true. One can observe two approaches to the study of MPECs: to restrict the attention to problems constrained by a nonlinear complementarity problem and to study the Lagrange function and behavior of the corresponding multipliers; or to impose a rather strict assumption that the lower problem attains (locally) a unique solution. The latter restriction enables us to apply successfully the so-called *implicit programming approach*.

In this thesis, we investigate stationarity concepts tailored to MPECs and EPECs and the connection between the various stationarity concepts. Due to the structural dependence of EPECs on MPECs, we naturally build upon known results about MPECs. We pay the main attention to a subclass of MPECs constrained by a *nonlinear complementarity problem* since this is the case of currently known applications of EPECs.

One of the main aims was to construct a bridge between stationarity conditions resulting from the above mentioned approaches. To this end we use many results from [45], [39] and [36]. However, the structure of our considered problem is slightly different, hence we decided to present most of the results with full proofs. This is done in Chapter 2. The main attention is paid to the so-called *Clarke stationarity* and *C-stationarity*, both based on application of Clarke generalized calculus. These two stationarity concepts are of particular importance to EPECs.

In Chapter 3 we give mathematical formulation of EPEC. Interestingly, the study of this class of problems was boosted by modeling of conflicting behavior of agents in deregulated electricity markets; we devote a separate section to several source problems which are currently of high scientific interest. We aim to address the question of existence of Clarke and C-stationary points and also of solutions to EPECs in mixed strategies.

Chapter 4 is devoted to MOPECs. We derive necessary optimality conditions and using the novel subdifferential calculus for set-valued mappings by Mordukhovich we establish existence of solutions to these problems.

In the last chapter, Chapter 5, several numerical methods are presented. All known algorithms to find solution to EPEC depend directly on techniques to solve MPECs numerically, in some cases due to very strong assumptions imposed on the data of EPEC. For this reason we attempt to derive an alternative algorithm based on the homotopy method tailored specifically to a special subclass of EPECs. Finally, an effective numerical technique to solve MOPECs is developed.

Parts of the original work which could be found in this thesis have already appeared in separate publications [8], [9] and [31] and working papers [10] and [11], some previous results by the author have been completely reworked and generalized to fit the structure of this thesis or complemented with additional results. Other sources have been also used throughout the thesis when appropriate or necessary. In each case, this is carefully documented.

Chapter 2

Mathematical Program with Equilibrium Constraints (MPEC)

In this chapter we investigate MPECs and associated first order necessary optimality conditions. In the center of focus of this chapter are stationarity concepts for MPECs with equilibrium constraints in the form of a nonlinear complementarity problem. We discuss the relations between stationarity concepts, in particular, of those based on Clarke generalized calculus. Also, we discuss the qualification conditions which are essential in deriving necessary optimality conditions for MPECs of the considered structure.

2.1 Mathematical formulation

An MPEC is an optimization problem with two sets of players; one leader trying to solve an upper-level minimization problem and one or more lower-level players, followers, trying to reach a parameterized (by the upper-level decision variable) Nash equilibrium by solving a lower-level equilibrium problem among themselves.

More precisely, this problem is defined as follows. Let (x, y) denote the multistrategy composed from the strategies $x \in \mathbb{R}^{l_1}$ of the leader and multistrategy $y \in \mathbb{R}^{ml_2}$ of m followers. Suppose that $\varphi : \mathbb{R}^{l_1+ml_2} \to \mathbb{R}$ is the objective function of the leader and $\kappa \subset \mathbb{R}^{l_1+ml_2}$ is a nonempty and closed set of constraints. For the feasible strategy x, let the set of solutions to the lower-level equilibrium problem, denoted by $S(x)$, be closed.

Definition 2.1. *(solution to abstract MPEC)*
An admissible multistrategy vector $(\bar{x}, \bar{y}) \in \mathbb{R}^{l_1+ml_2}$ *is a* solution to an abstract MPEC *if* $(\bar{x}, \bar{y})$ *is a solution to the following optimization problem*

$$\begin{aligned} \mathop{minimize}_{x,y} \quad & \varphi(x, y) \\ subject\ to \quad & y \in S(x), \\ & (x, y) \in \kappa. \end{aligned} \tag{2.1}$$

The solution to the lower problem represents an equilibrium condition and $S(x)$ specifies the set of such equilibria. This is the reason for the term "equilibrium constraints" in MPEC.

Note that the minimization in mathematical program (2.1) is considered in both variables, x and y, and hence we implicitly assume the so-called *optimistic* (or weak) *formulation* of MPEC. By the term optimistic we mean that whenever the lower problem has multiple solutions for a given x, the lower-level players choose one of the "best" in the sense that it minimizes the upper-level objective for a fixed x. We can explicitly express this in the reformulation of (2.1) to

$$\underset{x}{\text{minimize}} \quad \varphi^o(x), \tag{2.2}$$

where

$$\varphi^o(x) := \inf\{\varphi(x,y) \mid y \in S(x), (x,y) \in \kappa\}. \tag{2.3}$$

In a similar way we can obtain a *pessimistic* (or strong) *formulation*, assuming that the lower-level players choose one of the "worst" multistrategies with respect to the upper-level objective when multiple options are possible. Replacing "inf" by "sup" in (2.3) hence results in a "min-max" formulation of MPEC.

Observe that we can equivalently rewrite the constraints in (2.1) in a compact form

$$(x,y) \in \kappa \cap \operatorname{Gph} S.$$

The set $\kappa \cap \operatorname{Gph} S$ is hence called the *feasible region of MPEC* (2.1).

Since the mathematical program (2.1) is generally nonconvex due to its hierarchical structure, in order to guarantee the existence of its solution we need to impose additional restrictions on the data.

Theorem 2.2. *Let φ be lower semicontinuous,* Gph S *be closed and there exist a constant $c \in \mathbb{R}$ such that the set*

$$\Xi_c = \{(x,y) \in \kappa \cap \operatorname{Gph} S \mid \varphi(x,y) \leq c\}$$

is nonempty and bounded. Then MPEC (2.1) possesses a solution.

Proof. The existence of solution is due to the classical Bolzano-Weierstrass theorem. For details, see [39, Proposition 1.1]. □

Let $\kappa = U \times \mathbb{R}^{ml_2}$, where U is a closed set of feasible strategies of the leader and let $V^1, \ldots, V^m \subset \mathbb{R}^{l_2}$ denote closed convex sets of admissible strategies of followers. Let $f^j : \mathbb{R}^{l_1+ml_2} \to \mathbb{R}, j = 1, \ldots, m$, denote the individual objective of the jth follower and assume that for each $j = 1, \ldots, m$, the objectives f^j are continuously differentiable on an open set containing $U \times \Omega$, where $\Omega := \times_{j=1}^m V^j$. Finally, define

$$F(x,y) := \begin{pmatrix} \nabla_{y^1} f^1(x,y) \\ \vdots \\ \nabla_{y^m} f^m(x,y) \end{pmatrix}.$$

Then we can replace the equilibrium constraint $y \in S(x)$ in (2.1) by the equivalent generalized equation

$$0 \in F(x,y) + N(y;\Omega). \tag{2.4}$$

Thus an admissible multistrategy vector $(\bar{x},\bar{y}) \in \mathbb{R}^{l_1+ml_2}$ is a solution to MPEC if $(\bar{x},\bar{y})$ is a solution to the following optimization problem

$$\begin{aligned} \underset{x,y}{\text{minimize}} \quad & \varphi(x,y), \\ \text{subject to} \quad & 0 \in F(x,y) + N(y;\Omega), \\ & x \in U. \end{aligned} \tag{2.5}$$

This particular problem belongs to a broad subclass of problems of MPECs (2.1) with the *solution map* in the form

$$S(x) = \{y \in \mathbb{R}^{ml_2} | 0 \in f(x,y) + Q(x,y)\}$$

with function $f : \mathbb{R}^{l_1+ml_2} \to \mathbb{R}^{ml_2}$ and multifunction $Q : \mathbb{R}^{l_1+ml_2} \rightrightarrows \mathbb{R}^{ml_2}$. Mathematical program

$$\begin{aligned} \underset{x,y}{\text{minimize}} \quad & \varphi(x,y) \\ \text{subject to} \quad & 0 \in f(x,y) + Q(x,y), \\ & (x,y) \in \kappa \end{aligned} \tag{2.6}$$

covers optimization problems constrained by classical variational inequalities and complementarity problems. In this thesis we are particularly interested in the latter, i.e., the subclass of MPECs given by the mathematical programs

$$\begin{aligned} \underset{x,y}{\text{minimize}} \quad & \varphi(x,y) \\ \text{subject to} \quad & 0 \le F^1(x,y) \perp F^2(x,y) \ge 0, \\ & x \in U, \end{aligned} \tag{2.7}$$

with functions $F^1, F^2 : \mathbb{R}^{l_1+ml_2} \to \mathbb{R}^{ml_2}$ continuously differentiable on an open set containing $U \times \mathbb{R}^{ml_2}$. To emphasize the presence of complementarity constraints, we refer to (2.7) as to the *mathematical program with complementarity constraints* (MPCC).

For a deeper insight to the analysis of MPECs and MPCCs, we refer the readers to the monographs [25], [39] and [30].

Another class of hierarchical problems with one upper-level player are *bilevel programs*. These problems are characterized by the lower problem in the form of optimization problem

$$\begin{aligned} \underset{y}{\text{minimize}} \quad & f(x,y) \\ \text{subject to} \quad & y \in V(x) \end{aligned} \tag{2.8}$$

with the solution map

$$S(x) = \underset{y \in V(x)}{\arg\min} f(x,y).$$

Note that bilevel programs constitute a subclass of MPECs in the sense of Definition 2.1. Thus just like in the case of an abstract MPEC, if the solution to the lower-level problem is not unique, the upper-level objective function is not well determined and hence the problem is *ill-possed*. The optimistic reformulation is the usual way how to overcome this ill-possedness.

On the other hand, MPEC (2.6) can be understood as the generalization of a bilevel program only when the lower problem is replaced by its necessary and sufficient optimality conditions, either represented by the generalized equation, variational inequality or Karush-Kuhn-Tucker (KKT) conditions in the form of complementarity problem, entering the upper problem as constraints. Note, that this is possible if the problem (2.8) is convex and also some constraint qualification, e.g, Slater constraint qualification, is satisfied. Otherwise, one can detect stationary points which are not even feasible in the original bilevel program.

A bilevel program is in turn a special case of a hierarchical mathematical program which possesses multiple levels of optimization. Such multilevel mathematical programs are useful in modeling of hierarchical decision making processes and optimization of engineering designs, see [25, Chapter 1.2] and references therein.

Though on the first glance it might look appealing, the equivalent reformulation of (2.8) to the form

$$\begin{aligned} z &\in V(x), \\ f(x,z) &\leq \inf \{ f(x,y) \mid y \in V(x) \} \end{aligned} \tag{2.9}$$

does not ease the investigation of the bilevel problems. This is due to the fact that the second constraint in (2.9) does not satisfy any constraint qualification. For more on this subject, see early work [35], a recent paper [14] and the references therein. For other relations between bilevel programs or MPECs and other well-known optimization problems, solution algorithms and applications, see [13] and the references therein.

2.2 Necessary optimality conditions via nonlinear programming

Some MPECs can be converted to the following form

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \varphi(x,y) \\ \text{subject to} \quad & y = S(x), \\ & x \in U. \end{aligned} \tag{2.10}$$

Assume that $\varphi : \mathbb{R}^{l_1+ml_2} \to \mathbb{R}$ and $S : \mathbb{R}^{l_1} \to \mathbb{R}^{ml_2}$ are locally Lipschitz continuous functions and that $U \subset \mathbb{R}^{l_1}$ is a closed set. Then, if we set $h(x) := \varphi \circ \Phi(x)$ with

$$\Phi(x) := \begin{pmatrix} x \\ S(x) \end{pmatrix},$$

the MPEC (2.10) turns out to be a nonlinear program (NLP)

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & h(x) \\ \text{subject to} \quad & x \in U, \end{aligned} \tag{2.11}$$

where $h : \mathbb{R}^{l_1} \to \mathbb{R}$ is locally Lipschitz continuous function. If $\bar{x}$ is a local minimizer of (2.11), then one has

$$0 \in \partial h(\bar{x}) + N(\bar{x}; U). \tag{2.12}$$

Using the formula for upper approximation of limiting subdifferential of composite function, the necessary optimality conditions for MPEC (2.10) are as follows.

Theorem 2.3. *Let $(\bar{x}, \bar{y})$ be a local minimizer of (2.10). Then there exist vectors $(u^*, v^*) \in \partial\varphi(x, y)$ such that*

$$0 \in u^* + D^*S(\bar{x})(v^*) + N(\bar{x}; U). \tag{2.13}$$

Proof. For proof see [38, Theorem 1.6]. □

From now on, assume φ to be continuously differentiable. Thus the generalized equation (2.13) attains the form

$$0 \in \nabla_x \varphi(\bar{x}, \bar{y}) + D^*S(\bar{x})(\nabla_y \varphi(\bar{x}, \bar{y})) + N(\bar{x}; U). \tag{2.14}$$

In MPECs, the set U has frequently implicit structure and hence to obtain necessary conditions in terms of the original data of the problem one needs to use the chain rule to compute upper approximation of $N(\bar{x}; U)$ under suitable constraint qualification.

In accordance with nonlinear programming, the generalized equation (2.14) defines a natural *stationary concept*. However, in most cases we may not be able to compute the coderivative $D^*S(\bar{x})(\nabla_y \varphi(\bar{x}, \bar{y}))$ exactly. Then we have to confine ourself with its upper approximation and thus weaker stationarity conditions.

For S locally single-valued around $\bar{x}$ and locally Lipschitz, one such possible upper approximation can be $(\bar{\partial} S(\bar{x})^\top \nabla_y \varphi(\bar{x}, \bar{y})$ or even its upper approximation. Clearly, this leads to still weaker stationarity conditions.

2.3 Mathematical program with complementarity constraints

Let us take a closer look at the mathematical program (2.7). Note that for a special case when $F^2(x, y) := y$, MPCC attains the form of (2.5) with $\Omega = \mathbb{R}^{ml_2}_+$ since

$$S(x) = \{y \in \mathbb{R}^{ml_2} | 0 \le F^1(x, y) \perp F^2(x, y) \ge 0\} \tag{2.15}$$

$$= \{y \in \mathbb{R}^{ml_2} | 0 \in F^1(x, y) + N(F^2(x, y); \mathbb{R}^{ml_2}_+)\}. \tag{2.16}$$

There are also other ways how to express the solution map S which assigns $x \in \mathbb{R}^{l_1}$ the solution set of the *nonlinear complementarity problem* (NCP)

$$\begin{aligned} &\text{find} && y \\ &\text{such that} && 0 \le F^1(x,y) \perp F^2(x,y) \ge 0, \end{aligned} \tag{2.17}$$

e.g., via the so-called *Pang NCP function*

$$S(x) = \{y \in \mathbb{R}^{ml_2} | 0 = \min\{F_i^1(x,y), F_i^2(x,y)\},\ i = 1, \ldots, ml_2\}, \tag{2.18}$$

or using the graph of normal cone mapping

$$S(x) = \left\{ y \in \mathbb{R}^m | 0 \in \begin{pmatrix} F^2(x,y) \\ -F^1(x,y) \end{pmatrix} \in \ \mathrm{Gph} N(\cdot; \mathbb{R}^m_+) \right\}. \tag{2.19}$$

Another possibility is to work with an enhanced version of the solution map, S^e, in which we introduce extra variable $\nu = F^1(x,y)$ and obtain

$$S^e(x) = \left\{ (y,\nu) \in \mathbb{R}^m \times \mathbb{R}^m | 0 \in \begin{pmatrix} F^1(x,y) - \nu \\ F^2(x,y) \end{pmatrix} + N(y,\nu; \mathbb{R}^m \times \mathbb{R}^m_+) \right\}. \tag{2.20}$$

The multifunction S^e is related to the the solution map S by the following relationship

$$S^e(x) = \begin{pmatrix} S(x) \\ F^1(x, S(x)) \end{pmatrix}.$$

2.3.1 Stationarity conditions for MPCCs

We can look at the MPCC (2.7) as a special constrained mathematical program having additionally to a general constraint set U also finitely many functional constraints of inequality and equality types. From this perspective we can work with a whole class of stationary concepts for MPCCs which are centered around Lagrange function. For obvious reasons these are sometimes called *KKT-type stationarity concepts*.

First, let us introduce the sets of indices related to activities of constraints in complementarity problem (2.17) at $(\bar{x}, \bar{y})$

$$\begin{aligned} I^+(\bar{x},\bar{y}) &= \{i \in \{1,\ldots,ml_2\} | F_i^1(\bar{x},\bar{y}) > 0, F_i^2(\bar{x},\bar{y}) = 0\}, \\ L(\bar{x},\bar{y}) &= \{i \in \{1,\ldots,ml_2\} | F_i^1(\bar{x},\bar{y}) = 0, F_i^2(\bar{x},\bar{y}) > 0\}, \\ I^0(\bar{x},\bar{y}) &= \{i \in \{1,\ldots,ml_2\} | F_i^1(\bar{x},\bar{y}) = 0, F_i^2(\bar{x},\bar{y}) = 0\}. \end{aligned}$$

If there is no doubt about the reference point, we write only I^+, L and I^0. The index set I^0 is usually called the *index set of biactive inequality constraints*. For brevity, we denote $a^+ = |I^+(\bar{x},\bar{y})|$ and $a^0 = |I^0(\bar{x},\bar{y})|$.

Consider the following auxiliary nonlinear program

$$\begin{aligned} &\underset{x,y}{\text{minimize}} && \varphi(x,y) \\ &\text{subject to} && F^1(x,y) \ge 0, \ \ F^2(x,y) \ge 0, \\ &&& x \in U, \end{aligned} \tag{2.21}$$

which results from the MPCC (2.7) by ignoring the complementarity structure of constraints. The first order optimality conditions of the NLP (2.21) are as follows:
There exist multipliers (λ^1, λ^2) and a vector $\xi \in N(x;\Omega)$ such that

$$\begin{aligned}
&0 = \nabla_x \varphi(x,y) - \sum_{i=1}^{ml_2} \lambda_i^1 \nabla_x F_i^1(x,y) - \sum_{i=1}^{ml_2} \lambda_i^2 \nabla_x F_i^2(x,y) + \xi,\\
&0 = \nabla_y \varphi(x,y) - \sum_{i=1}^{ml_2} \lambda_i^1 \nabla_y F_i^1(x,y) - \sum_{i=1}^{ml_2} \lambda_i^2 \nabla_y F_i^2(x,y),\\
&0 \le F^1(x,y) \perp \lambda^1 \ge 0,\\
&0 \le F^2(x,y) \perp \lambda^2 \ge 0,\\
&\xi \in N(x;U).
\end{aligned} \tag{2.22}$$

Set $G(x,y) = (F^1(x,y))^\top F^2(x,y)$. Then similarly to the conditions above, the first order optimality conditions of the MPCC (2.7) are given by:
There exist multipliers $(\lambda^1, \lambda^2, \lambda^G)$ and a vector $\xi \in N(x;\Omega)$ such that

$$\begin{aligned}
&0 = \nabla_x \varphi(x,y) - \sum_{i=1}^{ml_2} \lambda_i^1 \nabla_x F_i^1(x,y) - \sum_{i=1}^{ml_2} \lambda_i^2 \nabla_x F_i^2(x,y) - \lambda^G \nabla_x G(x,y) + \xi,\\
&0 = \nabla_y \varphi(x,y) - \sum_{i=1}^{ml_2} \lambda_i^1 \nabla_y F_i^1(x,y) - \sum_{i=1}^{ml_2} \lambda_i^2 \nabla_y F_i^2(x,y) - \lambda^G \nabla_y G(x,y),\\
&G(x,y) = 0\\
&F^1_{L\cup I^0}(x,y) = 0, \quad F^1_{I^+}(x,y) > 0,\\
&F^2_{I^+\cup I^0}(x,y) = 0, \quad F^2_{L}(x,y) > 0,\\
&\lambda^1_{I^+} = 0, \quad \lambda^1_{I^0} \ge 0,\\
&\lambda^2_{L} = 0, \quad \lambda^2_{I^0} \ge 0,\\
&\xi \in N(x;U).
\end{aligned} \tag{2.23}$$

Now, since

$$(\nabla G(x,y))^\top = F^1(x,y)^\top \nabla F^2(x,y) + F^2(x,y)^\top \nabla F^1(x,y),$$

let us rearrange conditions (2.23), setting

$$\lambda_L^{F^1} = \lambda_L^1 + \lambda^G F_L^2(x,y), \qquad \lambda_{I^+}^{F^2} = \lambda_{I^+}^2 + \lambda^G F_{I^+}^1(x,y), \tag{2.24}$$

$$\lambda_{I^+\cup I^0}^{F^1} = \lambda_{I^+\cup I^0}^1, \qquad \lambda_{L\cup I^0}^{F^2} = \lambda_{L\cup I^0}^2. \tag{2.25}$$

Due to the nature of index sets I^+, L and I^0, this yields the following representation of the first order optimality conditions:

There exist multipliers $(\lambda^{F^1}, \lambda^{F^2})$ and a vector $\xi \in N(x;\Omega)$ such that

$$
\begin{aligned}
&0 = \nabla_x \varphi(x,y) - \sum_{i \in L \cup I^0} \lambda_i^{F^1} \nabla_x F_i^1(x,y) - \sum_{i \in I^+ \cup I^0} \lambda_i^{F^2} \nabla_x F_i^2(x,y) + \xi, \\
&0 = \nabla_y \varphi(x,y) - \sum_{i \in L \cup I^0} \lambda_i^{F^1} \nabla_y F_i^1(x,y) - \sum_{i \in I^+ \cup I^0} \lambda_i^{F^2} \nabla_y F_i^2(x,y), \\
&\lambda_{I^0}^{F^1} \geq 0, \quad \lambda_{I^0}^{F^2} \geq 0, \\
&\xi \in N(x;U).
\end{aligned}
\tag{2.26}
$$

Following the terminology coined in [45], the conditions (2.26) are called *strong stationarity conditions.* The investigation of MPCCs gave rise to a whole series of stationary concepts tailored to MPCCs. Their respective conditions differ only in requirements imposed on vectors $\lambda_{I^0}^{F^1}$ and $\lambda_{I^0}^{F^2}$. In this respect, the weakest stationarity concept involves no restrictions on biactive multipliers.

Definition 2.4. *(weakly, C-, M- and strongly stationary point)*
Let $(\bar{x}, \bar{y})$ be feasible for the MPCC (2.7). Then we call the point $(\bar{x}, \bar{y})$

i) weakly stationary *(or critical) if there exist multipliers $(\lambda^{F^1}, \lambda^{F^2})$ and a normal $\xi \in N(x;U)$ such that the conditions*

$$
\begin{aligned}
&0 = \nabla_x \varphi(\bar{x},\bar{y}) - \sum_{i \in L \cup I^0} \lambda_i^{F^1} \nabla_x F_i^1(\bar{x},\bar{y}) - \sum_{i \in I^+ \cup I^0} \lambda_i^{F^2} \nabla_x F_i^2(\bar{x},\bar{y}) + \xi, \\
&0 = \nabla_y \varphi(\bar{x},\bar{y}) - \sum_{i \in L \cup I^0} \lambda_i^{F^1} \nabla_y F_i^1(\bar{x},\bar{y}) - \sum_{i \in I^+ \cup I^0} \lambda_i^{F^2} \nabla_y F_i^2(\bar{x},\bar{y}),
\end{aligned}
\tag{2.27}
$$

are satisfied.

ii) C-stationary *if it is a weakly stationary point and, additionally, $\lambda_i^{F^1} \lambda_i^{F^2} \geq 0$ for all $i \in I^0$.*

iii) M-stationary *if it is a weakly stationary point and, additionally, either $\lambda_i^{F^1} > 0$ and $\lambda_i^{F^2} > 0$, or $\lambda_i^{F^1} \lambda_i^{F^2} = 0$ for all $i \in I^0$.*

iv) strongly stationary *if it is a weakly stationary point and, additionally, $\lambda_{I^0}^{F^1} \geq 0$, $\lambda_{I^0}^{F^2} \geq 0$.*

In the above definition, "M" and "C" stands for Mordukhovich and Clarke, respectively. Note that if $I^0 = \emptyset$, i.e., in the (lower-level) *strict complementarity case*, strong, M-, C- and weak stationarity concepts coincide. Also, the restrictions imposed upon biactive multipliers directly result in the following chain of implications

$$\text{strong stationarity} \Rightarrow \text{M-stationarity} \Rightarrow \text{C-stationarity} \Rightarrow \text{weak stationarity.}$$

Clearly, Slater constraint qualification can never hold at any feasible point of (2.7). It is well known that at any feasible point, also linear independence constraint qualification (LICQ) or Mangasarian-Fromowitz constraint qualification (MFCQ) are violated.

This phenomenon is closely related to the geometry of the complementarity structure of constraints and results in the unbounded set of Lagrangian multipliers. This leaves the conventional numerical optimization methods with a possibility of failure of convergence to a solution.

In [45] one can find suitable variants of both LICQ and MFCQ for MPCCs with geometric constraints given by finitely many functional constraints of the inequality and equality types. Then we say that the MPCC (2.7) satisfies the *MPEC linear independence constraint qualification* (MPEC-LICQ) and the *MPEC Mangasarian-Fromowitz constraint qualification* (MPEC-MFCQ) at a feasible point $(\bar{x}, \bar{y})$ if the auxiliary nonlinear program

$$\begin{aligned} \underset{x,y}{\text{minimize}} \quad & \varphi(x,y) \\ \text{subject to} \quad & F^1_{L\cup I^0}(x,y) = 0, \;\; F^1_{I^+}(x,y) \geq 0, \\ & F^2_{I^+\cup I^0}(x,y) = 0, \;\; F^2_L(x,y) \geq 0, \\ & x \in U \end{aligned} \tag{2.28}$$

satisfies LICQ and MFCQ at $(\bar{x}, \bar{y})$, respectively. The feasible region of the NLP (2.28) is a subset of the feasible region of the MPCC (2.7) locally around $(\bar{x}, \bar{y})$. So, every minimizer of the MPCC is also a local minimizer of the corresponding NLP (2.28). This is the reason why this program is called *tightened nonlinear program* (TNLP). Note that there is a whole list of constraint qualifications tailored specifically to MPCCs, with MPEC-LICQ and MPEC-MFCQ among the strongest ones, cf. [18].

However, unlike in [45] or [18], we do not impose at this point any structural requirements on the set U of geometric constraints, thus we need to work with *generalized* versions of the respective constraint qualifications.

Definition 2.5. *(MPEC generalized LICQ and MFCQ)*
The MPCC (2.7) is said to satisfy

i) the MPEC generalized LICQ *(MPEC-GLICQ) at a feasible point $(\bar{x}, \bar{y})$ if the relation*

$$\begin{pmatrix} (\nabla_x F^2_{I^+\cup I^0}(\bar{x},\bar{y}))^\top & (\nabla_x F^1_{L\cup I^0}(\bar{x},\bar{y}))^\top \\ (\nabla_y F^2_{I^+\cup I^0}(\bar{x},\bar{y}))^\top & (\nabla_y F^1_{L\cup I^0}(\bar{x},\bar{y}))^\top \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \in \begin{pmatrix} -N(\bar{x}; U) \\ 0 \end{pmatrix} \tag{2.29}$$

with $(\tilde{u}, \tilde{v}) \in \mathbb{R}^{a^+ + a^0} \times \mathbb{R}^{ml_2 - a^+}$ implies $(\tilde{u}, \tilde{v}) = 0$.

ii) the MPEC generalized MFCQ *(MPEC-GMFCQ) at a feasible point $(\bar{x}, \bar{y})$ if the relation (2.29) with $(\tilde{u}, \tilde{v}) \in \mathbb{R}^{a^+ + a^0} \times \mathbb{R}^{ml_2 - a^+}$ such that for each $i \in I^0$ either $\tilde{u}_i \tilde{v}_i = 0$ or $\tilde{u}_i < 0$ and $\tilde{v}_i < 0$, implies $(\tilde{u}, \tilde{v}) = 0$.*

Note that $0 \in N(\bar{x}; U)$, hence (2.29) implies in particular

$$\left(\nabla F^2_{I^+\cup I^0}(\bar{x},\bar{y}))^\top \tilde{u} + \nabla F^1_{L\cup I^0}(\bar{x},\bar{y}))^\top \tilde{v} = 0, \;\; (\tilde{u}, \tilde{v}) \in \mathbb{R}^{a^+ + a^0} \times \mathbb{R}^{ml_2 - a^+} \right) \Rightarrow (\tilde{u}, \tilde{v}) = 0.$$

This is, however, true only if all the gradient vectors $\nabla F^1_i(\bar{x},\bar{y}), \nabla F^2_j(\bar{x},\bar{y}), i \in I^+ \cup I^0, j \in L \cup I^0$ are linearly independent. Thus MPEC-GLICQ is a proper generalization of linear independence constraint qualification for MPCCs. Clearly, MPEC-GLICQ implies MPEC-GMFCQ, since the latter restricts the values of $(\tilde{u}, \tilde{v})$.

It turns out that MPEC-GMFCQ is just strong enough for M-stationarity conditions to be necessary optimality conditions. The following theorem is a modified version of [36, Theorem 3.1] where the statement is proved for the MPEC (2.5) with $\Omega = \mathbb{R}^{ml_2}_+$.

Theorem 2.6. *Let $(\bar{x}, \bar{y})$ be a local minimizer of the MPCC (2.7). If MPEC-GMFCQ holds at $(\bar{x}, \bar{y})$ then there exist multipliers $\lambda^{F^1}, \lambda^{F^2}$ and $\xi \in N(\bar{x}; U)$ such that (2.27) hold and either $\lambda^{F^1}_i > 0$ and $\lambda^{F^2}_i > 0$, or $\lambda^{F^1}_i \lambda^{F^2}_i = 0$ for all $i \in I^0$. In particular, $(\bar{x}, \bar{y})$ is M-stationary.*

Proof. When MPEC-GMFCQ holds we can compute an upper approximation of the normal cone to the feasible region

$$\left\{ (x,y) \in U \times \mathbb{R}^{ml_2} | \begin{pmatrix} F^2(x,y) \\ -F^1(x,y) \end{pmatrix} \in \text{Gph } N(\cdot; \mathbb{R}^{ml_2}_+) \right\}.$$

Recalling the first order necessary optimality conditions for nonlinear programs, $(\bar{x}, \bar{y})$ thus satisfies conditions

$$\begin{aligned} 0 \in & \nabla\varphi(\bar{x}, \bar{y}) + \\ & + \begin{pmatrix} (\nabla_x F^2(\bar{x}, \bar{y})^\top & -(\nabla_x F^1(\bar{x}, \bar{y})^\top \\ (\nabla_y F^2(\bar{x}, \bar{y})^\top & -(\nabla_y F^1(\bar{x}, \bar{y})^\top \end{pmatrix} N(F^2(\bar{x}, \bar{y}), -F^1(\bar{x}, \bar{y}); \text{Gph } N(\cdot, \mathbb{R}^{ml_2}_+)) \\ & + N(\bar{x}; U) \times \{0\}. \end{aligned}$$

Take into account that

$$N(F^2(\bar{x}, \bar{y}), -F^1(\bar{x}, \bar{y}); \text{Gph } N(\cdot, \mathbb{R}^{ml_2}_+)) = \bigtimes_{i=1}^{ml_2} N(F^2_i(\bar{x}, \bar{y}), -F^1_i(\bar{x}, \bar{y}); \text{Gph } N(\cdot, \mathbb{R}_+))$$

and that

$$N(F^2_i(\bar{x}, \bar{y}), -F^1_i(\bar{x}, \bar{y}); \text{Gph } N(\cdot, \mathbb{R}_+)) = \begin{cases} \{0\} \times \mathbb{R}, & i \in L, \\ \mathbb{R} \times \{0\}, & i \in I^+, \\ (\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\}) \cup (\mathbb{R}_- \times \mathbb{R}_+), & i \in I^0. \end{cases}$$

Now, consider arbitrary $(u,v) \in N(F^2(\bar{x}, \bar{y}), -F^1(\bar{x}, \bar{y}); \text{Gph } N(\cdot, \mathbb{R}^{ml_2}_+))$ and set $\lambda^{F^1} := v$ and $\lambda^{F^2} := -u$. Then we arrive exactly at M-stationarity conditions. This completes the proof. □

The M-stationarity conditions are clearly the proper counterpart of Mordukhovich stationarity known from nonlinear programming, hence the choice for the name of the stationarity concept.

To prove directly that under MPEC-GLICQ local minimizers of (2.7) are C-stationary, one just needs to properly modify [45, Lemma 1], although this statement follows from Theorem 2.6. We present here the respective modification because we will use partial results from the proof later in the text.

Theorem 2.7. *Let $(\bar{x}, \bar{y})$ be a local minimizer of the MPCC (2.7). If MPEC-GLICQ holds at $(\bar{x}, \bar{y})$ then there exist multipliers $\lambda^{F^1}, \lambda^{F^2}$ and $\xi \in N(\bar{x}; U)$ such that conditions (2.27) hold and $\lambda_i^{F^1}\lambda_i^{F^2} \geq 0$ for all $i \in I^0$. In particular, $(\bar{x}, \bar{y})$ is C-stationary.*

Proof. Let us rewrite the MPCC (2.7) as

$$\begin{aligned} \text{minimize} \quad & \varphi(x, y) \\ \text{subject to} \quad & 0 = \min\{F_i^1(x, y), F_i^2(x, y)\}, \quad i = 1, \ldots, ml_2, \\ & x \in U. \end{aligned}$$

From [30, Theorem 5.19 (ii)] and [29, Theorem 3.36] we get the following version of *Fritz John conditions*. There exist multipliers $r \geq 0, \lambda_i^{\min}, i = 1, \ldots, ml_2$, not all zero, and $\xi \in N(\bar{x}; U)$ such that

$$\begin{aligned} 0 &= r\nabla_x\varphi(\bar{x}, \bar{y}) + \sum_{i=1}^{ml_2} \lambda_i^{\min} c_i + \xi, \\ 0 &= r\nabla_y\varphi(\bar{x}, \bar{y}) + \sum_{i=1}^{ml_2} \lambda_i^{\min} d_i, \end{aligned} \tag{2.30}$$

with

$$(c_i, d_i) \in \bar{\partial} \min\{F_i^1(x, y), F_i^2(x, y)\} = \begin{cases} \nabla F_i^1(\bar{x}, \bar{y}), & i \in L, \\ \operatorname{conv}\{\nabla F_i^1(\bar{x}, \bar{y}), \nabla F_i^2(\bar{x}, \bar{y})\}, & i \in I^0, \\ \nabla F_i^2(\bar{x}, \bar{y}), & i \in I^+. \end{cases}$$

For every $i \in I^0$ there is $\alpha_i \in [0, 1]$ such that

$$\begin{aligned} c_i &= \alpha_i \nabla_x F^1(\bar{x}, \bar{y}) + (1 - \alpha_i)\nabla_x F^2(\bar{x}, \bar{y}), \\ d_i &= \alpha_i \nabla_y F^1(\bar{x}, \bar{y}) + (1 - \alpha_i)\nabla_y F^2(\bar{x}, \bar{y}). \end{aligned}$$

Set

$$\lambda_i^{F^1} = \begin{cases} -\lambda_i^{\min}, & i \in L, \\ -\alpha_i\lambda_i^{\min}, & i \in I^0, \\ 0, & i \in I^+, \end{cases}$$

$$\lambda_i^{F^2} = \begin{cases} 0, & i \in L, \\ (1 - \alpha_i)\lambda_i^{\min}, & i \in I^0, \\ -\lambda_i^{\min}, & i \in I^+. \end{cases}$$

Then, since $\alpha_i \in [0, 1]$, we have $\lambda_i^{F^1}\lambda_i^{F^2} = \alpha_i(1 - \alpha_i)(\lambda_i^{\min})^2 \geq 0$ for each $i \in I^0$.

This results in the following conditions which differ from C-stationarity conditions only in the presence of a nonnegative multiplier r.

$$\begin{aligned}
&0 = r\nabla_x\varphi(\bar{x},\bar{y}) - \sum_{i\in L\cup I^0} \lambda_i^{F^1}\nabla_x F_i^1(\bar{x},\bar{y}) - \sum_{i\in I^+\cup I^0} \lambda_i^{F^2}\nabla_x F_i^2(\bar{x},\bar{y}) + \xi,\\
&0 = r\nabla_y\varphi(\bar{x},\bar{y}) - \sum_{i\in L\cup I^0} \lambda_i^{F^1}\nabla_y F_i^1(\bar{x},\bar{y}) - \sum_{i\in I^+\cup I^0} \lambda_i^{F^2}\nabla_y F_i^2(\bar{x},\bar{y}),\\
&\lambda_i^{F^1}\lambda_i^{F^2} \geq 0, \quad i\in I^0,\\
&\xi \in N(\bar{x};U).
\end{aligned} \tag{2.31}$$

Assume now, that $r = 0$. Then the first two lines of (2.31) may be written as

$$\begin{aligned}
-\xi &= -\sum_{i\in L\cup I^0} \lambda_i^{F^1}\nabla_x F_i^1(\bar{x},\bar{y}) - \sum_{i\in I^+\cup I^0} \lambda_i^{F^2}\nabla_x F_i^2(\bar{x},\bar{y}),\\
0 &= -\sum_{i\in L\cup I^0} \lambda_i^{F^1}\nabla_y F_i^1(\bar{x},\bar{y}) - \sum_{i\in I^+\cup I^0} \lambda_i^{F^2}\nabla_y F_i^2(\bar{x},\bar{y}).
\end{aligned}$$

Setting $\tilde{u} = -\lambda_{I^+\cup I^0}^{F^2}$ and $\tilde{v} = -\lambda_{L\cup I^0}^{F^1}$, from MPEC-GLICQ we get $\lambda_{I^+\cup I^0}^{F^2} = \lambda_{L\cup I^0}^{F^1} = 0$. This implies also $\lambda_i^{\min} = 0$ for all $i = 1,\dots,ml_2$. The latter is, of course, a contradiction to the statement that multipliers $r \geq 0, \lambda_i^{\min}, i = 1,\dots,ml_2$, are not all simultaneously zero.

Hence, $r \neq 0$ and scaling yields $r = 1$. This completes the proof. □

It turns out that to prove the above statement directly, MPEC-GMFCQ is insufficient to prevent the case of vanishing multiplier r. Nevertheless, recall that M-stationarity implies C-stationarity, hence MPEC-GMFCQ implies C-stationarity of local minimizers. This is the statement of the following corollary.

Corollary 2.8. *Let $(\bar{x},\bar{y})$ be local minimizer of MPEC (2.7). If MPEC-GMFCQ holds at $(\bar{x},\bar{y})$ then there exist multipliers $\lambda^{F^1},\lambda^{F^2}$ and $\xi \in N(\bar{x};U)$ such that (2.27) hold and $\lambda_i^{F^1}\lambda_i^{F^2} \geq 0$ for all $i\in I^0$. In particular, $(\bar{x},\bar{y})$ is C-stationary.*

Note also that MPEC-GLICQ does not provide uniqueness of multipliers if $U \neq \emptyset$. The following example shows that MPEC-GLICQ can be satisfied and yet there may be at least two different sets of multipliers satisfying C-stationarity conditions.

Example 2.9. Consider an MPCC

$$\begin{aligned}
\underset{x_1,x_2,y}{\text{minimize}} \quad & 2x_1 + 2x_2 + y\\
\text{subject to} \quad & 0 \leq x_1 - x_2 - y \perp y \geq 0,\\
& x_1, x_2 \geq 0.
\end{aligned}$$

at the feasible point $(\bar{x}_1,\bar{x}_2,\bar{y}) = (0,0,0)$. Then conditions

$$\begin{aligned}
u &\geq 0,\\
-u &\geq 0,\\
-u + v &= 0,
\end{aligned}$$

imply $u = v = 0$ and hence MPEC-GLICQ holds. On the other hand one can easily check that there are multiple sets of vectors $(\lambda^{F^1}, \lambda^{F^2}, \xi_1, \xi_2)$ with $(\xi_1, \xi_2) \in \mathbb{R}^2_-$ satisfying the conditions (2.27), e.g., (1,2,-1, -3) or (2,3,0, -4).

Clearly, our reference point is even strongly stationary. In fact, it is the unique global minimizer of our MPCC. △

2.3.2 Implicit programming approach and Clarke stationarity

In this section we consider an alternative approach to MPECs. We are particularly interested in various criteria under which the lower-level complementarity problem locally defines an *implicit function*. Most of the results in this section follow directly from [39], although, for slightly different structure of an MPCC. Using the combination of the calculus of Mordukhovich and of Clarke, however, we derive stronger optimality conditions then in [39]. Only when we believe it is appropriate, we present the the full proof.

Consider the generalized equation (2.4) with the solution map

$$S(x) = \{y \in \mathbb{R}^{ml_2} | 0 \in F(x,y) + N(y;\Omega)\}.$$

In what follows we work with the following condition of Robinson [43] concerning the multivalued map $\Sigma : \mathbb{R}^{ml_2} \rightrightarrows \mathbb{R}^{ml_2}$ generated by partial linearization of $F(\bar{x}, \bar{y})$ in (2.4).

Definition 2.10. *(Strong regularity condition)*
Let $\bar{y} \in S(\bar{x})$. Suppose that there exist neighborhoods $\mathcal{V}$ of $\bar{y}$ and $\mathcal{O}$ of $0 \in \mathbb{R}^{ml_2}$ such that the map $\xi \to \Sigma(\xi) \cap \mathcal{V}$ is single-valued and Lipschitz continuous on $\mathcal{O}$, where

$$\Sigma(\xi) = \{y \in \mathbb{R}^{ml_2} | \xi \in F(\bar{x}, \bar{y}) + \nabla_y F(\bar{x}, \bar{y})(y - \bar{y}) + N(y;\Omega)\}.$$

Then we say that the generalized equation (2.4) is strongly regular *at $(\bar{x}, \bar{y})$ or that at this point the generalized equation (2.4) satisfies the* strong regularity condition *(SRC).*

The strong regularity condition plays an important role in implicit programming mainly due to the following result.

Theorem 2.11. *Let the generalized equation (2.4) be strongly regular $(\bar{x}, \bar{y})$. Then there is a neighborhood $\mathcal{U}$ of $\bar{x}$ and $\mathcal{V}$ of $\bar{y}$ such that the map $\sigma(x) = S(x) \cap \mathcal{V}$ is single-valued and locally Lipschitz continuous on $\mathcal{U}$.*

Proof. For proof see [43]. □

For Ω being a convex polyhedral set we get a useful characterization of the strong regularity condition.

Theorem 2.12. *Let Ω be a convex polyhedron. Then the following statements are equivalent.*

i) The generalized equation (2.4) is strongly regular at $(\bar{x}, \bar{y})$.

ii) The generalized equation

$$\xi \in \nabla_y F(\bar{x},\bar{y})\eta + N(\eta; K(\bar{y} - F(\bar{x},\bar{y}),\bar{y})) \tag{2.32}$$

is single-valued on $\mathbb{R}^{ml_2}$.

Proof. See, e.g., [39, Theorem 5.3]. $\square$

We can apply Theorem 2.12 also to to the underlying generalized equation in (2.20). This enables us to derive rather simple linear algebraic criteria for single-valuedness and Lipschitz behavior of the map σ around $\bar{x}$. Note that the third argument $\bar{\nu}$ of the generalized equation in (2.20) is uniquely determined by $\bar{x}$ and $\bar{y}$ via relation $\bar{\nu} = F^1(\bar{x},\bar{y})$. Thus we can refer just to the point $(\bar{x},\bar{y})$.

If SRC holds at $(\bar{x},\bar{y})$, then there exist neighborhoods $\mathcal{U}$ of $\bar{x}$ and $\mathcal{V}$ of $\bar{y}$ and a Lipschitz continuous map $\sigma : \mathcal{U} \to \mathbb{R}^m \times \mathbb{R}^m$ such that

$$\sigma(\bar{x}) = (\bar{y}, F^1(\bar{x},\bar{y})) \text{ and } \sigma(x) = S^e(x) \cap (\mathcal{V} \times F^1(x,\mathcal{V})) \text{ for all } x \in \mathcal{U}.$$

The map σ can be split into two Lipschitz operators σ_y and σ_ν which correspond, locally around $\bar{x}$, to the $y-$ and $\nu-$component of the solution to the underlying generalized equation in (2.20). Moreover, it suffices to analyze just the operator σ_y since

$$\sigma_\nu(x) = F^1(x, \sigma_y(x)) \text{ for all } x \in \mathcal{U}.$$

The criterion of SRC for the generalized equation in (2.20) is stated in the following theorem.

Theorem 2.13. *Denote by* $Z(x,y)$ *an* $(ml_2 + a^+ + a^0) \times (ml_2 + a^+ + a^0)$ *matrix given by*

$$Z(x,y) = \begin{pmatrix} \nabla_y F^1(x,y) & -E_{I^+}^\top & -E_{I^0}^\top \\ \nabla_y F^2_{I^+}(x,y) & 0 & 0 \\ \nabla_y F^2_{I^0}(x,y) & 0 & 0 \end{pmatrix}.$$

Then the generalized equation in (2.20) is strongly regular at $(\bar{x},\bar{y})$ *if and only if the generalized equation*

$$\xi \in Z(\bar{x},\bar{y})\eta + N(\eta; \mathbb{R}^{ml_2+a^+} \times \mathbb{R}^{a^0}_+)$$

possesses a unique solution η *for all* $\xi \in \mathbb{R}^{ml_2+a^++a^0}$.

Proof. In this case, the generalized equation (2.32) attains the form

$$\xi \in \begin{pmatrix} \nabla_y F^1(\bar{x},\bar{y}) & -E \\ \nabla_y F^2(\bar{x},\bar{y}) & 0 \end{pmatrix} \eta + N(\eta; K)$$

with

$$\begin{aligned} K &= \{(u,v) \in \mathbb{R}^{ml_2} \times \mathbb{R}^{ml_2} | v_{L\cup I^0} \geq 0\} \cap \{(u,v) \in \mathbb{R}^{ml_2} \times \mathbb{R}^{ml_2} | v_L = 0\} \\ &= \{(u,v) \in \mathbb{R}^{ml_2} \times \mathbb{R}^{ml_2} | v_{I^0} \geq 0, v_L = 0\}. \end{aligned}$$

Hence for $(u,v) \in K$ we have

$$N(u,v;K) = \{\eta^* = (u^*,v^*) \in \mathbb{R}^{ml_2} \times \mathbb{R}^{ml_2} | u^* = 0, v^*_{I^+} = 0, 0 \leq -v^*_{I^0} \perp v_{I^0} \geq 0\}.$$

Now observe that the columns of matrix

$$\begin{pmatrix} -E \\ 0 \end{pmatrix}$$

corresponding to inactive inequality constraints of F^2 can be removed because the components v_i vanish for $i \in L$. The same columns of

$$\begin{pmatrix} F^2(\bar{x},\bar{y}), & 0 \end{pmatrix}$$

can be omitted since the components $v^*_i, i \in L$, are free and these rows do not restrict variable η.

This completes the proof. □

The application of [39, Lemma 5.6] to the statement of Theorem 2.13 yields a linear algebraic characterization of the strong regularity of the generalized equation in (2.20) at $(\bar{x},\bar{y})$. Recall that a square matrix A is called a *P-matrix* if all its principal subdeterminants are positive.

Theorem 2.14. *The following statements are equivalent:*

i) The generalized equation in (2.20) is strongly regular at $(\bar{x},\bar{y})$.

ii) The matrix
$$\begin{pmatrix} \nabla_y F^1(\bar{x},\bar{y}) & -E^\top_{I^+} \\ \nabla_y F^2_{I^+}(\bar{x},\bar{y}) & 0 \end{pmatrix}$$
is nonsingular and its Schur complement in $Z(\bar{x},\bar{y})$ *is a P-matrix.*

Proof. The claim follows from [39, Lemma 5.6]. □

Since SRC ensures local single-valuedness of the solution map, we are able to characterize the local properties of S by the *generalized Jacobian* of σ_y at the reference point, or at least by its upper approximation. Provided that locally around $\bar{x}$, the Lipschitz operator σ_y is a PC^1 *function*, the computation of an upper approximation of $\bar{\partial}\sigma_y$ is rather simple.

The continuity of σ_y around $\bar{x}$ provides us with the stability of index sets of active constraints.

Lemma 2.15. *Let SRC hold at* $(\bar{x},\bar{y})$. *Then there is a neighborhood* $\mathcal{U}$ *of* $\bar{x}$ *such that*

$$I^+(\bar{x},\bar{y}) \subset I^+(x,\sigma_y(x)) \quad \text{and} \quad L(\bar{x},\bar{y}) \subset L(x,\sigma_y(x)), \;\; \forall x \in \mathcal{U}.$$

Proof. The proof immediately follows from the continuity of σ_y. □

Hence, in the neighborhood $\mathcal{U}$ of $\bar{x}$ for each point $x \in \mathcal{U}$ there is a subset M of index set I^0 such that

$$\begin{aligned} &F_i^1(x,y) \geq 0, \;\; F_i^2(x,y) = 0 \quad \text{for } i \in I^+ \cup M, \\ &F_i^1(x,y) = 0, \;\; F_i^2(x,y) \geq 0 \quad \text{for } i \in L \cup (I^0 \setminus M). \end{aligned} \tag{2.33}$$

Ignoring the inequalities in (2.33) results in system of nonlinear equations

$$\begin{aligned} &F_i^1(x,y) = 0 \quad \text{for } i \in L \cup (I^0 \setminus M), \\ &F_i^2(x,y) = 0 \quad \text{for } i \in I^+ \cup M. \end{aligned} \tag{2.34}$$

To this system we can apply the classical implicit function theorem, provided the matrix

$$\begin{pmatrix} \nabla_y F^1_{L \cup (I^0 \setminus M)}(\bar{x},\bar{y}) \\ \nabla_y F^2_{I^+ \cup M}(\bar{x},\bar{y}) \end{pmatrix}$$

is nonsingular. However, this is implied by SRC.

Denote the elements of the family $\mathcal{P}(I^0(\bar{x},\bar{y}))$ of all subsets of $I^0(\bar{x},\bar{y})$ by $M_i(\bar{x},\bar{y})$ where indices i run in a suitable index set $\mathbb{K}(\bar{x},\bar{y})$.

Lemma 2.16. *Let SRC hold at $(\bar{x},\bar{y})$. Then*

i) for every $i \in \mathbb{K}(\bar{x},\bar{y})$ the matrix

$$D^i(\bar{x},\bar{y}) = \begin{pmatrix} \nabla_y F^1_{L \cup (I^0 \setminus M_i)}(\bar{x},\bar{y}) \\ \nabla_y F^2_{I^+ \cup M_i}(\bar{x},\bar{y}) \end{pmatrix}$$

is regular and

ii) either

$$\det\, D^i(\bar{x},\bar{y}) > 0 \textit{ for all } i \in \mathbb{K}(\bar{x},\bar{y})$$

or

$$\det\, D^i(\bar{x},\bar{y}) < 0 \textit{ for all } i \in \mathbb{K}(\bar{x},\bar{y}).$$

Proof. From Theorem 2.14 we have

$$\det \begin{pmatrix} \nabla_y F^1(\bar{x},\bar{y}) & -E^\top_{I^+} \\ \nabla_y F^2_{I^+}(\bar{x},\bar{y}) & 0 \end{pmatrix} \neq 0.$$

Clearly, the application of the Laplace's formula for computation of determinants yields

$$\det\, D^1(\bar{x},\bar{y}) := \det \begin{pmatrix} \nabla_y F^1_{L \cup I^0}(\bar{x},\bar{y}) \\ \nabla_y F^2_{I^+}(\bar{x},\bar{y}) \end{pmatrix} \neq 0,$$

where $M_1(\bar{x},\bar{y}) := \emptyset \in \mathcal{P}(I^0(\bar{x},\bar{y}))$.

From Theorem 2.14 we also know that the matrix

$$\begin{pmatrix} \nabla_y F^2_{I^0}(\bar{x},\bar{y}) & 0 \end{pmatrix} \begin{pmatrix} \nabla_y F^1(\bar{x},\bar{y}) & -E^\top_{I^+} \\ \nabla_y F^2_{I^+}(\bar{x},\bar{y}) & 0 \end{pmatrix}^{-1} \begin{pmatrix} E^\top_{I^0} \\ 0 \end{pmatrix} \tag{2.35}$$

is a P-matrix, i.e., each of its $2^{a^0} - 1$ major submatrices (including itself) has a positive determinant. However, for every $i \in \mathbb{K}(\bar{x},\bar{y}) \setminus \{1\}$ there is a major submatrix of (2.35) such that its determinant can be expressed as

$$\frac{\det\ D^i(\bar{x},\bar{y})}{\det\ D^1(\bar{x},\bar{y})}.$$

Hence, the sign of determinants det $D^i(\bar{x},\bar{y})$ for all $i \in \mathbb{K}(\bar{x},\bar{y}) \setminus \{1\}$ is determined by the sign of det $D^1(\bar{x},\bar{y})$. This proves both parts of the lemma. □

As a corollary of Lemma 2.16 i) we have that locally around $\bar{x}$, σ_y is a PC^1 function. Denote by $\sigma_i, i \in \mathbb{K}(\bar{x},\bar{y})$ the implicit functions specified by systems of equations (2.34). An upper approximation of $\bar{\partial}\sigma_y$ then takes the form

$$\bar{\partial}\sigma_y(\bar{x}) \subset \operatorname{conv}\{\nabla\sigma_i(\bar{x}) | i \in \mathbb{K}(\bar{x},\bar{y})\},$$

cf. [46, Proposition A.4.1]. We summarize this in the following theorem.

Theorem 2.17. *Assume that SRC holds at $(\bar{x},\bar{y})$. Then*

$$\bar{\partial}\sigma_y(\bar{x}) \subset \operatorname{conv}\{\mathcal{B}^i(\bar{x},\bar{y}) | i \in \mathbb{K}(\bar{x},\bar{y})\} \tag{2.36}$$

where $\mathcal{B}^i(\bar{x},\bar{y}), i \in \mathbb{K}(\bar{x},\bar{y})$, is a unique solution of the system of equations in Π

$$D^i(\bar{x},\bar{y})\Pi = -\begin{pmatrix} \nabla_x F^1_{L\cup(I^0\setminus M_i)}(\bar{x},\bar{y}) \\ \nabla_x F^2_{I^+\cup M_i}(\bar{x},\bar{y}) \end{pmatrix}.$$

Proof. See [39, Theorem 6.17] □

In order to obtain precise formula for the generalized Jacobian and to replace inclusions with equalities, additional assumptions are needed.

Lemma 2.18. *Let the assumptions of Theorem 2.17 be fulfilled. Assume that $l_1 \geq a^0$ and that every collection of at most $l_1 + ml_2$ rows of the matrix*

$$\begin{pmatrix} \nabla F^1_{L\cup I^0}(\bar{x},\bar{y}) \\ \nabla F^2_{I^+\cup I^0}(\bar{x},\bar{y}) \end{pmatrix}$$

is linearly independent. Then for each $i \in \mathbb{K}(\bar{x},\bar{y})$

$$\mathcal{B}^i(\bar{x},\bar{y}) \in \bar{\partial}\sigma_y(\bar{x}).$$

Proof. See [39, Proposition 6.19]. □

Note that the linear independence condition in the above lemma is implied by MPEC-GLICQ.

For the formulation of the stationary conditions below we use the technique of the so-called *adjoint equations*. This technique works as follows. Consider a vector $q \in \mathbb{R}^m$, matrix $A \in \mathbb{R}^m \times \mathbb{R}^m$ and matrices $P, B \in \mathbb{R}^m \times \mathbb{R}^n$ with $AP = B$. If $\bar{p}$ solves the adjoint equation

$$A^\top p = q,$$

then

$$P^\top q = B^\top \bar{p}.$$

Definition 2.19. *(Clarke stationarity conditions)*
Let $(\bar{x}, \bar{y})$ be a feasible point for the MPCC (2.7) and let SRC hold at $(\bar{x}, \bar{y})$. Then we call $(\bar{x}, \bar{y})$ Clarke stationary *if it satisfies*

$$0 \in \nabla_x \varphi(\bar{x}, \bar{y}) - \operatorname{conv} \left\{ \left(\begin{array}{c} \nabla_x F^1_{L \cup (I^0 \setminus M_i)}(\bar{x}, \bar{y}) \\ \nabla_x F^2_{I^+ \cup M_i}(\bar{x}, \bar{y}) \end{array} \right)^\top p^i(\bar{x}, \bar{y}) | i \in \mathbb{K}(\bar{x}, \bar{y}) \right\} + N(\bar{x}; U), \quad (2.37)$$

where $p^i(\bar{x}, \bar{y})$ are the unique *solutions of*

$$\left(\begin{array}{c} \nabla_y F^1_{L \cup (I^0 \setminus M_i)}(\bar{x}, \bar{y}) \\ \nabla_y F^2_{I^+ \cup M_i}(\bar{x}, \bar{y}) \end{array} \right)^\top p = \nabla_y \varphi(\bar{x}, \bar{y}). \quad (2.38)$$

In the next theorem we show that the strong regularity condition is sufficient for Clarke stationarity conditions (2.37) and (2.38) to be necessary first order optimality conditions.

Theorem 2.20. *Let $(\bar{x}, \bar{y})$ be a local solution of the MPCC (2.7). Let SRC hold at $(\bar{x}, \bar{y})$ and for all $i \in \mathbb{K}(\bar{x}, \bar{y})$ the vectors $p^i(\bar{x}, \bar{y})$ be the unique solutions of (2.38). Then conditions (2.37) are fulfilled. In particular, the point $(\bar{x}, \bar{y})$ is Clarke stationary.*

Proof. The considered MPCC can be on the neighborhood $\mathcal{U}$ of $\bar{x}$ reduced to

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & \varphi(x, y) \\ \text{subject to } & y = \sigma_y(x), \\ & x \in U \cap \mathcal{U}. \end{aligned}$$

From Theorem 2.17 one has

$$\bar{\partial} \sigma_y(\bar{x}) \subset \operatorname{conv}\{\mathcal{B}^i(\bar{x}, \bar{y}) | i \in \mathbb{K}(\bar{x}, \bar{y})\},$$

where $\mathcal{B}^i(\bar{x}, \bar{y}), i \in \mathbb{K}(\bar{x}, \bar{y})$, is a unique solution of linear matrix equation in Π

$$D^i(\bar{x}, \bar{y})\Pi = - \left(\begin{array}{c} \nabla_x F^1_{L \cup (I^0 \setminus M_i)}(\bar{x}, \bar{y}) \\ \nabla_x F^2_{I^+ \cup M_i}(\bar{x}, \bar{y}) \end{array} \right).$$

From [30, Proposition 5.3], the relation between the limiting and the Clarke subdifferentials and generalized Jacobian chain rule [7, Theorem 2.6.6] we get

$$0 \in \nabla_x \varphi(\bar{x},\bar{y}) + \ \mathrm{conv}\left\{(\mathcal{B}^i(\bar{x},\bar{y}))^\top \nabla_y \varphi(\bar{x},\bar{y}) | i \in \mathbb{K}(\bar{x},\bar{y})\right\} + N(\bar{x};U).$$

The application of the technique of adjoint equations completes the proof. □

Let us turn our attention briefly to the MPEC (2.5) with $\Omega = \mathbb{R}^{ml_2}_+$. As mentioned above, this MPEC can be reformulated as an MPCC

$$\begin{aligned} \underset{x,y}{\text{minimize}} \quad & \varphi(x,y) \\ \text{subject to} \quad & 0 \le F(x,y) \perp y \ge 0, \\ & x \in U. \end{aligned} \tag{2.39}$$

Recall that

$$\begin{aligned} I^+(\bar{x},\bar{y}) &= \{i \in \{1,\dots,ml_2\} | F_i(\bar{x},\bar{y}) > 0\}, \\ L(\bar{x},\bar{y}) &= \{i \in \{1,\dots,ml_2\} | y_i > 0\}, \\ I^0(\bar{x},\bar{y}) &= \{i \in \{1,\dots,ml_2\} | F_i(\bar{x},\bar{y}) = y_i = 0\}. \end{aligned} \tag{2.40}$$

For the generalized equation

$$0 \in F(x,y) + N(y;\mathbb{R}^{ml_2}_+), \tag{2.41}$$

the counterpart to Theorem 2.14 attains the following form.

Theorem 2.21. *The following statements are equivalent*

i) The generalized equation (2.41) is strongly regular at $(\bar{x},\bar{y})$.

ii) The matrix $\nabla_y F_{L,L}(\bar{x},\bar{y})$ *is nonsingular and its Schur complement in the matrix*

$$\begin{pmatrix} \nabla_y F_{L,L}(\bar{x},\bar{y}) & \nabla_y F_{L,I^0}(\bar{x},\bar{y}) \\ \nabla_y F_{I^0,L}(\bar{x},\bar{y}) & \nabla_y F_{I^0,I^0}(\bar{x},\bar{y}) \end{pmatrix}$$

is a P-matrix.

Proof. See [39, Theorem 5.9]. □

Clarke stationarity conditions for the MPCC (2.39) under SRC reduce to

$$0 \in \nabla_x \varphi(\bar{x},\bar{y}) - \ \mathrm{conv}\left\{(\nabla_x F_{L\cup(I^0\setminus M_i)}(\bar{x},\bar{y}))^\top p^i(\bar{x},\bar{y}) | i \in \mathbb{K}(\bar{x},\bar{y})\right\} + N(\bar{x};U), \tag{2.42}$$

where $p^i(\bar{x},\bar{y})$ are the unique solutions of

$$(\nabla_y F_{L\cup(I^0\setminus M_i),L\cup(I^0\setminus M_i)}(\bar{x},\bar{y}))^\top p = \nabla_y \varphi_{L\cup(I^0\setminus M_i)}(\bar{x},\bar{y}). \tag{2.43}$$

2.3.3 Equivalence of Clarke and C-stationarity

In this section we will closely investigate the relation between above defined concepts of Clarke stationarity and C-stationarity. Note that these concepts are not defined for the same class of MPCCs. We are not able to work with Clarke stationarity without the assumption of strong regularity. This condition is, unfortunately, insufficient for both concepts to coincide; MPEC-GLICQ needs to be fulfilled as well.

First, notice that MPEC-GLICQ and SRC, both implying extra requirements on the data of complementarity constraints, are generally unrelated conditions even for $U = \mathbb{R}^{l_1}$. We show this by means of simple examples.

Example 2.22. Consider for some objective function $\varphi(x_1, x_2, y_1, y_2)$ the following pair of complementarity constraints

$$\begin{array}{rclccccccccl} 0 & \leq & -x_1 & & +2y_1 & -y_2 & \perp & & & +y_1 & & \geq & 0, \\ 0 & \leq & & -x_2 & -3y_1 & +2y_2 & \perp & -x_1 & -x_2 & & +y_2 & \geq & 0 \end{array}$$

and a reference point $(x_1, x_2, y_1, y_2) = (0, 0, 0, 0)$.

The matrix

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

is regular and

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is a P-matrix. Hence, due to Lemma 2.18, the strong regularity condition holds at (0,0,0,0). On the other hand, the vectors

$$\begin{pmatrix} -1 \\ 0 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

are linearly dependent and hence MPEC-GLICQ is violated. △

Example 2.23. Consider for some objective function $\varphi(x_1, x_2, y_1, y_2)$ the following pair of complementarity constraints

$$\begin{array}{rclccccccl} 0 & \leq & -x_1 & & +y_1 & -2y_2 & \perp & +y_1 & & \geq & 0, \\ 0 & \leq & & -x_2 & -2y_1 & +y_2 & \perp & & +y_2 & \geq & 0 \end{array}$$

and a reference point $(x_1, x_2, y_1, y_2) = (0, 0, 0, 0)$. MPEC-GLICQ is clearly satisfied and

$$\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$$

is a regular matrix. However,

$$-\frac{1}{3} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

is not P-matrix and hence SRC is violated at (0,0,0,0). △

Recall, that the conditions for Clarke stationarity involve an upper approximation of $\bar{\partial}\sigma_y(x)$. If we are able to compute this object precisely (see Lemma 2.18), the computation is invariant to the concrete representation of the solution mapping. If this is not the case, we may end up with different upper approximations.

Assume that the SRC condition is satisfied at $(\bar{x},\bar{y})$ and let us compute an upper approximation of the generalized Jacobian $\bar{\partial}\sigma(\bar{x})$ where σ is given by (2.18). This time we will apply the calculus of generalized differentiation to (2.18).

Denote by $\Phi(x,y)$ the vector mapping such that $\Phi_i(x,y) = \min\{F_i^1(x,y), F_i^2(x,y)\}, i = 1,\dots,ml_2$. To apply [29, Theorem 4.32d] we need to guarantee validity of the following constraint qualification

$$0 \in D^*\Phi(\bar{x},\bar{y})(y^*) \Rightarrow y^* = 0,$$

or equivalently, since Φ is Lipschitz continuous for F^1, F^2 continuously differentiable, using the *scalarization formula* [29, Theorem 3.28],

$$0 \in \partial \langle y^*, \Phi(\bar{x},\bar{y})\rangle \Rightarrow y^* = 0. \tag{2.44}$$

Then

$$\begin{aligned}
D^*\sigma(\bar{x},\bar{y})(y^*) \subset & \bigcup_{u\in\mathbb{R}^{ml_2}} \{x^* | (x^*,-y^*) \in D^*\Phi(\bar{x},\bar{y})(u)\} = \\
= & \bigcup_{u\in\mathbb{R}^{ml_2}} \{x^* | (x^*,-y^*) \in \partial \langle u, \Phi(\bar{x},\bar{y})\rangle\} = \\
= & \bigcup_{u\in\mathbb{R}^{ml_2}} \left\{x^* | (x^*,-y^*) \in \partial \sum_{i=1}^{ml_2} u_i\Phi_i(\bar{x},\bar{y})\right\} \subset \\
\subset & \bigcup_{u\in\mathbb{R}^{ml_2}} \left\{x^* | (x^*,-y^*) \in \sum_{i=1}^{ml_2} \partial(u_i\Phi_i(\bar{x},\bar{y}))\right\},
\end{aligned} \tag{2.45}$$

where the last inclusion is due to [29, Theorem 3.36]. Denote for every $i = 1,\dots,ml_2$,

$$\begin{aligned}
I_i(\bar{x},\bar{y}) &= \{j \in \{1,2\} | F_i^j(\bar{x},\bar{y}) = \Phi_i(\bar{x},\bar{y})\},\\
\Lambda_i(\bar{x},\bar{y}) &= \{(\lambda_i^1,\lambda_i^2) \in \mathbb{R}_+^2 | \lambda_i^1 + \lambda_i^2 = 1, \lambda_i^j(F_i^j(\bar{x},\bar{y}) - \Phi_i(\bar{x},\bar{y})) = 0, j \in \{1,2\}\}.
\end{aligned}$$

Then from [29, Theorem 3.36] we have

$$\partial(u_i\Phi_i(\bar{x},\bar{y})) = \begin{cases} \bigcup_{(\lambda_i^1,\lambda_i^2)\in\Lambda_i(\bar{x},\bar{y})} \left\{\sum_{j\in I_i(\bar{x},\bar{y})} \lambda_i^j u_i \nabla F_i^j(\bar{x},\bar{y})\right\} & u_i \le 0,\\ \bigcup_{j\in I_i(\bar{x},\bar{y})} \{u_i\nabla F_i^j(\bar{x},\bar{y})\} & u_i \ge 0. \end{cases} \tag{2.46}$$

From (2.46), clearly, MPEC-GLICQ implies the constraint qualification (2.44).

Using the above upper approximation (2.45) together with (2.46), we can now show that under SRC and MPEC-GLICQ, Clarke stationarity conditions imply C-stationarity conditions.

Theorem 2.24. *Let $(\bar{x}, \bar{y})$ be a feasible point for the MPCC (2.7) such that SRC and MPEC-GLICQ are satisfied. Then, if $(\bar{x}, \bar{y})$ is a Clarke stationary point of the MPCC, there exist Lagrange multipliers $\lambda^{F^1}, \lambda^{F^2}$ and a normal vector $\xi \in N(\bar{x}; U)$ such that $(\bar{x}, \bar{y}, \lambda^{F^1}, \lambda^{F^2}, \xi)$ satisfies C-stationarity conditions of the MPCC.*

Proof. MPEC-GLICQ imply the linear independence assumption from Lemma 2.18. Hence under SRC we have

$$\bar{\partial}\sigma_y(\bar{x}) = \operatorname{conv}\{\mathcal{B}^i | i \in \mathbb{K}(\bar{x}, \bar{y}\},$$

where $\mathcal{B}^i$ are the unique solutions to the matrix equation in variable Π

$$\begin{pmatrix} \nabla_y F^1_{L\cup(I^0\setminus M_i)}(\bar{x}, \bar{y}) \\ \nabla_y F^2_{I^+\cup M_i}(\bar{x}, \bar{y}) \end{pmatrix} \Pi = - \begin{pmatrix} \nabla_x F^1_{L\cup(I^0\setminus M_i)}(\bar{x}, \bar{y}) \\ \nabla_x F^2_{I^+\cup M_i}(\bar{x}, \bar{y}) \end{pmatrix}.$$

Since we can compute the generalized Jacobian to σ_y at $\bar{x}$ precisely, the Clarke stationary conditions are equivalent to

$$0 \in \nabla_x \varphi(\bar{x}, \bar{y}) + \bar{\partial}\sigma(\bar{x})^\top \nabla_y \varphi(\bar{x}, \bar{y}) + N(x; U). \tag{2.47}$$

Using the relation between coderivatives and Clarke generalized Jacobians for single-valued mappings, which amounts to

$$(\bar{\partial}\sigma(\cdot))^\top y^* = \ \operatorname{conv} D^*\sigma(\cdot)(y^*) \quad \text{for all } y^* \in \mathbb{R}^{ml_2},$$

together with (2.45) and (2.46), we get

$$\begin{aligned} &\left(\bar{\partial}\sigma(\bar{x})\right)^\top y^* \subset \\ &\subset \bigcup_{\substack{u \in \mathbb{R}^{ml_2} \\ (\lambda_i^1, \lambda_i^2) \in \Lambda_i(\bar{x}, \bar{y}) \\ i=1,\dots,ml_2}} \left\{ \sum_{i=1}^{ml_2} \sum_{j \in I_i(\bar{x}, \bar{y})} u_i \lambda_i^j \nabla_x F_i^j(\bar{x}, \bar{y}) \middle| 0 = y^* + \sum_{i=1}^{ml_2} \sum_{j \in I_i(\bar{x}, \bar{y})} u_i \lambda_i^j \nabla_y F_i^j(\bar{x}, \bar{y}) \right\}. \end{aligned} \tag{2.48}$$

Taking $y^* = \nabla_y \varphi(\bar{x}, \bar{y})$ and inserting (2.48) to (2.47), we get for some $u \in \mathbb{R}^{ml_2}$, $(\lambda_i^1, \lambda_i^2) \in \Lambda_i(\bar{x}, \bar{y}), i = 1, \dots, ml_2$, and $\xi \in N(\bar{x}; U)$

$$\begin{aligned} 0 =& \nabla_x \varphi(\bar{x}, \bar{y}) + \sum_{i=1}^{ml_2} \sum_{j \in I_i(\bar{x}, \bar{y})} u_i \lambda_i^j \nabla_x F_i^j(\bar{x}, \bar{y}) + \xi, \\ 0 =& \nabla_y \varphi(\bar{x}, \bar{y}) + \sum_{i=1}^{ml_2} \sum_{j \in I_i(\bar{x}, \bar{y})} u_i \lambda_i^j \nabla_y F_i^j(\bar{x}, \bar{y}). \end{aligned}$$

Now, for each $i = 1, \dots, ml_2$, set $\lambda_i^{F^j} := -u_i \lambda_i^j$ for $j \in I_i(\bar{x}, \bar{y})$ and $\lambda_i^{F^j} := 0$ otherwise. If $I_i(\bar{x}, \bar{y}) = \{1, 2\}$, which corresponds to $i \in I^0(\bar{x}, \bar{y})$, we observe that

$$\lambda_i^{F^1} \lambda_i^{F^2} = (u_i)^2 \lambda_i^1 \lambda_i^2 \geq 0$$

since $(\lambda_i^1, \lambda_i^2) \in \Lambda_i(\bar{x}, \bar{y})$. Hence, we arrived at C-stationarity conditions. In particular, $(\bar{x}, \bar{y})$ is a C-stationary point to MPCC. $\square$

Now, we show that also the opposite implication holds. The proof involves computation of a solution to a system of linear equations derived from the C-stationarity conditions and rearranging the terms to obtain Clarke stationary conditions. To be able to compute the solution of the system of linear equations, we first need the following auxiliary linear algebraic result about rows of *adjunct matrices* and determinants.

Consider $2k$ vectors $x^{i,j} \in \mathbb{R}^k, i = 1, \dots k, j = 1, 2$ such that for each i there is j that $x^{i,j} \neq 0$ (in other words, for each i either $x^{i,1}$ or $x^{i,2}$ is a nonzero vector) and such that every collection of at most k nonzero vectors $x^{i,j}$ is linearly independent. Denote by A a $k \times 2k$ matrix with columns composed of all vectors $x^{i,j}, i = 1, \dots, k, j = 1, 2$.

Suppose also that we are given $2k$ nonzero constants $\gamma^{i,j} \in \mathbb{R}$. Denote by A^γ a $k \times k$ matrix which ith column is given by $\gamma^{i,1}x^{i,1} + \gamma^{i,2}x^{i,2}$ and by $s \in \mathbb{R}^k$ a vector composed of 1's and 2's such that $s_i = j$ only if $x^{i,j} \neq 0$ for $i = 1, \dots, k, j = 1, 2$. Then there is 2^{a^0} such vectors, $s^1, \dots, s^{2^{a^0}}$, where $a^0 = k - |\{i = 1, \dots, k | \exists j : x^{i,j} = 0\}|$.

Consider the operation $*$ such that $(A * s)$ is a matrix with the ith column given by $x^{i,j}$ for $s_i = j$.

Lemma 2.25. *For each $i = 1, \dots, k$, the following relation holds for the ith row of adjunct matrix Adj A^γ*

$$(\text{Adj } A^\gamma)_i = \frac{1}{\gamma^{i,1} I_{[x^{i,1} \neq 0]} + \gamma^{i,2} I_{[x^{i,2} \neq 0]}} \sum_{l=1}^{2^{a^0}} \left(\prod_{j=1}^{k} \gamma^{j,s_j^l} \right) \text{Adj } (A * s^l)_i.$$

Moreover,

$$\det A^\gamma = \sum_{l=1}^{2^{a^0}} \left(\prod_{i=1}^{k} \gamma^{i,s_i^l} \right) \det (A * s^l).$$

Proof. Both parts of the statement follow directly from the basic rules for computation of determinants:
i) if A is an $n \times n$ matrix with one of the columns $x = x^1 + x^2$, where $x^1, x^2 \in \mathbb{R}^n$, then

$$\det A = \det A^1 + \det A^2.$$

The matrices A^1 and A^2 in the above formula are obtained from matrix A by replacing its column x by vectors x^1 and x^2, respectively;
ii) if we multiply a column (or row) of matrix A by a constant c, then the determinant of such matrix is equal to $c(\det A)$.

Without loss of generalization let $k = 2$, $a^0 = 1$, $s^1 = (1,1)^\top$, $s^2 = (1,2)^\top$ and

$$A^\gamma = \begin{pmatrix} \gamma^{1,1}x_1^{1,1} & \gamma^{2,1}x_1^{2,1} + \gamma^{2,2}x_1^{2,2} \\ \gamma^{1,1}x_2^{1,1} & \gamma^{2,1}x_2^{2,1} + \gamma^{2,2}x_2^{2,2} \end{pmatrix}.$$

The first row of adjunct matrix is composed of $(\text{Adj } A^\gamma)_{11}$ and $(\text{Adj } A^\gamma)_{12}$. Clearly,

$$\begin{aligned}(\text{Adj } A^\gamma)_{11} &= \gamma^{2,1}x_2^{2,1} + \gamma^{2,2}x_2^{2,2} = \\ &= \frac{1}{\gamma^{1,1}}\gamma^{1,1}\gamma^{2,1}(\text{Adj } (A * s^1))_{11} + \frac{1}{\gamma^{1,1}}\gamma^{1,1}\gamma^{2,2}(\text{Adj } (A * s^2))_{11}, \\ (\text{Adj } A^\gamma)_{12} &= -\gamma^{2,1}x_1^{2,1} - \gamma^{2,2}x_1^{2,2} = \\ &= \frac{1}{\gamma^{1,1}}\gamma^{1,1}\gamma^{2,1}(\text{Adj } (A * s^1))_{12} + \frac{1}{\gamma^{1,1}}\gamma^{1,1}\gamma^{2,2}(\text{Adj } (A * s^2))_{12}.\end{aligned}$$

Similarly,

$$\begin{aligned}(\text{Adj } A^\gamma)_{21} &= -\gamma^{1,1}x_2^{1,1} = \\ &= \frac{1}{\gamma^{2,1}+\gamma^{2,2}}\gamma^{1,1}\gamma^{2,1}(\text{Adj } (A * s^1))_{21} + \frac{1}{\gamma^{2,1}+\gamma^{2,2}}\gamma^{1,1}\gamma^{2,2}(\text{Adj } (A * s^2))_{21}, \\ (\text{Adj } A^\gamma)_{22} &= \gamma^{1,1}x_1^{1,1} = \\ &= \frac{1}{\gamma^{2,1}+\gamma^{2,2}}\gamma^{1,1}\gamma^{2,1}(\text{Adj } (A * s^1))_{22} + \frac{1}{\gamma^{2,1}+\gamma^{2,2}}\gamma^{1,1}\gamma^{2,2}(\text{Adj } (A * s^2))_{22}.\end{aligned}$$

Finally,

$$\begin{aligned}\det A^\gamma = {} & \det \begin{pmatrix} \gamma^{1,1}x_1^{1,1} & \gamma^{2,1}x_1^{2,1} \\ \gamma^{1,1}x_2^{1,1} & \gamma^{2,1}x_2^{2,1}\end{pmatrix} + \det \begin{pmatrix} \gamma^{1,1}x_1^{1,1} & \gamma^{2,2}x_1^{2,2} \\ \gamma^{1,1}x_2^{1,1} & \gamma^{2,2}x_2^{2,2}\end{pmatrix} = \\ & = \gamma^{1,1}\gamma^{2,1} \det (A * s^1) + \gamma^{1,1}\gamma^{2,2} \det (A * s^2).\end{aligned}$$

The computations above for the general case are analogous and yield the desired formulas. □

Theorem 2.26. *Let $(\bar{x}, \bar{y})$ be a feasible point for the MPCC (2.7) such that both SRC and MPEC-GLICQ are satisfied. Then, if there exist Lagrange multipliers $\lambda^{F^1}, \lambda^{F^2}$ and a normal $\xi \in N(\bar{x}; U)$ such that $(\bar{x}, \bar{y}, \lambda^{F^1}, \lambda^{F^2}, \xi)$ satisfies C-stationarity conditions of the MPCC, then $(\bar{x}, \bar{y})$ is also a Clarke stationary point of the MPCC.*

Proof. Since $(\bar{x}, \bar{y}, \lambda^{F^1}, \lambda^{F^2}, \xi)$ satisfies C-stationarity conditions, from the proof of Theorem 2.7 it follows that for each $i = 1, \ldots, ml_2$, there exist $\beta_i \in [0, 1]$ and $\zeta_i \in \mathbb{R}$ such that

$$\lambda_i^{F^1} = \beta_i\zeta_i, \qquad i \in L \cup I^0, \tag{2.49}$$

$$\lambda_i^{F^2} = (1 - \beta_i)\zeta_i, \quad i \in I^+ \cup I^0, \tag{2.50}$$

and

$$\begin{aligned}0 \in \nabla\varphi(\bar{x}, \bar{y}) &- \sum_{i \in L}\beta_i\nabla F_i^1(\bar{x}, \bar{y})\zeta_i - \sum_{i \in I^0}(\beta_i\nabla F_i^1(\bar{x}, \bar{y}) + (1 - \beta_i)\nabla F_i^2(\bar{x}, \bar{y}))\zeta_i - \\ &- \sum_{i \in I^+}(1 - \beta_i)\nabla F_i^2(\bar{x}, \bar{y})\zeta_i + N(\bar{x}; U) \times \{0_{ml_2}\}.\end{aligned} \tag{2.51}$$

The last ml_2 rows of (2.51) form a system of ml_2 linear equations in ml_2 variables $\zeta_1, \ldots, \zeta_{ml_2}$. Its system matrix is regular due to MPEC-GLICQ, hence, there is a unique solution $\bar{\zeta} = (\bar{\zeta}_1, \ldots, \bar{\zeta}_{ml_2})$. Using the above auxiliary algebraic results to compute the lines of the inverse to the system matrix we derive the formulas for each component of $\bar{\zeta}$.

We can apply Lemma 2.25 with $k = ml_2, a^0 = |I^0|, \gamma^{i,1} = \beta_i, \gamma^{i,2} = (1-\beta_i), i = 1, \ldots, ml_2$, and

$$x^{i,1} = \begin{cases} \nabla_y F_i^1(\bar{x}, \bar{y}), & i \in L \cup I^0, \\ 0, & i \in I^+, \end{cases}$$
$$x^{i,2} = \begin{cases} \nabla_y F_i^2(\bar{x}, \bar{y}), & i \in I^+ \cup I^0, \\ 0, & i \in L. \end{cases}$$

Hence the ith component of the solution, $\bar{\zeta}_i, i = 1, \ldots, ml_2$, is given by

$$\begin{aligned} \bar{\zeta}_i &= \frac{1}{\det\ A_\gamma}(\mathrm{Adj}\ A^\gamma)_i \nabla_y \varphi(\bar{x}, \bar{y}) = \\ &= \frac{\sum_{l=1}^{2^{a^0}} \left(\prod_{j=1}^{ml_2} \gamma^{j,s_j^l}\right) \mathrm{Adj}\ (A * s^l)_i \nabla_y \varphi(\bar{x}, \bar{y})}{\left(\gamma^{i,1} I_{[i \in L \cup I^0]} + \gamma^{i,2} I_{[i \in I^+ \cup I^0]}\right) \sum_{l=1}^{2^{a^0}} \left(\prod_{j=1}^{ml_2} \gamma^{j,s_j^l}\right) \det\ (A * s^l)}. \end{aligned} \tag{2.52}$$

Note that due to SRC and Lemma 2.16, each $\bar{\zeta}_i, i = 1, \ldots, ml_2$, is well defined.

Next we rearrange the terms in (2.52) to recover simple formulas for $\bar{\zeta}$ in terms of the coefficients of convex combination from relation (2.37) and coordinates of vectors $\bar{p}^j(\bar{x}, \bar{y}), j = 1, \ldots, 2^{a^0}$, which solve (2.38). Setting

$$\alpha_j = \frac{\left(\prod_{i=1}^{ml_2} \gamma^{i,s_i^j}\right)\ \det\ (A * s^j)}{\sum_{l=1}^{2^{a^0}} \left(\prod_{i=1}^{ml_2} \gamma^{i,s_i^l}\right)\ \det\ (A * s^l)}, \quad j = 1, \ldots, 2^{a^0}, \tag{2.53}$$

we have $\alpha_j \geq 0$ for each j and $\sum_{j=1}^{2^{a^0}} \alpha_j = 1$. Recalling that

$$\bar{p}_i^j = \frac{1}{\det\ (A * s^j)}(\mathrm{Adj}\ (A * s^j))_i \nabla \varphi(\bar{x}, \bar{y}), \tag{2.54}$$

application of (2.52), (2.53) and (2.54) to the first l_1 rows of (2.51) results in

$$0 \in \nabla_x \varphi(\bar{x}, \bar{y}) - \sum_{j=1}^{2^{a^0}} \alpha_j (B * s^j) \bar{p}^j + N(\bar{x}; U) \tag{2.55}$$

due to relations

$$\beta_i \bar{\zeta}_i = \sum_{j=1}^{2^{a^0}} \alpha_j \bar{p}_i^j, \qquad i \in L, \tag{2.56}$$

$$(1-\beta_i)\bar{\zeta}_i = \sum_{j=1}^{2^{a^0}} \alpha_j \bar{p}_i^j, \qquad i \in I^+, \tag{2.57}$$

$$\beta_i \bar{\zeta}_i = \sum_{j: s^{i,j}=1} \alpha_j \bar{p}_i^j, \quad i \in I^0, \tag{2.58}$$

$$(1-\beta_i)\bar{\zeta}_i = \sum_{j: s^{i,j}=2} \alpha_j \bar{p}_i^j, \quad i \in I^0, \tag{2.59}$$

where B can be derived from A by replacing each $\nabla_y F_i^j(\bar{x}, \bar{y})$ with $\nabla_x F_i^j(\bar{x}, \bar{y})$.

Clearly, (2.54) and (2.55) are Clarke stationarity conditions to the MPCC. In particular, $(\bar{x}, \bar{y})$ is Clarke stationary. □

To illustrate the importance of each of the assumptions, SRC and MPEC-GLICQ, we present two examples. Strong regularity is a key ingredient in the definition of Clarke stationarity. If SRC is violated, the adjoint equation may not have a solution or, on the other hand, may have multiple solutions.

To emphasize the need for verification of SRC, the first of two examples shows that Clarke stationarity conditions can indeed be satisfied even if SRC is violated. This, of course, does not mean that the corresponding point is Clarke stationary. Also, in the absence of strong regularity, C-stationarity conditions are satisfied independently of validity of the Clarke stationarity conditions.

Example 2.27. Consider the following MPCC:

$$\begin{array}{ll} \text{minimize} & -x_1 - x_2 - 2y_1 - 2y_2 \\ \text{subject to} & \\ & \begin{array}{lllllllll} 0 & \leq & -x_1 & & +y_1 & -2y_2 & \perp & +y_1 & & \geq & 0, \\ 0 & \leq & & -x_2 - 2y_1 & & +y_2 & \perp & & +y_2 & \geq & 0. \end{array} \end{array}$$

From Example 2.23 we already know that SRC is violated at (0,0,0,0), while MPEC-GLICQ is satisfied. Nevertheless, we can still compute vectors $\bar{p}^i, i = 1, \ldots, 4$,

$$\bar{p}^1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \quad \bar{p}^2 = \begin{pmatrix} -6 \\ -2 \end{pmatrix}, \quad \bar{p}^3 = \begin{pmatrix} -2 \\ 6 \end{pmatrix}, \quad \bar{p}^4 = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$$

and Clarke stationarity conditions

$$0 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} - \alpha_1 \begin{pmatrix} -2 \\ -2 \end{pmatrix} - \alpha_2 \begin{pmatrix} 0 \\ 2 \end{pmatrix} - \alpha_3 \begin{pmatrix} 2 \\ 0 \end{pmatrix} - \alpha_4 \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

are satisfied for $\alpha = (0, 1/2, 1/2, 0)$. However, with respect to Definition 2.19, (0,0,0,0) is not Clarke stationary.

Clearly, C-stationarity conditions are violated since the corresponding (unique) multipliers are $\lambda_1^{F_1} = 1, \lambda_2^{F_1} = 1, \lambda_1^{F_2} = -1$ and $\lambda_2^{F_2} = -1$. Replacing the objective with $x_1 + x_2 - 2y_1 - 2y_2$, (0,0,0,0) becomes C-stationary with multipliers (-1, -1, -3, -3) while Clarke stationarity conditions are again satisfied despite of violation of SRC, this time for $\alpha = (0, 1/2, 1/2, 0)$. △

The second example shows that MPEC-GLICQ plays important role for the validity of multiplier-sign conditions. In absence of MPEC-GLICQ, Clarke stationarity generally implies only weak stationarity.

Example 2.28. Consider the following MPCC:

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & \end{array}$$

$$\begin{array}{ccccccccccc} 0 & \le & -x_1 & & +2y_1 & y_2 & \perp & & & +y_1 & & \ge & 0, \\ 0 & \le & & -x_2 & -3y_1 & +2y_2 & \perp & -x_1 & -x_2 & & +y_2 & \ge & 0. \end{array}$$

SRC holds at (0,0,0,0), while MPEC-GLICQ is violated, cf. Example 2.22. Note that in this case the MPEC multipliers are not uniquely determined and the point (0,0,0,0) is critical with multipliers $(\lambda_1^{F^1}, \lambda_2^{F^1}, \lambda_1^{F^2}, \lambda_2^{F^2}) = (\lambda, \lambda, \lambda, -\lambda)$, where λ is an arbitrary real constant. Clearly, for any set of MPEC multipliers, C-stationarity conditions about the common signs of biactive multipliers are violated. Hence the point (0,0,0,0) is just weakly stationary.

All $\bar{p}^i, i = 1, \ldots, 4$, are equal to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and Clarke stationarity conditions are trivially satisfied. Hence, the point (0,0,0,0) is Clarke stationary.

Note, that the relations (2.56)-(2.59) are violated. △

Chapter 3

Equilibrium Problem with Equilibrium Constraints (EPEC)

As a natural generalization of an MPEC we can introduce an equilibrium concept also to the upper level. For that we need to increase the number of leaders and change the structure of the problem accordingly. In this way one obtains the so-called *equilibrium problem with equilibrium constraints* consisting of several, mutually coupled MPECs. The term “equilibrium problem” in EPEC refers to the fact that this problem is no longer a single minimization problem under equilibrium constraints.

In this chapter we present four source problems to illustrate the application of EPECs. Further we discuss the question of existence of solutions to EPECs in mixed strategies and, based on the results of the previous chapter, also the existence of Clarke and C-stationary points. The cooperative behavior of leaders will be discussed in the following chapter devoted to multiobjective problems with equilibrium constraints.

3.1 Mathematical formulation

Assume that we have to do with n leaders and m followers. Analogously to MPECs, multistrategy $y \in \mathbb{R}^{ml_2}$ consists from all followers' strategies. Let x^i denote the strategy of the ith leader. Again, to distinguish the strategies, objective functions, feasible sets, etc., of each player we use the upper indices.

The multistrategy $x := (x^1, x^2, \ldots, x^n)$ contains the strategies of all leaders. Suppose that the behavior of the leaders is described by their individual objectives $\varphi^i : \mathbb{R}^{nl_1+ml_2} \to \mathbb{R}, i = 1, \ldots, n$. Let the nonequilibrium constraints in the problem of each leader i concern the strategies of all players, i.e., $\kappa^i \subset \mathbb{R}^{nl_1+ml_2}, i = 1, \ldots, n$.

Definition 3.1. *(Solution to abstract noncooperative EPEC)*
A vector of admissible strategies $(\bar{x}, \bar{y}) \in \mathbb{R}^{nl_1+ml_2}$ *is a* solution to an abstract EPEC *if*

$(\bar{x}, \bar{y})$ *solves simultaneously abstract MPECs*

$$\begin{aligned} &\underset{x^i, y}{\text{minimize}} && \varphi^i(x^i, \bar{x}^{-i}, y) \\ &\text{subject to} && y \in S(x^i, \bar{x}^{-i}), \qquad i = 1, \ldots, n, \\ &&& (x^i, \bar{x}^{-i}, y) \in \kappa^i, \end{aligned} \tag{3.1}$$

where the solution mapping S *depends on strategies of all leaders.*

In other words, $(\bar{x}, \bar{y})$ solves the abstract EPEC if for each $i = 1, \ldots, n$, $(\bar{x}^i, \bar{y})$ belongs to the set of local solutions to ith abstract MPEC in (3.1) in variables (x^i, y).

Note that each x^i acts simultaneously as a decision variable in the MPEC of the ith leader and as a parameter in the remaining ones. It is obvious, that since we define the EPEC as a series of n MPECs linked together via upper-level and lower-level variables, all problematic features of MPECs discussed in the previous section are inevitably inherited to EPECs as well.

Define for each $i = 1, \ldots, n$, and a fixed admissible multistrategy $\bar{x}^{-i}$ the multifunction

$$S_{\bar{x}^{-i}}(x^i) = S(x^i, \bar{x}^{-i}).$$

Note that x^i enters the lower-level problem as a fixed parameter as well. Thus, MPECs in (3.1) share the same equilibrium constraints. However, due to the fact that Gph S may not be consistent with each set of nonequilibrium constraints κ^i, even the existence of a feasible point to EPEC might be uncertain.

From now on, we consider only EPECs in which separation of the nonequilibrium constraints is possible, including the constraints on multistrategy y into the lower-level problem and considering the nonequilibrium constraints only on strategies $x^i, i = 1, \ldots, n$. I.e., $\kappa^i = U^i \times \mathbb{R}^{(n-1)l_1} \times \mathbb{R}^{ml_2}$, where $U^i \subset \mathbb{R}^{l_1}$ is the individual feasible set of leader i.

Note that even then, the lower-level strategy y in the EPEC (3.1) is shared across all MPECs. If S is not single-valued, the formulation of EPEC (3.1) could be called *multioptimistic*. However, such EPECs are in many cases *ill-possed*. We explain this on the following EPEC composed of only two MPECs.

Let the MPEC of the first and the second leader be given by

$$\inf_{x^1 \in U^1} \inf_{y \in S(x^1, \bar{x}^2)} \varphi^1(x^1, \bar{x}^2, y), \tag{3.2}$$

$$\inf_{x^2 \in U^2} \inf_{y \in S(\bar{x}^1, x^2)} \varphi^2(\bar{x}^1, x^2, y), \tag{3.3}$$

respectively. Formulated in this form, clearly, the EPEC is ill-possed if S is not single-valued. The argmin sets

$$\underset{y \in S(\bar{x}^1, \bar{x}^2)}{\arg\min}\ \varphi^1(\bar{x}^1, \bar{x}^2, y),$$

$$\underset{y \in S(\bar{x}^1, \bar{x}^2)}{\arg\min}\ \varphi^2(\bar{x}^1, \bar{x}^2, y)$$

may not have a common element, inevitably resulting in nonexistence of a solution to the EPEC. On the other hand, this does not cause any problem if the solution to the lower problem has single-valued and multi-valued components and the upper-level objectives depend only on the single-valued part of the solution of the lower problem.

As an alternative, we can consider optimistic (or pessimistic) formulations with respect to one particular leader. We analyze this possibility in detail in Section 3.3.

As in the MPEC case consider y to be feasible, provided its components belong to the sets $V^j \subset \mathbb{R}^{l_2}, j = 1, \ldots, m$, and let the followers act according to their objectives $f^j : \mathbb{R}^{nl_1+ml_2} \to \mathbb{R}, j = 1, \ldots, m$.

In what follows, we presume that assumption (A0) below holds.

(A0) Let each objective $f^j, j = 1, \ldots, m$, be continuously differentiable on an open set containing $\bigtimes_{i=1}^n U^i \times \Omega$ and let Ω be closed.

Denote by $\omega := \bigtimes_{i=1}^n U^i \subset \mathbb{R}^{nl_1}$ the set of feasible leaders' strategies. Then, for a given vector $\bar{x} \in \omega$ and given strategies y^{-j}, the optimal strategy of the jth follower amounts to a solution of the optimization problem

$$\begin{aligned} &\underset{y^j}{\text{minimize}} \quad f^j(\bar{x}, y^j, \bar{y}^{-j}) \\ &\text{subject to } y^j \in V^j. \end{aligned} \tag{3.4}$$

A solution map

$$S(x) := \{y \in \mathbb{R}^{ml_2} \mid 0 \in F(x, y) + N(y; \Omega)\},$$

where

$$F(x, y) := \begin{pmatrix} \nabla_{y^1} f^1(x, y) \\ \vdots \\ \nabla_{y^m} f^m(x, y) \end{pmatrix},$$

is then a multifunction that maps a feasible $x \in \omega$ to Nash equilibria of problems (3.4) for $j = 1, \ldots, m$.

This allows us to modify the definition of the solution to the abstract EPEC accordingly: a vector of admissible strategies $(\bar{x}, \bar{y}) \in \mathbb{R}^{nl_1+ml_2}$ is a solution to EPEC if for each $i = 1, \ldots, n$, $(\bar{x}^i, \bar{y})$ belongs to the set of local solutions to the MPEC

$$\begin{aligned} &\underset{x^i, y}{\text{minimize}} \quad \varphi^i(x^i, \bar{x}^{-i}, y) \\ &\text{subject to} \quad 0 \in F(x^i, \bar{x}^{-i}, y) + N(y; \Omega), \\ &\qquad\qquad\quad x^i \in U^i. \end{aligned} \tag{3.5}$$

Similarly to the MPEC case, when equilibrium constraints of an EPEC are in the form of a nonlinear complementarity problem we call such problem an *equilibrium problem* *with complementarity constraints* (EPCC). In particular, a vector of admissible strategies $(\bar{x}, \bar{y}) \in \mathbb{R}^{nl_1+ml_2}$ is a solution to an EPCC if $(\bar{x}, \bar{y})$ solves simultaneously n MPCCs

$$\begin{aligned} &\underset{x^i, y}{\text{minimize}} && \varphi^i(x^i, \bar{x}^{-i}, y) \\ &\text{subject to} && 0 \le F^1(x^i, \bar{x}^{-i}, y) \perp F^2(x^i, \bar{x}^{-i}, y) \ge 0, \qquad i = 1, \ldots, n, \\ &&& x^i \in U^i \end{aligned} \tag{3.6}$$

with functions $F^1, F^2 : \mathbb{R}^{nl_1+ml_2} \to \mathbb{R}$ continuously differentiable on an open set containing $\omega \times \mathbb{R}^{ml_2}$.

One of the possible ways how to further simplify the MPEC structure and hence in our case also the structure of EPEC is to consider assumptions under which the solution map becomes single-valued. This enables us to invoke the implicit programming approach. Hence we impose the following essential assumption.

(A1) For each $i = 1, \ldots, n$, and for all admissible multistrategies $\bar{x}^{-i}$, the map $S_{\bar{x}^{-i}}$ is single-valued and locally Lipschitz continuous on an open set containing U^i.

Under assumption (A1) one can rewrite every problem (3.5) for $i = 1, \ldots, n$, to the form

$$\begin{aligned} &\underset{x^i}{\text{minimize}} && \theta^i(x^i, \bar{x}^{-i}) \\ &\text{subject to} && x^i \in U^i, \end{aligned} \tag{3.7}$$

where functions $\theta^i : \omega \to \mathbb{R}$ are defined by

$$\theta^i(x) = \varphi^i(x, S(x)).$$

We may refer to functions $\theta^i, i = 1, \ldots, n$, as to loss functions of the reduced game only among leaders, keeping the consistency of terminology from non-hierarchical games.

The problem (3.7) is now without any hierarchical structure and one can apply the theory of noncooperative Nash games, to compute local Nash equilibria which form noncooperative solutions to EPEC. Since verification of assumption (A1) requires checking the properties of n multivalued mappings, we work with its modified version.

(A1') S is single-valued and locally Lipschitz continuous on an open set containing ω.

The latter assumption now involves only one multifunction. Note that (A1') implies assumption (A1).

3.2 Source problems

It is a very important question whether EPECs can actually be used in modeling of problems with real-world applications. From the survey of available works on EPECs it may appear that the problem of deregulated electricity markets, which primarily motivated the introduction of EPEC as a new class of hierarchical problems, is the only discussed application. This was one of the reasons why we decided to include this section to the thesis, despite the fact it does not contain any mathematical results of the author and the reader can easily skip it and proceed directly to the next section with exception of the oligopolistic market model which arises in numerical study presented in Section 5.3.

On the other hand, this section might serve as an inspiration to researches interested in EPECs. This is especially true in the case of the last problem presented in this section (traffic equilibrium problem with private toll roads). Also, note that the list of source problems presented here is definitely not exhaustive.

The notation in this section differs from the rest of the thesis. We present each problem using the notation of the source references.

3.2.1 Oligopolistic market problem

Consider an *oligopolistic market* model with $n+m$ firms producing a homogeneous product and attempting to maximize their profits; see, e.g., [32] and [39]. Let $x^i \in \mathbb{R},\ i = 1, 2, \dots, n$, denote the *production* of the ith leader and let $y^j \in \mathbb{R},\ j = 1, 2, \dots, m$, be the production of the jth follower.

Assume that the multistrategy vector x of the leaders' productions belongs to some closed subset ω of $\mathbb{R}^n$. Let

$$T = \sum_{i=1}^{n} x^i + \sum_{j=1}^{m} y^j$$

denote the *overall production on the market*, and let $p : \operatorname{int}\mathbb{R}_+ \to \operatorname{int}\mathbb{R}_+$ be the so-called *inverse demand curve* that assigns T the price at which consumers are willing to purchase. The objectives of leaders can now be written in the form

$$\varphi^i(x, y) := c^i(x^i) - x^i p(T),\ i = 1, 2, \dots, n,$$

and similarly the objectives of followers attain the form

$$f^j(x, y) := c^{n+j}(y^j) - y^j p(T),\ j = 1, 2, \dots, m,$$

where the functions $c^k : \mathbb{R}_+ \to \mathbb{R}_+, k = 1, \dots, n+m$, represent the production costs.

Concerning the data of the problem, suppose that

(A2) (i) the functions $c^{n+j}, j = 1, 2, \dots, m$, are convex and twice continuously differentiable;

(ii) p is twice continuously differentiable and strictly convex on $\operatorname{int}\mathbb{R}_+$;

(iii) $\vartheta p(\vartheta)$ is a concave function of ϑ;

(iv) ω does not contain the zero vector.

When all parts of assumption (A2) are fulfilled, then for all $j = 1, 2, \ldots, m$, the objective function of the jth follower is convex with respect to y^j; cf. [32]. Thus we have

$$F(x, y) = \begin{pmatrix} \nabla c^{n+1}(y^1) - p(T) - y^1 \nabla p(T) \\ \vdots \\ \nabla c^{n+m}(y^m) - p(T) - y^m \nabla p(T) \end{pmatrix} \tag{3.8}$$

and, as proved in [39, Lemma 12.2], the corresponding partial Jacobian $\nabla_y F(x, y)$ is positive definite at each feasible pair (x, y). This implies that the Robinson strong regularity condition is fulfilled and assumption (A1') holds true at each feasible pair (x, y).

Note that this model comprises both EPECs and multiobjective equilibrium problems with equilibrium constraints, see Chapter 4, based on the behavior leaders. Later in Chapter 5 we present numerical results for this oligopolistic market problem with cooperative leaders.

3.2.2 Forward-spot market model

In [49, Chapter 4] one can find a *two-period forward market model* where each player solves a nonconvex MPEC, and the whole problem can be formulated as an EPEC. In fact it is another possibility how to modify the Cournot-Nash model to get the EPEC structure, this time introducing the second period to the game instead of the second level to the hierarchy of the game. In two-period model, each player is trying to maximize his or her profits in both periods, deciding about his or her production which is available only in the first period, about a forward position, a part of his production which he or she will sell in the second period according to a contract made in the first period and thus also about spot sales in the second period.

Let us denote by $f = (f_1, \ldots, f_n)$ the *forward position vector*. For fixed forward positions $\bar{f}$ and fixed *production quantities* $\bar{x}_{-i}(\bar{f})$ of the other $n-1$ producers, the producers face a Cournot-Nash game (in production quantities) in the second period. Thus, the ith producer chooses his or her production quantity x_i in order to maximize his or her profits in the second period. Hence, $\bar{x}_i(\bar{f})$ is the solution to the following maximization problem in variable x_i

$$\underset{x_i \geq \bar{f}_i}{\text{maximize}} \; p\left(x_i + \sum_{j \neq i} \bar{x}_j(\bar{f})\right)(x_i - \bar{f}_i) - c_i(x_i), \tag{3.9}$$

where $p(\cdot)$ is the *spot price* (inverse demand function) in the second period and $c_i(\cdot)$ is the *cost function* of the ith producer.

In the first period, the producers are playing a Cournot-Nash game in forward quantities. Thus, assuming the forward positions of the other producers $\bar{f}_{-i}$ are fixed, the ith producer chooses his forward position so as to maximize his overall profit function which is given as a sum of revenue of sales of forwards for the forward price in the first period and optimal payoff in the second period. Under *perfect foresight* this results in solving

$$\underset{f_i \geq 0}{\text{maximize}} \;\; p\left(\sum_{j=1}^{n} x_j(f_i, \bar{f}_{-i})\right) x_i(f_i, \bar{f}_{-i}) - c_i(x_i(f_i, \bar{f}_{-i})). \tag{3.10}$$

Let us propose the following assumption:

(A3) Let the inverse demand function be linear

$$p(z) = a - bz, \;\; a, b > 0 \;\; \text{for } z \geq 0$$

and for $i = 1, \ldots, n$, the production cost functions of the ith producer be in the form

$$c_i(z) = c_i z, \;\; c_i > 0.$$

Then under (A3) one can reformulate the above problem given by (3.9) and (3.10) for each $i = 1, \ldots, n$, as a system of coupled MPCCs in variables (f_i, s, θ_i)

$$\begin{aligned} \text{maximize} \quad & (\theta_i - c_i)(f_i + s_i) \\ \text{subject to} \quad & \theta_i = a - b\left(f_i + \sum_{j=1}^{n} s_j + \sum_{j \neq i} \bar{f}_j\right), \\ & 0 \leq s \perp c - \theta_i e + bs \geq 0, \\ & f_i \geq 0, \end{aligned} \tag{3.11}$$

where $s = (s_1, \ldots, s_n)$, s_i denoting the *spot sales* of the ith producer in the second period, $c = (c_i, \ldots, c_n)$ and e denotes the vector of all ones. Now we can clearly see an EPEC structure of the whole two-period forward-spot market model. In the forward market, every producer is leader, while in the spot market, every producer is in the role of follower already with the knowledge of the every producer's forward position in the first period.

3.2.3 Deregulated electricity market model

An important issue in all *deregulated electricity markets* is the market power of participants such as generators, large utilities, or providers of ancillary services. The transportation of power from a *generation node* (source) to a *consumption node* (sink) is governed by the *Kirkhoff laws*. Laically speaking, power flows along the paths of the least resistance. So, transmission of power is different from the transportation of the ordinary commodity in a spatial market. The location and quantity of any injection or withdrawal of power determines the actual *transmission capacity* of any link in electric network. As a result, the key issue in the overall design is how a network (grid) operator dispatches electricity.

In this section, which is based on [22], we show how we can model via EPECs the so-called *pool-type market problems* as operated in Australia, New Zealand and some parts of United States, where the *independent system operator* (ISO) dispatches electricity from generators to consumers by maximizing "social welfare" (minimizing "social cost") based on the cost/utility functions that are bid by generators/consumers.

Suppose an electric network with $N+1$ nodes labeled $0, \ldots, N$, and a set L of links, where the link between node i and j is written ij. For the sake of simplicity, assume that there is a single generator or consumer at any node i. *Bidders* (generators and retailers) have complete information about the network, the ISO's operation procedure and all other participants' cost/utility functions. The ISO, taking account of the network, solves a social cost minimization problem assuming the bids are truthful, announcing a dispatch for each bidder and possibly distinct prices at each node. Consumers pay generators according to the scheduled dispatch and nodal prices. The market is then cleared according to each player's binding bid.

Each player's actual cost or utility function is a quadratic function in quantity q_i, either cost, $A_i q_i + B_i q_i^2$ $(q_i \geq 0)$, or utility, $-A_i q_i - B_i q_i^2$ $(q_i \leq 0)$, where each A_i and B_i is assumed to be positive. A consumer at node i is dispatched a quantity in the range $[0, \frac{A_i}{2B_i}]$ where his or her actual utility function is increasing. We can hence let the generators (consumers) bid their supply (demand) functions to the ISO in the form of a pair of coefficients (a_i, b_i). Their bids, $a_i q_i + b_i q_i^2$ $(q_i \geq 0)$, or $-a_i q_i - b_i q_i^2$ $(q_i \leq 0)$, then may naturally differ from the actual cost or utility function of bidder i.

The ISO solves the following problem of minimizing the social cost, over all $N+1$ nodes:

$$
\begin{aligned}
\underset{q_0,\ldots,q_N}{\text{minimize}} \quad & \sum_{i=0}^{N} (a_i q_i + b_i q_i^2) \\
\text{subject to} \quad & q_0 + q_1 + \cdots + q_N = 0, && (3.12) \\
& -C_{ij} \leq \sum_{k=0}^{N} \phi_{ij,k} q_k \leq C_{ij}, \;\; i<j, ij \in L, && (3.13) \\
& q_i \geq 0, \;\; i: \text{ generator}, \;\; q_i \leq 0, \;\; i: \text{ consumer}, && (3.14)
\end{aligned}
$$

where C_{ij} denotes the transmission limit on the link ij and $\phi_{ij,k}$ (distribution factor) denotes the contribution of injection (withdrawal) at node k to the link ij. The distribution factors are determined by the network's physical properties. The optimal solution to this problem is denoted by $q = (q_0(a,b), \ldots, q_N(a,b))$.

Let us denote the Lagrange multipliers corresponding to the optimal solution such that $\lambda, \underline{\mu}_{ij}, \overline{\mu}_{ij}$ and ν_i are the multiplier corresponding to (3.12), (3.13) (left hand side and right hand side inequality) and (3.14), respectively. The optimal quantities q_k commit the ISO to paying (charging) the kth player a price that is consistent with their bid supply (demand) function:

$$p_k = a_k + 2b_k q_k.$$

Additionally, if $q_k \neq 0$, the ISO in effect sets p_k equal to

$$p_k = -\lambda - \sum_{i<j, ij \in L} (\overline{\mu}_{ij} - \underline{\mu}_{ij}) \phi_{ij,k},$$

as it comes from the ISO's KKT conditions. Note that when binding transmission quantities are missing, p_k equals the shadow price $(-\lambda)$ of the requirement that electricity generated equals electricity consumed.

Let us now consider the behavior of profit-maximizing player on the market described above. Given that bidder i's price is $a_i + 2b_i q_i$, then the profit maximization problem is:

$$\begin{aligned}
&\underset{a_i, b_i}{\text{maximize}}\, (a_i + 2b_i q_i)q_i - (A_i q_i + B_i q_i^2) \\
&\text{subject to } \; \underline{A}_i \le a_i \le \overline{A}_i \\
&\qquad\qquad\quad \underline{B}_i \le b_i \le \overline{B}_i \\
&\qquad\qquad\quad q_i \text{ such that } q = (q_0, \ldots, q_N) \text{ solves the ISO's minimization} \\
&\qquad\qquad\quad \text{problem given the other participants' bids } (a_{-i}, b_{-i}).
\end{aligned} \tag{3.15}$$

The constants $\underline{A}_i, \overline{A}_i$ and $\underline{B}_i, \overline{B}_i$, assumed to satisfy

$$0 < \underline{A}_i \le A_i \le \overline{A}_i \text{ and } 0 < \underline{B}_i \le B_i \le \overline{B}_i$$

are lower and upper bounds for a_i and b_i that are based on industry knowledge and are imposed by the ISO.

The problem (3.15) is a bilevel programming problem, where the lower-level problem is that q must solve the optimal power flow problem of the ISO.

Since the ISO's problem is a strictly convex quadratic problem, one can replace the constraints in (3.15) with the KKT conditions of the ISO's problem, that is, to reformulate the bilevel problem for the ith bidder as an MPEC. Hence, a dispatch $q(a, b)$ solves the problem if and only if there exist multipliers corresponding to the constraints that satisfy the usual KKT conditions at $q(a, b)$. Given the other participants' bids (a_{-i}, b_{-i}), bidder i's problem becomes

$$\begin{aligned}
&\underset{a_i, b_i, q, \lambda, \underline{\mu}, \overline{\mu}, \nu}{\text{maximize}}\; (a_i + 2b_i q_i)q_i - (A_i q_i + B_i q_i^2) \\
&\text{subject to } \; \underline{A}_i \le a_i \le \overline{A}_i \\
&\qquad\qquad\quad \underline{B}_i \le b_i \le \overline{B}_i \\
&\text{(ISO-KKT)} \begin{cases}
\text{where, for each } j = 0, \ldots, N, \text{ and } mn \in L \text{ with } m < n: \\
a_j + 2b_j q_j + \lambda + \sum_{i<k, ik \in L} \phi_{ik,j}(\overline{\mu}_{ik} - \underline{\mu}_{ik}) - \nu_j = 0 \\
q_0 + \cdots + q_N = 0 \\
0 \le C_{mn} + \sum_{k=0}^{N} \phi_{mn,k} q_k \perp \underline{\mu}_{mn} \ge 0 \\
0 \le C_{mn} - \sum_{k=0}^{N} \phi_{mn,k} q_k \perp \overline{\mu}_{mn} \ge 0 \\
0 \le q_j \perp \nu_j \ge 0 \;\; \text{if bidder } j \text{ is a generator} \\
0 \ge q_j \perp \nu_j \le 0 \;\; \text{if bidder } j \text{ is a consumer.}
\end{cases}
\end{aligned} \tag{3.16}$$

The game based on all the participants' problems (3.16), for $i = 0, \ldots N$, gives rise to an EPCC.

3.2.4 Traffic equilibrium problem with private toll roads

Recently, a lot of effort is being put into investigation of models of a road system which is partially provided also by a private sector. The private sector would build and maintain roads and cover its costs by charging toll. Primarily motivated by profitability, private investors are believed to be more efficient; they build and operate facilities at less cost then a public sector. There is a steadily growing discussion on this topic, see, e.g., [52] and references therein.

To formulate the model mathematically, we use the standard notation used in *traffic equilibrium models.*

Consider a *transportation network* G which is given by a pair of sets $\mathcal{N}$, the set of nodes, and $\mathcal{A}$, the set of arcs (links between ordered pairs od distinct nodes) and we write $G = (\mathcal{N}, \mathcal{A})$.

Denote by W the set of *origin-destination* (OD) pairs in G. For every $w \in W$, let R_w denote the set of all paths connecting OD pair $w \in W$. The set of all routes is then given as $R = \bigcup_{w \in W} R_w$. Naturally, we assume that our network is connected, i.e., for each pair of nodes there is a route between them.

Denote by F_r the flow on route $r \in R$ and let v_a denote the flow on link $a \in \mathcal{A}$. Introducing the incidence matrix Δ with elements

$$\delta_{ar} = \begin{cases} 1 & \text{if path } r \text{ uses link } a, \\ 0 & \text{otherwise,} \end{cases}$$

then

$$v_a = \sum_{r \in R} \delta_{ar} F_r, \tag{3.17}$$

or, using the vectors v of all link flows and F of all path flows, $v = \Delta F$.

The traffic on a transportation network $G = (\mathcal{N}, \mathcal{A})$ is in equilibrium if the so-called *Wardrop user equilibrium principle* holds. This principle states that for each OD pair $w \in W$, every user of the network G will choose the route between OD pair w which has the minimal costs. Moreover, routes with costs higher that the minimum will have no flows.

Denote by C_r the costs experienced by a persons using route $r \in R$ which is a function of the flow on r and by D_w traffic demand between OD pair $w \in W$ which is a function of the minimum OD travel costs

$$\mu_w = \min_{r \in R_w} C_r, \quad w \in W.$$

Then the Wardrop principle can be written as

$$0 \le C_r(F) - \mu_w \perp F_r \ge 0, \ \ \forall r \in R_w, w \in W, \tag{3.18}$$

$$\sum_{r \in R_w} F_r = D_w(\mu), \ \ \forall w \in W, \tag{3.19}$$

$$\mu_w \ge 0, \ \ \forall w \in W \tag{3.20}$$

with μ being a vector with components $\mu_w, w \in W$.

Suppose for simplicity that each firm provides single toll road (single link on the network), and denote by J the subset of $\mathcal{A}$ corresponding to the set of toll roads on the network. For every link $a \in J$, the corresponding private subject can choose a level of capacity y_a on the link a and toll charge u_a. The remaining links $a \in \mathcal{A}, a \notin J$ are free of charge to use and have fixed capacity $y_a = C_a$.

Consider, for further simplicity, additivity assumption on costs: the costs on route r are simply the sum of the costs of each arc a comprising the route r. In our setting, the costs C_r are of the following form

$$C_r = \sum_{a \in \mathcal{A}} \alpha \delta_{ar} t_a(v_a, y_a) + \sum_{a \in J} \delta_{ar} u_a, \tag{3.21}$$

where α is the value of time which transfers time into monetary units and $t_a(v_a, y_a)$ denotes a travel time on arc a subject to the flow v_a and capacity y_a. With respect to (3.17) the above defined costs truly depend on route flows and we add also natural dependence on the transportation capacity of the arcs.

However, this formulation is questionable since each individual values time in a different way. This can be partially remedied by use of the nonadditive travel costs, see, e.g., [1]. Here, we will suffice with the additive formula (3.21).

The decision problem of each firm $a \in J$ is to maximize its profits, given as the difference toll revenues and building and maintenance costs by choosing an appropriate level of capacity and toll charge on the link it operates. Let the costs of firm $a \in J$ be given by $\eta I_a(y_a)$, where parameter η is common to each toll firm and for simplicity the unit period project costs do not include variable costs of road use.

Putting all parts of the model together, the traffic equilibrium model with private toll roads is given as a system of maximization problems for $a \in J$

$$\begin{array}{ll} \underset{u_a, y_a}{\text{maximize}} & \sum_{r \in R} \delta_{ar} F_r u_a - \eta I_a(y_a) \\ \text{subject to} & 0 \leq \sum_{a \in \mathcal{A}} \alpha \delta_{ar} t_a((\Delta F)_a, y_a) + \sum_{a \in J} \delta_{ar} u_a - \mu_w \perp F_r \geq 0, \quad \forall r \in R_w, w \in W, \\ & \sum_{r \in R_w} F_r = D_w(\mu), \qquad \forall w \in W, \\ & \mu_w \geq 0, \qquad \forall w \in W. \end{array} \tag{3.22}$$

Each such problem (3.22) is clearly of an MPCC structure and all of them are linked together via upper-level decision variables and a solution to (3.18)-(3.20). Hence they constitute an EPCC.

3.3 Existence of solutions

The reformulation (3.7) plays an important role in application of the implicit programming approach to EPECs. This technique to reduce a bilevel program to the upper-level optimization problem under assumption (A1') is widely used for MPECs, see [39]. In terms of EPECs, we reduce our problem via implicit programming to a generally nonconvex Nash game. This enables us to use existence results developed for Nash games. Let us recall the concept of mixed strategies (or mixed solutions).

Definition 3.2. *(mixed strategy)*
A mixed strategy *for a player with a set of admissible strategies U is a probability measure μ in the set U, i.e., it is a nonnegative and σ-additive measure on U with $\mu(U) = 1$.*

The set U is usually called a *set of pure strategies* (or an action space). In the case when this probability measure degenerates to a Dirac measure, a probability measure that assigns a singleton the measure 1, we arrive at the pure strategy.

A mixed strategy is usually interpreted on the concept of repeated games. The player then no longer plays each time only one particular strategy but he or she plays all strategies from his or her action space U and the frequencies with which pure strategies are played will converge to the probability distribution generated by his or her mixed strategy. In what follows, we allow only leaders to play mixed strategies.

Let us consider the following two assumptions concerning the admissible sets $U^i, i = 1, \ldots, n$, of leaders

(A4) for each $i = 1, \ldots, n$, the set U^i is compact;

(A5) for each $i = 1, \ldots, n$, the set U^i is convex;

and two assumptions imposed on their cost functions

(A6) for each $i = 1, \ldots, n$, function $\theta^i : \omega \to \mathbb{R}$ is continuous on an open set containing ω;

(A7) for each $i = 1, \ldots, n$, function $\theta^i : \omega \to \mathbb{R}$ is strictly convex in variable x^i for all values of $x^k \in U^k, k \neq i$.

Now let us recall the Nash theorem [2, Theorem 2.12] which states the existence of a solution to a non-hierarchical n-person Nash game under assumptions (A4)-(A7). We have already mentioned that the hierarchical structure causes *loss of the convexity* of the cost function of the upper-level player. Thus, the assumption (A7) is generally not satisfied for EPECs. If this was the case, an MPEC could have multiple local optima, the existence of which could be guaranteed, e.g., by Theorem 2.2. However, violation of assumption (A7) may result in non-existence of any (pure strategy) solution of an EPEC, cf. Example 3.1 below.

On the other hand, mixed strategy concept of solutions appears well justified for EPECs with implicit structure because the existence of a mixed solution to Nash equilibrium game can be achieved under validity of much weaker assumptions.

Theorem 3.3. *Let the assumptions (A4) and (A6) hold. Then the n-person game specified by mathematical programs (3.7), $i = 1, \ldots, n$, admits at least one equilibrium point in mixed strategies.*

Proof. For proof see [41]. □

Before we provide a similar existence theorem for EPEC composed of mathematical programs (3.5), we present an example of EPEC with two leaders and one follower. This example shows how easily the solution in pure strategies may not exist even in the case when convexity assumptions appear to be satisfied. It also vindicates the need to investigate conditions ensuring the existence of a solution in mixed strategies.

Example 3.4. Consider the following three-person game on $[0, 2]^2 \times [-2, 2]$ with cost functions

$$\begin{aligned}
\varphi^1(x^1, x^2, x^3) &= (x^1 - x^2)^2 - x^1 + x^2 + x^3, \\
\varphi^2(x^1, x^2, x^3) &= (x^1 - x^2)^2 - 2(x^3)^2, \\
\varphi^3(x^1, x^2, x^3) &= (x^3)^2 - 2x^1x^3 + 2x^2x^3.
\end{aligned}$$

As a three person Nash game, assumptions (A4)-(A7) are satisfied and consequently there is a solution in pure strategies.

Consider now the situation when player 1 and player 2 become the leaders. The game can be reduced to a game only among the leaders with cost functions

$$\begin{aligned}
\theta^1(x^1, x^2) &= (x^1 - x^2)^2, \\
\theta^2(x^1, x^2) &= -(x^1 - x^2)^2,
\end{aligned}$$

with the solution map of the follower specified by

$$S(x^1, x^2) = x^1 - x^2, (x^1, x^2) \in [0, 2]^2.$$

Even if the cost function of the second player is now non-convex, according to Theorem 3.3, a solution in mixed strategies exists. Based on the results in [6], the solution of this EPEC is

$$\begin{aligned}
\bar{x}^1 &= 1 && \text{with probability } 1, \\
\bar{x}^2 &= \begin{cases} 0 & \text{with probability } 1/2, \\ 2 & \text{with probability } 1/2, \end{cases} \\
\bar{x}^3 &= \bar{x}^1 - \bar{x}^2 && \text{with probability } 1.
\end{aligned}$$

We purposely write the solution of the third player in the above form to emphasize that he or she always plays a pure strategy, although actually it is 1 with probability $\frac{1}{2}$ and -1 with probability $\frac{1}{2}$.

Clearly, since this is the only solution in mixed strategies, the considered EPEC does not have any pure strategy solution. △

It is apparent that existence theory of n-person Nash games can be easily applied to EPECs when assumption (A1') holds and we do not allow followers to play mixed strategies. We state the existence result for mixed solutions to EPECs in the following theorem.

Theorem 3.5. *Let the assumptions (A1') and (A4) hold and let for all $i = 1, \ldots, n$, functions φ^i be continuous on an open set containing $\omega \times \mathbb{R}^{ml_2}$. Then the EPEC (3.5) possesses at least one solution in mixed strategies.*

Proof. Assumption (A1') together with continuity of functions $\varphi^i, i = 1, \ldots, n$, guarantee validity of assumption (A6). It remains to apply Theorem 3.3. □

The assumption (A1') plays a crucial role in the above theorem and the existence of solutions to the EPEC can be guaranteed via rather lenient conditions. It turns out that in the absence of (A1'), the situation gets much more complicated, as explained next.

Consider that (A1') does not hold simply because $S(x)$ is not single valued for some $x \in \omega$. Let us consider first the case when this is true for strategies of just one follower. Without loss of generality we can omit the remaining followers and analyze a multi-leader-single-follower game.

When the lower problem is not uniquely solvable, we may apply the *optimistic hypothesis*, this time, however, only *with respect to one of the leaders*. I.e, we may replace the loss function of the kth leader for a chosen $k = 1, \ldots, n$, by the *marginal function*

$$\psi^k(x) = \inf_{y \in S(x)} \varphi^k(x, y). \tag{3.23}$$

We can associate to this marginal function the so-called *marginal map*

$$M_k^o(x) = \{u \in S(x) | \varphi^k(x, u) = \inf_{y \in S(x)} \varphi^k(x, y)\}.$$

This marginal function reflects an expectation of leader k that the follower will try to help him or her to achieve the best outcome. Similarly, using

$$\Phi^k(x) = \sup_{y \in S(x)} \varphi^k(x, y),$$

one speaks of the *pessimistic formulation with respect to leader k*. The corresponding marginal map

$$M_k^p(x) = \{u \in S(x) | \varphi^k(x, u) = \sup_{y \in S(x)} \varphi^k(x, y)\}$$

expresses a reasonable expectation of the kth leader that the follower, if there is a chance to choose from several strategies leading to the same (optimal) outcome, will try to harm leader k as much as possible.

The optimistic position can be expected, e.g., in the case when the follower participates in profits of the kth leader. Compared to the realistically applicable pessimistic position, the optimistic position may violate legislative constraints (in some cases the cooperation is forbidden by legislation) or natural constraints (cooperation is not possible, e.g., in so-called games against the nature), for the discussion see [12].

Let us first investigate a situation when the kth leader is able to persuade the follower to select an optimal solution which accommodates his purposes best. Due to the presence of other leaders we need to investigate the behavior of the suggested couple of players carefully.

To ensure continuity of marginal function (3.23) of the kth leaders, it is sufficient to ensure existence of the so-called *continuous selection* of S. This is guaranteed if S is a continuous multifunction.

It is known that ensuring lower semicontinuity of S without single-valuedness may be problematic or quite restrictive. The possible lack of lower semicontinuity of the solution mapping may lead to a very unstable solution, cf. [12].

One of the suitable ways to test both lower and upper semicontinuity is the use of the powerful *Mordukhovich criterion*

$$D^*S(x,y)(0) = \{0\} \tag{3.24}$$

which ensures the Aubin property of multifunction S around a point (x, y). Expressing (3.24) in terms of the initial data of our problem (3.5), together with the qualification condition from [27, Theorem 6.10] we obtain the condition

$$0 \in (\nabla_y F(x,y))^T w + D^*N(y, -F(x,y); \Omega)(w) \Rightarrow w = 0. \tag{3.25}$$

Unfortunately, this condition forces S to be single-valued and locally Lipschitz continuous whenever Ω is polyhedral, see [15]. If $\nabla_x F$ is not surjective, we can avoid this drawback by replacing the condition (3.25) with the condition on calmness of multifunction

$$P(q) = \{(x,y) \mid (y, -F(x,y)) + q \in \operatorname{Gph} N(\cdot; \Omega)\},$$

see [21, Theorem 6] and related results therein.

Similarly to S, the marginal map need not be single-valued for some values of x. On the other hand, no matter what $y \in M_k^o(x)$ is chosen by the follower, the value of the marginal function remains the same. However, this is no longer true for the cost functions of other leaders; their values may be, of course, influenced by the choice of a marginal selection $\sigma_k^o(x)$ from $M_k^o(x)$.

The objectives of the remaining leaders can be expressed in the form

$$\psi^i(x) = \varphi^i(x, \sigma_k^o(x)), \ i \neq k.$$

To ensure the continuity of all $\psi^i, i \neq k$, the existence of a continuous selection of M_k^o is not sufficient. Even if such a selection exists, the follower may have no intention to play it and nor can any leader force him or her to do so. Hence we need to impose additional assumptions under which the marginal map is single-valued. This is the main difference from the analysis of the optimistic formulation of an MPEC.

The respective sufficient conditions are stated in the lemma below.

Lemma 3.6. *Let $\varphi(x,y)$ be continuous and strictly convex function in y for all x and let S be continuous, convex- and compact-valued multifunction. Then function*

$$M^o(x) = \underset{y\in S(x)}{\arg\min}\, \varphi(x,y)$$

is single-valued and continuous.

Proof. The single-valuedness of M^o is clear from the assumptions.

Assume that by contradiction there is a sequence $x_i \to \bar{x}$ with $\bar{y} = M^o(\bar{x}) \subset S(\bar{x})$ and $y_i = M^o(x_i)$ such that $y_i \to y^0 \neq \bar{y}$. The condition $y_i = M^o(x_i)$ is equivalent to $(y_i \in S(x_i), \varphi(x_i, y_i) = \inf_{z\in S(x_i)} \varphi(x_i, z))$. According to [3, Theorem 1.4.16], under our assumptions the marginal function $\inf_{z\in S(x_i)} \varphi(x_i, z)$ is continuous and hence for $x_i \to \bar{x}$ converges to $\inf_{z\in S(\bar{x})} \varphi(\bar{x}, z)$ which equals to $\varphi(\bar{x}, y^0)$ with $y^0 \in S(\bar{x})$. This means that $y^0 = M^o(\bar{x})$ which is in contradiction with $\bar{y} \neq y^0$. □

Note that, in fact, we analyze the problem as a hierarchical three-level game where on the new, middle level the follower selects from the solution map strategies that belong to the single-velued marginal map.

We can now state the conditions ensuring the existence of a solution to the optimistic formulation of EPEC with respect to the kth leader.

Theorem 3.7. *Let assumption (A4) hold, S be convex- and compact-valued, functions $\varphi^i, i = 1,\ldots,n$, be continuous on an open set containing $\omega \times \mathbb{R}^{ml_2}$, φ^k be strictly convex in y for all values of $x \in \omega$. Further, assume that for all $(\bar{x},\bar{y}) \in \mathrm{Gph}\, S$ the multifunction P is calm at $(0,\bar{x},\bar{y})$ and let the condition*

$$0 \in (\nabla_y F(\bar{x},\bar{y}))^\top w + D^*N(\bar{y}, -F(\bar{x},\bar{y}); \Omega)(w) \Rightarrow w \in \mathrm{Ker}(\nabla_x F(\bar{x},\bar{y}))^\top \tag{3.26}$$

hold. Then the EPEC composed of problems (3.7), $i = 1,\ldots,n$, with functions θ^i replaced by ψ^i admits a solution in mixed strategies.

Proof. According to Theorem 3.3, we need to ensure that functions $\theta^i, i = 1,\ldots,n$ are continuous on ω.

To this end we first invoke the result [3, Theorem 1.4.16] stating that the marginal function θ^k is continuous if φ^k is continuous and S is compact-valued and continuous. For continuity of S we apply the results from [21, Theorem 6 and formula (39)] to $h(x,y) = (y, -F(x,y)), \Lambda = \mathrm{Gph}\, N(;\Omega), \Theta = \mathbb{R}^{ml_2}$ and $v = (u, -w)$ which ensures the Aubin property of S around each point from $\mathrm{Gph} S$.

Applying Lemma 3.6 we get $M^o_k = \sigma^o_k$ single-valued and continuous. This implies that the cost functions $\psi^i, i \neq k$, are continuous which completes the proof. □

Compact-valuedness of S can be obtained, e.g., by requiring the set Ω to be compact and function $F(x,y)$ to be continuous. To ensure the convex-valuedness of S, we need Ω to be a convex set and $F(x,y)$ monotone in y for all admissible values of x, i.e.,

$$\left\langle F(x,y^1) - F(x,y^2), y^1 - y^2 \right\rangle \geq 0, \ \ \forall y^1, y^2 \in \Omega.$$

It is clear from the proof of Theorem 3.7 that the continuity of the marginal function is not sensitive to whether we consider optimistic or pessimistic formulation of the problem with respect to the kth leader. However, where needed, convexity of $\varphi^k(x,y)$ has to be replaced by concavity.

Lemma 3.8. *Let $\varphi(x,y)$ be continuous and strictly concave function in y for all values of x and let S be continuous and convex- and compact-valued multifunction. Then the function*

$$M^p(x) = \arg\max_{y\in S(x)} \varphi(x,y)$$

is single-valued and continuous.

Proof. The proof is analogous to the proof of Lemma 3.6. □

Similarly to optimistic bilevel problem, the objectives of the remaining leaders are in the form

$$\Phi^i(x) = \varphi^i(x, \sigma_k^p(x)),\ i \neq k,$$

where σ_k^p is a marginal selection from M_k^p. Now, we can obtain results corresponding to Theorem 3.7.

Theorem 3.9. *Let the assumption (A4) hold, S be convex- and compact-valued, functions $\varphi^i, i = 1,\ldots,n$, be continuous on an open set containing $\omega \times \mathbb{R}^{ml_2}$, φ^k be strictly concave in y for all x. Further, assume that for all $(\bar{x},\bar{y}) \in \operatorname{Gph} S$ the multifunction P is calm at $(0,\bar{x},\bar{y})$ and let the condition*

$$0 \in (\nabla_y F(\bar{x},\bar{y}))^\top w + D^*N(\bar{y}, -F(\bar{x},\bar{y}); \Omega)(w) \Rightarrow w \in \operatorname{Ker}(\nabla_x F(\bar{x},\bar{y}))^\top$$

hold. Then the EPEC composed of problems (3.7), $i = 1,\ldots,n$, with functions θ^i replaced by Φ^i admits a solution in mixed strategies.

Proof. The proof is analogous to the proof of Theorem 3.7. □

One can easily find examples to see that the continuity of the solution map is not necessary for continuity of the marginal selection. On the other hand, the Aubin property seems too restrictive for ensuring lower semicontinuity. It would be worth investigating more precise criteria to achieve the results of Theorems 3.7 and 3.9.

Let us now briefly discuss the general case with multiple followers and with the multi-valued lower level solution mapping. Without loss of generality assume just the game with two followers, both able to respond to the leaders' strategies by playing more than just one optimal reaction. Naturally, each follower in question behaves independently and influences the range of multiple rational reactions of the other one. This makes the treatment of a general situation extremely difficult, e.g., when one follower aims to harm one particular leader and the second follower tries to do the same to another leader.

We may avoid the above mentioned complications if the decision to select the common strategy is not taken by the followers themselves, e.g., this could be the role of an independent entity on separate middle level in the hierarchy of decision making. We aim to address these issues in detail in our future analysis of particular situations.

3.4 Stationarity concepts and existence of stationary points

In this section we present conditions associated with suitable stationarity concepts for EPECs. These conditions are connected to the respective necessary optimality conditions for MPECs due to the structural dependence of EPECs on MPECs.

First, take a look at the EPEC for which each of n MPECs can be formulated as the mathematical program (3.7). For this EPEC we can derive stationarity conditions analogous to conditions (2.14).

Theorem 3.10. *Let $(\bar{x}^1, \dots, \bar{x}^n, \bar{y})$ be a solution of the EPEC composed of n MPECs (3.7). Then for each $i = 1, \dots, n$, there exists a vector $\xi^i \in N(\bar{x}^i; U^i)$ such that*

$$0 \in \nabla_{x^i}\varphi^i(\bar{x}^i, \bar{x}^{-i}, \bar{y}) + D^* S_{\bar{x}^{-i}}(\bar{x}^i)(\nabla_y \varphi^i(\bar{x}^i, \bar{x}^{-i}, \bar{y})) + \xi^i. \tag{3.27}$$

Proof. It suffices to apply Theorem 2.13 to each MPEC (3.7). □

The system of conditions (3.27) amounts to a natural stationary concept for EPECs. As we emphasized in the previous chapter, in most cases we are unable to compute the coderivative term in (3.27) precisely, and consider weaker conditions, replacing the coderivative term with suitable upper approximation.

Let us now focus on the case when the lower-level solution map S is given by the NCP (2.17), i.e., on the EPCC (3.6).

To proceed to the EPCC counterparts of MPCC stationarities, we will make use of the EPEC versions of the MPEC generalized Mangasarian-Fromowitz constraint qualification and MPEC generalized linear independence constraint qualification.

Definition 3.11. *(EPEC generalized MFCQ and LICQ)*
We say that the EPCC (3.6) satisfies

i) EPEC generalized MFCQ *(EPEC-GMFCQ) at a feasible point $(\bar{x}^1, \dots, \bar{x}^n, \bar{y})$ if for each $i = 1, \dots, n$, the MPCC in (3.6) satisfies MPEC-GMFCQ at $(\bar{x}^i, \bar{x}^{-i}, \bar{y})$.*

ii) EPEC generalized LICQ *(EPEC-GLICQ) at a feasible point $(\bar{x}^1, \dots, \bar{x}^n, \bar{y})$ if for each $i = 1, \dots, n$, the MPCC in (3.6) satisfies MPEC-GLICQ at $(\bar{x}^i, \bar{x}^{-i}, \bar{y})$.*

Clearly, as it was true for the MPEC versions of the respective constraint qualifications, EPEC-GLICQ implies EPEC-GMFCQ.

Analogously, one can define KKT-type stationarity conditions for EPCCs by those for MPCCs.

Definition 3.12. *(strongly, M-, C- and weakly stationary point for EPCC)*
Let $(\bar{x}^1, \dots, \bar{x}^n, \bar{y})$ be feasible for the EPCC (3.6). Then we call $(\bar{x}^1, \dots, \bar{x}^n, \bar{y})$ strongly (M-, C- and weakly) stationary point for the EPCC *(3.6) if for each $i = 1, \dots, n$, $(\bar{x}^i, \bar{x}^{-i}, \bar{y})$ is strongly (M-, C- and weakly) stationary point for the MPCC in (3.6).*

Also for EPCCs we have the chain of implications

strong stationarity ⇒ M-stationarity ⇒ C-stationarity ⇒ weak stationarity.

Following the same arguments used in the proof of Theorem 2.6 we get the following results.

Theorem 3.13. *Let $(\bar{x}^1, \ldots, \bar{x}^n, \bar{y})$ be a solution of the EPCC (3.6). If EPEC-GMFCQ holds at $(\bar{x}^1, \ldots, \bar{x}^n, \bar{y})$ then it is M-stationary point for the EPCC and thus also C-stationary point for the EPCC.*

Proof. It suffices to apply Theorem 2.6 to MPCC in (3.6), for each $i = 1, \ldots, n$. □

The complementarity constraints of each MPCC in (3.6), $i = 1, \ldots, n$, can be equivalently reformulated as the generalized equation

$$0 \in \begin{pmatrix} F^1(x^i, \bar{x}^{-i}, y) - \nu^i \\ F^2(x^i, \bar{x}^{-i}, y) \end{pmatrix} + N(y, \nu^i; \mathbb{R}^{ml_2} \times \mathbb{R}^{ml_2}_+). \tag{3.28}$$

Whenever for every $i = 1, \ldots, n$, the generalized equation (3.28) is strongly regular at $(\bar{x}^i, \bar{y})$, we can apply Theorem 2.17 and the technique of adjoint equations to derive the Clarke stationary conditions for EPCC.

Definition 3.14. *(Clarke stationarity conditions to EPCC)*
Let $(\bar{x}^1, \ldots, \bar{x}^n, \bar{y})$ be a feasible point for the EPCC (3.6) and let each generalized equation (3.28) be strongly regular at $(\bar{x}^i, \bar{y}), i = 1, \ldots, n$. Then we call $(\bar{x}^1, \ldots, \bar{x}^n, \bar{y})$ Clarke stationary for the EPCC *if for each $i = 1, \ldots, n$, the following conditions*

$$0 \in \nabla_{x^i}\varphi^i(\bar{x}, \bar{y}) - \operatorname{conv}\left\{ \begin{pmatrix} \nabla_{x^i} F^1_{L\cup(I^0\setminus M_j)}(\bar{x}, \bar{y}) \\ \nabla_{x^i} F^2_{I^+\cup M_j}(\bar{x}, \bar{y}) \end{pmatrix}^\top p^{ij}(\bar{x}, \bar{y}) | j \in \mathbb{K}(\bar{x}, \bar{y}) \right\} + N(\bar{x}^i; U^i), \tag{3.29}$$

are satisfied, where $p^{ij}(\bar{x}, \bar{y})$ are the unique solutions of the adjoint equations

$$\begin{pmatrix} \nabla_y F^1_{L\cup(I^0\setminus M_j)}(\bar{x}, \bar{y}) \\ \nabla_y F^2_{I^+\cup M_j}(\bar{x}, \bar{y}) \end{pmatrix}^\top p = \nabla_y \varphi^i(\bar{x}, \bar{y}). \tag{3.30}$$

Theorem 3.15. *Let $(\bar{x}^1, \ldots, \bar{x}^n, \bar{y})$ be a solution of the EPCC (3.6). Let each generalized equation (3.28) be strongly stationary at $(\bar{x}^i, \bar{x}^{-i}, \bar{y})$ and for all $j \in \mathbb{K}(\bar{x}, \bar{y})$ the vectors $p^{ij}(\bar{x}, \bar{y})$ be the unique solutions of (3.30). Then for each $i = 1, \ldots, n$, conditions (3.29) are fulfilled. In particular, the point $(\bar{x}, \bar{y})$ is Clarke stationary for the EPCC.*

Proof. The statement follows from Theorem 2.20 applied separately to each MPCC in (3.6). □

As a consequence of Theorems 2.24 and 2.26, under SRC for each of n generalized equations (3.28) and EPEC-GLICQ satisfied, we verify also the equivalence of Clarke and C-stationarity for EPCCs. This is the statement of the following theorem.

Theorem 3.16. *Let $(\bar{x}^1, \ldots, \bar{x}^n, \bar{y})$ be a feasible point for the EPCC (3.6). Let EPEC-GLICQ hold at $(\bar{x}_1, \ldots, \bar{x}_n, \bar{y})$ and for each $i = 1, \ldots, n$, let SRC hold for the generalized equation (3.28) at $(\bar{x}^i, \bar{x}^{-i}, \bar{y})$. Then the point $(\bar{x}^1, \ldots, \bar{x}^n, \bar{y})$ is Clarke stationary for the EPCC if and only if it is C-stationary for the EPCC.*

Note that the assumption of the strong regularity of each generalized equation (3.28) at the reference point can be replaced by stronger assumption of the strong regularity of the generalized equation

$$0 \in \begin{pmatrix} F^1(x^1, \ldots, x^n, y) - \nu \\ F^2(x^1, \ldots, x^n, y) \end{pmatrix} + N(y, \nu; \mathbb{R}^{ml_2} \times \mathbb{R}^{ml_2}_+).$$

at the reference point.

In [37] one can find an *existence theorem* for Clarke stationary points for EPECs. This theorem can be reformulated for EPCCs as follows.

Theorem 3.17. *Let assumptions (A1'), (A4) and (A5) be fulfilled and suppose that for each $i = 1, \ldots, n$ the multifunctions $\Gamma^i : x \rightrightarrows \bar{\partial} S_{x^{-i}}(x^i)$ are upper semicontinuous on ω. Then the EPCC (3.6) possesses a Clarke stationary point.*

Proof. To prove that there is a Clarke stationary point for the EPCC it is sufficient to show that the generalized equation

$$0 \in C(x) + N(x; \omega) \tag{3.31}$$

with the multifunction

$$C(x) = \begin{pmatrix} \nabla_{x^1} \varphi^1(x, S(x)) \\ \vdots \\ \nabla_{x^n} \varphi^n(x, S(x)) \end{pmatrix} + \begin{pmatrix} \Gamma^1(x)^\top \\ \vdots \\ \Gamma^n(x)^\top \end{pmatrix} \begin{pmatrix} \nabla_y \varphi^1(x, S(x)) \\ \vdots \\ \nabla_y \varphi^n(x, S(x)) \end{pmatrix}$$

has a solution. For details see [37, Theorem 3.3]. $\square$

To our knowledge, Theorem 3.17 provided up to now the only result derived for EPECs concerning the existence of (at least) stationary points. As a corollary of Theorems 3.16 and 3.17 we can present the following existence result.

Corollary 3.18. *Let assumptions (A4) and (A5) be fulfilled and for each $i = 1, \ldots, n$, the generalized equation (3.28) be strongly regular and MPEC-GLICQ hold at every feasible point of the MPCC in (3.6). Further, let the solution map of each generalized equation (3.28), $i = 1, \ldots, n$, be single-valued and multifunctions Γ^i be upper semicontinuous on ω. Then the EPCC (3.6) possesses a C-stationary point.*

Recall that even though the respective generalized equations are assumed to be strongly regular at each feasible point, the solution map may not be single-valued. Strong regularity implies single-valuedness only locally on the neighborhood of the point from the graph of S. For global unicity of the solution to the generalized equation one can suppose, e.g., strict monotonicity of the single-valued part of the generalized equation, see [39, Theorem 4.4].

Note that, e.g., when the complementarity constraints are in the form of a linear complementarity problem, the assumption of upper semicontinuity of the multifunctions $\Gamma^i, i = 1, \ldots, n$, is automatically satisfied since the solution maps $S_{x^{-i}}$ do not depend on x^{-i}.

We can modify the conditions in the above theorem and corollary by weakening the compactness assumptions, for in many applications the sets $U^i, i = 1, \ldots, n$, (and thus also the set ω) are unbounded. To this end, we present the following two modifications of Theorem 3.17. Before we proceed with the statement and its proof, recall that a set $V \subset \mathbb{R}^n$ is said to be *contractible*, if there is a point $x^0 \in V$ and a continuous function $g : V \times [0,1] \to V$, such that

$$g(x,0) = x \text{ and } g(x,1) = x^0 \text{ for each } x \in V.$$

Theorem 3.19. *Let assumptions (A1') and (A5) be fulfilled. Further, suppose that there is a convex set E with nonempty interior such that*

i) the set $\omega \cap E$ is nonempty and compact;

ii) multifunctions $\Gamma^i, i = 1, \ldots, n$, restricted to $\omega \cap E$ are upper semicontinuous;

iii) for each $x \in \omega \cap \mathrm{bdry}(E)$ there is an $x^0 \in \omega \cap \mathrm{int}(E)$ such that

$$\langle y, x - x^0 \rangle \geq 0 \quad \textit{for all } y \in C(x).$$

Then the EPCC (3.6) possesses a Clarke stationary point.

Proof. To prove the existence of a Clarke stationary point to EPCC (3.6) it suffices to apply [17, Theorem 3.2] to the generalized equation (3.31). All but assumptions (iii)(b) and (iii)(c) of [17, Theorem 3.2] follow directly from the statement of the theorem.

Clearly, the multifunction C is nonempty-, convex- and compact-valued. Further, $C(x)$ is contractible for each $x \in \omega \cap E$, since $C(x)$ is a convex set. Thus condition (iii)(c) of [17, Theorem 3.2] is satisfied. The condition ii) implies upper semicontinuity of C restricted to $\omega \cap E$ and hence also assumption (iii)(b) of [17, Theorem 3.2] holds. This completes the proof. □

The second modification involves generalization of *coercivity for set-valued maps.*

Theorem 3.20. *Let the sets $U_i, i = 1, \ldots, n$, be closed and convex (possibly unbounded) and let assumption (A1') be fulfilled. Further, suppose that multifunctions $\Gamma^i, i = 1, \ldots, n$, are upper semicontinuous on ω and that there is an $x^0 \in \omega$ such that*

$$\lim_{\|x\| \to +\infty} \left(\inf_{y \in C(x)} \frac{\langle y, x - x^0 \rangle}{\| x \|} \right) = +\infty$$

Then the EPCC (3.6) possesses a Clarke stationary point.

Proof. Analogously to Theorem 3.19, it suffices to apply [17, Corollary 3.1] to the generalized equation (3.31). □

Similarly to Corollary 3.18, on the basis of Theorems 3.19 and 3.20 we can derive existence results also for C-stationary points to EPCC using Theorem 3.16.

Chapter 4

Multiobjective Problem with Equilibrium Constraints (MOPEC)

In the previous chapter we have focused on EPECs in which the upper problem admits a structure of a Nash game, i.e., when leaders act noncooperatively. We can investigate also the opposite sort of "extreme" situation when leaders cooperate by solving a multiobjective optimization problem. This brings us to the study of a class of *multiobjective problems with equilibrium constraints.*

In this chapter the main attention is paid to MOPECs with equilibrium constraints in the form of *mixed complementarity problem* (MCP). We discuss the existence of solutions to this problem and derive its necessary optimality conditions.

4.1 Mathematical formulation

Following the notation introduced in previous chapters, suppose yet again that our problem involves n leaders and m followers. In the multiobjective problem with equilibrium constraints, the behavior of leaders is not only described by their individual objectives $\varphi^i : \mathbb{R}^{nl_1+ml_2} \to \mathbb{R}, i = 1, 2, \ldots, n$, but this time also by a closed convex cone $K \subset \mathbb{R}^n$ that specifies an *ordering* of $\mathbb{R}^n$ in the standard way:

$$z^1 \preceq z^2 \;\Leftrightarrow\; z^2 - z^1 \in K. \tag{4.1}$$

When discussing solutions to multiobjective problems, we speak of *Pareto optimal points.* As it is common in standard optimization literature, we distinguish two notions of (generalized) Pareto optimality. Denote by φ the map from $\mathbb{R}^{nl_1+ml_2}$ to $\mathbb{R}^n$ such that $\varphi := (\varphi^1, \varphi^2, \ldots, \varphi^n)^\top$. Then for the (unconstrained) multiobjective problem

$$\text{minimize } \varphi(x, y)$$

with respect to partial ordering induced by a cone K, a multistrategy $(\bar{x}, \bar{y})$ is called *strongly Pareto optimal* if there is no multistrategy (x, y) such that $(x, y) \neq (\bar{x}, \bar{y})$ and

$$\varphi(\bar{x}, \bar{y}) - \varphi(x, y) \in K.$$

However, to our purposes, a weaker notion of Pareto optimality proves to be more suitable.

A multistrategy $(\bar{x}, \bar{y})$ is called *weakly Pareto optimal* if there is no multistrategy (x, y) such that

$$\varphi(\bar{x}, \bar{y}) - \varphi(x, y) \in \mathrm{rint} K.$$

Note that for $K = \mathbb{R}^n_+$ we arrive at the standard notions of Pareto optimality.

Now we are able to define a weak Pareto solution of an abstract MOPEC.

Definition 4.1. *(weak Pareto solution of abstract MOPEC)*
A multistrategy $(\bar{x}, \bar{y}) \in \mathbb{R}^{nl_1} \times \mathbb{R}^{ml_2}$ is a weak Pareto solution of an abstract MOPEC

$$\begin{aligned} &\underset{K}{\mathit{minimize}} && \varphi(x, y) \\ &\mathit{subject\ to} && y \in S(x), \\ & && (x, y) \in \kappa, \end{aligned} \tag{4.2}$$

if $\bar{y} \in S(\bar{x})$ and there is a neighborhood $\mathcal{U}$ of $(\bar{x}, \bar{y})$ such that for all $(x, y) \in \mathcal{U} \cap \kappa$ with $y \in S(x)$ we have

$$\varphi(\bar{x}, \bar{y}) - \varphi(x, y) \notin \mathrm{rint} K. \tag{4.3}$$

Here again, S denotes the solution map to the lower problem for given multistrategy of all leaders and κ denotes the set of nonequilibrium constraints.

Also in this case, we implicitly assume the optimistic formulation of the problem. If the multifunction S is not single-valued, it is clear from the above definition that the problem (4.2) is still well-defined, contrary to the EPEC case.

As before, $S(x)$ can be, e.g., a solution set to the equilibrium problem in the form of generalized equation

$$0 \in F(x, y) + Q(x, y) \tag{4.4}$$

with single-valued function F and multifunction Q which in many practical cases is a normal cone mapping.

Now, let us specify the behavior of players on the lower level, just as we did when discussing EPECs, but with a slight modification in the structure of followers' feasible sets. Let the multistrategies $y \in \mathbb{R}^{ml_2}$ be feasible provided that all of its components $y^j, j = 1, \ldots, m$, belong to given boxes (intervals) $\mathbb{I}^j \subset \mathbb{R}^{l_2}$.

Let the followers act according to their objectives $f^j : \mathbb{R}^{nl_1 + ml_2} \to \mathbb{R}, j = 1, 2, \ldots, m$. Hence, for a given multistrategies $\bar{x} \in \omega$ and $\bar{y}^{-i}$ the strategy of the jth follower amounts to a solution of the optimization problem

$$\begin{aligned} &\underset{y^j}{\text{minimize}} && f^j(\bar{x}, \bar{y}^j, \bar{y}^{-j}) \\ &\text{subject to} && y^j \in \mathbb{I}^j. \end{aligned} \tag{4.5}$$

Recall that we assume that the assumption (A0) from page 35 is fulfilled. Given $\bar{x}$, a corresponding multistrategy $\bar{y}$ amounts this time to a solution of the *mixed complementarity problem* defined by

$$0 \in F(\bar{x}, y) + N(y; \mathbb{I}), \tag{4.6}$$

where

$$\mathbb{I} := \bigtimes_{j=1}^{m} \mathbb{I}^j.$$

Now, the solution map S is given by

$$S(x) := \{y \in \mathbb{R}^{ml_2} |\, 0 \in F(x,y) + N(y;\mathbb{I})\}.$$

Thus we arrive at a special MOPEC, where the behavior of the followers is described by an MCP of the type above. According to the accepted terminology, we therefore speak of a *multiobjective problem with complementarity constraints*, abbreviated to MOPCC.

Let again $\kappa = \omega \times \mathbb{R}^{ml_2}$, where ω amounts to $\bigtimes_{i=1}^{n} U^i$ with $U^i \subset \mathbb{R}^{l_1}$ being the set of feasible strategies of leader i.

Definition 4.2. *(weak Pareto solution of MOPCC)*
A multistrategy pair $(\bar{x}, \bar{y}) \in \omega \times \mathbb{I}$ is declared to be a weak Pareto solution of an MOPCC *if*

$$0 \in F(\bar{x},\bar{y}) + N(\bar{y};\mathbb{I}),$$

and there is a neighborhood $\mathcal{U}$ of $(\bar{x},\bar{y})$ such that for all $(x,y) \in \mathcal{U} \cap (\omega \times \mathbb{I})$ for which

$$0 \in F(x,y) + N(y;\mathbb{I})$$

we have

$$\varphi(\bar{x},\bar{y}) - \varphi(x,y) \notin \mathrm{rint} K. \tag{4.7}$$

We can summarize the corresponding multiobjective optimization problem as follows:

$$\begin{aligned} \underset{K}{\text{minimize}} \quad & \varphi(x,y) \\ \text{subject to} \quad & 0 \in F(x,y) + N(y;\mathbb{I}), \\ & x \in \omega. \end{aligned} \tag{4.8}$$

Note that in [28], [30], and [53] one can find results for substantially more general concepts of multiobjective optimization.

4.2 Existence of weak Pareto solutions

In this section we pay attention to conditions ensuring the existence of optimal solutions to MOPECs (4.8). For this we apply recent results involving *subdifferential calculus for set-valued mappings* and new conditions of *coercivity* and *Palais-Smale* type each of which ensures the existence of optimal solutions to set-valued optimization problems with non-compact feasible sets [4, 5]. Multiobjective (or vector-valued) optimization problems can be viewed as a special case of the set-valued optimization.

First, we need to introduce some important extensions of the notion of subdifferential to set-valued mappings, developed in [4, 5].

Consider a set-valued mapping $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and the partial ordering on $\mathbb{R}^m$ specified by a nonempty closed cone $K \subset \mathbb{R}^m$. Using (4.1) we can define the *generalized epigraph* of H with respect to the partial ordering by

$$\mathrm{epi}H = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | y \in H(x) + K\}. \tag{4.9}$$

Note that $\mathrm{epi}H = \mathrm{Gph}H$ if $K = \{0\}$. Otherwise we have the strict inclusion $\mathrm{Gph}H \subset \mathrm{epi}H$.

By means of the generalized epigraph (4.9) we can associate with H and K the *epigraphical multifunction* $\mathcal{E}_H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ given by

$$\mathcal{E}_H(x) = \{y \in \mathbb{R}^m | y \in H(x) + K\}.$$

The limiting and singular subdifferentials of multivalued mapping H are generated by the coderivative of its epigraphical multifunction.

Definition 4.3. *(limiting and singular subdifferentials of a multifunction)*
Let a multifunction H map $\mathbb{R}^n$ into $\mathbb{R}^m$ with partial ordering on $\mathbb{R}^m$ induced by a cone K and let $(\bar{x}, \bar{y}) \in epiH$. Then the limiting subdifferential of the multifunction *H at $(\bar{x}, \bar{y})$ is defined by*

$$\partial_K H(\bar{x}, \bar{y}) := \{x^* \in D^*\mathcal{E}_H(\bar{x}, \bar{y})(z^*) | -z^* \in N(0; K), \|z^*\| = 1\} \tag{4.10}$$

and the singular subdifferential *of H at $(\bar{x}, \bar{y})$ is defined by*

$$\partial_K^\infty H(\bar{x}, \bar{y}) := D^*\mathcal{E}_H(\bar{x}, \bar{y})(0). \tag{4.11}$$

Note that in case of extended real valued functions on $\mathbb{R}^n$ the subdifferentials (4.10) and (4.11) reduce to the classical limiting and singular subdifferentials, respectively, provided $K = \mathbb{R}^+$, i.e., with the standard order on $\mathbb{R}$.

Let us now define the set-valued counterpart of coercivity condition.

Definition 4.4. *(coercivity of set-valued mappings)*
We say that the multifunction $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ satisfies the coercivity condition *if there is a compact set $\Theta \subset \mathbb{R}^n$ such that*

$$\left.\begin{matrix} x \in \mathbb{R}^n \setminus \Theta \\ y \in H(x) \end{matrix}\right\} \Rightarrow \exists\, (u, v) \in \mathrm{Gph}\ H \ \ \textit{with} \ \ u \in \Theta \ \ \textit{and} \ \ v \leq y. \tag{4.12}$$

For the definition of set-valued counterpart of Palais-Smale condition we need to introduce a generalization of boudedness from below.

Definition 4.5. *(quasiboudedness from below)*
For a set-valued mapping $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and a set $\Xi \subset \mathbb{R}^m$, we say that F is quasibounded from below *with respect to Ξ if there is a bounded set M such that*

$$H(\Xi) \subset M + K,$$

where K is the cone specifying the generalized order optimality and $H(\Xi) = \bigcup_{x \in \Xi} H(x)$.
A set $A \subset \mathbb{R}^m$ is quasibounded from below if the constant mapping $H(x) \equiv A$ has this property.

Definition 4.6. *(subdifferential Palais-Smale condition)*
A set-valued mapping $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ satisfies the subdifferential Palais-Smale condition *if any sequence $\{x^{(k)}\} \subset \mathbb{R}^n$ such that there are*

$$y^{(k)} \in H(x^{(k)}) \text{ and } x^{(k)*} \in \partial_K H(x^{(k)}, y^{(k)}) \text{ with } \|x^{(k)*}\| \to 0 \text{ for } k \to \infty$$

contains a convergent subsequence provided that $\{y^{(k)}\}$ is quasibounded from below.

Recall the classical Palais-Smale condition for differentiable real-valued function $\varphi : \mathbb{R}^n \to \mathbb{R}$. Let $\{x^{(k)}\} \subset \mathbb{R}^n$ be a sequence such that $\{\varphi(x^{(k)})\}$ is bounded from below and $\|\nabla\varphi(x^{(k)})\| \to 0$ as $k \to \infty$. Then $\{x^{(k)}\}$ contains a convergent subsequence.

It is known, that the Palais-Smale condition implies coercivity for C^1 functions and locally Lipschitz functions. In [4] the authors presented an existence result for the constrained set-valued optimization problem

$$\begin{aligned} \underset{K}{\text{minimize}} \quad & H(x) \\ \text{subject to} \quad & x \in \Xi, \end{aligned} \tag{4.13}$$

first under coercivity condition imposed on H and later under a corresponding version of the subdifferential Palais-Smale condition.

Whenever the constraint set Ξ of the set-valued optimization problem (4.13) involves also equilibrium constraints, we arrive at a class of problems called *set-valued optimization problems with equilibrium constraints* (SOPECs). These problems are formally defined as

$$\begin{aligned} \underset{K}{\text{minimize}} \quad & H(x) \\ \text{subject to} \quad & 0 \in G(x) + Q(x), \\ & x \in \kappa, \end{aligned} \tag{4.14}$$

where H, G and Q are generally set-valued mappings. Note that for H and G single-valued we get an MOPEC.

Let us denote by Min $H(x)$ the *collection of minimal points* of the set $H(x)$ defined by

$$\operatorname{Min} H(x) = \{\bar{y} \in H(x) | \bar{y} - y \notin K \text{ whenever } y \in H(x) \setminus \{\bar{y}\}\}.$$

Similarly, replacing K with $\operatorname{rint} K \neq \emptyset$ we obtain the *collection of weakly minimal points.*

To obtain a version of subdifferential Palais-Smale condition for the constraint case (4.13), it suffices to consider Definition 4.6 with a restriction of H to Ξ

$$H_\Xi(x) = H(x) + \Delta(x, \Xi) \quad \text{with} \quad \Delta(x, \Xi) = \begin{cases} 0 \in \mathbb{R}^m & \text{if } x \in \Xi, \\ \emptyset & \text{otherwise.} \end{cases}$$

The following proposition taken from [4] and [5] states the conditions for the existence of weak minimizers to the problem (4.13).

Proposition 4.7. *Let $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be quasibounded from below with respect to Ξ and have a closed epigraph. Let the sets Ξ and $\operatorname{Min} H(x)$ for $x \in \Xi$ be closed and let*

$$\text{for every } x \in \Xi \text{ and } y \in H(x) \text{ there is } \bar{y} \in \operatorname{Min} H(x) \text{ with } \bar{y} \leq y. \tag{4.15}$$

Then the set-valued optimization problem (4.13) admits a weak minimizer in each of the following cases:

i) Let the constraint set Ξ be compact.

ii) Let the multifunction H satisfy the coercivity condition.

iii) Let the following version of the subdifferential Palais-Smale condition hold: Any sequence $\{x^{(k)}\} \subset \Xi$ such that there are

$$y^{(k)} \in H(x^{(k)}) \text{ and } x^{(k)*} \in \partial_K H(x^{(k)}, y^{(k)}) + N(x^{(k)}; \Xi) \text{ with } \|x^{(k)*}\| \to 0 \text{ for } k \to \infty$$

contains a convergent subsequence provided that $\{y^{(k)}\}$ is quasibounded from below. In addition, assume that for every $(x, y) \in \operatorname{Gph} H$ with $x \in \Xi$ the qualification condition

$$\partial_K^\infty H(x, y) \cap (-N(x; \Xi)) = \{0\} \tag{4.16}$$

is satisfied.

Proof. The first and the second assertion follow from [4, Theorem 4.1] and the third statement is a finite-dimensional counterpart of [5, Theorem 3.2]. □

Let us now apply the previous proposition to the MOPCC (4.8). This enables us to state the following the conditions ensuring the existence of weak Pareto solutions to the considered MOPCC.

Theorem 4.8. *Let φ be lower-semicontinuous and let the mapping $\tilde{\varphi}(x, y, z) := \varphi(x, y)$ be quasibounded from below with respect to*

$$\operatorname{Gph}(-F) \cap (\mathbb{R}^{nl_1} \times \operatorname{Gph} N(\cdot; \mathbb{I})) \cap (\omega \times \mathbb{R}^{ml_2} \times \mathbb{R}^{ml_2}),$$

F be a continuous function and ω and $\mathbb{I}$ be closed sets. Then the MOPCC (4.8) admits a weak Pareto solution in each of the following cases:

i) The sets ω and $\mathbb{I}$ are compact.

ii) The function φ is coercive.

iii) Let the following version of the subdifferential Palais-Smale condition be satisfied: any sequence $\{(x^{(k)}, y^{(k)})\} \subset \omega \times \mathbb{R}^{ml_2}$ such that there are $z^{(k)} = -F(x^{(k)}, y^{(k)}) \in N(y^{(k)}; \mathbb{I})$, $y^{(k)*} \in \mathbb{R}^{ml_2}$, $\|x^{(k)*}\| \to 0$ with

$$x^{(k)*} \in \partial_K \varphi(x^{(k)}, y^{(k)}) + D^* F(x^{(k)}, y^{(k)}, -z^{(k)})(y^{(k)*}) + \{0\} \times D^* N(y^{(k)}, z^{(k)}; \mathbb{I})(y^{(k)*}) + N(x^{(k)}; \omega) \times \{0\} \tag{4.17}$$

contains a convergent subsequence provided that $\{z^{(k)}\}$ is quasibounded from below.

In addition, whenever $x \in \omega, z = -F(x, y) \in N(y; \mathbb{I})$ and $y^* \in \mathbb{R}^{ml_2}$, let the following qualification conditions be fulfilled:

$$-\partial_K^\infty \varphi(x, y) \cap (D^* F(x, y, -z)(y^*) + \{0\} \times D^* N(y, z; \mathbb{I})(y^*) + N(x; \omega) \times \{0\}) = \{0\}, \tag{4.18}$$

$$\left.\begin{array}{l} x^{1*} \in D^* F(x, y, -z)(y^*) \\ x^{2*} \in \{0\} \times D^* N(y, z; \mathbb{I})(y^*) \\ x^{3*} \in N(x; \omega) \times \{0\} \\ \ \ 0 = x^{1*} + x^{2*} + x^{3*} \end{array}\right\} \Rightarrow \left\{ \begin{array}{l} \ \ y^* = 0 \\ x^{1*} = x^{2*} = x^{3*} = 0 \end{array}\right. \tag{4.19}$$

Proof. Clearly, for the MOPCC (4.8) it suffices to set $H := \tilde{\varphi}$ and

$$\Xi := \text{ Gph } (-F) \cap (\mathbb{R}^{nl_1} \times \text{Gph } N(\cdot; \mathbb{I})) \cap (\omega \times \mathbb{R}^{ml_2} \times \mathbb{R}^{ml_2}).$$

Observe that for F continuous, ω and $\mathbb{I}$ compact, the set Ξ is also compact. Also, for single-valued functions the condition (4.15) is trivially satisfied. This proves the first statement.

The second statement follows immediately from the proposition above.

It remains to prove the third statement. Taking into account the relationship

$$(x^*, y^*) \in N(x, y, -z;\ \text{Gph } (-F)) \Leftrightarrow (x^*, -y^*) \in N(x, y, z;\ \text{Gph } F)$$

and applying the intersection rule for limiting normal cones [29, Corollary 3.37], we get the inclusion

$$\begin{aligned} &N(x, y, z; \Xi) \subset \\ &\quad \subset N(x, y, z;\ \text{Gph } (-F)) + N(x, y, z; \mathbb{R}^{nl_1} \times \text{Gph } N(\cdot; \mathbb{I})) + N(x, y, z; \omega \times \mathbb{R}^{ml_2} \times \mathbb{R}^{ml_2}) \end{aligned}$$

provided the qualification condition (4.19) holds, where we set $z = -F(x, y) \in N(y; \mathbb{I})$ and applied the definition of coderivative.

Observe that

$$\partial_K^\infty \tilde{\varphi}(x, y, z) = \partial_K^\infty \varphi(x, y) \times \{0\}.$$

Also, the inclusion $(x^*, 0) \in N(x, y, z; \Xi)$ implies

$$x^* \in D^*F(x, y, -z)(y^*) + \{0\} \times D^*N(y, z; \mathbb{I})(y^*) + N(x; \omega) \times \{0\}$$

with some $y^* \in \mathbb{R}^{ml_2}$.

This also implies the qualification condition (4.18) from (4.16) and the version of subdifferential Palais-Smale condition with (4.17) which completes the proof. $\square$

4.3 Necessary optimality conditions

In the recent book by Mordukhovich [30], the whole section 5.3.5 is devoted to necessary optimality conditions for MOPECs with equilibrium constraints given by the generalized equation (4.4) on infinite-dimensional spaces. Hence we kindly refer the reader to [30] and present here only the specification of these necessary optimality conditions to the MOPCC (4.8).

Let $(\bar{x}, \bar{y})$ be a weak Pareto solution of the MOPCC (4.8) and assume that φ is locally Lipschitz continuous around $(\bar{x}, \bar{y})$. Further suppose that

$$\mathbb{I}^i = [a^i, b^i]$$

and that $a^i_j < b^i_j$ for all $j = 1, \ldots, l_2$, and $i = 1, \ldots, m$. Similarly to previous cases when we were dealing with complementarity constraints, let $k := (i-1)l_2 + j, 0 < j \leq l_2$, and let us employ the following index sets:

$$\begin{aligned} L(y) :=& \{k \in \{1, \ldots ml_2\} | a^i_j < y_k < b^i_j\}, \\ I^+_1(x, y) :=& \{k \in \{1, \ldots ml_2\} | F_k(x, y) > 0\}, \\ I^+_2(x, y) :=& \{k \in \{1, \ldots ml_2\} | F_k(x, y) < 0\}, \\ I^0_1(x, y) :=& \{k \in \{1, \ldots ml_2\} | y_k = a^i_j, F_k(x, y) = 0\}, \\ I^0_2(x, y) :=& \{k \in \{1, \ldots ml_2\} | y_k = b^i_j, F_k(x, y) = 0\} \end{aligned} \tag{4.20}$$

related to the constraint $y \in \mathbb{I}$. As before, the arguments x, y will be omitted whenever it cannot lead to a confusion. To simplify the notation, put

$$I^+(x, y) := I^+_1(x, y) \cup I^+_2(x, y), \ \ I^0(x, y) := I^0_1(x, y) \cup I^0_2(x, y).$$

The following optimality conditions represent a modification of [53, Theorem 5.29]. Note that we do not need to assume the single-valuedness of S in this statement.

Theorem 4.9. *Let $(\bar{x}, \bar{y})$ be a weak Pareto solution of MOPCC.*

i) Then there exist vectors $\bar{z} \in K^-, \bar{u}, \bar{v} \in \mathbb{R}^{ml_2}$, not simultaneously equal to zero, such that the following hold:

$$\bar{u}_L = 0 \ \ \text{and} \ \ \bar{v}_{I^+} = 0,$$

for $k \in I_1^0(\bar{x},\bar{y})$ either $\bar{u}_k\bar{v}_k = 0$ or $\bar{u}_k < 0$ and $\bar{v}_k > 0$,
for $k \in I_2^0(\bar{x},\bar{y})$ either $\bar{u}_k\bar{v}_k = 0$ or $\bar{u}_k > 0$ and $\bar{v}_k < 0$,

and one has the inclusion

$$0 \in D^*\varphi(\bar{x},\bar{y})(-\bar{z}) + \begin{pmatrix} -(\nabla_x F(\bar{x},\bar{y}))^\top \bar{v} + N(\bar{x};\omega) \\ \bar{u} - (\nabla_y F(\bar{x},\bar{y}))^\top \bar{v} \end{pmatrix}. \tag{4.21}$$

ii) Assume further that either F is affine and ω is convex polyhedral, or the constraint qualification

$$\left.\begin{array}{r} -(\nabla_x F(\bar{x},\bar{y}))^\top v \in -N(\bar{x},\omega), \\ u - (\nabla_y F(\bar{x},\bar{y}))^\top v = 0, \\ u_L = 0, v_{I^+} = 0, \\ \text{for } k \in I_1^0(\bar{x},\bar{y}) \text{ either } u_k v_k = 0 \text{ or } u_k < 0 \text{ and } v_k > 0, \\ \text{for } k \in I_2^0(\bar{x},\bar{y}) \text{ either } u_k v_k = 0 \text{ or } u_k > 0 \text{ and } v_k < 0, \end{array}\right\} \Rightarrow v = 0 \tag{4.22}$$

is fulfilled. Then $\bar{z} \neq 0$.

Proof. To justify the first statement, we rewrite our MCP

$$0 \in F(x,y) + N(y;\mathbb{I}) \tag{4.23}$$

to the form

$$\Phi(x,y) \in \Lambda,$$

where $\Phi(x,y) = \begin{pmatrix} y \\ -F(x,y) \end{pmatrix}$ and $\Lambda = \text{Gph } N(\cdot;\mathbb{I})$. To proceed, we first apply to problem (4.8) the general results on multiobjective optimization involving weak Pareto optimality, see [28] and [30, Theorem 5.80], and then use a calculus rule to compute the basic normal cone to the constraint set defined by

$$M := \left\{(x,y) \in \mathbb{R}^{nl_1+ml_2} \mid \Phi(x,y) \in \Lambda\right\};$$

see, e.g., [27, Theorem 6.10]. This allows us, by taking into account the special structure of the mapping Φ and the set Λ, to reduce calculations to computing the normal cone $N(\bar{y}, -F(\bar{x},\bar{y});\Lambda)$ of the graphical set $\Lambda = \text{Gph } N(\cdot;\mathbb{I})$. The latter has been done in [36, Lemma 2.2]. To complete the proof of the first part of the theorem, it remains to observe that $N(0;K) = K^-$.

To derive the second part of the theorem, the "qualified" form of the necessary optimality conditions with $\bar{z} \neq 0$, we invoke the result from [28, Theorem 3.2]. This gives us a vector $\bar{z} \in N(0;K) \setminus \{0\}$ satisfying

$$0 \in D^*\varphi(\bar{x},\bar{y})(-\bar{z}) + N(\bar{x},\bar{y};\Delta),$$

where $\Delta = \{(x,y) \in \omega \times \mathbb{R}^{ml_2} \mid \Phi(x,y) \in \Lambda\}$. Assuming now the constraint qualification (4.22) imposed in the theorem, we conclude by [27, Theorem 6.10] that the normal cone $N(\bar{x},\bar{y};\Delta)$ is included into the set given as the second term on the right-hand side of (4.21).

If, as an alternative to the constraint qualification (4.22), F is affine and ω is convex polyhedral, we observe that the map

$$M(p) = \{(x,y) \in \omega \times \mathbb{R}^{ml_2} \mid \Phi(x,y) + p \in \Lambda\}$$

happens to be calm at $(0,\bar{x},\bar{y})$. Since we clearly have $\Delta = M(0)$, the desired representation of $N(\bar{x},\bar{y};\Delta)$ is now provided by [20, Theorem 4.1]. This completes the proof of the theorem. □

Note that the constraint qualification (4.22) imposed in Theorem 4.9 is exactly the respective MPEC-GMFCQ.

Let us localize the assumption (A1') from page 36 as follows:

(A1") S is single-valued and locally Lipschitz continuous on a neighborhood of $\bar{x}$.

The latter enables us to derive the following qualified form of necessary optimality conditions for the MOPCC under consideration.

Theorem 4.10. *Let $(\bar{x},\bar{y})$ be a weak Pareto solution of MOPCC. Suppose that assumption (A1") is fulfilled and that the modified constraint qualification*

$$\left.\begin{aligned} &-(\nabla_x F(\bar{x},\bar{y}))^\top v = 0\\ &u - (\nabla_y F(\bar{x},\bar{y}))^\top v = 0\\ &u_L = 0, v_{I^+} = 0\\ &\text{for } k \in I_1^0(\bar{x},\bar{y}) \text{ either } u_k v_k = 0 \text{ or } u_k < 0 \text{ and } v_k > 0\\ &\text{for } k \in I_2^0(\bar{x},\bar{y}) \text{ either } u_k v_k = 0 \text{ or } u_k > 0 \text{ and } v_k < 0 \end{aligned}\right\} \Rightarrow v = 0 \tag{4.24}$$

holds true. Then there exists a nonzero vector $\bar{z} \in K^-$ and multipliers $\bar{u},\bar{v} \in \mathbb{R}^{ml_2}$ satisfying the relationships

$$\bar{u}_L = 0 \text{ and } \bar{v}_{I^+} = 0,$$

$$\text{for } k \in I_1^0(\bar{x},\bar{y}) \text{ either } \bar{u}_k\bar{v}_k = 0 \text{ or } \bar{u}_k < 0 \text{ and } \bar{v}_k > 0,$$

$$\text{for } k \in I_2^0(\bar{x},\bar{y}) \text{ either } \bar{u}_k\bar{v}_k = 0 \text{ or } \bar{u}_k > 0 \text{ and } \bar{v}_k < 0,$$

as well as the inclusion (4.21).

Proof. Our problem can be rewritten to the form of the multiobjective program

$$\begin{aligned} &\underset{K}{\text{minimize}} \;\; \Theta(x)\\ &\text{subject to} \;\; x \in \omega, \end{aligned} \tag{4.25}$$

where $\Theta(x) := \varphi(x, S(x))$. By virtue of (A1”), the map Θ is locally Lipschitz continuous around $\bar{x}$. It follows from [28, Theorem 3.2] the existence of a nonzero vector $\bar{z} \in K^-$ satisfying

$$0 \in D^*\Theta(\bar{x})(-\bar{z}) + N(\bar{x}; \omega).$$

It remains therefore to compute the coderivative of the map Θ. From [27, Theorem 5.1] we have the inclusion

$$D^*\Theta(\bar{x})(w^*) \subset \{a + D^*S(\bar{x})(b) | (a, b) \in D^*\varphi(\bar{x}, \bar{y})(w^*)\}$$

for all $w^* \in \mathbb{R}^n$ due to the assumptions imposed on φ and S. To compute the coderivative of S, we rewrite our MCP to the form (4.23) and employ again [27, Theorem 6.10]. It follows from this result that under the constraint qualification (4.24) we have the inclusion

$$\begin{aligned} D^*S(\bar{x})(y^*) \subset \quad & \{-(\nabla_x F(\bar{x}, \bar{y}))^\top v | -y^* = u - (\nabla_x F(\bar{x}, \bar{y}))^\top v, u_L = 0, v_{I^+} = 0, \\ & \text{for } k \in I_1^0(\bar{x}, \bar{y}) \text{ either } u_k v_k = 0 \text{ or } u_k < 0 \text{ and } v_k > 0 \\ & \text{and for } k \in I_2^0(\bar{x}, \bar{y}) \text{ either } u_k v_k = 0 \text{ or } u_k > 0 \text{ and } v_k < 0\} \end{aligned}$$

for all $y^* \in \mathbb{R}^{ml_2}$. The latter allows us to conclude that

$$D^*\Theta(\bar{x})(\bar{z}) \subset \{a - (\nabla_x F(\bar{x}, \bar{y}))^\top v | 0 = b + u - (\nabla_y F(\bar{x}, \bar{y}))^\top v, (a, b) \in D^*\varphi(\bar{x}, \bar{y})(\bar{z})\},$$

that $u_L = 0$ and $v_{I^+} = 0$, that either $u_k v_k = 0$ or $u_k < 0$ and $v_k > 0$ for $k \in I_1^0(\bar{x}, \bar{y})$, and that either $u_k v_k = 0$ or $u_k > 0$ and $v_k < 0$ for $k \in I_2^0(\bar{x}, \bar{y})$. This is exactly what we need, whence the proof is complete. □

We conclude this section with several remarks.

It follows from the proof of Theorem 4.10 that the modified constraint qualification (4.24) is needed only for the computation of $D^*S(\bar{x})$ in terms of F. If F is affine, we do not need any constraint qualification at all; cf. [21, Theorem 6]. The modified constraint qualification (4.24) from Theorem 4.10 is less restrictive than the constraint qualification (4.22) from Theorem 4.9. The reason is that the MCP under consideration is not coupled with the constraint $x \in \omega$.

To ensure the localized assumption (A1”), it is sufficient to suppose that the MCP satisfies the strong regularity condition at $(\bar{x}, \bar{y})$. However, SRC ensures simultaneously the constraint qualification (4.22) of Theorem 4.9, cf. [36, Proposition 2.6].

Chapter 5

Solution Methods for EPECs and MOPECs

The previous chapters have focused on theoretical aspects of MPECs, EPECs and MOPECs. In this final chapter we discuss numerical methods to obtain solutions to EPECs and MOPECs. We first give a brief overview of existing approaches. The main part of this chapter is devoted to the generalization of a homotopy method to search for C-stationary points of EPCCs with simple structure. Final section concerns the solution method for MOPECs which invokes implicit programming approach.

5.1 Overview

5.1.1 Diagonalization methods

The first approaches used by researches to solve problems from the EPEC class are the *diagonalization type methods* based on algorithms developed specifically for MPECs. The main idea is to solve one MPEC at a time and repeat this procedure *cyclically* for every MPEC. The computed solutions are then used to update the multistrategy vector of leaders until a fixed point of this operation is found.

In [49] one can find a detailed description of the *nonlinear Jacobi* and the *nonlinear Gauss-Seidel diagonalization methods*. The former one works as follows:

1) Choose a feasible starting multistrategy $(x^{(0)}, y^{(0)})$ of the EPEC, maximum number of iterations $J \in \mathbb{N}$ and accuracy tolerance $\varepsilon > 0$ and set $k = 1$;

2) For each $i = 1, \dots, n$, fix $x^{-i,(k-1)}$ and solve the MPEC of the ith leader. Denote the solutions for leaders of these problems as $x^{i,(k)}$;

3) Check the accuracy tolerance, i.e., if $||x^{i,(k)} - x^{i,(k-1)}|| < \varepsilon$ for each $i = 1, \dots, n$, then STOP and declare $(x^{(k)}, y^{(k)})$ as the solution, else go to step 4;

4) If $k < J$, then increase k by one and go to step 2 else report that the procedure failed to find a solution.

The Gauss-Seidel method improves the Jacobi method since the "updated information" about the vector of leaders' multistrategies is used immediately after solving each MPEC and not just after completed cycle. Hence Gauss-Seidel method works as follows:

1) Choose a feasible starting multistrategy $(x^{(0)}, y^{(0)})$ of the EPEC, maximum number of iterations $J \in \mathbb{N}$ and accuracy tolerance $\varepsilon > 0$ and set $k = 1$;

2) For each $i = 1, \ldots, n$, fix $(x^{1,(k)}, \ldots, x^{i-1,(k)}, x^{i+1,(k-1)}, \ldots, x^{n,(k-1)})$ and solve the MPEC for the ith leader. Denote the solution of this problem as $x^{i,(k)}$.

3) Check the accuracy tolerance, i.e., if $||x^{i,(k)} - x^{i,(k-1)}|| < \varepsilon$ for each $i = 1, \ldots, n$, then STOP and declare $(x^{(k)}, y^{(k)})$ as the solution, else go to step 4;

4) If $k < J$, then increase k by one and go to step 2 else report that the procedure failed to find a solution.

For details on these methods, see [22] or [49].

5.1.2 Sequential nonlinear complementarity method

The algorithms based on the diagonalization method can be generally used for EPEC with the equilibrium constraints in arbitrary form. A *sequential NCP method*, introduced in [49], was designed for EPCCs with the equilibrium constraint governed by the NCP (2.17).

After a *regularization of the complementarity condition*, these constraints amount to

$$\begin{aligned} F^1(x^i, \bar{x}^{-i}, y) &\geq 0, \\ F^2(x^i, \bar{x}^{-i}, y) &\geq 0, \\ F_j^1(x^i, \bar{x}^{-i}, y) F_j^2(x^i, \bar{x}^{-i}, y) &\leq t, \quad j = 1, \ldots, ml_2, \end{aligned} \tag{5.1}$$

where $t > 0$. Replacing the complementarity constraints by (5.1), we transform the MPCC into a regularized nonlinear program. This is done simultaneously for each MPCC. The sequential NCP method then can be described as solving, under assumption of EPEC-LICQ, a sequence of mixed complementarity problems for $t \searrow 0$. These mixed complementarity problems correspond to collection of first-order KKT systems of each regularized NLP.

A detailed comparison of the numerical performance of the above mentioned methods on randomly generated EPECs in the form of EPCCs with quadratic objectives and linear complementarity constraints, can be found in [49].

5.1.3 Price-consistent NCP method

In [24] another special type of EPEC is considered. The equilibrium constraints represent the first order conditions of the following optimization problems of the followers

$$\begin{aligned} \underset{y^j}{\text{minimize}} \quad & f^j(\bar{x}, y^j, \bar{y}^{-j}) \\ \text{subject to} \quad & c^j(\bar{x}, y^j, \bar{y}^{-j}) \geq 0, \\ & y^j \geq 0. \end{aligned} \tag{5.2}$$

Again, using the solution mapping S of the lower problem, the ith leader is trying to solve the MPEC in the form

$$\begin{aligned} \underset{x^i, y}{\text{minimize}} \quad & \varphi^i(x^i, \bar{x}^{-i}, y) \\ \text{subject to} \quad & d^i(x^i, \bar{x}^{-i}, y) \geq 0, \\ & x^i \geq 0, \\ & y \in S(x^i, \bar{x}^{-i}). \end{aligned} \tag{5.3}$$

Note that in [24] the upper-level objectives as well as the constraints are allowed to depend also on multipliers of the constraints $c^j(x, y) \geq 0$. Here, to be consistent with the previous parts of the thesis, we restrict ourselves to the above considered case.

To achieve a price consistent restriction of the EPEC composed of MPECs (5.3), the following assumptions are considered.

(A8) For each $i = 1, \ldots, n$, the nonequilibrium constraints of the ith leader consist of a set of constraints of the form $d^i(x^i) \geq 0$ and a set of constraints $d(x, y) \geq 0$ common for all leaders;

(A9) For each $i = 1, \ldots, n$, the objective function of the ith leader includes a term dependent on x only and a term common for all leaders, i.e., the objective is of the form $\varphi^i(x, y) = \varphi^i(x) + \varphi(x, y)$;

If we strengthen the assumption (A9) such that each upper-level objective is of the form

$$\varphi^i(x, y) = \varphi^i(x^i) + \varphi(x, y),$$

then we say that the EPEC composed of MPECs (5.3) is *completely separable*.

Generally, this problem entails three sets of players; the leaders, the followers and the markets. The markets decide about the multipliers (shadow prices) of the (resource) constraints. When considering the so-called *price-consistent problem*, the multipliers associated with the common constraints are set to be the same. This allows us to eliminate a large number of multipliers and to reduce significantly the size of the problem.

Under assumptions (A8) and (A9), we can reduce the price-consistent EPEC to the following problem

$$
\begin{aligned}
\bar{x}^i &\in \left\{ \operatorname*{arg\,min}_{x^i \geq 0, d^i(x^i) \geq 0} \ \varphi^i(x^i, \bar{x}^{-i}) + \varphi(x^i, \bar{x}^{-i}, \bar{y}) - d(x^i, \bar{x}^{-i}, \bar{y})^\top \bar{\lambda} - (h(x^i, \bar{x}^{-i}, \bar{y}, \bar{z}) - \bar{s})^\top \bar{\mu} \right\}, \\
& \quad i = 1, \ldots, n, \\
(\bar{y}, \bar{z}, \bar{s}) &\in \left\{ \operatorname*{arg\,min}_{y \geq 0, z \geq 0, s \geq 0} \ \varphi(\bar{x}, y) - d(\bar{x}, y)^\top \bar{\lambda} - (h(\bar{x}, y, z) - s)^\top \bar{\mu} + \bar{\sigma}(y^\top \ z^\top) s \right\}, \\
(\bar{\lambda}, \bar{\mu}, \bar{\sigma}) &\in \left\{ \operatorname*{arg\,min}_{\lambda \geq 0, \mu, \sigma \geq 0} \ d(\bar{x}, \bar{y})^\top \lambda + (h(\bar{x}, \bar{y}, \bar{z}) - \bar{s})^\top \mu - \sigma(\bar{y}^\top, \ \bar{z}^\top) \bar{s} \right\},
\end{aligned}
\tag{5.4}
$$

where

$$
h(x, y, z) = \begin{pmatrix} \nabla_{y^1} f^1(x, y) - \nabla_{y^1} c^1(x, y) z^1 \\ \vdots \\ \nabla_{y^m} f^m(x, y) - \nabla_{y^m} c^m(x, y) z^m \\ c^1(x, y) \\ \vdots \\ c^m(x, y) \end{pmatrix}.
$$

Assume in addition the complete separability. Then the following result relates the price consistent restriction (5.4) to a special MPEC.

Proposition 5.1. *Assume that the EPEC (5.3) is completely separable. Then the first order optimality conditions of (5.4) are equivalent to the strong stationarity conditions of the following MPEC:*

$$
\begin{aligned}
\operatorname*{minimize}_{x} \quad & \sum_{i=1}^{n} \varphi^i(x^i) + \varphi(x, y) \\
\text{subject to} \quad & d^i(x^i) \geq 0, \quad i = 1, \ldots, n, \\
& d(x, y) \geq 0, \\
& h(x, y, z) - s = 0, \\
& 0 \leq \begin{pmatrix} y \\ z \end{pmatrix} \perp s \geq 0.
\end{aligned}
\tag{5.5}
$$

Proof. For proof see [24, Proposition 5.1]. □

The problem above can be interpreted as finding one particular solution to a multiobjective optimization problem. Since price consistency is a restriction, any solution to (5.4) or (5.5) is a solution to the original problem. Clearly, in this way one may not be able to find a solution even in the case when it exists.

For detailed comparison of the diagonalization methods, sequential nonlinear complementarity method and a numerical solution method of the price consistent restriction of EPEC, see [24, Section 6].

5.2 Homotopy method for computation of C-stationary points to EPCCs

None of the above methods is without a significant drawback. The sequence of points produced by diagonalization methods may not converge. Moreover, even if there is a limit to this sequence and the lower problem is not uniquely solvable, the limit point may not be a solution of EPEC. The sequential nonlinear complementarity method leads to solving a sequence of large and complex complementarity problems. The price-consistent method depends on highly restrictive assumptions on the structure of the problem, which in some applications could not be justified.

In this section we intend to design the first numerical method tailored specifically to the structure of the EPCC. However, even we could not avoid to impose some restrictive assumptions. The most crucial restriction concerns the dimension of the lower problem; this helps us to slightly simplify the description of the proposed algorithm.

Let us turn our attention to the simplest form of EPCC (3.6) with convex-quadratic objective functions φ^i constrained only by the lower problem in the form of a *one-dimensional* linear complementarity problem

$$\begin{aligned} &\text{for a given vector } x \text{ find } y \in \mathbb{R} \\ &\qquad\text{such that } 0 \leq Ax + by + a \perp y \geq 0 \end{aligned} \tag{5.6}$$

with a row vector $A \in \mathbb{R}^{1 \times nl}$ and real constants a, b.

Further we assume that $b > 0$, which is sufficient for the linear complementarity problem (5.6) to be uniquely solvable, [33]. This problem corresponds to the necessary and sufficient first order optimality conditions of the convex-quadratic parametric optimization problem

$$\begin{aligned} \underset{y}{\text{minimize}} \quad & \frac{1}{2} by^2 + (Ax + a)y \\ \text{subject to} \quad & y \geq 0. \end{aligned} \tag{5.7}$$

Hence, in this section we aim to analyze and propose a numerical method for the class of EPCCs associated with n convex-quadratic MPCCs, where the ith mathematical program, $i = 1, \dots, n$, has the form

$$\begin{aligned} \underset{x^i, y}{\text{minimize}} \quad & \frac{1}{2} (x^\top, y) Q^i \begin{pmatrix} x \\ y \end{pmatrix} + (c^i)^\top \begin{pmatrix} x \\ y \end{pmatrix} \\ \text{subject to} \quad & 0 \leq Ax + by + a \perp y \geq 0 \end{aligned} \tag{5.8}$$

with the symmetric matrix $Q^i \in \mathbb{R}^{nl_1+1} \times \mathbb{R}^{nl_1+1}$ and the vector $c^i \in \mathbb{R}^{nl_1+1}$. Further we assume that the square submatrix which results from Q^i by deletion of rows and columns with indices corresponding to parameters x^{-i} is positive definite.

In [42], the authors introduced two versions of a piecewise affine homotopy method which search for C-stationary points of convex-quadratic mathematical programs with linear complementarity constraints. The methods performed surprisingly well and its elegant geometric interpretation inspired us to consider modification of the homotopy method I, [42], tailored to the EPCC above.

We admit that EPCCs with only one follower with scalar decision variable seem too restrictive. Also, in the view of restrictions we impose upon the data, cf. the next section, nontrivial description of the algorithm and the fact that the proposed numerical method may not find any C-stationary point even if there is one, the practical use our modified homotopy method is questionable.

On the other hand, during the process we gained a detailed, previously unknown information about the structure of the sets of stationary points and solutions to the considered class of problems.

5.2.1 Parameter-free problem

Analogously to the general case (2.40), we can define the index sets $I^+(\bar{x},\bar{y}), L(\bar{x},\bar{y})$ and $I^0(\bar{x},\bar{y})$ associated with problem (5.6), setting $F(x,y) = Ax + by + a$.

For the ith MPCC (5.8), $i = 1,\dots,n$, and a feasible point $(\bar{x},\bar{y})$ we can explicitly write down *MPEC-LICQ* as the following condition: The $(l_1+1)\times 2$ matrix

$$\begin{pmatrix} (A^\top_{L\cup I^0})_{x^i} & 0_{I^+\cup I^0} \\ b_{L\cup I^0} & 1_{I^+\cup I^0} \end{pmatrix}$$

has the full column rank. The EPEC-LICQ is then said to hold at $(\bar{x},\bar{y})$, if MPEC-LICQ holds at $(\bar{x},\bar{y})$ for each MPCC (5.8), $i = 1,\dots,n$.

Denote by λ^i and μ^i the multipliers of the ith mathematical program (5.8) corresponding to the constraints $Ax + by + a \geq 0$ and $y \geq 0$, respectively. Let us write

$$Q^i = \begin{pmatrix} Q^i_{xx} & Q^i_{xy} \\ Q^i_{yx} & Q^i_{yy} \end{pmatrix}.$$

Then the stationarity conditions for the program (5.8), cf. Definition 2.4, consist of conditions (2.27) in form of a system of linear equations

$$0 = \begin{pmatrix} (Q^i_{xx})_{x^i} & (Q^i_{xy})_{x^i} & -(A^\top_{L\cup I^0})_{x^i} & 0 \\ Q^i_{yx} & Q^i_{yy} & -b_{L\cup I^0} & -1_{I^+\cup I^0} \\ A_{L\cup I^0} & b_{L\cup I^0} & 0 & 0 \\ 0 & 1_{I^+\cup I^0} & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{\lambda}^i_{L\cup I^0} \\ \bar{\mu}^i_{I^+\cup I^0} \end{pmatrix} + \begin{pmatrix} c^i_{x^i} \\ c^i_y \\ a_{L\cup I^0} \\ 0 \end{pmatrix}, \tag{5.9}$$

and, additionally, the respective conditions on multipliers $\bar{\lambda}_{I^0}$ and $\bar{\mu}_{I^0}$.

If MPEC-LICQ holds true, strong stationarity conditions and hence also all other types of stationarity conditions are the first order necessary optimality conditions [45, Theorem 7(1)].

The collection of conditions (5.9) for each $i = 1,\dots,n$, together into one system produces a non-square system of linear equations. Recall that we assume $b > 0$ and thus the variable y is uniquely determined by the vector x. We can therefore treat the variable y

in each MPCC separately, denoting it by y^i. This allows us to work with the following square system of linear equations where, implicitly, variables y^i for all $i = 1, \dots, n$ attain the same value:

$$0 = \begin{pmatrix} Q_{xx} & Q_{xy} & -\tilde{A}_{L\cup I^0} & 0 \\ Q_{yx} & Q_{yy} & -\tilde{B}^\top_{L\cup I^0} & -E^\top_{I^+\cup I^0} \\ \bar{A}_{L\cup I^0} & \tilde{B}_{L\cup I^0} & 0 & 0 \\ 0 & E_{I^+\cup I^0} & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{x} \\ \tilde{y} \\ \bar{\lambda}_{L\cup I^0} \\ \bar{\mu}_{I^+\cup I^0} \end{pmatrix} + \begin{pmatrix} c_x \\ c_y \\ \tilde{a}_I \\ 0 \end{pmatrix}, \tag{5.10}$$

where

$$Q_{xx} = \begin{pmatrix} (Q^1_{xx})_{x^1} \\ \vdots \\ (Q^n_{xx})_{x^n} \end{pmatrix} \in \mathbb{R}^{nl\times nl}, Q_{yx} = \begin{pmatrix} Q^1_{yx} \\ \vdots \\ Q^n_{yx} \end{pmatrix} \in \mathbb{R}^{n\times nl},$$

$$Q_{xy} = \operatorname{diag}((Q^1_{xy})_{x^1}, \dots, (Q^n_{xy})_{x^n}) \in \mathbb{R}^{nl\times n},\ Q_{yy} = \operatorname{diag}(Q^1_{yy}, \dots, Q^n_{yy}) \in \mathbb{R}^{n\times n},$$

$$\bar{A}_{L\cup I^0} = \begin{pmatrix} A_{L\cup I^0} \\ \vdots \\ A_{L\cup I^0} \end{pmatrix} \in \mathbb{R}^{(1-a^+)n\times nl},\ \tilde{A}_{L\cup I^0} = \operatorname{diag}((A^\top_{L\cup I^0})_{x^1}, \dots, (A^\top_{L\cup I^0})_{x^n}) \in \mathbb{R}^{nl\times(1-a^+)n},$$

$$\tilde{B}_{L\cup I^0} = \operatorname{diag}(b_{L\cup I^0}, \dots, b_{L\cup I^0}) \in \mathbb{R}^{(1-a^+)n\times n},\ E_{I^+\cup I^0} = \operatorname{diag}(1_{I^+\cup I^0}, \dots, 1_{I^+\cup I^0}) \in \mathbb{R}^{(a^++a^0)n\times n},$$

$$\tilde{y} = \begin{pmatrix} \bar{y}^1 \\ \vdots \\ \bar{y}^n \end{pmatrix} \in \mathbb{R}^n,\ \tilde{\lambda}_{L\cup I^0} = \begin{pmatrix} \bar{\lambda}^1_{L\cup I^0} \\ \vdots \\ \bar{\lambda}^n_{L\cup I^0} \end{pmatrix} \in \mathbb{R}^{(1-a^+)n},\ \tilde{\mu}_{I^+\cup I^0} = \begin{pmatrix} \bar{\mu}^1_{I^+\cup I^0} \\ \vdots \\ \bar{\mu}^n_{I^+\cup I^0} \end{pmatrix} \in \mathbb{R}^{(a^++a^0)n},$$

$$c_x = \begin{pmatrix} c^1_{x^1} \\ \vdots \\ c^n_{x^n} \end{pmatrix} \in \mathbb{R}^{nl},\ c_y = \begin{pmatrix} c^1_y \\ \vdots \\ c^n_y \end{pmatrix} \in \mathbb{R}^n \text{ and }\ \tilde{a}_{L\cup I^0} = \begin{pmatrix} a_{L\cup I^0} \\ \vdots \\ a_{L\cup I^0} \end{pmatrix} \in \mathbb{R}^{(1-a^+)n}.$$

To understand better the structure of the system (5.10) see the next example.

Example 5.2. Consider the EPCC consisting of the following two MPCCs with parameters $\alpha, \beta \in \mathbb{R}$:

$$\begin{aligned} &\underset{x^1\in\mathbb{R}^1, y\in\mathbb{R}^1}{\text{minimize}} && \frac{1}{2}(x^1,\, x^2,\, y)^\top \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ y \end{pmatrix} + (1,\, 0,\, \alpha)^\top \begin{pmatrix} x^1 \\ x^2 \\ y \end{pmatrix} \\ &\text{subject to} && 0 \le 2x^1 + 2x^2 + y - 2 \perp y \ge 0, \end{aligned}$$

$$\begin{aligned} &\underset{x^2\in\mathbb{R}^1, y\in\mathbb{R}^1}{\text{minimize}} && \frac{1}{2}(x^1,\, x^2,\, y)^\top \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ y \end{pmatrix} + (0,\, 1,\, \beta)^\top \begin{pmatrix} x^1 \\ x^2 \\ y \end{pmatrix} \\ &\text{subject to} && 0 \le 2x^1 + 2x^2 + y - 2 \perp y \ge 0, \end{aligned}$$

E.g., at the feasible point $(\bar{x}^1, \bar{x}^2, \bar{y}) = (1, 0, 0)$ both constraints are active, hence $I^0 = \{1\}$ and the system (5.10) becomes

$$0 = \begin{pmatrix} 2 & 1 & 1 & 0 & -2 & 0 & 0 & 0 \\ 2 & 3 & 0 & 2 & 0 & -2 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 2 & 0 & 3 & 0 & -1 & 0 & -1 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{x}^1 \\ \bar{x}^2 \\ \bar{y}^1 \\ \bar{y}^2 \\ \bar{\lambda}^1 \\ \bar{\lambda}^2 \\ \bar{\mu}^1 \\ \bar{\mu}^2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ \alpha \\ \beta \\ -2 \\ -2 \\ 0 \\ 0 \end{pmatrix}. \tag{5.11}$$

$\triangle$

Recall the definition of the KKT-type stationarity concepts for EPCCs: Let $(\bar{x}, \bar{y})$ be a feasible point for the EPCC associated with the MPCCs (5.8). Then we call $(\bar{x}, \bar{y})$

i) *weakly stationary* if there exist Lagrange multipliers $\bar{\lambda}, \bar{\mu}$ such that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu})$ satisfies conditions (5.10);

ii) *C-stationary*, if there exist Lagrange multipliers $\bar{\lambda}, \bar{\mu}$ such that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu})$ satisfies conditions (5.10) and, additionally, $\bar{\lambda}^i_{I^0}\bar{\mu}^i_{I^0} \geq 0, i = 1, \ldots, n$;

iii) *M-stationary*, if there exist Lagrange multipliers $\bar{\lambda}, \bar{\mu}$ such that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu})$ satisfies conditions (5.10) and, additionally, either $\bar{\lambda}^i_{I^0} > 0$ and $\bar{\mu}^i_{I^0} > 0$ or $\bar{\lambda}^i_{I^0}\bar{\mu}^i_{I^0} = 0, i = 1, \ldots, n$;

iv) *strongly stationary*, if there exist Lagrange multipliers $\bar{\lambda}, \bar{\mu}$ such that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu})$ satisfies conditions (5.10) and, additionally, $\bar{\lambda}^i_{I^0} \geq 0$ and $\bar{\mu}^i_{I^0} \geq 0, i = 1, \ldots, n$.

The following proposition shows that under EPEC-LICQ the set of strongly stationary points of EPCC coincide with the set of solutions to EPCC.

Proposition 5.3. *Let $(\bar{x}, \bar{y})$ be a local equilibrium point of EPCC. If EPEC-LICQ holds at $(\bar{x}, \bar{y})$, then it a strongly stationary point with unique multipliers. Conversely, a strongly stationary point $(\bar{x}, \bar{y})$ is a local equilibrium point of EPCC.*

Proof. The first statement of the proposition follows from [45, Theorem 7(1)] applied to each ith MPCC, $i = 1, \ldots, n$. Since Lagrangian of each MPCC is strictly convex, the second statement is implied by [45, Theorem 7(2)]. $\square$

The following two assumptions imposed on the data of the EPCC are crucial for the homotopy method to execute each step in a "regular" way.

(A10) The EPEC-LICQ holds at each feasible point of EPCC.

(A11) Consider two matrices

$$\begin{pmatrix} Q_{xx} & Q_{xy} & -\tilde{A}_L \\ Q_{yx} & Q_{yy} & -\tilde{B}_L^\top \\ \bar{A}_L & \tilde{B}_L & 0 \end{pmatrix}, \quad \begin{pmatrix} Q_{xx} & Q_{xy} & 0 \\ Q_{yx} & Q_{yy} & -E_{I^+}^\top \\ 0 & E_{I^+} & 0 \end{pmatrix}$$

and all matrices

$$\begin{pmatrix} Q_{xx} & Q_{xy} & -(\tilde{A}_{I^0}^\top)_I^\top & 0 \\ Q_{yx} & Q_{yy} & -(\tilde{B}_{I^0})_I^\top & -(E_{I^0})_J^\top \\ (\bar{A}_{I^0})_1 & (\tilde{B}_{I^0})_1 & 0 & 0 \\ 0 & E_{I^0} & 0 & 0 \end{pmatrix},$$

where the index sets index sets $I \subset \{1, \ldots, n\}$ and $J \subset \{1, \ldots, n\}$ fulfill $|I| + |J| = n + 1$. Then we suppose that all these matrices are nonsingular.

When we say that some condition imposed upon data or some property of data holds in *generic* sense, we mean that it holds for all data in an open and dense subset of the data space. This notion of "typical data" is particularly attractive if the data space is endowed with a topology. One of the possibilities how to prove that some condition holds in a generic sense is to show that data which do not satisfy such condition or data with undesired property lie in the union of finitely many smooth manifolds of positive codimensions.

Alternatively, if the data space is endowed with a measure, the property holds in a generic sense whenever it holds for almost all data with respect to this measure, cf. [48].

Although the above assumptions on data of the EPCC might appear too restrictive, both hold in generic sense.

Proposition 5.4. *Assumption (A10) holds for all (A, b, a) from some open and dense subset M^* of the set $M = \{(A, b, a) \in \mathbb{R}^{1\times(nl+1)} \times \mathbb{R}^1 \times \mathbb{R}^1\}$.*

Proof. The validity of EPEC-LICQ in generic sense is an immediate consequence of [48, Theorem 3(1)], which states that MPEC-LICQ holds true in generic sense. □

Proposition 5.5. *Assumption (A2) holds for all (Q, A, b, a) from some open and dense subset $N^\#$ of the set $N = \{(Q, A, b, a) \in \mathbb{R}^{(nl+1)\times(nl+1)} \times \mathbb{R}^{1\times(nl+1)} \times \mathbb{R}^1 \times \mathbb{R}^1\}$.*

Proof. The statement follows from the fact that the set of all matrices $M \in \mathbb{R}^{m\times n}$ of rank $r \leq \min\{m, n\}$ is a smooth manifold of codimension $(m - r)(n - r)$ in $\mathbb{R}^{m\times n}$, cf. [23]. Thus, each square matrix is nonsingular in generic sense. This completes the proof. □

In view of Propositions 5.4 and 5.5, we presume that from now on assumptions (A10) and (A11) hold.

Similarly to [42] we can define a nondegenerate C-stationary point of EPCC as follows.

Definition 5.6. *Let $(\bar{x}, \bar{y})$ be a C-stationary point of the EPCC with multipliers $\bar{\lambda}$ and $\bar{\mu}$. Then we call $(\bar{x}, \bar{y})$* nondegenerate *if for each $i = 1 \ldots, n$, and $j \in I^0$ the sign conditions imposed on multipliers are satisfied with strict inequality, i.e., $\lambda_j^i \mu_j^i > 0$.*

The above condition is usually called the *upper-level strict complementarity.* Now, for a nondegenerate C-stationary point, we can introduce the following generalization of the concept of a C-index from [42].

Definition 5.7. *The* C-index *of a nondegenerate C-stationary point* $(\bar{x}, \bar{y})$ *is the sum of negative components of the vector* $\bar{\lambda}$ *(or, equivalently,* $\bar{\mu}$*).*

Clearly, a nondegenerate C-stationary point is strongly stationary if and only if its C-index vanishes.

5.2.2 A oneparametric problem

Let us modify our EPCC in such a way that it will include a onedimesional real-valued parameter t. The parametric problem EPCC(t) will then consist of n oneparametric MPCCs, where the ith MPCC(t), $i = 1, \ldots, n$, is defined by

$$\begin{aligned} \underset{x^i, y}{\text{minimize}} \quad & \frac{1}{2}(x^\top, y)Q^i \begin{pmatrix} x \\ y \end{pmatrix} + (d^i(t))^\top \begin{pmatrix} x \\ y \end{pmatrix} \\ \text{subject to} \quad & 0 \le Ax + By + a \perp y \ge 0, \end{aligned} \tag{5.12}$$

where $d^i(t) := d^i + t(c^i - d^i), i = 1, \ldots, n$, for some vectors $d^i \in \mathbb{R}^{nl_1+1}$ and $t \in \mathbb{R}$. Later we will describe how the vectors $d^i = d^i(0), i = 1, \ldots, n$, are constructed.

The C-stationarity conditions of the EPCC(t) consist of

$$0 = \begin{pmatrix} Q_{xx} & Q_{xy} & -\tilde{A}_{L\cup I^0} & 0 \\ Q_{yx} & Q_{yy} & -\tilde{B}^\top_{L\cup I^0} & -E^\top_{I^+\cup I^0} \\ \bar{A}_{L\cup I^0} & \tilde{B}_{L\cup I^0} & 0 & 0 \\ 0 & E_{I^+\cup I^0} & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ \tilde{y} \\ \lambda_{L\cup I^0} \\ \mu_{I^+\cup I^0} \end{pmatrix} + \begin{pmatrix} d_x(t) \\ d_y(t) \\ \tilde{a}_{L\cup I^0} \\ 0 \end{pmatrix}, \tag{5.13}$$

$$0 \le \lambda^i_{I^0}\mu^i_{I^0}, \quad i = 1, \ldots, n, \tag{5.14}$$

where the vectors $d_x(t)$ and $d_y(t)$ are composed of components of vectors $d^1(t), \ldots, d^n(t)$ in the following way

$$d_x(t) = \begin{pmatrix} (d^1(t))_{x^1} \\ \vdots \\ (d^n(t))_{x^n} \end{pmatrix} \text{ and } d_y(t) = \begin{pmatrix} (d^1(t))_y \\ \vdots \\ (d^n(t))_y \end{pmatrix}.$$

Note that, by choosing $t = 1$, we arrive at the original EPCC and its corresponding C-stationarity conditions.

Let us introduce the following sets:

$$\begin{aligned} \Sigma_{sol} &= \{(t, x, y) \in \mathbb{R} \times \mathbb{R}^{nl_1} \times \mathbb{R}^1 | (x, y) \text{ is a solution to EPCC}(t)\} \\ \Sigma_{S-stat} &= \{(t, x, y) \in \mathbb{R} \times \mathbb{R}^{nl_1} \times \mathbb{R}^1 | (x, y) \text{ is a strongly stationary point of EPCC}(t)\} \\ \Sigma_{C-stat} &= \{(t, x, y) \in \mathbb{R} \times \mathbb{R}^{nl_1} \times \mathbb{R}^1 | (x, y) \text{ is a C-stationary point of EPCC}(t)\} \end{aligned}$$

As mentioned above, we have the relation $\Sigma_{sol} \subset \Sigma_{S-stat} \subset \Sigma_{C-stat}$ and due to assumption (A10), the first inclusion becomes equality.

For oneparametric as well as nonparametric EPCCs, it does not hold that all C-stationary points are nondegenerate in generic sense, see Figure 5.1 below. In our analysis, we are particularly interested in the following class of singular points.

Definition 5.8. *For $\bar{t} \in \mathbb{R}$ a C-stationary point $(\bar{x}, \bar{y})$ of EPCC($\bar{t}$) with multipliers $\bar{\lambda}, \bar{\mu}$ is called the* codimension n singularity *(co-n-singularity) if the following conditions hold:*

i) Exactly n entries of the vector $(\bar{\lambda}_{I^0}, \bar{\mu}_{I^0})$ vanish.

ii) If $I \subset \{1, \dots, n\}$ and $J \subset \{1, \dots, n\}$ are index sets such that $\bar{\lambda}_I \neq 0, \bar{\mu}_J \neq 0$ and $|I| + |J| = n$, then the matrix

$$\begin{pmatrix} c_x - d_x & Q_{xx} & Q_{xy} & -(\tilde{A}_{I^0}^\top)_I^\top & 0 \\ c_y - d_y & Q_{yx} & Q_{yy} & -(\tilde{B}_{I^0})_I^\top & -(E_{I^0})_J^\top \\ 0 & (\bar{A}_{I^0})_1 & (\tilde{B}_{I^0})_1 & 0 & 0 \\ 0 & 0 & E_{I^0} & 0 & 0 \end{pmatrix}$$

is nonsingular.

Further, we call the co-n-singular C-stationary point $(\bar{x}, \bar{y})$

i) 0-singularity, *if $I \neq \emptyset, J \neq \emptyset$ and $I \cap J = \emptyset$;*

ii) i-singularity, *if $|I \cap J| = i$;*

iii) exit point, *if either $I = \emptyset$ or $J = \emptyset$.*

Note that each co-n-singularity falls to exactly one of the above mentioned categories.

The homotopy method traces the set Σ_{C-stat}, searching for C-stationary points of the original problem. In order to design such algorithm, we have to understand the structure of the set Σ_{C-stat}, in particular its local structure around co-n-singularities. In the following we show that around each type of co-n singularity, Σ_{C-stat} admits different structure.

0-singularity

Fix a $\bar{t} \in [0, 1]$ and consider first the 0-singular C-stationary point $(\bar{x}, \bar{y})$ of EPCC($\bar{t}$). If $I \subset \{1, \dots, n\}$ and $J \subset \{1, \dots, n\}$ are index sets uniquely defined by conditions $\bar{\lambda}_I \neq 0, \bar{\mu}_J \neq 0$ and $|I| + |J| = n$, let I^c and J^c denote the complement of I and J in $\{1, \dots, n\}$, respectively.

Then Σ_{C-stat} can be locally around $(\bar{x}, \bar{y})$ described by means of the following n systems of equations

$$0 = H^{\lambda_j}(t, x, y, \lambda_{I \cup \{j\}}, \mu_J) =$$

$$= \begin{pmatrix} Q_{xx} & Q_{xy} & -(\tilde{A}_{I^0}^\top)_{I \cup \{j\}}^\top & 0 \\ Q_{yx} & Q_{yy} & -(\tilde{B}_{I^0})_{I \cup \{j\}}^\top & -(E_{I^0})_J^\top \\ (\bar{A}_{I^0})_1 & (\tilde{B}_{I^0})_1 & 0 & 0 \\ 0 & E_{I^0} & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ \tilde{y} \\ \lambda_{I \cup \{j\}} \\ \mu_J \end{pmatrix} + \begin{pmatrix} d_x(t) \\ d_y(t) \\ \tilde{a}_{I \cup \{j\}} \\ 0 \end{pmatrix},$$

for each $j \in I^c$ and

$$0 = H^{\mu_j}(t, x, y, \lambda_I, \mu_{J\cup\{j\}}) =$$
$$= \begin{pmatrix} Q_{xx} & Q_{xy} & -(\tilde{A}_{I^0}^\top)_I^\top & 0 \\ Q_{yx} & Q_{yy} & -(\tilde{B}_{I^0})_I^\top & -(E_{I^0})_{J\cup\{j\}}^\top \\ (\bar{A}_{I^0})_1 & (\tilde{B}_{I^0})_1 & 0 & 0 \\ 0 & E_{I^0} & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ \tilde{y} \\ \lambda_I \\ \mu_{J\cup\{j\}} \end{pmatrix} + \begin{pmatrix} d_x(t) \\ d_y(t) \\ \tilde{a}_I \\ 0 \end{pmatrix},$$

for each $j \in J^c$.

Clearly, for $j \in I^c$ we have $H^{\lambda_j}(\bar{t}, \bar{x}, \bar{y}, \bar{\lambda}_{I\cup\{j\}}, \bar{\mu}_J) = 0$ and for $j \in J^c$ we have $H^{\mu_j}(\bar{t}, \bar{x}, \bar{y}, \bar{\lambda}_I, \bar{\mu}_{J\cup\{j\}}) = 0$. Moreover, each system matrix in nonsingular due to the assumption (A11).

Hence, locally around $\bar{t}$ there exist for each $j \in I^c$ *locally unique linear functions* $(x^{\lambda_j}(t), y^{\lambda_j}(t), \lambda^{\lambda_j}(t), \mu^{\lambda_j}(t))$ such that

$$(x^{\lambda_j}(\bar{t}), y^{\lambda_j}(\bar{t}), \lambda^{\lambda_j}(\bar{t}), \mu^{\lambda_j}(\bar{t})) = (\bar{x}, \bar{y}, \bar{\lambda}_{I\cup\{j\}}, \bar{\mu}_J)$$

and

$$H^{\lambda_j}(t, x^{\lambda_j}(t), y^{\lambda_j}(t), \lambda^{\lambda_j}(t), \mu^{\lambda_j}(t)) = 0. \tag{5.15}$$

Analogously there exist locally unique linear functions $(x^{\mu_j}(t), y^{\mu_j}(t), \lambda^{\mu_j}(t), \mu^{\mu_j}(t))$ for each $j \in J^c$.

Around the 0-singularity $(\bar{t}, \bar{x}, \bar{y})$, only a part of the set

$$\Sigma^{\lambda_j} := \{(t, x^{\lambda_j}(t), y^{\lambda_j}(t)) | t - \bar{t} \in (-\epsilon, \epsilon)\}$$

for some $\epsilon > 0$ belongs to the set Σ_{C-stat}. This is that part of Σ^{λ_j}, denoted by $\Sigma_+^{\lambda_j}$, where the sign of multiplier $\lambda_j^{\lambda_j}(t)$ is the same the sign of multiplier $\mu_j^{\lambda_j}(t)$. Analogously, the part the sets

$$\Sigma^{\mu_j} := \{(t, x^{\mu_j}(t), y^{\mu_j}(t)) | t - \bar{t} \in (-\epsilon, \epsilon)\},$$

which belongs to Σ_{C-stat} is denoted by $\Sigma_+^{\mu_j}$.

Theorem 5.9. *At a 0-singular C-stationary point $(\bar{x}, \bar{y})$ of the EPCC($\bar{t}$) with multipliers $\bar{\lambda}, \bar{\mu}$, for each $j \in I$ the linear function $(x^{\lambda_j}(t), y^{\lambda_j}(t), \lambda^{\lambda_j}(t), \mu^{\lambda_j}(t))$ intersects transversally at $(\bar{t}, \bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu})$ with each linear function $(x^{\lambda_k}(t), y^{\lambda_k}(t), \lambda^{\lambda_k}(t), \mu^{\lambda_k}(t)), k \in I \setminus \{j\}$ and $(x^{\mu_k}(t), y^{\mu_k}(t), \lambda^{\mu_k}(t), \mu^{\mu_k}(t)), k \in J$.*

Also for each $j \in J$ the linear function $(x^{\mu_j}(t), y^{\mu_j}(t), \lambda^{\mu_j}(t), \mu^{\mu_j}(t))$ intersects transversally at $(\bar{t}, \bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu})$ with each linear function $(x^{\lambda_k}(t), y^{\lambda_k}(t), \lambda^{\lambda_k}(t), \mu^{\lambda_k}(t)), k \in I$ and $(x^{\mu_k}(t), y^{\mu_k}(t), \lambda^{\mu_k}(t), \mu^{\mu_k}(t)), k \in J \setminus \{j\}$.

Proof. It is sufficient to show that $\dot{\lambda}_j^{\lambda_j}(\bar{t}) := \frac{d}{dt}\lambda_j^{\lambda_j}(\bar{t}) \neq 0$. Since $\dot{\lambda}_j^{\lambda_k}(\bar{t}) = 0, k \in I \setminus \{j\}$ and $\dot{\lambda}_j^{\mu_k}(\bar{t}) = 0, k \in J$, this would mean that the linear function $(x^{\lambda_j}(t), y^{\lambda_j}(t), \lambda^{\lambda_j}(t), \mu^{\lambda_j}(t))$ does not point into the same direction as any of the other linear functions.

Take derivatives with respect to t in (5.15). This yields

$$0 = \begin{pmatrix} Q_{xx} & Q_{xy} & -(\tilde{A}_{I^0}^\top)^\top_{I\cup\{j\}} & 0 \\ Q_{yx} & Q_{yy} & -(\tilde{B}_{I^0})^\top_{I\cup\{j\}} & -(E_{I^0})^\top_J \\ (\bar{A}_{I^0})_1 & (\tilde{B}_{I^0})_1 & 0 & 0 \\ 0 & E_{I^0} & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}^{\lambda_j}(\bar{t}) \\ \dot{\tilde{y}}^{\lambda_j}(\bar{t}) \\ \dot{\lambda}^{\lambda_j}_{I\cup\{j\}}(\bar{t}) \\ \dot{\mu}^{\lambda_j}_J(\bar{t}) \end{pmatrix} + \begin{pmatrix} c_x - d_x \\ c_y - d_y \\ 0 \\ 0 \end{pmatrix}.$$

This system of linear equations can be equivalently rewritten to

$$0 = \begin{pmatrix} c_x - d_x & Q_{xx} & Q_{xy} & -(\tilde{A}_{I^0}^\top)^\top_{I\cup\{j\}} & 0 \\ c_y - d_y & Q_{yx} & Q_{yy} & -(\tilde{B}_{I^0})^\top_{I\cup\{j\}} & -(E_{I^0})^\top_J \\ 0 & (\bar{A}_{I^0})_1 & (\tilde{B}_{I^0})_1 & 0 & 0 \\ 0 & 0 & E_{I^0} & 0 & 0 \\ 0 & 0 & 0 & (e^j)^\top_{I\cup\{j\}} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \dot{x}^{\lambda_j}(\bar{t}) \\ \dot{\tilde{y}}^{\lambda_j}(\bar{t}) \\ \dot{\lambda}^{\lambda_j}_{I\cup\{j\}}(\bar{t}) \\ \dot{\mu}^{\lambda_j}_J(\bar{t}) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \dot{\lambda}^{\lambda_j}_j(\bar{t}) \end{pmatrix},$$

where e^j denotes the jth unit vector of basis in $\mathbb{R}^n$.

By Laplace formula applied to the last row, the latter system matrix is nonsingular, since the matrix

$$\begin{pmatrix} c_x - d_x & Q_{xx} & Q_{xy} & -(\tilde{A}_{I^0}^\top)^\top_I & 0 \\ c_y - d_y & Q_{yx} & Q_{yy} & -(\tilde{B}_{I^0})^\top_I & -(E_{I^0})^\top_J \\ 0 & (\bar{A}_{I^0})_1 & (\tilde{B}_{I^0})_1 & 0 & 0 \\ 0 & 0 & E_{I^0} & 0 & 0 \end{pmatrix}$$

is nonsingular for a co-n-singularity. Hence, $\dot{\lambda}^{\lambda_j}_j(\bar{t})$ cannot vanish. This proves the first part.

The proof of the second statement is analogous. □

Theorem 5.10. *On a neighborhood of a 0-singular C-stationary point $(\bar{x}, \bar{y})$ of the EPCC$(\bar{t})$ with multipliers $\bar{\lambda}, \bar{\mu}$, the set Σ_{C-stat} coincides with convex hull of the sets $\Sigma_+^{\lambda_j}$, $j \in I^c$, and $\Sigma_+^{\mu_j}$, $j \in J^c$. Moreover, all interior points of such convex hull share the same value of the C-index.*

Proof. Without loss of generality, it suffices to show that for $j, k \in J$ the convex hull of $\Sigma_+^{\lambda_j}$ and $\Sigma_+^{\lambda_k}$ belongs to Σ_{C-stat}.

Take $\alpha \in (0,1)$ and points $(t_1, x^{(1)}, y^{(1)}) \in \Sigma_+^{\lambda_j}$, $(t_2, x^{(2)}, y^{(2)}) \in \Sigma_+^{\lambda_k}$. Then we need to show that also

$$(t_\alpha, x^\alpha, y^\alpha) := \alpha(t_1, x^{(1)}, y^{(1)}) + (1-\alpha)(t_2, x^{(2)}, y^{(2)}) \in \Sigma_{C-stat}.$$

The point $(t_1, x^{(1)}, y^{(1)}, \lambda^{(1)}, \mu^{(1)})$ solves (5.13), where multipliers $\lambda^{(1)}, \mu^{(1)}$, uniquely determined by nonvanishing entries given by $\lambda^{\lambda^j}_{I\cup\{j\}}(t_1), \mu^{\lambda^j}_J(t_1)$, respectively, satisfy conditions (5.14). Analogously, the point $(t_2, x^{(2)}, y^{(2)}, \lambda^{(2)}, \mu^{(2)})$ solves (5.13), where multipliers

$\lambda^{(2)}, \mu^{(2)}$, uniquely determined by nonvanishing entries given by $\lambda^{\lambda^k}_{I\cup\{k\}}(t_2), \mu^{\lambda^k}_J(t_2)$, respectively, satisfy conditions (5.14).

Then, clearly, conditions (5.14) are satisfied for $\lambda^\alpha = \alpha\lambda^{(1)} + (1-\alpha)\lambda^{(2)}$ and $\mu^\alpha = \alpha\mu^{(1)} + (1-\alpha)\mu^{(2)}$. It remains to show that also $(t_\alpha, x^\alpha, y^\alpha, \lambda^\alpha, \mu^\alpha)$ solves (5.13). To prove the latter statement, it suffices now to recall that $d_x(t)$ and $d_y(t)$ is linear in t.

Taking any $\alpha \notin [0,1]$, conditions (5.14) are violated for $\lambda^\alpha = \alpha\lambda^{(1)} + (1-\alpha)\lambda^{(2)}$.

This finishes the proof of both parts of the theorem. □

i-singularity

At the i-singularity, let k be an index such that $\bar\lambda_k = \bar\mu_k = 0$. Then locally around $(\bar t, \bar x, \bar y)$ the whole sets Σ^{λ_k} and Σ^{μ_k} belong to Σ_{C-stat}. This is due to the fact that $\mu_k^{\lambda_k}(t) = 0$ and $\lambda_k^{\mu_k}(t) = 0$ for each $t \in (-\epsilon, \epsilon)$ and the respective kth sign condition on biactive multipliers is thus satisfied regardless of the signs of $\lambda_k^{\lambda_k}(t)$ and $\mu_k^{\mu_k}(t)$, respectively.

Theorem 5.9 clearly holds also for i-singularity. Then convex hull of the sets $\Sigma_+^{\lambda_j}, j \in I^c\backslash\{k\}, \Sigma_+^{\mu_j}, j \in J^c\backslash\{k\}, \{(t, x^{\lambda_k}(t), y^{\lambda_k}(t)) | t-\bar t \in [0,\epsilon)\}$ and $\{(t, x^{\mu_k}(t), y^{\mu_k}(t)) | t-\bar t \in [0,\epsilon)\}$ as well as convex hull of the sets $\Sigma_+^{\lambda_j}, j \in I^c\backslash\{k\}, \Sigma_+^{\mu_j}, j \in J^c\backslash\{k\}, \{(t, x^{\lambda_k}(t), y^{\lambda_k}(t)) | t-\bar t \in (-\epsilon, 0]\}$ and $\{(t, x^{\mu_k}(t), y^{\mu_k}(t)) | t-\bar t \in (-\epsilon, 0]\}$ belongs to the set Σ_{C-stat}. We summarize this in the following theorem.

Theorem 5.11. *On a neighborhood of an i-singular C-stationary point $(\bar x, \bar y)$ of the EPCC($\bar t$) with multipliers $\bar\lambda, \bar\mu$, the set Σ_{C-stat} coincides with a union of 2^i convex hulls of parts of sets $\Sigma^{\lambda_j}, j \in I^c$ and $\Sigma^{\mu_j}, j \in J^c$ specified above. Moreover, all interior points of each such convex hull share the same value of the C-index.*

Proof. The proof follows from the same arguments used in the proof of Theorem 5.10 and the observations above. □

Exit point

Note that there are only two possible exit points $(\bar t, \bar x, \bar y)$. At the first one with $\bar\lambda = 0$, all sets $\Sigma_+^{\lambda_j}, j = 1, \ldots, n$, belong to the set Σ_{C-stat}.

Moreover, the same is true for the feasible part of the set

$$\Sigma^{I^+} = \{(t, x^{I^+}(t), y^{I^+}(t)) | t-\bar t \in (-\epsilon, \epsilon)\}$$

for some $\epsilon > 0$, where the locally unique linear function $(x^{I^+}(t), y^{I^+}(t), 0, \mu^{I^+}(t))$ is defined by the regular system of equations

$$0 = \begin{pmatrix} Q_{xx} & Q_{xy} & 0 \\ Q_{yx} & Q_{yy} & -E_{I^+}^\top \\ 0 & E_{I^+} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ \mu_{I^+} \end{pmatrix} + \begin{pmatrix} d_x(t) \\ d_y(t) \\ 0 \end{pmatrix}.$$

Analogously, at the other exit point with $\bar\mu = 0$, the sets $\Sigma_+^{\mu_j}, j = 1, \ldots, n$, and feasible part of the set Σ^L belong to the set Σ_{C-stat}.

Theorem 5.12. *At an exit point $(\bar{x}, \bar{y})$ of the EPCC$(\bar{t})$ with multipliers $\bar{\lambda}, \bar{\mu}$, the statement of Theorem 5.9 holds true. Moreover, either the linear function $(x^{I^+}(t), y^{I^+}(t), 0, \mu^{I^+}(t))$ intersects at $(\bar{t}, \bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu})$ transversally with the linear functions $(x^{\lambda_j}(t), y^{\lambda_j}(t), \lambda^{\lambda_j}(t), \mu^{\lambda_j}(t))$, $j = 1, \ldots, n$ or the linear function $(x^L(t), y^L(t), \lambda^L(t), 0))$ intersects at $(\bar{t}, \bar{x}, \bar{y}, \bar{\lambda}, \bar{\mu})$ transversally with $(x^{\mu_j}(t), y^{\mu_j}(t), \lambda^{\mu_j}(t), \mu^{\mu_j}(t))$, $j = 1, \ldots, n$.*

Proof. Using the arguments from the proof of Theorem 5.9, we can prove that at the exit point with $\bar{\lambda} = 0$, for each $j = 1, \ldots n$, the derivative $\dot{\lambda}_j^{\lambda_j}(\bar{t}) \neq 0$ while $\dot{\lambda}_j^{I^+} = 0$. Similarly, at the second exit point for each $j = 1, \ldots n$, the derivative $\dot{\mu}_j^{\mu_j}(\bar{t}) \neq 0$ while $\dot{\mu}_j^L = 0$. □

Theorem 5.13. *On a neighborhood of the exit point $(\bar{x}, \bar{y})$ of the EPCC$(\bar{t})$ with multipliers $\bar{\lambda} = 0$ and $\bar{\mu} \neq 0$, the set Σ_{C-stat} coincides with a union of the feasible part of Σ^{I^+} and convex hull of the sets $\Sigma_+^{\lambda_j}, j = 1, \ldots, n$.*

On a neighborhood of the exit point $(\bar{x}, \bar{y})$ of the EPCC$(\bar{t})$ with multipliers $\bar{\lambda} \neq 0$ and $\bar{\mu} = 0$, the set Σ_{C-stat} coincides with a union of the feasible part of Σ^L and the convex hull of the sets $\Sigma_+^{\mu_j}, j = 1, \ldots, n$.

Proof. The proof follows from the same arguments used in the proof of Theorem 5.10 and observations above. □

Clearly, the C-index of nondegenerate C-stationary points can change only at co-n-singularities which are not 0-singular. We show the change of C-index on the EPCC from Example 5.1 with one particular setting of parameters α, β.

Example 5.2. (continued) Consider the EPCC from Example 5.1 with $\alpha = 3/2$ and $\beta = 1/2$ and suppose that $d_x(t) = (-6, -10)^\top$ and $d_y(t) = (-3, -5)^\top$.

Then one can find exactly six co-2-singularities of the EPCC(t): two exit points, (1/3, 2/3, 0) at $t = 2/3$ and (1, 0, 0) at $t = 0$ with multipliers (λ, μ) equal (0,0,1, 1/3) and (-2, -4, 0, 0), respectively; two 0-singularities, (25/9, -16/9, 0) at $t = 8/9$ and (1,0,0) at $t = 4/7$ with multipliers (2,0,0,-8/9) and (0,-6/7,4/7,0), respectively; and two 1-singularities, (1, 0, 0) at $t = 8/11$ and (17/9, -8/9, 0) at $t = 4/9$ with multipliers (6/11, 0, 8/11, 0) and (0,-2, 0, -4/9), respectively.

All co-2-singularities are depicted on Figure 5.1 in multiplier spaces; exit points as red bullets, 0-singularities as black bullets and 1-singularities as green bullets. The shaded area is the set of all possible biactive multipliers of C-stationary points to EPCC(t).

The interior points of the bounded piece correspond to multipliers of C-stationary points with C-index 1. The 1-singularity (1, 0, 0) at $t = 8/11$ connects this piece with the one with interior points with vanishing C-index. The other 1-singularity connects it with the piece with interior points with C-index 2. The latter two pieces are connected to the parts of the set Σ_{C-stat} of points with vanishing C-index by exit points.

Note on Figure 5.1 that slight shifts of the dashed lines corresponding to small perturbations to the data eliminate neither the co-2-singularities nor the remaining singular C-stationary points on the border of the shaded area. △

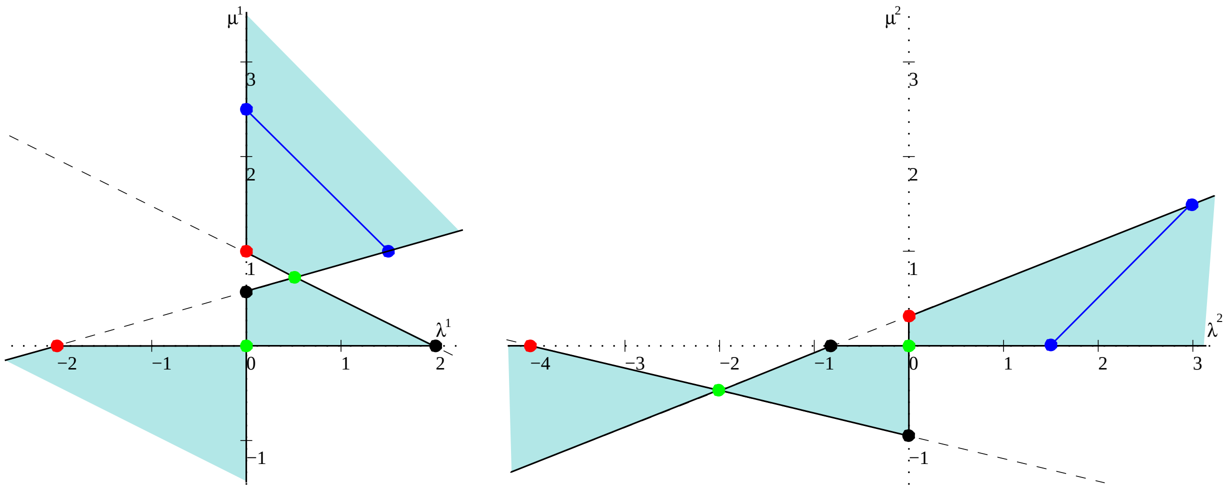

Figure 5.1: Co-2-singularities of the EPCC(t)

5.2.3 Homotopy method

The basic idea of the homotopy method we are about to describe in detail is to formulate an artificial EPCC by modifying (jointly) objective functions of all MPCCs (5.8) such that a chosen feasible point $(\bar{x}, \bar{y})$ becomes strongly stationary. The parameter t then creates a connection between the original and the artificial problem.

Let $(\bar{x}, \bar{y})$ be a feasible point of EPCC and L, I^+ and I^0 be the associated index sets. Based on the structure of the index sets we construct the vector $d(0) = (d_x(0)^\top, d_y(0)^\top)^\top$.

If $L = \{1\}$, then we set $\bar{\mu} := 0$ and choose a vector $\bar{\lambda}$ with arbitrary strictly positive components. If $I^+ = \{1\}$, then we set $\bar{\lambda} := 0$ and $\bar{\mu}$ with arbitrary strictly positive components. If $I^0 = \{1\}$, we set either $\bar{\mu} := 0$ and choose a multiplier vector $\bar{\mu}$ with arbitrary strictly positive components or vice versa. In either case, we use the following formula to compute the vector $d(0)$.

$$d(0) := -\begin{pmatrix} Q_{xx} & Q_{xy} \\ Q_{yx} & Q_{yy} \end{pmatrix} \begin{pmatrix} \bar{x} \\ \tilde{y} \end{pmatrix} + \begin{pmatrix} \tilde{A}_{L\cup I^0} \\ \tilde{B}^\top_{L\cup I^0} \end{pmatrix} \bar{\lambda} + \begin{pmatrix} 0 \\ E^\top_{I^+\cup I^0} \end{pmatrix} \bar{\mu}. \tag{5.16}$$

Then $(\bar{x}, \bar{y})$ is a solution of EPCC(0). To obtain vector $d(t)$, we set

$$d(t) = \begin{pmatrix} d_x(t) \\ d_y(t) \end{pmatrix} := \begin{pmatrix} d_x(0) \\ d_y(0) \end{pmatrix} + t\left(\begin{pmatrix} c_x \\ c_y \end{pmatrix} - \begin{pmatrix} d_x(0) \\ d_y(0) \end{pmatrix}\right). \tag{5.17}$$

The homotopy method traces the set Σ_{C-stat} starting at $t = 0$. Note that if for the initial feasible point the complementarity constraint is biactive, the method starts at one of the two exit points.

Overview of the homotopy method I for MPCCs

Before we proceed to the homotopy method in detail, let us summarize the homotopy method I from [42] which searches for C-stationary points of convex-quadratic mathematical programs with linear complementarity constraints.

The program (5.8) can be converted to the following convex-quadratic MPCC in variable $z = \begin{pmatrix} x^i \\ y \end{pmatrix}$

$$\begin{aligned} &\text{minimize } \frac{1}{2} z^\top \bar{Q} z + \bar{c}^\top z \\ &\text{subject to } 0 \leq \bar{A} z + \bar{a} \perp \bar{B} z + \bar{b} \geq 0, \end{aligned} \tag{5.18}$$

where

$$\bar{Q} = \begin{pmatrix} Q^i_{x^i x^i} & Q^i_{x^i y} \\ Q^i_{y x^i} & Q^i_{yy} \end{pmatrix}, \quad \bar{c} = \begin{pmatrix} c^i_{x^i} \\ c^i_y \end{pmatrix} + 2 \begin{pmatrix} Q_{x^i, x^{-i}} \\ Q_{y, x^{-i}} \end{pmatrix} \bar{x}^{-i},$$
$$\bar{A} = (A_{x^i}, b), \quad \bar{B} = (0, 1), \quad \bar{a} = (A^\top_{x^{-i}})^\top \bar{x}^{-i} + a, \quad \bar{b} = 0.$$

For the purposes of this summary of the homotopy method I, consider the general problem (5.18) with matrices $\bar{A}, \bar{B} \in \mathbb{R}^{ml_2 \times (l_1 + ml_2)}$.

Let MPEC-LICQ hold at each feasible point of (5.18). Given a feasible point $\bar{z}$ of the MPCC (5.18), put $\bar{\lambda}_{I^+} = 0, \bar{\mu}_L = 0$ and

$$\bar{d} = -\bar{Q}\bar{z} + \bar{A}^\top_{L \cup I^0} \bar{\lambda}_{L \cup I^0} + \bar{B}^\top_{I^+ \cup I^0} \bar{\mu}_{I^+ \cup I^0}$$

with some strictly positive values of components of vectors $\bar{\lambda}_{L \cup I^0}, \bar{\mu}_{I^+ \cup I^0}$.

Then $\bar{z}$ is a local minimizer of the program

$$\begin{aligned} &\text{minimize } \frac{1}{2} z^\top \bar{Q} z + (\bar{d} + t(\bar{c} - \bar{d}))^\top z \\ &\text{subject to } 0 \leq \bar{A} z + \bar{a} \perp \bar{B} z + \bar{b} \geq 0 \end{aligned} \tag{5.19}$$

for $t = 0$. Locally around the point (t, z, λ, μ), C-stationary points of MPCC$(t + \tau)$ and their corresponding multipliers are given by the path

$$\begin{pmatrix} z(\tau) \\ \lambda(\tau) \\ \mu(\tau) \end{pmatrix} = \begin{pmatrix} z \\ \lambda \\ \mu \end{pmatrix} + \tau \begin{pmatrix} \dot{z} \\ \dot{\lambda} \\ \dot{\mu} \end{pmatrix}$$

with

$$\begin{pmatrix} \bar{Q} & -\bar{A}^\top_{L \cup I^0} & -\bar{B}^\top_{I^+ \cup I^0} \\ \bar{A}_{L \cup I^0} & 0 & 0 \\ \bar{B}_{I^+ \cup I^0} & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{z} \\ \dot{\lambda} \\ \dot{\mu} \end{pmatrix} = \begin{pmatrix} \bar{c} - \bar{d} \\ 0 \\ 0 \end{pmatrix}.$$

At the start, t is set to zero and the method traces the homotopy path in the direction of increasing t. The steplength is then determined as the minimal positive value of $\bar{\tau}$ for which one of the following inequalities vanishes

$$\begin{aligned} \bar{A}_i z(\tau) + \bar{a}_i &> 0, \; i \in I^+, \\ \bar{B}_j z(\tau) + \bar{b}_j &> 0, \; j \in L, \\ \lambda_i(\tau) &\neq 0, \; i \in I^0, \\ \mu_j(\tau) &\neq 0, \; j \in I^0. \end{aligned}$$

This value can be easily determined using the ratios

$$
\begin{aligned}
q_i &= -\frac{\bar{A}_i z + \bar{a}_i}{\bar{A}_i \dot{z}}, \quad i \in I^+,\\
q_i &= -\frac{\lambda_i}{\dot{\lambda}_i}, \quad i \in I^0,\\
r_j &= -\frac{\bar{B}_j z + \bar{b}_j}{\bar{B}_j \dot{z}}, \quad j \in L,\\
r_j &= -\frac{\mu_j}{\dot{\mu}_j}, \quad j \in I^0.
\end{aligned}
$$

Then, if moving forward in t, the method takes the steplength

$$\bar{\tau} = \min(\{q_i \cap (0, 1-t), i \in I^+ \cup I^0, r_j \cap (0, 1-t), j \in L \cup I^0\}).$$

If this minimum is taken over the empty set, the value $t = 1$ can be reached directly and the method terminates with a C-stationary point of the MPCC.

If the minimum is attained at some $q_i, i \in I^0$, then $\lambda_i(t + \bar{\tau})$ vanishes and biactivity of constraint i is dropped (i.e., we put the index i to the set I^+). The sign of $\mu_i(t + \bar{\tau})$ then decides about the direction in t for the next step: if $\mu_i(t + \bar{\tau}) < 0$, the direction changes. If the minimum is attained at some $q_i, i \in I^+$, then we add the biactivity of the constraint i (i.e., we put index i to the set I^0) and the sign of multiplier $\mu_i(t + \bar{\tau})$ determines the direction of the next step. For ratios r_j we proceed analogously.

If the method currently proceeds in t backwards, the next step is the maximal negative value of the ratios

$$\bar{\tau} = \max(\{q_i \cap (-\infty, 0), i \in I^+ \cup I^0, r_j \cap (-\infty, 0), j \in L \cup I^0\}).$$

If this maximum is taken over the empty set, an infinite step could be taken to $t \searrow -\infty$ and the method thus terminates without a solution. Else, analogous changes in activities are performed.

The method described above depends on the knowledge of initial feasible point $\bar{z}$ of MPCC. The following Phase I.a approach uses the homotopy method itself to provide a feasible point.

Consider the following auxiliary problem in variables $z \in \mathbb{R}^{l_1 + m l_2}$ and $s \in \mathbb{R}$

$$
\begin{aligned}
&\text{minimize } \frac{1}{2}s^2\\
&\text{subject to } 0 \le (\bar{A}, \;\; u - \bar{a}) \begin{pmatrix} z \\ s \end{pmatrix} + \bar{a} \perp (\bar{B}, \;\; v - \bar{b}) \begin{pmatrix} z \\ s \end{pmatrix} + \bar{b} \ge 0
\end{aligned}
\tag{5.20}
$$

for some chosen vectors $u, v \in \mathbb{R}^{m l_2}$ with $0 \le u \perp v \ge 0$. Note that the point $(z, s) = (0, 1)$ is always feasible. Hence we can try to apply the homotopy method I to (5.20). If a solution point $(\bar{z}, 0)$ is found, $\bar{z}$ is a feasible point for MPCC.

If $l_1 + 1 \leq ml_2$, the first $l_1 + ml_2 + 1$ components of $(u^\top, v^\top)$ are set to zero and the remaining components are set to one. However, the Hessian of the objective is only positive semidefinite and thus the method may not succeed in some cases. Then, Phase I.b approach is guaranteed to provide either a feasible point or verification of inconsistency.

In Phase I.b, sometimes called the *disjunctive approach* [25], we check, using the Phase I of the simplex method, all 2^{ml_2} polyhedral pieces of the feasible region. Each such piece is determined by an index set $I \subset \{1, \dots, ml_2\}$ and conditions

$$\bar{A}_I z + \bar{a}_I = 0, \quad \bar{A}_{I^c} z + \bar{a}_{I^c} \geq 0, \tag{5.21}$$

$$\bar{B}_{I^c} z + \bar{b}_{I^c} = 0, \quad \bar{B}_I z + \bar{b}_I \geq 0. \tag{5.22}$$

If all polyhedral pieces are inconsistent, then the considered MPCC is also inconsistent.

Now, we modify this homotopy method I to the EPCC composed of MPCCs (5.8), using the knowledge about the structure of the set Σ_{C-stat} around co-n-singular points.

Phase I for EPCC

Analogously to Phase I procedure for MPCCs, we can compute an initial feasible point of the EPCC either via application of the homotopy method I for MPCCs to an auxiliary program or via checking each polyhedral piece of the feasible region of EPCC.

The problem (5.20) now takes the form of an MPCC in variables x, y and s

$$\begin{aligned} &\text{minimize } \frac{1}{2}s^2 \\ &\text{subject to } 0 \leq (A,\, b,\, u - \bar{a}) \begin{pmatrix} x \\ y \\ s \end{pmatrix} + \bar{a} \perp (0,\, 1,\, v) \begin{pmatrix} x \\ y \\ s \end{pmatrix} \geq 0 \end{aligned} \tag{5.23}$$

for some chosen scalars $u, v \in \mathbb{R}$ with $0 \leq u \perp v \geq 0$. Again, the point $(x, y, s) = (0, 0, 1)$ is always feasible. Hence, we can try to apply the homotopy method I to (5.23). If a solution point $(\bar{x}, \bar{y}, 0)$ is found, $(\bar{x}, \bar{y})$ is feasible point of the EPCC.

Similarly, if Phase I.a fails to provide a feasible point, we can apply Phase I.b. In our case it is enough to check, using the Phase I of the simplex method, just 2 polyhedral pieces of the feasible region. The first one is determined by conditions

$$Ax + by + a = 0, \quad y \geq 0,$$

while the second one by conditions

$$Ax + by + a \geq 0, \quad y = 0.$$

Overview of the algorithm

From the analysis of the structure of the set Σ_{C-stat} around co-n-singularities, it is clear that the set Σ_{C-stat} consists of finitely many convex polyhedral pieces: (one-dimensional) halflines corresponding to index sets I^+ and L and n-dimensional polyhedral sets corresponding to index set I^0. It is thus sufficient to design an algorithm which traces all one-dimensional faces of each such convex polyhedral piece; such procedure would give us full information about the set Σ_{C-stat}, see Example 5.1.

The description of the algorithm to trace the biactive part of the set Σ_{C-stat} is significantly more complicated then in the homotopy method I for MPCCs. We make use of the following lists of points or vectors:

"untreated exit points":	the list of visited exit points for which the corresponding set Σ_+^L or $\Sigma_+^{I^+}$ was not yet traced
"multiplier signs":	the list of vectors of signs of biactive multipliers, uniquely determining each convex polyhedral piece of biactive part of the set Σ_{C-stat}
"co-n-singularities":	the list of visited co-n-singularities
"i-singularities":	the list of visited i-singularities
"biactive C-stationary points":	the list of found C-stationary points in the biactive part of the set Σ_{C-stat}
"new directions":	the list of directions in which the next step can be made from the current iterate
"new multiplier signs":	the list of vectors of signs of biactive multipliers, uniquely determining polyhedral pieces connected by i-singularity to previously traced polyhedral piece of the set Σ_{C-stat}

At the start of the method, all lists above are empty.

First, we describe the steps of the method based on the initial structure of the index sets.

Starting the method at $(\bar{x}, \bar{y})$ and $t = 0$ with $L = \{1\}$ or $I^+ = \{1\}$, the method traces the set Σ_+^L or $\Sigma_+^{I^+}$ in the direction of increasing t up to the respective exit point. In the former case, we compute the ratio

$$r = -\frac{y}{\dot{y}}$$

with

$$\begin{pmatrix} Q_{xx} & Q_{xy} & -\tilde{A}_L \\ Q_{yx} & Q_{yy} & -\tilde{B}_L^\top \\ \bar{A}_L & \tilde{B}_L & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} c_x - d_x(0) \\ c_y - d_y(0) \\ 0 \end{pmatrix}.$$

For $r \leq 0$ or $r \geq 1$ we then make a step into $\bar{t} = 1$ and terminate with the solution, else take a step into $\bar{t} = r$ and add activity of the constraint $y \geq 0$. In latter case we compute the ratio

$$q = -\frac{Ax + by + a}{A\dot{x} + b\dot{y}}$$

with

$$\begin{pmatrix} Q_{xx} & Q_{xy} & 0 \\ Q_{yx} & Q_{yy} & -E_{I+}^\top \\ 0 & E_{I+} & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\mu} \end{pmatrix} = \begin{pmatrix} c_x - d_x(0) \\ c_y - d_y(0) \\ 0 \end{pmatrix}.$$

For $q \leq 0$ or $r \geq 1$ then make a step into $t = 1$ and terminate with the solution, else take a step into $t = q$ and add activity of the constraint $Ax + by + a \geq 0$.

Starting the method at $(\bar{x}, \bar{y})$ and $t = 0$ with $I^0 = \{1\}$, we add the point $(\bar{x}, \bar{y})$ to the list "untreated exit points", otherwise proceed in the same way as if we got to one of the exit points by a step described above. The reason for this is that the method traces the set Σ_+^L or $\Sigma_+^{I^+}$ at the end of the procedure unless it was already traced the the step above.

Each step of the algorithm in the biactive case proceeds by tracing line segments between two neighboring co-n-singularities or half lines emanating from each co-n-singularity. Each such line can be generated by fixing $n-1$ vanishing multipliers. In the former case these fixed vanishing multipliers are common to both co-n-singularities. If index sets of free multipliers are I and J, cf. assumption (A11), and we are moving in t forward, the method takes the steplength

$$\bar{\tau} = \min(\{-\frac{\lambda^j}{\dot{\lambda}^j} \cap (0, 1-t), j \in I, -\frac{\lambda^i}{\dot{\lambda}^i} \cap (0, 1-t), j \in J\}),$$

where the vectors $\dot{\lambda}_I$ and $\dot{\mu}_I$ are given by the solution of

$$\begin{pmatrix} Q_{xx} & Q_{xy} & -(\tilde{A}_{I^0}^\top)_I^\top & 0 \\ Q_{yx} & Q_{yy} & -(\tilde{B}_{I^0})_I^\top & -(E_{I^0})_J^\top \\ (\tilde{A}_{I^0})_1 & (\tilde{B}_{I^0})_1 & 0 & 0 \\ 0 & E_{I^0} & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\lambda}_I \\ \dot{\mu}_J \end{pmatrix} = \begin{pmatrix} c_x - d_x(0) \\ c_y - d_y(0) \\ 0 \\ 0 \end{pmatrix}. \tag{5.24}$$

If the minimum is taken over the empty set, $t = 1$ can be reached directly.

If we are moving in t backwards, the steplength is determined by

$$\bar{\tau} = \max(\{-\frac{\lambda^j}{\dot{\lambda}^j} \cap (-\infty, 0), j \in I, -\frac{\lambda^j}{\dot{\lambda}^j} \cap (-\infty, 0), j \in J\}).$$

Now we describe how the algorithm proceeds in the biactive case. First, we add the vector of signs of the nonzero multiplier vector to the list "multiplier signs". We label the exit point to be the "starting point" and initiate the following recursive procedure called "SearchStep":

1) If the current iterate is already in the list "co-n-singularities", terminate "SearchStep", else add the iterate to the list.

2) If $t = 1$, and the iterate for the current vector of multiplier signs is not on the list "biactive C-stationary points", add it to the list with information about the multiplier signs and terminate "SearchStep".

3) If the current iterate is an i-singularity, and not in the list "i-singularities", add it to the list.

4) If the current iterate is an exit point not in the list "untreated exit points" and is not labeled as "starting point", add it to the list.

5) Put all possible n directions, determined by the index sets I and J, from the current iterate to the list "new directions". As long as the list is nonempty, execute step 6).

6) For the first direction in the list, determine the direction of the step in t by the sign of the derivative in variable t of that free multiplier which is vanishing at the current iterate and the corresponding component of the vector of sings of multipliers. If they coincide, the method proceeds with the step forward in t, else we proceed backward in t. Find the steplength. If the next step has a finite length, initiate the procedure "SearchStep" for the new iterate. Delete the first entry from the list "new directions".

When the first call of "SearchStep" terminates, we have successfully finished the analysis of the first convex polyhedral patch of the set Σ_{C-stat}. Then, until the list "i-singularities" is empty, repeat the following steps:

1) Determine the list "new multiplier signs".

2) Until the list "new multiplier signs" is empty, repeat the following. If its first entry is not in the list "multiplier signs", add it to the list "multiplier signs", label the first entry in the list "i-singularities" to be the "starting point", empty the list "co-n-singularities" and initiate "SearchStep". Delete the first entry in the list "new multiplier signs".

3) Delete the first entry in the list "i-singularities".

Now, if the list "untreated exit points" is nonempty, it suffices to check Σ_+^L and $\Sigma_+^{I^+}$ not yet investigated .

Following the set of rules above, the algorithm clearly never traces the same convex polyhedral piece of the set Σ_{C-stat} twice. However, it either terminates after one step at a nonbiactive C-stationary point or traces only polyhedral pieces of the set Σ_{C-stat} connected by i-singular and exit points with $t < 1$.

Theorem 5.14. *Let the data of the EPCC associated with n MPCCs (5.8) satisfy both assumptions (A1) and (A2). Then the following assertions hold:*

i) The algorithm terminates after finitely many steps.

ii) If the list "biactive C-stationary points" is nonempty, the set of all detected biactive C-stationary points consists of the union of convex hulls of points from the list "biactive C-stationary points" with the same corresponding vector of signs of multipliers. Moreover, interior points of each such convex hull consist of nondegenerate C-stationary points with the same C-index.

Proof. The statement of part i) follows due to the rules described above. There are only finitely many co-n-singularities and each convex polyhedral piece of the set Σ_{C-stat} is traced at most once.

The second statement follows from Theorems 5.10, 5.11 and 5.13. □

Example 5.2. (continued) Let us choose the initial feasible point $(\bar{x}^1, \bar{x}^2, \bar{y}, \bar{\lambda}^1, \bar{\lambda}^2, \bar{\mu}^1, \bar{\mu}^2)$ $= (2, 2, 0, 0, 0, 1, 1)$. Then the computation of $d(0)$ according to (5.16) yields (-6, -10, -3, -5)$^\top$. The application of the homotopy method described above results in the following three C-stationary points (x^1, x^2, y): within the biactive case the algorithm finds points (-2, 3, 0) and (1, 0, 0) with multipliers $(\lambda^1, \lambda^2, \mu^1, \mu^2)$ equal to (0,3, 5/2, 3/2) and (3/2, 3/2, 1, 0), respectively, and a nondegenerate C-stationary point (10/3, -8/3, 2/3) with the multiplier vector (17/6, 1/2, 0, 0).

The set of C-stationary points then consists of the union of the point (10/3, -8/3, 2/3) and convex hull of points (-2, 3, 0) and (1, 0, 0). Note that since each point is even strongly stationary, the set of C-stationary points coincides with the set of solutions of the EPCC.

On Figure 5.1, the blue bullets and all points on the blue line correspond to multipliers of C-stationary points of the EPCC within the biactive case. △

5.2.4 Numerical results

We have tested the performance of the homotopy method for EPCCs associated with $n = 2, \ldots, 7$ MPCCs with convex-quadratic objective functions and with one linear complementarity constraint. For each such problem we considered $l_1 = 1, 10$ and 50 variables on on the upper-level. For each combination of (n, l_1) we run the method on hundred randomly generated test problems. The algorithm was implemented in Matlab 6.5 and tests were performed on a 2.8GHz PC with 1GB RAM. The results are summarized in Table 5.1.

The columns in Table 5.1 denote the following:

I.a:	number of problems, for which Phase I.a succeeded
C:	number of problems, for which at least one C-stationary point was found
M:	number of problems, for which at least one M-stationary point was found
S:	number of problems, for which at least one solution was found
biac:	number of problems, for which the method entered the biactive case
#C-s:	total number of detected C-stationary points
#M-s:	total number of detected M-stationary points
#S-s:	total number of detected solutions
#n-biact:	total number of detected nonbiactive stationary points
∅cpu:	average CPU-time for solved problems in seconds
∅biac C-s:	average number of computed C-stationary points in the biactive case

Table 5.1: Numerical results for homotopy method

n	l	I.a	C	M	S	biac	#C-s	#M-s	#S-s	#n-biac	∅cpu	∅biac C-s
2	1	86	62	60	50	53	93	83	59	44	0.065	0.925
2	10	85	56	56	43	51	94	79	51	31	0.075	1.235
2	50	78	63	62	46	52	105	89	61	29	0.158	1.462
3	1	76	66	64	53	52	197	141	83	45	0.116	2.923
3	10	85	67	63	40	59	292	177	75	28	0.151	4.475
3	50	89	71	67	45	65	320	191	82	30	0.528	4.462
4	1	83	65	61	44	57	551	298	83	38	0.309	9.000
4	10	80	87	84	56	76	1191	596	163	39	0.732	15.158
4	50	81	83	80	47	74	1369	580	96	38	4.722	17.987
5	1	81	76	72	53	70	1718	817	117	48	0.949	23.857
5	10	85	85	83	39	79	3657	1277	106	35	5.331	45.848
5	50	82	93	92	54	82	5601	1849	187	44	20.018	67.7683
6	1	80	80	73	49	68	7937	1920	138	43	6.136	116.088
6	10	78	92	89	55	81	15962	4818	182	49	30.147	196.457
6	50	92	97	96	55	91	26650	6478	370	43	130.664	292.385
7	1	81	89	84	52	82	57354	9112	533	43	109.848	698.915
7	10	85	98	97	48	91	111419	17564	493	41	353.154	1223.934
7	50	89	98	98	56	94	178385	23136	727	46	1196.385	1897.223

We conclude this section with several remarks.

For each tested problem we applied first the phase I.a. If it failed to produce a feasible point of EPCC, the first polyhedral piece in phase I.b yielded a starting point for our homotopy method. The first piece corresponds in our case to that part of feasible set for which the constraint $y \geq 0$ is active. We could have, of course, started immediately with phase I.b, since for $m = 1$ this procedure involves checking only 2 pieces and is thus not that costly as in case of a high number of complementarity constraints.

With higher values of n, the method is more likely to find a C-stationary point. Moreover, only for a very small number of test problems for which a C-stationary point was found the method failed to find also an M-stationary point. The strongly stationary points, in our case already the solutions to EPCCs, were found for each tested combination of (n, l_1) roughly for 50 percent of randomly generated test problems.

The obtained results indicate an interesting fact that EPCCs may posses *huge number of solutions*. This brings up several important issues. The most serious one is the impact of this large cardinality of the solution set on concrete decision making processes and interpretation of these solutions with respect to the input data. Also, recall the numerical methods from previous part of this chapter. In the view of our analysis, the question arises, to which specific solution these methods converge?

5.3 Numerical method for MOPCCs

In this section we propose and describe a numerical method to solve the MOPCCs considered in Chapter 4 based on the implicit programming approach, cf. [25]. The reformulation (4.25) of our MOPCC plays a crucial role in this approach.

Note that the optimality conditions (4.21) can be used for numerical purposes only in the case when the index sets (4.20) at the solution can be guessed or well estimated. Alternatively, one can employ a nonsmooth multiobjective optimization method, e.g., the online multiobjective optimization software WWW-NIMBUS 4.1. For details about NIMBUS we refer to [26] and to the web page `http://nimbus.mit.jyu.fi`.

The implicit programming approach has been developed in connection with the Stackelberg situation in [40], [39] with the usage of a standard bundle method in nonsmooth optimization. In this section we describe a variant of this approach, which can be used for the numerical solution of the class of MOPCCs under consideration. As a test problem we take an example from [40], see below.

Under the assumption (A1') we readily observe that $(\bar{x}, \bar{y}) \in \omega \times \mathbb{I}$ is a weak Pareto solution of MOPCC whenever

$$\bar{y} = S(\bar{x})$$

and there is a neighborhood $\mathcal{U}$ of $(\bar{x}, S(\bar{x}))$ such that (with $\Theta(x) = \varphi(x, S(x))$) the relation

$$\Theta(x) - \Theta(\bar{x}) \in \operatorname{rint} K,$$

does not hold for any $(x, S(x)) \in \mathcal{U} \cap (\omega \times \mathbb{I})$. We face a new game only among the leaders without any hierarchical structure in form of a multiobjective optimization problem.

In order to use NIMBUS for calculations, one has to provide an oracle which is able to compute the function values of each leader's objective and the matrix of subgradients

$$\left(\hat{\xi}^1, \ldots, \hat{\xi}^n\right),$$

where $\hat{\xi}^i$ is an arbitrary element from the Clarke subdifferential $\bar{\partial}\Theta^i(x)$, $i = 1, \ldots, n$. Assume that all objectives φ^i, $i = 1, \ldots, n$, are continuously differentiable. Using the technique of adjoint equations, $\hat{\xi}^i$ is then computed by the formula

$$\hat{\xi}^i = \nabla_x \varphi^i(x, y) - (\nabla_x F_{L \cup (I^0 \setminus M)}(x, y))^\top \hat{\pi}^i,$$

where $\hat{\pi}^i$ is the unique solution of the adjoint equation

$$0 = (\nabla_y F_{L \cup (I^0 \setminus M), L \cup (I^0 \setminus M)}(x, y))^\top \pi - \nabla_y \varphi^i(x, y)_{L \cup (I^0 \setminus M)},$$

and where M is an arbitrary subset of $I^0(x, y) = I_1^0(x, y) \cup I_2^0(x, y)$, see the development in Section 2.3.2 leading to (2.42) and (2.43).

The followers' strategies for the given leaders' strategies can be computed by any existing method for the solution of MCP; we used the method proposed by Fukushima in [19] based on the sequential quadratic programming code NLPQL due to Schittkowski.

Table 5.2: Parameter specification for the production costs

	Firm 1	Firm 2	Firm 3
a_i	2	3	5
b_i	15	12	2

WWW-NIMBUS 4.1 works as follows. The user must specify the starting point of the procedure. Each time we have used NIMBUS, the starting point was set as the production quantities from the Stackelberg game. NIMBUS then computes a solution to the considered MOPCC which we call initial. This point is a projection of the starting point onto the set of effective points. Since this initial solution is rarely satisfactory, the user is asked to "guide the solver to a desired direction". In NIMBUS, this process is called "classification". The user can choose which of the function values should be decreased from the current level and which of the functions are less important. After submitting a new classification, NIMBUS provides a new optimal solution.

Consider now again an oligopolistic market model from Section 3.2.1 and further assume in the respective model that the leaders act cooperatively and that followers face production limitations given by $\mathbb{I} \subset \mathbb{R}^{ml_2}_{+}$. Then this model belongs to the family of MOPCCs discussed in Chapter 4.

Recall that whenever assumption (A2) is satisfied then at each feasible multistrategy (x, y) the assumption (A1") and, moreover, also the constraint qualification (4.24) holds true. Thus, given an optimal strategy pair $(\bar{x}, \bar{y})$, the necessary optimality conditions from Theorem 4.10 are satisfied.

If the function F (3.8) happens to be affine and ω is convex polyhedral, we get the optimality conditions from Theorem 4.9 without any constraint qualification, and so we do not need to impose conditions (i)-(iv) of assumption (A2). This situation occurs in the following illustrative example.

Example 5.15. Consider an example of three firms supplying some homogeneous product on the market with the linear demand function

$$p(T) = 20 - T$$

and assume that each firm has a linear production cost function in the form

$$c^i(x^i) = a_i x^i + b_i,\ i = 1, 2, 3,$$

with the coefficients given by Table 5.2.

Each firm aims to minimize its loss functions $\varphi^i(x^1, x^2, x^3),\ i = 1, 2, 3$, given by

$$\begin{aligned}
\varphi^1(x^1, x^2, x^3) &= 2x^1 + 15 - x^1(20 - x^1 - x^2 - x^3), \\
\varphi^2(x^1, x^2, x^3) &= 3x^2 + 12 - x^2(20 - x^1 - x^2 - x^3), \\
\varphi^3(x^1, x^2, x^3) &= 5x^3 + 2 - x^3(20 - x^1 - x^2 - x^3).
\end{aligned}$$

Table 5.3: Productions and profits - Cournot and Stackelberg games

		Firm 1	Firm 2	Firm 3
Cournot	Production	5.500	4.500	2.500
equilibrium	Profit	15.250	8.250	4.250
Stackelberg	Production	11.000	2.667	0.667
equilibrium	Profit	25.333	-4.889	-1.556
MOPCC	Production	5.000	4.953	2.524
stationary point	Profit	12.619	10.404	4.369

When Firm 1 and Firm 2 become the market leaders who act cooperatively, the resulting problem can be written in the form (4.8) with $y = x^3$. In this simple case we can even compute stationary points satisfying the necessary conditions (4.21). Assuming that the follower (Firm 3) produces $x^3 > 0$, we arrive at the system of four equations with six variables:

$$\begin{aligned}
0 &= z^1(-18 + 2x^1 + x^2 + x^3) + z^2x^2 - v, \\
0 &= z^1x^1 + z^2(-17 + x^1 + 2x^2 + x^3) - v, \\
0 &= z^1x^1 + z^2x^2 - 2v, \\
0 &= -15 + x^1 + x^2 + 2x^3.
\end{aligned}$$

If we fix the value of the multiplier z^1 (e.g., $z^1 = 1$), then for different settings of x^1 we get solutions to the system above. However, since the optimality conditions (4.21) are only necessary, an additional analysis is needed to verify that the obtained solution is a weak Pareto solution to the given MOPCC.

In Table 5.3 we present the comparison of the production quantities and profits of each firm when they play the Cournot game, the Stackelberg game with Firm 1 as the leader, and the MOPCC with Firm 1 and Firm 2 as the leaders; for brevity we present the comparison only for one possible stationary point. The optimal market prices for Cournot, Stackelberg, and MOPCC games are 7.5, 5.667 and 7.524, respectively. △

Next we consider a more realistic example of the oligopolistic market modeling and apply to its solution the numerical algorithm and computer codes described above.

Example 5.16. Let all the production cost functions be in the form

$$c^i(x^i) = b_ix^i + \frac{\beta_i}{1+\beta_i}K_i^{-\frac{1}{\beta_i}}(x^i)^{\frac{1+\beta_i}{\beta_i}},$$

where b_i, K_i and $\beta_i,\ i = 1, \dots, n+m$, are given positive parameters. Further, let

$$p(T) = 5000^{\frac{1}{\gamma}}T^{-\frac{1}{\gamma}},$$

Table 5.4: Parameter specification for the production costs

	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
b_i	2	8	6	4	2
K_i	5	5	5	5	5
β_i	1.2	1.1	1.0	0.9	0.8

Table 5.5: Productions and profits - Stackelberg game

Stackelberg	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
Production	99.5329	44.3804	45.8893	44.2806	40.2357
Profit	958.6347	284.6830	350.5039	393.2799	410.5319
$p(T) = 18.2270$					

with a positive parameter γ termed demand elasticity.

Each production cost function is convex and twice continuously differentiable on some open set containing the feasible set of strategies of a corresponding player. The inverse demand curve is twice continuously differentiable on int $\mathbb{R}_+$, strictly decreasing, and convex. Observe that the so-called *industry revenue curve*

$$Tp(T) = 5000^{\frac{1}{\gamma}} T^{\frac{\gamma-1}{\gamma}}$$

is concave on int $\mathbb{R}_+$ for $\gamma \geq 1$. We assume that at least one leader on the market is producing some positive production quantity. Hence all the above assumptions (i)–(iv) are fulfilled, and assumption (A1)' is fulfilled as well.

The data are taken from [40] and [39], where numerical tests are performed for $n+m = 5$, $\gamma \in [1, 2]$ and the parameters of the production cost function given by Table 5.4. For these data, Table 5.5 shows the productions and profits of all the firms for $\gamma = 1.0$ in the Stackelberg case, when Firm 1 is the only leader; clearly, Firm 1 dominates the market.

Consider next the MOPCC case, when Firm 5 (the second strongest producer) becomes the second leader. The results are displayed in Table 5.6. The first section corresponds to the initial solution given by NIMBUS, the second one describes the situation under a contract that is beneficial for both leaders. In the third section we show the initial solution given by NIMBUS in the case when a uniform upper bound was imposed on the productions of the followers. We have set the upper production bound to 49 to demonstrate the effect of presence of the active upper bounds.

Due to a great difference between the market power of Firm 1 and Firm 5 (see Table 5.5), the stronger leader Firm 1 has to sacrifice a part of its profit to the benefit of Firm 5. One can expect that the bigger the power difference between both leaders, the more the stronger leader has to sacrifice. Observe that also the remaining three firms significantly

Table 5.6: Productions and profits - Firm 1 and Firm 5 leaders

MOPCC, $\gamma = 1.0$	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
no upper production bound					
Production	62.8288	49.1867	49.7383	47.2930	42.4805
Profit	840.8600	378.3762	442.9064	478.9642	485.6284
$p(T) = 19.8786$					
no upper production bound					
Production	88.8892	46.7669	47.7940	45.7640	33.7368
Profit	978.8980	328.1489	393.6102	433.3945	410.9734
$p(T) = 19.0150$					
upper production bound 49					
Production	52.7198	49.0000	49.0000	48.7618	41.2945
Profit	784.5941	421.1413	483.4428	527.4281	517.9824
$p(T) = 20.7662$					

Table 5.7: Productions and profits - Stackelberg game, $b_1 = 10$

Stackelberg	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
$\gamma = 1.0$					
Production	55.5483	50.1342	50.5040	47.8997	42.9768
Profit	343.3453	400.008	463.9979	498.3845	502.6867
$p(T) = 20.2378$					

increased their profits. We could see this phenomenon already in Example 5.2. This improvement is even more noticeable in the case when their productions are limited.

We finish our analysis by modifying the input data. We alter the parameter specifications from Table 5.4 and set $b_1 = 10$ (instead of 2) to show the results of the situation when not necessarily the strongest producers pretend to become cooperative leaders. The elasticity parameter γ remains 1.0. For the Stackelberg situation with Firm 1 as the leader, the productions and profits of all the firms are shown in Table 5.7.

In Table 5.8 we present then the results for the MOPCC case when Firm 2 becomes the second leader. The first section of this table presents the respective initial solution given by NIMBUS. The second and the third sections represent situations when the contract between both leaders is more beneficial for Firm 1 and Firm 2, respectively. The last section of the table displays one acceptable solution in the case when the followers' productions are subject to a certain upper bound (namely, 50).

We can observe that with an agreement more beneficial for the stronger leader, the market price decreases but still exceeds the market price in the Stackelberg game.

Table 5.8: Productions and profits - games with two Leaders, $b_1 = 10$

MOPCC, $\gamma = 1.0$	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
Production	45.2558	42.5467	52.6768	49.6384	44.3219
Profit	357.8634	410.9407	529.9167	558.8654	555.3318
$p(T) = 21.3275$					
Production	50.8043	33.9370	53.0478	49.9381	44.5561
Profit	394.2186	357.6356	542.1837	570.0930	565.0856
$p(T) = 21.5254$					
Production	36.5470	50.1959	52.8052	49.7420	44.4028
Profit	311.8783	458.3726	534.1281	562.7208	558.6816
$p(T) = 21.3956$					
upper bound 50					
Production	42.6431	39.8824	50.0000	50.0000	45.0693
Profit	371.6054	419.1422	548.4432	592.5498	582.1324
$p(T) = 21.9689$					

Note that all the conclusions stated in connection with the previous set of data can be applied here as well. $\triangle$

Conclusion

In this thesis we have discussed several models with hierarchical structure in which an equilibrium problem arises either only on lower level or both levels.

In the chapter dedicated to MPECs, we presented the most important subclasses; with the main focus on MPCCs. Our main aim was to build a bridge between KKT-type stationarity concepts coined in [45] and optimality conditions derived in [39]. It was found that Clarke and C-stationarity conditions coincide if the underlying generalized equation of the respective (lower-level) solution map is strongly regular and MPEC-GLICQ holds true. The latter is a modified version of linear independence constraint qualification specifically tailored to the MPCC structure.

We have used this bridge to derive the existence conditions for C-stationary points of EPCCs. We realize that even if implicit programming approach can be applied, the resulting problem admits a structure of a nonconvex Nash game. We have therefore focused our attention on a generalized concept of solutions and investigated sufficient conditions for existence of solutions in mixed strategies.

Motivated by the oligopolistic market model, we investigated also the case of cooperative behavior of the upper-level players. Using the advanced subdifferential calculus for set-valued mappings and subdifferential Palais-Smale type condition, we obtained existence of solutions to MOPCCs.

Finally, we have focused our attention on approaches to solve EPECs and MOPECs numerically. The proposed generalization of the homotopy method I from [42], despite the strong limitation on the structure of considered problem, revealed rather complex structure of the sets stationarity points of EPCCs. This indicates that in general case the set of solutions (if non-empty) is composed not only of isolated points. We plan to analyze whether this phenomenon occurs solely due to linear-quadratic structure of considered problems and to test the numerical performance of the method for the EPCC with a greater number of complementarity constraints. Another important question is the sensitivity analysis of stationary points in EPCCs with respect to perturbations of the data. We intend to investigate these topics in future research.

Appendix A

Variational Analysis

Throughout the thesis we used many term and results we believed unnecessary to include directly in the text, assuming that the reader may be already familiar with the theory of nonlinear optimization, MPECs and generalized calculus of Mordukhovich. In this and the following appendices we intend to present the definitions and results which were required for our analysis.

A.1 Multifunctions

For maps F which assign subsets of $\mathbb{R}^m$ to points from $\mathbb{R}^n$, denoted by $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, we use the term *set-valued mapping* or simply *multifunction*. Both terms can be used interchangeably.

Definition A.1. *(domain and graph of a multifunction)*
For a set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, we call the sets

$$\text{Dom } F := \{x \in \mathbb{R}^n | F(x) \neq \emptyset\}, \tag{A.1}$$

$$\text{Gph } F := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | y \in F(x)\}, \tag{A.2}$$

the domain *of F and the* graph *of F, respectively.*

Definition A.2. *(basic properties of a multifunction)*
Let $x \in \mathbb{R}^n$. A multifunction $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is called

a) closed at x *if the following implication holds:*

$$\left.\begin{array}{r} x^{(i)} \xrightarrow{\text{Dom } F} x \\ y^{(i)} \in F(x^{(i)}) \\ y^{(i)} \to y \end{array}\right\} \Rightarrow y \in F(x).$$

b) upper semicontinuous at x *if for any neighborhood U of $F(x)$ there is $\eta > 0$ such that for all $x' \in \eta\mathbb{B}(x), F(x') \subset U$.*

c) lower semicontinuous at x *if for any* $y \in F(x)$ *and for any sequence of elements* $x^{(n)} \in$ Dom F *converging to* x*, there is a sequence of elements* $y^{(n)} \in F(x^{(n)})$ *converging to* y*.*

d) continuous at x *if it is both upper and lower semicontinuous at* x*.*

e) continuous *if it is continuous at every point* $x \in \mathbb{R}^n$*.*

f) convex-valued *if for each* $x \in$ Dom F *the set* $F(x)$ *is convex.*

g) closed-valued *if for each* $x \in$ Dom F *the set* $F(x)$ *is closed.*

Local Lipschitz continuity of single-valued functions can be naturally extended to multifunctions.

Definition A.3. *A set-valued mapping* $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *has* the Aubin property *around* $(\bar{x}, \bar{y}) \in$Gph F *if there are neighborhoods* V *of* $\bar{x}$ *and* W *of* $\bar{y}$ *and a constant* $k \geq 0$ *such that*

$$F(x') \cap W \subset F(x) + k\|x' - x\|\mathbb{B} \ \ \textit{for all } x, x' \in V.$$

Note that if F is not lower semicontinuous at $\bar{x}$ then there exist $\bar{y} \in F(\bar{x})$ such that Aubin property does not hold around $(\bar{x}, \bar{y})$.

Definition A.4. *A set-valued map* $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ *is said to be* calm at $(\bar{x}, \bar{y}) \in$ Gph F with modulus $\lambda \geq 0$ *if there are neighborhoods* V *of* $\bar{x}$ *and* W *of* $\bar{u}$ *such that*

$$F(x) \cap W \subset F(\bar{x}) + \lambda\|x - \bar{x}\|\mathbb{B} \ \ \textit{for all } x \in V.$$

Obviously, if F has the Aubin property around some point of its graph, it is also calm at this reference point .

A.2 Generalized differentiation

Definition A.5. *(limiting normal cone)*
Given $\Omega \subset \mathbb{R}^n$ *and* $\bar{x} \in$ cl Ω*, the* limiting *(or basic)* normal cone *to* Ω *at* $\bar{x}$ *is defined by*

$$N(\bar{x}; \Omega) = \operatorname*{Lim\,sup}_{x \to \bar{x}} \,[\text{cone } (x - \Pi(x; \Omega)]. \tag{A.3}$$

By convention, we set $N(\bar{x}; \Omega) := \emptyset$ if $\bar{x} \notin$ cl Ω.

The Euclidean projector onto cl Ω is given by

$$\Pi(x; \Omega) := \{w \in \text{cl } \Omega \mid \| x - w \| = \text{dist } (x; \Omega)\}\,.$$

In (A.3), the symbol "Lim sup" stands for the *Painlevé-Kuratowski upper* (or outer) *limit* that is defined for a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at a point $\bar{x}$ by

$$\operatorname*{Lim\,sup}_{x \to \bar{x}} F(x) := \{y \in \mathbb{R}^m | \exists x^{(k)} \to \bar{x}, \exists y^{(k)} \to y \text{ with } y^{(k)} \in F(x^{(k)})\}.$$

The limiting normal cone (A.3) is generally *nonconvex*. For a convex set Ω, however, it reduces to the normal cone in the sense of convex analysis. The normal cone can be equivalently represented as

$$N(\bar{x};\Omega) = \operatorname*{Lim\,sup}_{x \xrightarrow{\Omega} \bar{x}} \hat{N}(x;\Omega),$$

where the *Fréchet normal* (or prenormal) *cone* $\hat{N}(\cdot;\Omega)$ is defined by

$$\hat{N}(\bar{x};\Omega) := \begin{cases} \{x^* \in \mathbb{R}^n | \operatorname*{limsup}_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \leq 0\} & \text{for } \bar{x} \in \mathrm{cl}\Omega, \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that the prenormal cone is the negative polar cone to the *Bouligand-Severi contingent cone*

$$T(\bar{x};\Omega) = \operatorname*{Lim\,sup}_{t \searrow 0} \frac{\Omega - \bar{x}}{t}.$$

The limiting normal cone (A.3) cannot be dual to any tangent cone due to its nonconvexity: polar cones are always convex.

By the critical cone of Ω with respect to y and $x - y$ we understand the set

$$K(x,y;\Omega) = T(y;\Omega) \cap \{x-y\}^{\perp}.$$

Definition A.6. *(Clarke tangent and normal cones)*
Given $\Omega \subset \mathbb{R}^n$ and $\bar{x} \in cl\ \Omega$, the Clarke tangent cone *to Ω at $\bar{x}$ is defined by*

$$T_C(\bar{x};\Omega) = \operatorname*{Lim\,inf}_{\substack{x \xrightarrow{\Omega} \bar{x} \\ t \searrow 0}} \frac{\Omega - \bar{x}}{t}.$$

and the Clarke normal cone $N_C(\bar{x};\Omega)$ *is its negative polar cone.*

For an arbitrary set it obviously holds that

$$\hat{N}(x;\Omega) \subset N(x;\Omega) \subset N_C(x;\Omega),$$

where the inclusions can be replaced by equalities for convex set Ω.

For details on normal cones, we refer the reader to [44], [29] and [30].

The limiting and Clarke subdifferentials can be defined in terms of the respective normal cones.

Definition A.7. *(limiting and Clarke subdifferentials)*
Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz continuous. Then the limiting subdifferential *of f in $\bar{x} \in \mathbb{R}^n$ is given by*

$$\partial f(\bar{x}) := \{\xi | (\xi,-1) \in N(\bar{x}, f(\bar{x}); \text{epi } f)\}$$

and the Clarke subdifferential *of f in $\bar{x} \in \mathbb{R}^n$ is given by*

$$\bar{\partial} f(\bar{x}) := \{\xi | (\xi,-1) \in N_C(\bar{x}, f(\bar{x}); \text{epi } f)\}.$$

It immediately follows that

$$\partial f(x) \subset \bar{\partial} f(x),$$

and if f is a convex function the inclusion becomes equality.

For details on the calculus of Clarke and limiting subdifferentials, we refer the reader to [7], [29] and [30].

The extension of the Clarke subdifferential to locally Lipschitz continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$ is the generalized Jacobian.

Definition A.8. *(generalized Jacobian)*
Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitz continuous. Then the generalized Jacobian *of f at x is the subset of $\mathbb{R}^{m\times n}$ given by*

$$\bar{\partial} f(x) = \text{ conv } \{\lim_{i\to\infty} \nabla f(x^{(i)}) | x^{(i)} \to x, x^{(i)} \neq \Omega_f\},$$

where

$$\Omega_f := \{x | \nabla f(x) \ \textit{does not exist}\}.$$

For locally Lipschitz continuous functions, the set Ω_f has Lebesgue measure zero. If $m = 1$, the generalized Jacobian coincides with the transpose of the Clarke subdifferential. It is however common in literature to denote both objects by the same symbol.

Among the main derivative-like constructions for multifunctions are coderivatives. They provide a pointwise approximation of a set-valued mappings using elements of dual spaces.

Definition A.9. *(coderivative)*
Given a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and a point $(\bar{x}, \bar{y})$ from its graph the coderivative *$D^*F(\bar{x}, \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ of F at $(\bar{x}, \bar{y})$ is a set-valued map defined by*

$$D^*F(\bar{x}, \bar{y})(y^*) := \{x^* \in \mathbb{R}^n | (x^*, -y^*) \in N(\bar{x}, \bar{y};\ \text{Gph } F)\}, \tag{A.4}$$

where $\bar{y}$ in the notation $D^*F(\bar{x}, \bar{y})$ is omitted if F is single-valued at $\bar{x}$.

In general, $D^*F(\bar{x}, \bar{y})(\cdot)$ is a positively homogeneous closed multifunction at all points $\bar{x} \in$Dom $F, \bar{y} \in F(\bar{x})$ and it reduces to the adjoint Jacobian

$$D^*F(\bar{x})(y^*) = \{\nabla F(\bar{x})^\top y^*\}, y^* \in \mathbb{R}^m,$$

when F is single-valued and strictly differentiable at $\bar{x}$.

A.3 Variational inequality and complementarity problem

The following definitions are taken from [16].

Definition A.10. *(variational inequality)*
For a convex set $\Omega \subset \mathbb{R}^m$ and a map $f : \Omega \to \mathbb{R}^m$ the variational inequality *is a problem to find a point $x \in \Omega$ such that*

$$(y - x)^\top f(x) \geq 0 \ \ \forall y \in \Omega. \tag{A.5}$$

If Ω is closed and f continuous on an open set containing Ω, the set of solutions to variational inequality (A.5) is closed, possibly empty. Equivalently, the variational inequality can be written down using the normal cone in the form of generalized equation:

$$0 \in f(x) + N(x; \Omega).$$

When Ω is a cone, the variational inequality can be expressed in an equivalent form of a complementarity problem.

Definition A.11. *(complementarity problem)*
For a convex cone Ω and a map $f : \Omega \to \mathbb{R}^m$, the complementarity problem *is to find a point $x \in \mathbb{R}^m$ such that the conditions*

$$\Omega \ni x \perp f(x) \in \Omega^*,$$

where Ω^* is the dual (positive polar) cone of Ω.

Consider the following special case. When Ω is the nonnegative orthant of $\mathbb{R}^m$ the complementarity problem is referred to as the (classical) nonlinear complementarity problem.

Definition A.12. *(nonlinear complementarity problem)*
Given a map $f : \mathbb{R}^m_+ \to \mathbb{R}^m$, the nonlinear complementarity problem *is to find a point $x \in \mathbb{R}^m$ such that*

$$0 \leq x \perp f(x) \geq 0.$$

This model can be easily extended to a generalized complementarity problem involving two (or possibly more) functions $F^1, F^2 : \mathbb{R}^n \to \mathbb{R}^m$:

$$0 \leq F^1(x) \perp F^2(x) \geq 0.$$

Consider now a cone $\Omega \subset \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}_+, m_1 + m_2 = m$. We can formulate the following generalization of NCP.

Definition A.13. *(Mixed complementarity problem)*
Let g and h be two mappings from $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}_+$ into $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}_+$, respectively. The mixed complementarity problem *is to find a pair $(u, v) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ such that*

$$\begin{aligned} &g(u, v) = 0, \\ &0 \leq v \perp h(u, v) \geq 0. \end{aligned}$$

An important special case of the variational inequality (A.5) is the one with the set Ω given by

$$\Omega = \{x \in \mathbb{R}^m | a_i \leq x_i \leq b_i, i = 1, \ldots, m\},$$

with real constants a_i and b_i satisfying

$$-\infty \leq a_i < b_i \leq \infty, \ \ \forall i.$$

If all a_i and b_i are finite, we refer to it as to a box constrained variational inequality. With a and b the vectors with components a_i and b_i respectively, this variational inequality can be equivalently written down as

$$\begin{aligned} f(x) + y^+ - y^- &= 0, \\ 0 \leq y^+ \perp x - a &\geq 0, \\ 0 \leq y^- \perp x - b &\geq 0. \end{aligned}$$

Clearly, a box constrained variational inequality attains the form of MCP.

Definition A.14. *(C-function)*
A function $\Psi : \mathbb{R}^2 \to \mathbb{R}$ is called a C-function *(complementarity function), if for any pair $(a, b) \in \mathbb{R}^2$*

$$\Phi(a, b) = 0 \Leftrightarrow [(a, b) \geq 0, ab = 0].$$

Given any C-function Φ, the NCP can be equivalently reformulated to the equation form:

$$0 = \begin{pmatrix} \Phi(x_1, f_1(x)) \\ \vdots \\ \Phi(x_m, f_m(x)) \end{pmatrix}.$$

The simplest C-function is the minimum function $\Phi(a, b) := \min\{a, b\}, (a, b) \in \mathbb{R}^2$, in connection with NCP also called *Pang NCP function.* So, x solves the generalized NCP if and only if $\min\{F^1(x), F^2(x)\} = 0$.

We conclude this section with the definition of a face of a nonempty convex polyhedral set C.

Definition A.15. *(face of a convex polyhedral set)*
A subset C' of a convex polyhedral set $C \subset \mathbb{R}^n$ is called a face *of C, if it is convex and if for each line segment $[x, y] \subset C$ with $(x, y) \cap C' \neq \emptyset$ one has $x, y \in C'$.*

Appendix B

LICQ and MFCQ of Standard Nonlinear Program

Consider the (classical) nonlinear programming problem

$$\begin{aligned} \text{minimize}\, & f(x) \\ & g(x) \leq 0, \\ & h(x) = 0, \end{aligned} \tag{B.1}$$

with continuously differentiable functions $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^p$.

We introduce only the constraint qualifications which play role in our analysis of MPECs and EPECs, namely, the linear independence and Mangasarian-Fromowitz constraint qualifications.

Definition B.1. *(LICQ and MFCQ for nonlinear programs)*
Let $\bar{x}$ be a feasible point of the program (B.1). We say that

a) the linear independence constraint qualification *holds at $\bar{x}$ if the gradients*

$$\nabla g_i(\bar{x}),\ \forall i \in I_g,$$
$$\nabla h_j(\bar{x}),\ \forall j = 1, \ldots, p,$$

are linearly independent, with the set of the active inequality constraints constraints

$$I_g = \{i | g_i(\bar{x}) = 0\}.$$

b) the Mangasarian-Fromowitz constraint qualification *holds at $\bar{x}$ if the gradients*

$$\nabla h_j(\bar{x}),\ \ \forall j = 1, \ldots, p,$$

are linearly independent and there exists a vector $d \in \mathbb{R}^n$ such that

$$\nabla g_i(\bar{x})^\top d < 0,\ \ \forall i \in I_g,$$
$$\nabla h_j(\bar{x})^\top d = 0,\ \ \forall j = 1, \ldots, p,$$

We can reformulate the LICQ into into equivalent form:

$$\left(\sum_{i\in I_g}\mu_i\nabla g_i(\bar{x})+\sum_{j=1}^{p}\nu_j\nabla h_j(\bar{x})=0\right)\Rightarrow\begin{cases}\mu_i=0, & i\in I_g,\\ \nu_j=0, & j=1,\ldots,p.\end{cases}$$

The Mangasarian-Fromowitz constraint qualification can be equivalently stated in the form:

$$\left.\begin{aligned}&\sum_{i\in\mathcal{I}_g}\mu_i\nabla g_i(\bar{x})+\sum_{j=1}^{p}\nu_j\nabla h_j(\bar{x})=0\\ &\hspace{12em}\mu_i\geq 0,\ \ i\in I_g\end{aligned}\right\}\Rightarrow\begin{cases}\mu_i=0, & i\in I_g,\\ \nu_j=0, & j=1,\ldots,p.\end{cases}$$

Hence, in the Fritz John type of necessary optimality conditions for the nonlinear program (B.1), LICQ and MFCQ prevent the existence of the so-called abnormal (degenerate) multiplier.

Clearly,

$$\text{LICQ}\ \Rightarrow\ \text{MFCQ}.$$

Appendix C

Noncooperative Nash Games

Let us have n players and assume that each player $i, i = 1, \dots, n$ may choose to play a strategy x^i from his or her action space U^i. In infinite games, the action space for at least one of the players has infinitely many elements. We may simply assume that $U^i \subset \mathbb{R}^l$. Denote the feasible set of multistrategies $x := (x^1, \dots, x^n)$ by

$$\omega := \bigtimes_{i=1}^{n} U^i.$$

Let us also denote for the ith player by x^{-i} an element of $\omega_{-i} := \bigtimes_{j \neq i} U^j$, where x^{-i} stands for the strategies of the other players, over which he or she has no control in the absence of cooperation.

Definition C.1. *(decision rule)*
A decision rule *of the ith player is a multifunction $C^i : \omega_{-i} \rightrightarrows U^i$ which assigns to the multistrategies $x^{-i} \in \omega_{-i}$ determined by the other players, a strategy set $C^i(x^{-i}) \subset U^i$.*

From the previous definition an obvious question arises. Once each player has been identified with its own decision rule, under what assumptions there is a common multistrategy, the so called *consistent multistrategy* $x \in \omega$, such that

$$x^i \in C^i(x^{-i}), \;\; \forall i = 1, \dots, N.$$

Theorem C.2. *Let all n strategy sets $U^i, i = 1, \dots, n$, be convex and compact and all n decision rules C^i be upper semicontinuous multifunctions with nonempty, closed and convex values. Then there exists a consistent multistrategy.*

Proof. See [2, Theorem 12.1]. □

Let us now suppose that the decision rules are determined by *loss functions* $\varphi^i : \omega \to \mathbb{R}$. The associated decision rules, the so called *canonical decision rules* are defined by

$$\bar{C}^i(x^{-i}) := \left\{ x^i \in U^i | \varphi^i(x^i, x^{-i}) = \inf_{y^i \in U^i} \varphi^i(y^i, x^{-i}) \right\}.$$

This leads us to the definition of Nash equilibria.

Definition C.3. *(Nash equilibrium)*
A multistrategy $\bar{x} \in \omega$ which is consistent for the canonical decision rules is called a noncooperative Nash equilibrium.

Theorem C.4. *The following assertions are equivalent:*

a) *$\bar{x}$ is a noncooperative equilibrium;*

b) $\varphi^i(\bar{x}^i, \bar{x}^{-i}) \leq \varphi^i(x^i, \bar{x}^{-i}) \;\; \forall i = 1, \ldots, n, \;\; \forall x^i \in U^i$;

c) $\sum_{i=1}^n (\varphi^i(\bar{x}^i, \bar{x}^{-i}) - \varphi^i(x^i, \bar{x}^{-i})) \leq 0 \;\; \forall x \in \omega$.

Proof. See [2, Proposition 12.1]. □

One can often find the definition of Nash equilibria in terms of part b) of the Theorem C.4 which, in words, states that no individual deviation from the equilibrium strategy decreases the value of loss function of a player in question.

The following theorem provides a set of sufficient conditions under which a Nash equilibrium exists. This theorem is also known as Nash theorem.

Theorem C.5. *Let for each $i = 1, \ldots, n$, the sets U^i be convex and compact and the functions φ^i be continuous and convex in x^i for every $x^{-i} \in \omega_{-i}$. Then there exists a noncooperative equilibrium.*

Proof. The theorem follows from the Ky Fan's theorem. For more details on the proof see [2, Theorem 12.2]. □

Assume that for $i = 1, \ldots, n$, the loss functions φ^i are continuously differentiable. If there is a Nash equilibrium $\bar{x} \in \omega$, one observes that it is a solution of the generalized equation

$$0 \in F(x) + N(x; \omega), \tag{C.1}$$

where $F(x)$ denotes the vector composed of the partial gradients of $\nabla_{x^i}\varphi^i(x)$

$$F(x) = \begin{pmatrix} \nabla_{x^1}\varphi^1(x) \\ \vdots \\ \nabla_{x^n}\varphi^n(x) \end{pmatrix}.$$

To gain more details, we refer, e.g., to [2].

Bibliography

[1] R.P. Agdeppa, N. Yamashita and M. Fukushima (2007): The Traffic Equilibrium Problem with Nonadditive Costs and Its Monotone Mixed Complementarity Problem Formulation, *Transportation Research Part B* **41**(8), 862-864.

[2] J.-P. Aubin (1993): *Optima and Equilibria*, Springer, Berlin.

[3] J.-P. Aubin and H. Frankowska (1990): *Set-valued Analysis*, Birkhäuser, Boston.

[4] T. Q. Bao and B. S. Mordukhovich (2007): Variational principles for set-valued mappings with applications to multiobjective optimization, *Control and Cybernetics* **36**, 531-562.

[5] T. Q. Bao and B. S. Mordukhovich (2007): Existence of minimizers and necessary conditions in set-valued optimization with equilibrium constraints, *Appl. Math.* **26**, 453-472.

[6] T. Başar and G. J. Olsder (1982): *Dynamic Noncooperative Game Theory*, Academic Press Inc. (London) Ltd.

[7] F. H. Clarke (1983): *Optimization and Nonsmooth Analysis*, Wiley, New York.

[8] M. Červinka (2004): Necessary Conditions for Solutions to 2-Leader-and-1-Follower EPEC, *WDS'04 Proceedings of Contributed Papers, Part I*, 58-62.

[9] M. Červinka (2007): A note on existence of mixed solutions to equilibrim problems with equilibrium constraints, *Bulletin of the Czech econometric society* **14**(24), 27-44.

[10] M. Červinka (2008): On Equivalence and Existence of Clarke and C-stationary Points for Equilibrium Problems with Complementarity Constraints, working paper.

[11] M. Červinka and D. Ralph (2008): On Computation of C-stationary Points for Equilibrium Problems with Linear Complementarity Constraints via Homotopy Method, working paper.

[12] S. Dempe (2002): *Foundations of bi-level programming*, Kluwer Academic Publishers, Dordrecht, The Netherlands.

[13] S. Dempe (2003): Annotated Bibliography on Bilevel Programming and Mathematical Programs with Equilibrium Constraints, *Optimization* **52**, 333-359.

[14] S. Dempe, J. Dutta and B. S. Mordukhovich (2007): New necessary optimality conditions in optimistic bilevel programming, *Optimization* **56**, 577-604.

[15] A. L. Dontchev and R. T. Rockafellar (1996): Characterizations of Strong Regularity for Variational Inequalities over Polyhedral Convex Sets, *SIAM J. Optimization* **6**(4), 1087-1105.

[16] F. Facchinei and J.-S. Pang (2003): *Finite-Dimensional Variational Inequalities and Complementarity Problems, Volume I*, Springer Verlag, New York.

[17] S. C. Fang and E. L. Peterson (1982): Generalized Variational Inequalities, *Journal of Optimization Theory and Applications* **38**(3), 363-383.

[18] M. L. Flegel (2005): *Constraint Qualifications and Stationary Concepts for Mathematical Programs with Equilibrium Constraints*, Dissertation, University of Würzburg.

[19] M. Fukushima (1992): Equivalent differentiable optimization problems and descent methods for asymetric variational inequality problems, *Mathematical Programming* **53**, 99-110.

[20] R. Henrion, A. Jourani and J. V. Outrata (2002): On the calmness of a class of multifunctions, *SIAM Journal on Optimization* **13**, 603-618.

[21] R. Henrion and J. V. Outrata (2005): Calmness of constraint systems with applications, *Mathematical Programming Ser. B* **104**(2-3), 437-464.

[22] X. Hu, D. Ralph, E. K. Ralph, P. Bardsley and M. C. Ferris (2004): Electricity generation with looped transmission networks: Bidding to an ISO, *Judge Institute of Management*, Cambridge University, Research Paper No. 2004/16.

[23] H. T. Jongen, P. Jonker and F. Twilt (2000): *Nonlinear Optimization in Finite Dimensions*, Kluwer, Dordrecht.

[24] S. Leyffer and T. Munson (2005): Solving Multi-Leader-Follower Game, Preprint ANL/MCS-P1243-0405.

[25] Z.-Q. Luo, J.-S. Pang D. and Ralph (1996): *Mathematical Programs with Equilibrium Constraints*, Cambridge University Press, Cambridge.

[26] K. Miettinen and M. M. Mäkelä (2006): Synchronous Approach in Interactive Multiobjective Optimization, *European Journal of Operational Research* **170**(3), 909-922.

[27] B. S. Mordukhovich (1994): Generalized differential calculus for nonsmooth and set-valued mappings, *J. Math. Analysis and Applications* **183**, 250-288.

[28] B. S. Mordukhovich (2004): Equilibrium problems with equilibrium constraints via multiobjective optimization, *Optimization Methods and Software* **19**, 479-492.

[29] B. S. Mordukhovich (2005): *Variational Analysis and Generalized Differentiation Vol. 1: Basic Theory*, Springer, Berlin.

[30] B. S. Mordukhovich (2005): *Variational Analysis and Generalized Differentiation Vol. 2: Applications*, Springer, Berlin.

[31] B. S. Mordukhovich, J. V. Outrata and M. Červinka (2007): Equilibrium Problems with Complementarity Constraints: Case study with Applications to Oligopolistic Markets, *Optimization* **56**(4), 479-494.

[32] F. H. Murphy, H. D. Sherali and A. L. Soyster (1982): A mathematical programming approach for determining oligopolistic market equilibrium, *Mathematical Programming* **24**, 92-106.

[33] K. G. Murty (1988): *Linear Complementarity, Linear and Nonlinear Programming*, Helderman-Verlag.

[34] J. F. Nash (1950): Non-cooperative games, *Annals of Mathematics* **54**, 286-295.

[35] J. V. Outrata (1990): On the numerical solution of a class of Stackelberg problems, *Zeit. f. Operations research* **34**, 255-277.

[36] J. V. Outrata (1999): Optimality conditions for a class of mathematical programs with equilibrium constraints, *Mathematics of Operations Research* **24**, 627-644.

[37] J. V. Outrata (2004): A note on a class of Equilibrium Problems with Equilibrium Constraints, *Kybernetika* **40**, 585-594.

[38] J. V. Outrata (2006): Mathematical Programs with Equilibrium Constraints: Theory and Numerical Methods. *Nonsmooth Mechanics of Solids, CISM Courses and Lectures* **485**, (J. Haslinger and G.E. Stavroulakis, eds.), Springer, 221-274.

[39] J. V. Outrata, M. Kočvara and J. Zowe (1998): *Nonsmooth Approach to Optimization Problems with Equilibrium Constraints*, Kluwer Academic Publisher, Dordrecht, The Netherlands.

[40] J. V. Outrata and J. Zowe (1995): A numerical approach to optimization problems with variational inequality constraints, *Mathematical Programming* **68**, 105-130.

[41] G. Owen (1976): Existence of Equilibrium Pairs in Continuous Games, *Int. Journal of Game Theory* **5**, 97-105.

[42] D. Ralph and O. Stein (2006): Homotopy Methods for QPCCs, Preprint No. 120, Lehrstuhl C für Mathematik, RWTH Aachen.

[43] S. M. Robinson (1980): Strongly regular generalized equations, *Mathematics of Operations Research* **5**, 43-62.

[44] R. T. Rockafellar and R. J.-B. Wets (1998): *Variational Analysis*, Springer, Berlin.

[45] H. Scheel and S. Scholtes (2000): Mathematical Programs with Complementarity Constraints: Stationarity, Optimality and Sensitivity, *Mathematics of Operations Research* **25**(1), 1-22.

[46] S. Scholtes (1994): *Introduction to Piecewise Differential Equations*, Habilitation thesis, Institut für Statistik und Mathematische Wirtschaftstheorie, Universität Karlsruhe, Germany.

[47] S. Scholtes: On the existence and computation of EPEC solutions, a talk given at the ICCP Conference in Cambridge, July 2002.

[48] S. Scholtes and M. Stöhr (2001): How stringent is the linear independence assumption for mathematical programs with complementarity constraints?, *Mathematics of Operations Research* **26**, 851-863.

[49] C.-L. Su (2005): *Equilibrium Problems with Equilibrium Constraints: Stationarities, Algorithms and Applications*, Ph.D. thesis, Department of Management Science and Engineering, Standford University.

[50] J. von Neumann and O. Morgenstern (1944): *Theory of games and economic behavior*, Princeton University Press, Princeton.

[51] H. von Stackelberg (1934): *Marktform und Gleichgewicht*, Springer, Berlin.

[52] H. Yang, F. Xiao and H. Huang (2005): Competition and Equilibria of Private Toll Roads with Elastic Demand, working paper.

[53] J. J. Ye and Q. J. Zhu (2003): Multiobjective Optimization Problem with Variational Inequality Constraints, *Mathematical Programming* **96**(1), 139-160.