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Semilinear copulas

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Abstract

A family of copulas, called semilinear, is constructed starting with some assumptions about the linearity of the copulas along some segments of the unit square. This family contains some other known families of copulas (e.g., Cuadras–Augé, Fréchet) and has a nice statistical interpretation. Several construction methods are provided, especially concerning aggregation of semilinear copulas, and a special form of ordinal sum construction is introduced. Some results about related families of quasi-copulas and semicopulas are hence given.

Keywords: Copula; Quasi-copula; Semicopula

1. Introduction

A two-dimensional *copula* (a *copula*, for short) is a function C from $[0, 1]^2$ into $[0, 1]$ which satisfies the following properties:

(C1) $C(x, 0) = C(0, x) = 0$ for all $x \in [0, 1]$;
(C2) $C(x, 1) = C(1, x) = x$ for all $x \in [0, 1]$ and
(C3) for all x, x', y, y' in $[0, 1]$ with $x \leqslant x'$ and $y \leqslant y'$,

$$V_C([x, x'] \times [y, y']) := C(x', y') - C(x, y') - C(x', y) + C(x, y) \geqslant 0.$$

Conditions (C1) and (C2) express the *boundary properties* of a copula C, (C3) is the 2-*increasing property* of C, also called *moderate growth*, and $V_C([x, x'] \times [y, y'])$ is called the C-*volume* of the rectangle $[x, x'] \times [y, y']$ (see [22] for a thorough exposition).

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Copulas were introduced in a statistical context in order to join bivariate distribution functions (=d.f.'s) to their univariate marginal d.f.'s. In fact, according to *Sklar's theorem* [24], for each random pair (X, Y) there is a copula $C = C_{X,Y}$ (uniquely defined whenever X and Y are continuous) such that the joint distribution function $F_{X,Y}$ of (X, Y) may be represented, for all $x, y \in \overline{\mathbb{R}}$, under the form

$$F_{X,Y}(x, y) = C(F_X(x), F_Y(y)), \tag{1.1}$$

where F_X and F_Y are the d.f.'s of the random variables X and Y, respectively. Conversely, given a copula C and two univariate d.f.'s F_X and F_Y, the function $F_{X,Y}$ given by (1.1) is a bivariate d.f. with marginal d.f.'s F_X and F_Y.

Recently, the theory of copulas (and related operators) has received a growing interest from researchers interested in fuzzy set theory, especially in view of possible applications in preference modelling and similarities. For example, when the intersection of two fuzzy sets A and B on a finite universe X is defined pointwise, i.e., $(A \cap B)(x) = I(A(x), B(x))$, by means of an appropriate function I that generalizes the boolean conjunction, the function I may not to be associative, and, sometimes, it should satisfy some additional requirements like lipschitzianity and moderate growth [6,3,5,9,15]. To this end, having a great variety of copulas is of great importance in the choice of a suitable I.

Several methods for constructing copulas have been described in the literature (see, for instance, [16,22]); in particular, some of these make use of information of a geometric nature on the copula, such as a description of the graphs of horizontal, vertical and diagonal sections [10,17,18,22]. In the framework of *triangular norms* (t-norms, for short), constructions of this type have been considered several times: see [19,1]. In particular, the construction of a continuous Archimedean t-norm with given diagonal section is related to the Schröder functional equation. A construction closely connected with the idea of semilinear copulas was proposed by Mayor and Torrens [21], who characterized all continuous t-norms T that can be expressed in the form $T(x, y) = \max\{0, \delta_T(\max\{x, y\}) - |x - y|\}$. As a matter of fact, these t-norms are Bertino copulas (see [11] for more details).

The aim of this paper is to construct a class of copulas whose sections are linear on some specific segments of the unit square, and, therefore, they are called *semilinear*.

In Section 2, the class of semilinear copulas is introduced; this is, then, characterized in Section 3. Some remarks about the statistical interpretations of semilinear copulas are provided in Section 4. Some interesting examples, and, among them, a new construction of "ordinal sum" type, are given in Section 5. The study of the aggregation of several semilinear copulas is the object of Section 6. Finally, the related classes of semilinear semicopulas and quasi-copulas are considered (Section 7).

2. Definitions

First of all, recall that it follows from the definition that each copula C is a non-decreasing function of each of its arguments, and that it is 1-*Lipschitz*, i.e., for all x_1, x_2, y_1 and y_2 in $[0, 1]$,

$$|C(x_1, y_1) - C(x_2, y_2)| \leqslant |x_1 - x_2| + |y_1 - y_2|. \tag{2.1}$$

Moreover, for each copula C and for each (x, y) in $[0, 1]^2$

$$W(x, y) \leqslant C(x, y) \leqslant M(x, y),$$

where W and M are the lower and upper Fréchet–Hoeffding bounds, respectively, given by $W(x, y) = \max\{x+y-1, 0\}$, and $M(x, y) = \min\{x, y\}$ for all $(x, y) \in [0, 1]^2$. Another important copula is $\Pi(x, y) = xy$. Notice that W, Π, M are also *triangular norms* [19].

The diagonal section of a copula C is the function $\delta_C\colon [0, 1] \to [0, 1]$ given by $\delta_C(t) = C(t, t)$, which verifies the following properties:

(D1) $\delta_C(0) = 0$, $\delta_C(1) = 1$;
(D2) $\delta_C(u) \leqslant u$ for all $u \in [0, 1]$;

(D3) δ_C is non-decreasing and
(D4) δ_C is 2-*Lipschitz*, i.e., $|\delta_C(v) - \delta_C(u)| \leqslant 2|v - u|$ for all $u, v \in [0, 1]$.

The set of all functions $\delta : [0, 1] \to [0, 1]$ satisfying (D1)–(D4) will be denoted by $\mathcal{D}$. The elements of $\mathcal{D}$ will be called *diagonals*.

Definition 1. A copula C is called *lower semilinear* if, for all $x \in]0, 1]$, the mappings

$$h_x : [0, x] \to [0, 1], \quad h_x(t) := C(t, x),$$

$$v_x : [0, x] \to [0, 1], \quad v_x(t) := C(x, t),$$

are linear.

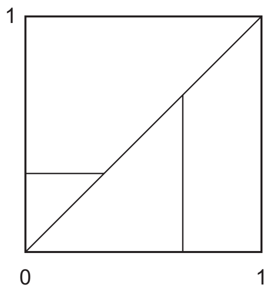
A copula C is called *upper semilinear* if, for all $x \in [0, 1[$, the mappings

$$h_x : [x, 1] \to [0, 1], \quad h_x(t) := C(t, x),$$

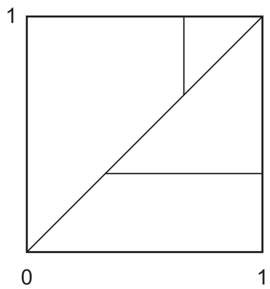
$$v_x : [x, 1] \to [0, 1], \quad v_x(t) := C(x, t),$$

are linear.

Roughly speaking, a copula is lower semilinear if its sections are linear on the segments joining any point of the diagonal of the unit square to the lower side and to the left side of the unit square, respectively. Analogously, a copula is upper semilinear if its sections are linear on the segments joining any point of the diagonal of the unit square to the upper side and to the right side of the unit square, respectively.



Lower semilinear copula



Upper semilinear copula

Thanks to the diagonal section, it is possible to give the explicit expressions for lower and upper semilinear copulas.

Lemma 2. *Let* $C : [0, 1]^2 \to [0, 1]$ *be a copula. Then the following statements are equivalent*:

(a) *C is lower semilinear and*
(b) *C is given by*

$$C(x, y) = \begin{cases} y\dfrac{\delta_C(x)}{x}, & y \leqslant x, \\ x\dfrac{\delta_C(y)}{y} & \textit{otherwise}, \end{cases} \tag{2.2}$$

where the convention $\frac{0}{0} := 0$ *is adopted.*

Proof. If C is lower semilinear, then Eq. (2.2) can be derived by the linear interpolation of the known values of the copula C. On the one hand, by interpolating between the values 0 at the point $(x, 0)$ and $\delta_C(x)$ at (x, x), and on the other hand, by interpolating between the values 0 at the point $(0, x)$ and $\delta_C(x)$ at (x, x). The rest of the proof is trivial. $\square$

By analogous considerations, one also has the following result.

Lemma 3. *Let* $C: [0,1]^2 \to [0,1]$ *be a copula. Then the following statements are equivalent*:

(a) *C is upper semilinear and*
(b) *C is given by*

$$C(x,y) = \begin{cases} (y-1)\dfrac{x-\delta_C(x)}{1-x} + x, & x \leqslant y, \\ (x-1)\dfrac{y-\delta_C(y)}{1-y} + y & \text{otherwise}, \end{cases} \tag{2.3}$$

where the convention $\frac{0}{0} := 1$ *is adopted.*

Notice that every upper semilinear copula C_U^δ with diagonal section δ is given by

$$C_U^\delta(x,y) = x + y - 1 + C_L^{\hat{\delta}}(1-x, 1-y),$$

where $C_L^{\hat{\delta}}$ is the lower semilinear copula determined by the diagonal section $\hat{\delta}(t) = 2t - 1 + \delta(1-t)$. Therefore, in the following, only "lower semilinear" copulas will be discussed, and they will be simply called "semilinear", if no confusion arises. Usually, the function C defined by (2.2) will be denoted by S_δ.

Observe that each semilinear copula S_δ is symmetric and its expression depends on the values of its diagonal section. It must be stressed that a diagonal $\delta \in \mathcal{D}$ need not be the diagonal section of a semilinear copula. For instance, let the function $\delta: [0,1] \to [0,1]$ be given by $\delta(x) := \max\{2x-1, 0\}$. Obviously, $\delta = \delta_W$ is in $\mathcal{D}$. Now, the function S_δ constructed from δ according to (2.2) is given by

$$S_\delta(x,y) = \begin{cases} 0 & \text{if } (x,y) \in \left[0, \dfrac{1}{2}\right]^2, \\ y\dfrac{2x-1}{x} & \text{if } y \leqslant x, x > \dfrac{1}{2}, \\ x\dfrac{2y-1}{y} & \text{otherwise}, \end{cases}$$

which is not a copula. It is easy to see that S_δ is a non-decreasing function from $[0,1]^2$ into $[0,1]$ with neutral element equal to 1. Since S_δ is not 1-Lipschitz, it is not a copula (in fact, S_δ is a 2-Lipschitz semicopula [13]).

3. Characterization

Here, some necessary and sufficient conditions on the diagonal δ are given in order to ensure that the function S_δ defined by (2.2) is a semilinear copula.

Theorem 4. *For a diagonal* $\delta \in \mathcal{D}$, *let the function* $S_\delta: [0,1]^2 \to [0,1]$ *be defined by*

$$S_\delta(x,y) := \begin{cases} y\dfrac{\delta(x)}{x} & \text{if } y \leqslant x, \\ x\dfrac{\delta(y)}{y} & \text{otherwise}, \end{cases} \tag{3.1}$$

where the convention $\frac{0}{0} := 0$ *is adopted. Then* S_δ *is a semilinear copula if*, *and only if*, *the functions* $\varphi_\delta, \eta_\delta:]0,1] \to [0,1]$ *defined by*

$$\varphi_\delta(x) := \frac{\delta(x)}{x}, \quad \eta_\delta(x) := \frac{\delta(x)}{x^2}$$

are non-decreasing and non-increasing, *respectively.*

Proof. Let S_δ be a semilinear copula. Since S_δ is non-decreasing in each place, φ_δ is non-decreasing. That η_δ is non-increasing is a consequence of the representation (3.1) and of the 2-increasing property of S_δ applied to the square $[u, v] \times [u, v]$, with $0 < u < v \leqslant 1$. In fact, $V_{S_\delta}([u, v] \times [u, v]) \geqslant 0$ is equivalent to

$$\delta(u) + \delta(v) \geqslant 2u \frac{\delta(v)}{v}.$$

This last inequality can be written in the form

$$\frac{\delta(u)}{u^2} \geqslant \frac{\delta(v)}{v^2} - \delta(v) \left(\frac{1}{u} - \frac{1}{v} \right)^2, \tag{3.2}$$

which, in its turn, is equivalent to

$$\eta_\delta(u) \geqslant \eta_\delta(v) - \delta(v) \frac{(v-u)^2}{u^2 v^2}. \tag{3.3}$$

First, notice that $\delta(u)$ is strictly positive for all $u > 0$. In the opposite case, there would exist $a > 0$ such that $\delta(a) = 0$; this would imply $S_\delta(x, y) = 0$ for all $(x, y) \in [0, a]^2$. Let $\overline{a}$ be the greatest such element, i.e.,

$$\overline{a} := \max\{x \in]0, 1[: \delta(x) = 0\}.$$

Choose $b \in]0, 1[$ such that $0.8b < \overline{a} < b$; then the S_δ-volume of the square $[0.8b, b]^2$ is

$$V_{S_\delta}([0.8b, b]^2) = \delta(b) - 1.6\delta(b) < 0,$$

which is a contradiction.

Suppose, if possible, that the function η_δ is not non-increasing, i.e., there exist u_0 and v_0 such that $0 < u_0 < v_0 \leqslant 1$ and $\eta_\delta(u_0) < \eta_\delta(v_0)$. Put

$$k := \frac{\eta_\delta(v_0) - \eta_\delta(u_0)}{v_0 - u_0}.$$

Then, for every $\varepsilon \in]0, v_0 - u_0]$ there are $u_\varepsilon, v_\varepsilon \in [u_0, v_0]$, $u_\varepsilon = v_\varepsilon - \varepsilon$, such that $\eta_\delta(v_\varepsilon) \geqslant \eta_\delta(u_\varepsilon) + k\varepsilon$. Take $\varepsilon \in]0, v_0 - u_0]$ such that

$$\varepsilon < k \frac{u_0^4}{\delta(v_0)}.$$

Then, applying (3.3) to u_ε and v_ε, yields

$$\eta_\delta(u_\varepsilon) \geqslant \eta_\delta(v_\varepsilon) - \frac{\delta(v_\varepsilon)}{u_\varepsilon^2 v_\varepsilon^2} \varepsilon^2 \geqslant \eta_\delta(u_\varepsilon) + k\varepsilon - \frac{\delta(v_\varepsilon)}{u_\varepsilon^2 v_\varepsilon^2} \varepsilon^2,$$

which implies

$$\varepsilon \geqslant k \frac{u_\varepsilon^2 v_\varepsilon^2}{\delta(v_\varepsilon)} \geqslant k \frac{u_0^4}{\delta(v_0)},$$

a contradiction. This proves that η_δ is non-increasing.

Conversely, let η_δ be non-increasing and φ_δ non-decreasing. Then for all u and v such that $0 < u < v \leqslant 1$ one has

$$\frac{\delta(u)}{u^2} \geqslant \frac{\delta(v)}{v^2},$$

which implies (3.2), i.e., that S_δ is 2-increasing on the square $[u, v]^2$. To conclude the proof, because of the symmetry of S_δ, it suffices to prove that S_δ is 2-increasing on rectangles $[x, x'] \times [y, y']$ with $x < x'$, $y < y'$ and $y' \leqslant x$. In such a case, since φ_δ is non-decreasing, one has

$$S_\delta(x', y') - S_\delta(x', y) - S_\delta(x, y') + S_\delta(x, y)$$
$$= \left(\frac{\delta(x')}{x'} - \frac{\delta(x)}{x}\right)(y' - y) = (\varphi_\delta(x') - \varphi_\delta(x))(y' - y) \geqslant 0.$$

Finally, the semilinearity of S_δ follows from Lemma 2. □

The set of all diagonals $\delta \in \mathcal{D}$ that satisfy the conditions of Theorem 4 will be denoted by $\mathcal{D}_S$:

$$\mathcal{D}_S = \{\delta \in \mathcal{D} : \varphi_\delta \text{ non-decreasing on}]0, 1], \eta_\delta \text{ non-increasing on}]0, 1]\}.$$

As each diagonal $\delta \in \mathcal{D}$ is 2-Lipschitz, and, therefore, absolutely continuous, the following claim can easily be proved.

Corollary 5. *Let $\delta \in \mathcal{D}$. Then the function S_δ defined by* (3.1) *is a semilinear copula if, and only if,*

$$\delta(x) \leqslant x\delta'(x) \leqslant 2\delta(x), \tag{3.4}$$

at all points $x \in]0, 1[$ where the derivative $\delta'(x)$ exists.

From (3.4) one has, for each $\delta \in \mathcal{D}_S$, $t \in]0, 1[$ and $x \in [t, 1]$,

$$\frac{\delta'(x)}{\delta(x)} \leqslant \frac{2}{x} \Longrightarrow \int_t^1 \frac{\delta'(x)}{\delta(x)}\, \mathrm{d}x \leqslant \int_t^1 \frac{2}{x} \Longrightarrow -\ln \delta(t) \leqslant -\ln t^2 \Longrightarrow \delta(t) \geqslant t^2.$$

The function $\delta_\Pi: [0, 1] \to [0, 1]$ given by $\delta_\Pi(t) = t^2$ is an element of $\mathcal{D}_S$ and thus it is the smallest element of $\mathcal{D}_S$. Consequently, $S_{\delta_\Pi} = \Pi$ is the smallest semilinear copula. On the other hand, the greatest element of $\mathcal{D}_S$ is the diagonal section of the upper Fréchet–Hoeffding bound M and, hence, the copula M is the greatest semilinear copula.

Example 6. For every $\theta \in [0, 1]$, let $\delta_\theta: [0, 1] \to [0, 1]$ be defined by

$$\delta_\theta(t) = t^{2-\theta}.$$

Then all the functions δ_θ, $\theta \in [0, 1]$, belong to $\mathcal{D}_S$. The family of semilinear copulas $(S_{\delta_\theta})_{\theta \in [0,1]}$ describes the Cuadras–Augé family of copulas [2].

Example 7. For every $\theta \in [0, 1]$, let $\delta_\theta: [0, 1] \to [0, 1]$ be given by

$$\delta_\theta(t) = \theta t^2 + (1 - \theta)t.$$

Then all the functions δ_θ, $\theta \in [0, 1]$, belong to $\mathcal{D}_S$. The family of semilinear copulas $(S_{\delta_\theta})_{\theta \in [0,1]}$ describes the Fréchet family of copulas, which are just convex linear combinations of the copulas Π and M [22].

4. Statistical comments about semilinear copulas

In [8], the first author studied the family of bivariate copulas C_f defined by

$$C_f(x, y) = \min\{x, y\} f(\max\{x, y\})$$

for a suitable function $f: [0, 1] \to [0, 1]$. It is easy to show that the class of semilinear copulas coincides with the above class by setting $f(t) = \delta(t)/t$. However, Theorem 4 is more general than the analogous characterization in [8,7], where the assumption of the differentiability for the diagonal is requested. Many statistical properties of semilinear copulas can be found in [7], and, hence, they will not be examined here. However, notice that the first idea of considering such copulas can be found in [2, Section 2.3].

It might be of interest to notice that semilinear copulas are, in fact, connected to another family of copulas introduced by Marshall [20] (see also [22]). Specifically, if $f, g: [0, 1] \to [0, 1]$ are continuous and increasing functions such that

$f(0) = g(0) = 0$, $f(1) = g(1) = 1$, and both $t \mapsto f(t)/t$ and $t \mapsto g(t)/t$ are decreasing, then Marshall showed that the mappings $C_{f,g}\colon [0,1]^2 \to [0,1]$ defined by

$$C_{f,g}(x, y) = xy \min\left\{\frac{f(x)}{x}, \frac{g(y)}{y}\right\} \tag{4.1}$$

are copulas (but the converse implication is not considered). By a simple change of notations, it is easily proved that the class of semilinear copulas coincides with the class of symmetric Marshall's copulas (which corresponds to the case $f = g$). But, more significantly, the family of Marshall's copulas is generated by semilinear copulas as shown in the following result.

Proposition 8. *The following statements are equivalent*:

(a) *C is a Marshall's copula*, *viz. it has the representation* (4.1) *and*
(b) *two semilinear copulas S_1 and S_2 exist such that*, *for all* $(x, y) \in]0, 1]^2$,

$$C(x, y) = \min\left\{\frac{S_1(xy, y)}{y}, \frac{S_2(x, xy)}{x}\right\}.$$

Proof. Marshall's copulas may be written in the form

$$C(x, y) = \min\{yf(x), xg(y)\}.$$

Now, it is enough to take

$$f(t) = \frac{\delta_{S_1}(t)}{t} \quad \text{and} \quad g(t) = \frac{\delta_{S_2}(t)}{t}$$

in order to conclude the proof. □

The preceding result together with the results of [20] allows to give the following statistical characterization of semilinear copulas.

Proposition 9. *If S_δ is a semilinear copula and*, *for a univariate d.f. F*, *$H(x, y) = S_\delta(F(x), F(y))$ is a bivariate d.f.*, *then there exist three independent random variables Z_1, Z_2 and Z_3 such that H is the joint distribution function of the random pair (X, Y)*, *where*

$$X = \max\{Z_1, Z_3\} \quad \textit{and} \quad Y = \max\{Z_2, Z_3\}.$$

In particular, it is not difficult to show that, given the functions φ_δ and η_δ defined in Theorem 4, $\varphi_\delta(F)$ is the distribution function of Z_1 and Z_2, and $1/\eta_\delta(F)$ is the distribution function of Z_3. Analogous statistical characterization of semilinear copulas can be given in terms of survival d.f.'s, also following [20].

Thus every semilinear copula S_δ describes the dependence structure of continuous r.v.'s X and Y that derive from a latent triple (Z_1, Z_2, Z_3), where Z_1 and Z_2 have a common distribution function.

For example, if one considers Z_1 and Z_2 uniformly distributed on $(0, 1)$ and Z_3 such that $F_{Z_3}(x) = x^{(1-\alpha)/\alpha}$ for all $x \in [0, 1]$, then the joint d.f. of the random pair $(\max\{Z_1, Z_3\}, \max\{Z_2, Z_3\})$ is a member of the Cuadras–Augé family.

On the other hand, if Z_1 and Z_2 have uniform distribution on $[0, 1]$ and Z_3 on $[0, 2]$, then the corresponding semilinear copula S is the ordinal sum of the copulas (A, M), where $A(x, y) = \sqrt{xy \min\{x, y\}}$, with respect to the partition $([0, \frac{1}{2}], [\frac{1}{2}, 1])$ (see [22] for more details on ordinal sums).

5. Examples

In this section, some examples of semilinear copulas are provided by imposing simple conditions on their diagonal sections.

Proposition 10. *Let S_δ be a semilinear copula.*

(a) *If there is an element $x_0 \in]0,1[$ such that $\delta(x_0) = x_0$, then $\delta(x) = x$ for all $x \in [x_0, 1]$, and $S_\delta(x, y) = \min\{x, y\}$ whenever $\max\{x, y\} \geqslant x_0$.*
(b) *If there is an element $x_0 \in]0,1[$ such that $\delta(x_0) = x_0^2$, then $\delta(x) = x^2$ for all $x \in [x_0, 1]$, and $S_\delta(x, y) = xy$ whenever $\max\{x, y\} \geqslant x_0$.*

Proof. (a) Let $x_0 \in]0,1[$ be such that $\delta(x_0) = x_0$. Since the function φ_δ is non-decreasing, one can conclude that, on $[x_0, 1]$,

$$1 = \frac{\delta(x_0)}{x_0} \leqslant \frac{\delta(x)}{x} \leqslant \frac{\delta(1)}{1} = 1,$$

namely, $\delta(x) = x$ for each $x \in [x_0, 1]$. From the representation (3.1) of semilinear copulas it follows that $S_\delta(x, y) = \min\{x, y\}$ if either x or y are in $[x_0, 1]$.

(b) Assume now that there exists $x_0 \in]0,1[$ such that $\delta(x_0) = x_0^2$; since the function η_δ is non-increasing, one has, for every $x \in [x_0, 1]$,

$$1 = \frac{\delta(x_0)}{x_0^2} \geqslant \frac{\delta(x)}{x^2} \geqslant \frac{\delta(1)}{1^2} = 1,$$

namely $\delta(x) = x^2$ for every $x \in [x_0, 1]$. Therefore $S_\delta(x, y) = xy$ whenever either x or y exceeds x_0. □

Example 11. For $\alpha \geqslant 1$, define a function $\delta_\alpha \colon [0, 1] \to [0, 1]$ via

$$\delta_\alpha(t) := t \min\{\alpha t, 1\}.$$

Then all the functions δ_α belong to $\mathcal{D}_S$. Notice that S_{δ_α} is the ordinal sum $(\langle 0, 1/\alpha, \Pi \rangle)$.

Now, some distinguished non-trivial examples of diagonal sections of semilinear copulas are introduced. All necessary calculations may be carried out by applying Corollary 5.

Proposition 12. (a) *A piecewise linear diagonal $\delta \in \mathcal{D}$ is an element of $\mathcal{D}_S$ if, and only if, one has, for every $i = 1, \ldots, n-1$,*

$$\frac{b_i}{a_i} \leqslant \frac{b_{i+1} - b_i}{a_{i+1} - a_i} \leqslant 2\frac{b_i}{a_i},$$

where $(a_0, b_0), \ldots, (a_n, b_n)$ are the end-points of the n segments forming the graph of δ such that

$$0 = a_0 < a_1 < \cdots < a_n = 1 \quad \textit{and} \quad 0 = b_0 < b_1 < \cdots < b_n = 1.$$

(b) *An absolutely monotone diagonal $\delta \in \mathcal{D}$, i.e., a function given by*

$$\delta(x) = \sum_{i=1}^{\infty} a_i x^i \ \ \textit{with } a_i \geqslant 0 \quad \textit{and} \quad \sum_{i=1}^{\infty} a_i = 1$$

is the diagonal section of a semilinear copula if, and only if,

$$\sum_{i=3}^{\infty} (i-1) a_i \leqslant a_1.$$

For example, for $n = 2$, part (a) of the previous proposition means that, when the graph of δ consists of two segments, the only “free” point (a_1, b_1) fulfils the condition

$$\frac{a_1}{2 - a_1} \leqslant b_1 \leqslant a_1.$$

The linearity of the semilinear copulas S_1 and S_2 on relevant segments leads to the equality

$$\delta_{S_2}(a)S_1(x,y) = \delta_{S_1}(a)S_2(x,y),$$

for all x, y, and a in $[0,1]$ such that $\max\{x,y\} = a$. This observation allows to introduce a notion of linear ordinal sum of semilinear copulas.

Proposition 13. *Let $\{a_i\}_{i=1}^n$ be a strictly decreasing finite sequence in $]0,1]$ with $a_1 = 1$. Let $\{S_i\}_{i=1}^n$ be a system of semilinear copulas and $\{\delta_i\}_{i=1}^n$ the corresponding system of diagonal sections. Then the function $S: [0,1]^2 \to [0,1]$ defined by*

$$S(x,y) := \frac{\delta_{i-1}(a_i)}{\delta_i(a_i)} S_i(x,y) \quad (x,y) \in [0,a_i]^2 \setminus]0,a_{i+1}[^2,$$

where, by convention, $\delta_0(1) := 1$ and $a_{n+1} := 0$, is a semilinear copula. This copula will be called the linear ordinal sum of the copulas *$\{S_i\}_{i=1}^n$ and will be denoted by*

$$S = (\langle a_i, S_i \rangle \mid i = 1, \ldots, n)_{\mathrm{L}}.$$

Observe that, for every semilinear copula S and for every strictly decreasing finite sequence $\{a_i\}_{i=1}^n$ in $]0,1]$ with $a_1 = 1$, one has

$$S = (\langle a_i, S \rangle \mid i = 1, \ldots, n)_{\mathrm{L}}.$$

Formally, one can define a linear ordinal sum only starting from a non-increasing sequence $\{a_i\}_{i=1}^n$. However, if $a_{i+1} = a_i$ for some i, then the $(i+1)$th summand may be omitted. Moreover, an infinite linear ordinal sum of semilinear copulas can also be introduced. Linear ordinal sums of semilinear copulas may be used in finding best possible bounds in the class of semilinear copulas with a fixed value at a point of the diagonal.

Proposition 14. *Let $x_0 \in]0,1[$ and $a \in [x_0^2, x_0]$. Then, for every semilinear copula S with $\delta_S(x_0) = a$, one has*

$$(\langle 1, \Pi \rangle, \langle a/x_0, M \rangle, \langle x_0, \Pi \rangle)_{\mathrm{L}} \leqslant S \leqslant (\langle 1, M \rangle, \langle x_0^2/a, \Pi \rangle, \langle x_0, M \rangle)_{\mathrm{L}}.$$

Proof. For a diagonal $\delta \in \mathcal{D}_S$ with $\delta(x_0) = a$, one has $\varphi_\delta(x_0) = a/x_0$, $\eta_\delta(x_0) = a/x_0^2$, and necessarily, $\varphi_\delta(x) \leqslant a/x_0$ on $]0, x_0[$, namely, for all $x \in [0, x_0]$,

$$\delta(x) \leqslant x\frac{a}{x_0}.$$

Similarly, $\eta_\delta(x) \geqslant a/x_0^2$ on $]0, x_0[$, i.e., for all $x \in [0, x_0]$,

$$\delta(x) \geqslant x^2 \frac{a}{x_0^2}.$$

Next, for all $x \in [x_0, 1]$ one has

$$\frac{a}{x_0} \leqslant \varphi_\delta(x) \leqslant 1 \quad \text{and} \quad 1 \leqslant \eta_\delta(x) \leqslant \frac{a}{x_0^2},$$

namely

$$\frac{a}{x_0}x \leqslant \delta(x) \leqslant x \quad \text{and} \quad x^2 \leqslant \delta(x) \leqslant \frac{a}{x_0^2}x^2.$$

Therefore, one has, for every $x \in [x_0, 1]$,

$$\max\left\{\frac{a}{x_0}x, x^2\right\} \leqslant \delta(x) \leqslant \min\left\{x, \frac{a}{x_0^2}x^2\right\},$$

and, summarizing, $\underline{\delta}_{a,x_0} \leqslant \delta \leqslant \overline{\delta}_{a,x_0}$, where

$$\underline{\delta}_{a,x_0}(x) := \begin{cases} \dfrac{a}{x_0^2}x^2 & \text{if } x \in [0, x_0], \\ \dfrac{a}{x_0}x & \text{if } x \in \left]x_0, \dfrac{a}{x_0}\right], \\ x^2 & \text{otherwise,} \end{cases}$$

and

$$\overline{\delta}_{a,x_0}(x) := \begin{cases} \dfrac{a}{x_0}x & \text{if } x \in [0, x_0], \\ \dfrac{a}{x_0^2}x^2 & \text{if } x \in \left]x_0, \dfrac{x_0^2}{a}\right], \\ x & \text{otherwise.} \end{cases}$$

As both functions $\underline{\delta}_{a,x_0}$ and $\overline{\delta}_{a,x_0}$ are in $\mathcal{D}_S$, and since the semilinear copulas corresponding to them are linear ordinal sums of the form

$$(\langle 1, \Pi\rangle, \langle a/x_0, M\rangle, \langle x_0, \Pi\rangle)_{\mathrm{L}} \quad \text{and} \quad (\langle 1, M\rangle, \langle x_0^2/a, \Pi\rangle, \langle x_0, M\rangle)_{\mathrm{L}},$$

the result follows. □

Notice that the above result improves the bounds given in [22, Theorem 3.2.3] for copulas with a given value at a point of the unit square (see also [10]).

6. Aggregation of semilinear copulas

If n diagonals $\delta_1, \delta_2, \ldots, \delta_n$ in $\mathcal{D}_S$ are given, then it may be asked what is, if any, the relationship between, on the one hand, the semilinear copula S_δ constructed from a function $\delta = A(\delta_1, \delta_2, \ldots, \delta_n)$, obtained by a pointwise aggregation of diagonals δ_i through an appropriate aggregation function A that ensures that δ belongs to $\mathcal{D}_S$ and, on the other one, the function obtained by the A-aggregation of the semilinear copulas $S_{\delta_1}, \ldots, S_{\delta_n}$.

Recall that, for $n \geqslant 2$, an *n-ary aggregation function* A on the interval $[0, \infty[$ is a function $A\colon [0, \infty[^n \to [0, \infty[$ that is non-decreasing in each variable, and such that $A(0, \ldots, 0) = 0$ and

$$\sup\{A(x_1, \ldots, x_n) \mid (x_1, \ldots, x_n) \in [0, \infty[^n\} = \infty.$$

An aggregation function A is said to be *homogeneous*, if one has, for all $x_1, \ldots, x_n$ and for every c in $[0, \infty[$,

$$A(cx_1, \ldots, cx_n) = cA(x_1, \ldots, x_n),$$

and A is said to be *idempotent*, if for all $x \in [0, \infty[$, $A(x, \ldots, x) = x$.

Proposition 15. *Let $A\colon [0, \infty[^n \to [0, \infty[$ be an idempotent homogeneous aggregation function. If $\delta_1, \delta_2, \ldots, \delta_n$ belong to $\mathcal{D}_S$, then the function $\delta = A(\delta_1, \ldots, \delta_n)$ defined, for all $x \in [0, 1]$, by*

$$\delta(x) := A(\delta_1(x), \delta_2(x), \ldots, \delta_n(x)),$$

is also in $\mathcal{D}_S$. Furthermore, the semilinear copula S_δ defined by (3.1) *is represented, for all $(x, y) \in [0, 1]^2$, by*

$$S_\delta(x, y) = A(S_{\delta_1}(x, y), \ldots, S_{\delta_n}(x, y)).$$

Proof. Let $\delta_1, \delta_2, \ldots, \delta_n$ belong to $\mathcal{D}_S$. It is easy to see that the mapping $\delta = A(\delta_1, \ldots, \delta_n)$ satisfies properties (D1)–(D3). Now, we prove that the functions φ_δ and η_δ are non-decreasing and non-increasing on $]0, 1]$, respectively, and then that δ is 2-Lipschitz, i.e., that condition (D4) is fulfilled. For every $x \in]0, 1]$,

$$\varphi_\delta(x) = \frac{\delta(x)}{x} = \frac{A(\delta_1(x), \ldots, \delta_n(x))}{x} = A\left(\frac{\delta_1(x)}{x}, \ldots, \frac{\delta_n(x)}{x}\right)$$

$$= A(\varphi_{\delta_1}(x), \ldots, \varphi_{\delta_n}(x)),$$

and similarly,

$$\eta_\delta(x) = A(\eta_{\delta_1}(x), \ldots, \eta_{\delta_n}(x)).$$

That φ_δ is non-decreasing and that η_δ is non-increasing on $]0, 1]$ follow from the analogous properties of the φ_{δ_i}'s and of the η_{δ_i}'s, and from the fact that A is non-decreasing in each variable.

Next, because of the monotonicity of η_δ, for every $u \in]0, 1]$ and for every $\varepsilon \in]0, u[$, one has

$$\frac{\delta(u)}{u^2} \leqslant \frac{\delta(u-\varepsilon)}{(u-\varepsilon)^2}.$$

Put $\delta(u) - \delta(u-\varepsilon) = \alpha$. Then

$$\delta(u)(u-\varepsilon)^2 \leqslant u^2(\delta(u) - \alpha),$$

which is equivalent to

$$\alpha \leqslant 2\varepsilon\frac{\delta(u)}{u} - \frac{\varepsilon^2\delta(u)}{u^2}.$$

Finally, on account of the monotonicity of φ_δ, one obtains

$$0 \leqslant \alpha \leqslant 2\varepsilon\frac{\delta(u)}{u} \leqslant 2\varepsilon\frac{\delta(1)}{1} = 2\varepsilon,$$

namely,

$$0 \leqslant \delta(u) - \delta(u-\varepsilon) \leqslant 2\varepsilon,$$

which proves the 2-Lipschitz property of δ on $]0, 1]$. Since $0 \leqslant \delta \leqslant id_{[0,1]}$, the 2-Lipschitz property of δ on $[0, 1]$ follows.

Further, for each $y \leqslant x$ and $x \neq 0$, one has

$$S_\delta(x, y) = y\frac{\delta(x)}{x} = \frac{y}{x}A(\delta_1(x), \ldots, \delta_n(x))$$

$$= A(\frac{y}{x}\delta_1(x), \ldots, \frac{y}{x}\delta_n(x)) = A(S_{\delta_1}(x, y), \ldots, S_{\delta_n}(x, y)).$$

The rest of the proof follows from the symmetry of semilinear copulas. □

Notice that the convex sums (weighted arithmetic means), the log-convex sums (weighted geometric means), the maximum and the minimum are aggregation functions on $[0, \infty[$ satisfying the assumptions of Proposition 15, and therefore the following result can be formulated.

Corollary 16. *The class $\mathcal{S}$ of all semilinear copulas is a convex and log-convex subclass of the class of all copulas $\mathcal{C}$. Moreover, $\mathcal{S}$ is also closed under suprema and infima and compact with respect to the topology of uniform convergence, i.e., it is a complete lattice.*

The result stated in Corollary 16 ought to be compared with those valid in general: while the set of all copulas is not closed under suprema and infima (see [23]), the class $\mathcal{S}$ of semilinear copulas is.

7. Semilinear quasi-copulas and semicopulas

The construction of semilinear copulas with prescribed diagonal section can also be applied to semicopulas and quasi-copulas, which are two generalizations of the notion of copula. Recall that a *semicopula* is a function $S: [0,1]^2 \to [0,1]$ that is non-decreasing in each variable and satisfies the boundary conditions (C1) and (C2) (see [13]). A *quasi-copula* $S: [0,1]^2 \to [0,1]$ is a semicopula satisfying the 1-Lipschitz condition (2.1) (see [14]).

The following characterization can be derived from [7].

Theorem 17. *For a diagonal $\delta \in \mathcal{D}$, let the function $S_\delta: [0,1]^2 \to [0,1]$ be defined by*

$$S_\delta(x,y) := \begin{cases} y\dfrac{\delta(x)}{x} & \text{if } y \leqslant x, \\ x\dfrac{\delta(y)}{y} & \text{otherwise}, \end{cases} \tag{7.1}$$

where the convention $\frac{0}{0} := 0$ is adopted. Then S_δ is a semilinear quasi-copula if, and only if, the function $\varphi_\delta:]0,1] \to [0,1]$ defined by

$$\varphi_\delta(x) := \frac{\delta(x)}{x}$$

is non-decreasing and satisfies

$$x_1 \cdot \frac{\varphi_\delta(x_2) - \varphi_\delta(x_1)}{x_2 - x_1} \leqslant 1$$

for every x_1, x_2 in $[0,1]$ with $x_1 < x_2$.

Since each diagonal $\delta \in \mathcal{D}$ is 2-Lipschitz, and, therefore, absolutely continuous, the following claim can easily be proved.

Corollary 18. *Let $\delta \in \mathcal{D}$. Then the function S_δ defined by* (7.1) *is a semilinear quasi-copula if, and only if,*

$$\delta(x) \leqslant x\delta'(x) \leqslant x + \delta(x),$$

at all points $x \in]0,1]$ where the derivative $\delta'(x)$ exists.

There are several semilinear quasi-copulas that are not copulas. For example, for every $x_0 \in]0,1[$, define the function $\delta_{x_0}: [0,1] \to [0,1]$ by

$$\delta_{x_0}(x) = \begin{cases} x_0 x & \text{if } x \leqslant x_0, \\ x - x_0 + x x_0 & \text{otherwise.} \end{cases}$$

Then $S_{\delta_{x_0}}$ is a proper semilinear quasi-copula.

Analogously, if one wishes to construct semilinear semi-copulas from diagonal sections, then one may start from the set $\mathcal{D}'$ containing all functions $\delta: [0,1] \to [0,1]$ satisfying (D1)–(D3), but not necessarily (D4). Evidently, $\mathcal{D}$ is a proper subset of $\mathcal{D}'$. The following result can be easily showed.

Theorem 19. *For a diagonal $\delta \in \mathcal{D}'$, let the function $S_\delta: [0,1]^2 \to [0,1]$ be defined by*

$$S_\delta(x,y) := \begin{cases} y\dfrac{\delta(x)}{x} & \text{if } y \leqslant x, \\ x\dfrac{\delta(y)}{y} & \text{otherwise}, \end{cases} \tag{7.2}$$

where the convention $\frac{0}{0} := 0$ *is adopted. Then* S_δ *is a semilinear semicopula if, and only if, the function* $\varphi_\delta\colon]0,1] \to [0,1]$ *defined by*

$$\varphi_\delta(x) := \frac{\delta(x)}{x}$$

is non-decreasing.

For example, for each $r \in [1, \infty[$, the function $\delta_r\colon [0,1] \to [0,1]$, defined by $\delta_r(x) = x^r$, is an element of $\mathcal{D}'$ and the corresponding semilinear semi-copula is given by

$$S_{\delta_r}(x, y) = \min\{x, y\}(\max\{x, y\})^{r-1}.$$

In [1, pp. 126–127], it was stressed that semicopulas of this type could be used in the study of generalizations of the classical symmetric difference of sets.

8. Conclusions

A family of copulas, called semilinear, has been constructed starting with some assumptions about the linearity of the copulas along some segments of the unit square. This family contains some other known families of copulas (e.g., Cuadras–Augé, Fréchet) and has a nice statistical interpretation. Several construction methods have been provided, especially concerning aggregation of semilinear copulas, and a special form of ordinal sum construction is introduced. Due to its great flexibility, this family may be used in many recent applications on fuzzy preference modelling and similarities. To this end, related families of quasi-copulas and semicopulas have been considered.

Finally, remark that a previous version of this manuscript has already originated some other investigations. Specifically, in [4], the idea of semilinearity is used in order to introduce a class of non-symmetric semilinear copulas. In [12], instead, the authors considered an extension of this family to the n-dimensional case ($n \geqslant 3$).

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References

[1] C. Alsina, M.J. Frank, B. Schweizer, Associative Functions. Triangular Norms and Copulas, World Scientific, Hackensack, 2006.
[2] C.M. Cuadras, J. Augé, A continuous general multivariate distribution and its properties, Comm. Statist. Theory Methods 10 (1981) 339–353.
[3] B. De Baets, H. De Meyer, Transitivity frameworks for reciprocal relations: cycle-transitivity versus FG-transitivity, Fuzzy Sets and Systems 152 (2005) 249–270.
[4] B. De Baets, H. De Meyer, R. Mesiar, Asymmetric semilinear copulas, Kybernetika (Prague) 43 (2007) 221–233.
[5] B. De Baets, S. Janssens, H. De Meyer, Meta-theorems on inequalities for scalar fuzzy set cardinalities, Fuzzy Sets and Systems 157 (2006) 1463–1476.
[6] S. Díaz, B. De Baets, S. Montes, Additive decomposition of fuzzy pre-orders, Fuzzy Sets and Systems 158 (2007) 830–842.
[7] F. Durante, A new class of symmetric bivariate copulas, J. Nonparam. Statist. 18 (2006) 499–510.
[8] F. Durante, A new family of symmetric bivariate copulas, C.R. Math. Acad. Sci. Paris 344 (2007) 195–198.
[9] F. Durante, E.P. Klement, R. Mesiar, C. Sempi, Conjunctors and their residual implicators: characterizations and construction methods, Mediterr. J. Math., in press.
[10] F. Durante, A. Kolesárová, R. Mesiar, C. Sempi, Copulas with given values on a horizontal and a vertical section, Kybernetika (Prague) 43 (2007) 209–220.
[11] F. Durante, R. Mesiar, C. Sempi, On a family of copulas constructed from the diagonal section, Soft Comput. 10 (2006) 490–494.
[12] F. Durante, J.J. Quesada-Molina, M. Úbeda-Flores, On a family of multivariate copulas for aggregation processes, Inform. Sci., accepted for publication.
[13] F. Durante, C. Sempi, Semicopulæ, Kybernetika (Prague) 41 (2005) 315–328.
[14] C. Genest, J.J. Quesada-Molina, J.A. Rodríguez-Lallena, C. Sempi, A characterization of quasi-copulas, J. Multivariate Anal. 69 (1999) 193–205.

[15] S. Janssens, B. De Baets, H. De Meyer, Bell-type inequalities for quasi-copulas, Fuzzy Sets and Systems 148 (2004) 263–278.
[16] H. Joe, Multivariate Models and Dependence Concepts, Chapman & Hall, London, 1997.
[17] E.P. Klement, A. Kolesárová, Extension to copulas and quasi-copulas as special 1-Lipschitz aggregation operators, Kybernetika (Prague) 41 (2005) 329–348.
[18] E.P. Klement, A. Kolesárová, R. Mesiar, C. Sempi, Copulas constructed from the horizontal section, Comm. Statist. Theory Methods, in press.
[19] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Academic Publishers, Dordrecht, 2000.
[20] A.W. Marshall, Copulas, marginals and joint distributions, In: L. Rüschendorf, B. Schweizer, M.D. Taylor (Eds.), Distribution Functions with Fixed Marginals and Related Topics, Institute of Mathematical Statistics, Lecture Notes-Monograph Series, Vol. 28, Hayward CA, 1996, pp. 213–222.
[21] G. Mayor, J. Torrens, On a family of t-norms, Fuzzy Sets and Systems 41 (1991) 161–166.
[22] R.B. Nelsen, An Introduction to Copulas, Springer, New York, 2006.
[23] R.B. Nelsen, M. Úbeda-Flores, The lattice-theoretic structure of sets of bivariate copulas and quasi-copulas, C.R. Math. Acad. Sci. Paris 341 (2005) 583–586.
[24] A. Sklar, Fonctions de répartition à n dimensions et leurs marges, Publ. Inst. Statist. Univ. Paris 8 (1959) 229–231.