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Can a stochastic cusp catastrophe model explain stock market crashes?

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ABSTRACT

This paper is the first attempt to fit a stochastic cusp catastrophe model to stock market data. We show that the cusp catastrophe model explains the crash of stock exchanges much better than other models. Using the data of U.S. stock markets we demonstrate that the crash of October 19, 1987, may be better explained by cusp catastrophe theory, which is not true for the crash of September 11, 2001. With the help of sentiment measures, such as the index put/call options ratio and trading volume (the former models the chartists, the latter the fundamentalists), we have found that the 1987 returns are bimodal, and the cusp catastrophe model fits these data better than alternative models. Therefore we may say that the crash has been led by internal forces. However, the causes for the crash of 2001 are external, which is also evident in much weaker presence of bifurcations in the data. In this case, alternative models explain the crash of stock exchanges better than the cusp catastrophe model.

1. Introduction

Unpredictable behavior of stock markets, especially unexpected crashes, has been a nightmare for the financial world ever since capital markets came into existence. Catastrophe theory attempts to unfold part of the information we might need to understand crash phenomena. It describes how small, continuous changes in control parameters, or independent variables influencing the state of the system, can have sudden, discontinuous effects on dependent variables. The theory is widely applicable as it can be used to describe a sudden collapse of a bridge under slowly mounting pressure, freezing of water when the temperature is gradually decreased, or the stability of black holes. In this paper, we apply the theory to sudden stock market changes that are known as crashes. Zeeman (1974) was the first to qualitatively describe the "unstable behavior of stock exchanges" by Thom (1975) catastrophe theory. We extend his ideas by incorporating a quantitative analysis in a stochastic setup.

This article provides an extension of contemporary knowledge as it puts the theory to test on financial data. As only a few papers deal with empirical applications of catastrophe theory – for instance in the field of physics, Aerts et al. (2003), Tamaki et al. (2003); chemistry, Wales (2001), biology, Torres (2001), and van Harten (2000); in the social sciences, Holyst et al. (2000); economics, Balasko (1978), Ho and Saunders (1980) or Jammemegg and Fischer (1986) – this paper contributes to that research. We build on Zeeman's qualitative description, but instead of using his model we use a

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randomly perturbed version, and the primary aim of this research is to answer the question of whether stochastic catastrophe models are capable of indicating stock market crashes.

The structure of this paper will be as follows. In the first part the basic principles of catastrophe theory will be introduced. Stochastic catastrophe theory and a review of statistical testing methods will be discussed. These methods have been crucial in the history of catastrophe theory, mainly in the hands of critics like Zahler and Sussman (1977) who widely criticized catastrophe theory for non-existence of methods enabling its statistical testing. In the second part, we argue that there exists a consistent theory for statistical testing and we discuss in detail Zeeman's main hypotheses about the instability of stock markets. The role of fundamentalists and chartists and their influence on stock market changes will also be discussed.

What we regard as the most significant aspect of this paper is estimating a cusp catastrophe on real-world financial data. Our key hypothesis is that the cusp catastrophe model is able to fit the data better than an alternative linear regression model, and/or nonlinear(logistic) model. We fit the catastrophe model to the data of the October 19, 1987 crash, known as Black Monday which was the greatest single-day loss (20.5%) that Wall Street has ever suffered in continuous trading. For comparison, we use another large crash, that of September 11, 2001. The final part is devoted to the hypothesis that while in 1987 the crash was caused by internal forces, in 2001 there were external forces, namely the 9/11 terrorist attack, that caused the crash. Thus the catastrophe model should fit the data of 1987 well, as bifurcations leading to instability are present. However, it should not perform better than linear regression does on the 2001 data. As Zeeman's original model considers returns of the stock market rather than prices, we follow his analysis, and use Standard and Poor's 500 index returns as the behavioral variable. As the control variables we use the measures of sentiment, precisely the OEX[1] put/call ratio which appears to be a very good measure of speculative money (i.e. in Bates, 1991; Finucane, 1991; or Wang et al., 2006) in the capital market, against the daily change of total trading volume, the ratio of advancing stocks volume and declining stocks volume, the Dow Jones Composite Bond Index, and a one-day lag of S&P 500 returns as good proxy for large, fundamental investors.

2. Catastrophe models

Catastrophe theory has been developed by the mathematician René Thom (1975) to help explain biological morphogenesis as one of the great mysteries confronting mathematical biology. The range of potential applications is, however, extremely broad as catastrophe theory is closely related to the theory of Taylor series approximations (Cobb and Zacks, 1985). Zeeman (1974) was the first to propose its application to stock market behavior. Although his work focused on qualitative descriptions rather than quantitative applications, his hypotheses were very interesting at that time. Unfortunately, catastrophe theory had to wait until its time came, mainly due to the spreading criticism led by Zahler and Sussman (1977) and Sussman and Zahler (1978a, b). Their arguments against catastrophe theories are based on excessive reliance on qualitative methods, inappropriate quantization in some applications, use of excessively restrictive or narrow mathematical assumptions, and nonexistence of statistical theory which would enable quantitative research to be performed on real-world data. Perhaps, the discussion is also ignited by the very name of the theory, which seems rather provocative; however, it has been chosen to emphasize one of the nontrivial aspects of the behavior of nonlinear dynamic models.[2]

Although Sussman and Zahler made some good points, their criticism has initiated debates that have persisted through several decades until now. The most recent contribution has been made by Rosser (2007), who in fact ridicules the previous criticisms. He summarizes the discussion and shows that the arguments which have caused the main incomprehension are at least petty. On the other hand, nonexistence of a statistical theory was clearly a problem in that time, which has led to reliance on qualitative methods. Statistical methods have thus quickly started to be the focus of the research. Cobb (1981) and Cobb and Watson (1980) provided a reliable method for estimation of the cusp catastrophe models based on maximum likelihood estimation (MLE). Two other methods have been developed: one by Guastello (1984) who used a simple regression technique, and the least-squares estimation method of Oliva et al. (1987) GEMCAT.[3] Finally, Hartelman (1997) proposed a consistent invariant stochastic catastrophe theory for empirical verification and testing. Poston and Stewart (1978), Guastello (1987) and Rosser (2007) provided a fairly comprehensive review of the related literature, while Rosser (2007) provided a good review of those few papers applying the model to business, finance and economics.

2.1. Basic framework

A key idea in catastrophe theory is that the system under study is driven toward an equilibrium state. Wagenmakers et al. (2005a) illustrated this by imagining the movement of a ball on a curved one-dimensional surface, as in Fig. 1. The ball represents the state of the system, whereas gravity represents the driving force.

[1] OEX are options with the Standard & Poor's 100 index underlying.
[2] Cobb and Zacks (1985).
[3] A general multivariate methodology for estimating catastrophe models.

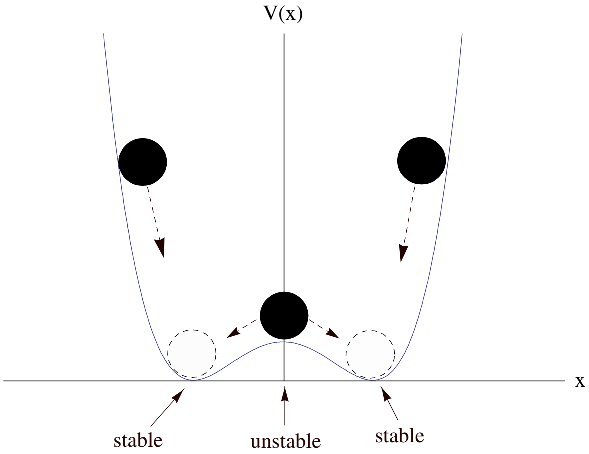

Fig. 1. Potential function $V(x)$ with two stable equilibrium points and one unstable equilibrium point.

Fig. 1 displays three possible equilibria. Two of the states are stable, meaning that the behavior of the system will remain relatively unaffected when the system is perturbed. One state is unstable, which means that only a small perturbation will drive the system toward a different state. Systems that are driven toward equilibrium values, such as the example using a ball, may be classified according to their configuration of critical points, more precisely points at which the first or second derivative equals zero. Qualitative behavior of the system is then driven by changes in critical points. Small changes in independent or control variables may lead to abrupt, discontinuous changes in state variable.

The behavior of the dynamical systems we want to study is completely determined by a so-called potential function. The potential function depends on behavioral and control variables. The behavioral variables describe the state of the system, while control variables determine the behavior of the system. The behavior of catastrophe models can become extremely complex, and according to the classification theory, we know seven different families of catastrophe models, based on the number of control and dependent variables, as in Thom (1975). We will focus on the so-called cusp catastrophe model as it is the simplest model that gives rise to sudden discontinuities. We use the phenomenological (P-bifurcation) approach.[4]

Let us assume one dependent variable Y. Then y_t represents the realization of a random variable Y_t, which evolves in time t for $t \in \langle 0, T \rangle$. The example of a ball from Fig. 1 may be quantified by postulating that the state of the system will change over time t according to

$$\frac{dy_t}{dt} = -\frac{dV(y_t;\alpha,\beta)}{dy_t}, \tag{1}$$

where $V(y_t;\alpha,\beta)$ is the potential function. When the right-hand side of (1) equals zero, the system is in equilibrium. The concept of a potential function is very general. For instance a potential function that is quadratic, i.e. $V(y_t;\alpha,\beta) = -\frac{1}{2}y_t^2 + \alpha y_t$, will yield the equilibria $dy_t/dt = y_t - \alpha = 0$ that describes a very simple response surface which is flat in every direction.

In contrast, the cusp model is based on the nonlinear deterministic dynamical system described by (1), where the behavior of y_t (rate of change of the stock market index in our case) will change over time t according to the derivative of the cusp potential function $V(y_t;\alpha,\beta)$ defined as

$$V(y_t;\alpha,\beta) = -\tfrac{1}{4}y_t^4 + \tfrac{1}{2}\beta y_t^2 + \alpha y_t, \tag{2}$$

which has equilibria at

$$\frac{dV(y_t;\alpha,\beta)}{dy_t} = -y_t^3 + \beta y_t + \alpha = 0, \tag{3}$$

where α and β are the control variables which determine the behavior of the system.

2.2. The cusp catastrophe response surface

The two dimensions of the control space, α and β, are factors which will depend upon the actual measured independent variables. Thus we need to introduce the independent variables into the analysis. Let us assume a set of n independent variables $\{X_1, X_2, \ldots, X_n\}$. Then x_i represents realizations of X_i, and control variables α_x and β_x, also called normal and

[4] Arnold (1998).

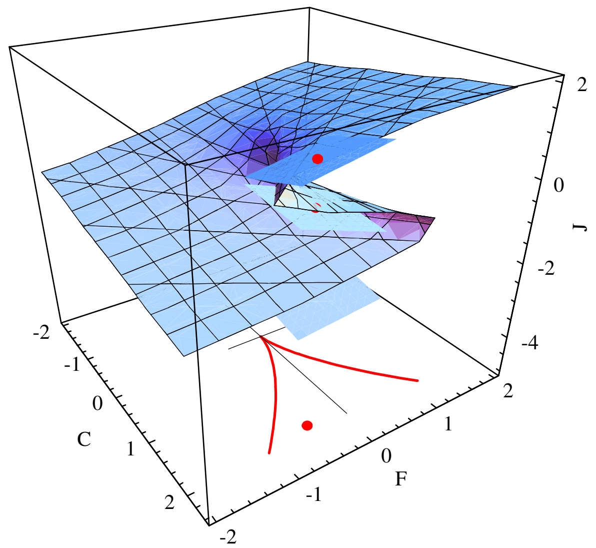

Fig. 2. The manifold of the cusp catastrophe model. Control parameters are chartists (C) and fundamentalists (F), state variable is the rate of change of the exchange market (J). There is also static example of equilibrium solution inside the cusp showing the bistability.

splitting factors, or asymmetry and bifurcation factors, respectively, are defined as a scalar-valued function of independent variables:

$$\alpha_x = \alpha_0 + \sum_{i=1}^{n} \alpha_i x_i, \tag{4}$$

$$\beta_x = \beta_0 + \sum_{i=1}^{n} \beta_i x_i. \tag{5}$$

These factors then determine the predicted values of y_t given x_i, meaning that for each value x_i there might be three predicted values of the state variable. Assuming the cusp potential function $V(y_t; \alpha, \beta)$ defined by (2), the predictions will be roots of the equation

$$\frac{dV(y_t; \alpha_x, \beta_x)}{dy_t} = -y_t^3 + \beta_x y_t + \alpha_x = 0, \tag{6}$$

which describes the cusp catastrophe response surface containing a smooth pleat. Fig. 2 illustrates this surface referring to Zeeman (1974) application of the cusp catastrophe theory to stock markets. A specific interpretation will be discussed later in the empirical part of this paper.

In the case of three roots, the central root among these three roots is called an "anti-prediction." In other words, it is the least probable state of the system. This feature is also clear from the bimodality of the probability density function (PDF) (Fig. 3) of y_t. Bimodality is another important aspect of the cusp model. It means that for one value of the control variables, two possible behavior points exist. The system might get into the hysteresis loop by jumping between these two possible equilibria points. When moving along the normal axis toward higher values of the splitting variable shown in Fig. 2, the jump from the upper sheet to the bottom sheet of the cusp surface occurs at a different value of the normal variable than the sudden jump from the bottom to the upper sheet does.

In addition, Cobb (1981) used λ and $\sigma > 0$ as the location and scale parameters, respectively; thus, we will consider the following form of equilibrium space:

$$\frac{dV(y_t; \alpha_x, \beta_x)}{dy_t} = -\left(\frac{y_t - \lambda}{\sigma}\right)^3 + \beta_x \left(\frac{y_t - \lambda}{\sigma}\right) + \alpha_x = 0, \tag{7}$$

where α_x and β_x are of the following forms: $\alpha_x = \alpha_0 + \sum_{i=1}^{n} \alpha_i x_i$ and $\beta_x = \beta_0 + \sum_{i=1}^{n} \beta_i x_i$. Hence, the statistical estimation problem is to estimate the $2n + 4$ parameters:

$$\{\lambda, \sigma, \alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n\} \tag{8}$$

from N observations of the $n + 1$ variables $\{y, x_1, \ldots, x_n\}$.

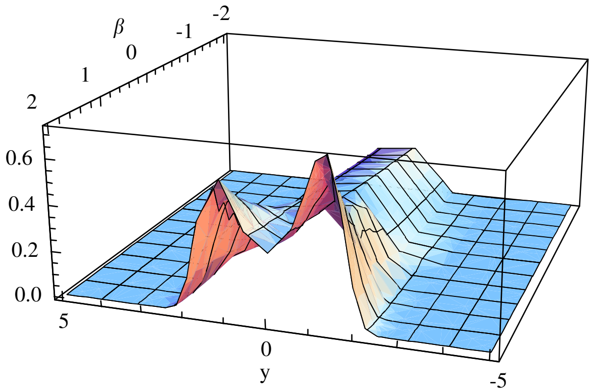

Fig. 3. PDF with parameters $\alpha = -0.1, \xi = -1.23, \lambda = 0, \sigma = 1$. The reader can observe how PDF changes from unimodal to bimodal with increasing bifurcation parameter β.

2.3. Stochastic dynamics and probability density function

From a dynamic system's point of view, (7) can be considered as the surface of the equilibrium points of a dynamic system of the state variable y_t, which follows the ordinary differential equation represented by (1)

$$dy_t = -\frac{dV(y_t; \alpha_x, \beta_x)}{dy_t}\, dt. \tag{9}$$

For real-world applications, it is necessary to add non-deterministic behavior into the system, as the current state of the system usually does not determine its next states entirely. We may obtain a stochastic form by superimposing an additive Gaussian white noise term.[5] The system is then described by a stochastic differential equation of the form

$$dy_t = -\frac{dV(y_t; \alpha_x, \beta_x)}{dy_t}\, dt + \sigma_{y_t}\, dW_t, \tag{10}$$

where $-dV(y_t; \alpha_x, \beta_x)/dy_t$ is called a drift or a deterministic part, representing the equilibrium state of the cusp catastrophe model, and $\sigma^2_{y_t}$ is an instantaneous variance of the process y_t. Here W_t is a standard Wiener process and $dW_t \sim N(0, dt)$. Cobb (1981), Cobb and Watson (1980), Hartelman (1997), Hartelman et al. (1998) and Wagenmakers et al. (2005) have established a link between PDF corresponding to the solution from (10) and a PDF corresponding to a limiting stationary stochastic process. They show that the PDF $f(y_t)$ converges in time to the PDF $f_S(y|x)$ as the dynamics of y_t are assumed to be much faster than changes in x_i. This has led to a definition of stochastic equilibrium state which is compatible with its deterministic counterpart. Instead of fitting the deterministic process where the equilibrium points of the system are of main interest, attention is drawn to the relative extremes of the conditional density function of y.

Following Hartelman (1997) and Wagenmakers et al. (2005), the limiting PDF of y is

$$f_S(y|x) = \xi \exp\left(-\frac{1}{4}\left(\frac{y-\lambda}{\sigma_y}\right)^4 + \frac{\beta_x}{2}\left(\frac{y-\lambda}{\sigma_y}\right)^2 + \alpha_x\left(\frac{y-\lambda}{\sigma_y}\right)\right). \tag{11}$$

The constant ξ normalizes the PDF, meaning that the integral of a normalized PDF over its entire range equals one. The modes and antimodes of the cusp catastrophe PDF can be obtained by solving the equation $df_S(y|x)/dy = 0$, which will yield exactly an implicit cusp surface Eq. (7). The parameters will be estimated by the estimation method developed by Hartelman (1997) and Wagenmakers et al. (2005).

As β_x changes from negative to positive, the PDF of y changes its shape from unimodal to bimodal, which is illustrated in Fig. 3 . It is also the reason why the β_x factor is called a *bifurcation* factor. For $\alpha_x = 0$, the PDF is symmetrical, while for other values it is asymmetric; thus, α_x is an *asymmetry* factor. By eliminating y from the canonical form (7) and its equation of double roots, one can obtain the statistic that discriminates between the unimodal and the bimodal cases. It is referred to as *Cardan's discriminant*:

$$\delta_x = \left(\frac{\alpha_x}{2}\right)^2 - \left(\frac{\beta_x}{3}\right)^3. \tag{12}$$

The PDF is bimodal resp. unimodal if δ_x is negative or nonnegative, respectively. Eq. (12) also determines the shape of the cusp, showing the locus of fold bifurcations which separate the region with two stable solutions from the region with one stable solution.

[5] Cobb and Watson (1980), Cobb (1981), Cobb and Zacks (1985), and Arnold (1998).

3. Empirical investigation of stock market crashes

The thoughtful reader has certainly noted that catastrophe theory models represent an extension to traditional models, i.e. linear ones, and therefore they have to satisfy the requirement of the empirical testability. In this part, we introduce measures by which we test whether the catastrophe model fits the data statistically better than a simple linear regression model, or alternative nonlinear models.

It should be noted that there is no single statistical test for rejection of the catastrophe model. Due to the multimodality of cusp catastrophe, traditional measure for the goodness of fit cannot be used. Let $\{\widehat{y}_i, \widehat{x}_{1,i}, \ldots, \widehat{x}_{n,i}\}_{i=1}^N$ be the estimates from empirical observations. Considered residual $\widehat{\varepsilon}_t = y_t - \widehat{y}_t$ can be determined only if the probability density function at time t is one-peaked; as the model generally offers more than one predicted value, it is difficult to find a tractable definition for a prediction error.

In testing, we follow Hartelman (1997), who created an application for estimating a cusp model, which we use in our empirical testing.[6] A comparison between the cusp and the linear regression models is made by means of a likelihood ratio test, which is asymptotically chi-squared distributed with degrees of freedom being equal to the difference in degrees of freedom for two compared models. As it may not be sufficient to reliably distinguish between catastrophe and non-catastrophe models, Hartelman (1997) also compares catastrophe model to a nonlinear logistic model:

$$y = \lambda + \frac{\sigma}{1 + e^{-(\alpha_x/\beta_x^2)}}, \tag{13}$$

where $\sigma > 0$. As the cusp catastrophe model and the logistic model are not nested, Akaike information criterion (AIC) and Bayesian information criterion (BIC) statistics are used in a testing routine to compare the models. Thus the approval of our hypothesis that the cusp catastrophe model better describes the data than the non-catastrophe model may be supported by the following[7]:

(1) The chi-square test should show that the likelihood of the cusp is significantly higher than that of the linear regression model.
(2) Akaike and Bayesian information criteria should be lower for the cusp catastrophe model.

Cobb (1992) also mentions Delay-R^2, and Maxwell-R^2 as measures of the goodness of fit. These pseudo-R^2 measures are of the following form:

$$1 - \frac{\sum_{t=0}^{T} \widehat{\varepsilon}_t^2}{\sum_{t=0}^{T} (\widehat{y}_t - \overline{y})^2}, \tag{14}$$

where $\overline{y}$ is the empirical mean of the measured data $\widehat{y}_t$, and $\sum_{t=0}^{T} \widehat{\varepsilon}_t^2$ is sum of squared errors. The error variance using Delay convention is determined by the mode that lies on the same side of the antimode as the observation. According to the Maxwell convention, expected value would be the mode for which the value of the probability density function is maximal. The pseudo-R^2 should be used with caution, because MLE does not really maximize it and asymptotic properties are not known. Thus we will use the pseudo-R^2 as a complementary test.

3.1. Uncertain behavior of stock exchanges

In the following part we will introduce the possible application of the proposed model to stock market analysis, which will later be tested on the data. Zeeman (1974) is the first who has clearly stated the hypothesis that the cusp catastrophe models are capable of explaining uncertain behavior of the stock market.[8] He also has merit in popularizing the ideas of catastrophe theory. After three decades, this paper is one of the first attempts to test[9] his qualitative description on the data from stock exchanges. First of all, we will briefly summarize the main ideas.

One of Zeeman's major hypotheses about market behavior is that there are two types of investors, namely fundamentalists and chartists. Fundamentalists are investors who act on the basis of estimates of economic factors such as supply and demand and before they invest in company stocks, they assess a growth potential. Chartists, on the other hand, base their investments upon behavior of the market. They use the charts of historical prices to predict future behavior. In other words, there are two variables C – chartists who represent the proportion of speculative money in the capital market, and F – fundamentalists who represent excess demand for stock. The simplest way of measuring the state of the stock

[6] The application is available at Han van der Maas's Website (http://users.fmg.uva.nl/hvandermaas/).

[7] Hartelman (1997).

[8] After the pioneering model of Zeeman (1974), a large amount of the literature on financial market models with fundamentalists and chartists has developed, e.g. Day and Huang (1990), Brock and Hommes (1998) and, more recently, Boswijk et al. (2007), Georges (2008), Bauer et al. (2009) and Evstigneev and Taksar (2009), among others. Recent state of the art surveys have been given in LeBaron (2006) and Hommes (2006).

[9] First to actually apply the catastrophe theory to explain stock market returns behavior on the financial data.

market is to choose an index I, in our analysis S&P 500 index. Let

$$J = \dot{I} = \frac{dI}{dt} \tag{15}$$

be a rate of change of the stock market index I. Then $J = 0$ represents a static market, $J > 0$ represents a bull market, and $J < 0$ represents a bear market. Let us apply Zeeman's hypothesis to (10). The state variable y_t will be the rate of change of the index, J; the fundamentalists, which are external driving forces, constitute an asymmetry control variable α_x, and chartists, which are more like part of the internal mechanism of the stock market, constitute a bifurcation control variable β_x. Naturally, there are many other external factors affecting the index I; however, an application of a dynamic relation between C, F and J described by the stochastic cusp catastrophe model (10) offers a solution to the matter. Zeeman also argues that the rate of change of the stock market J responds to changes in C and F much faster than C and F respond to changes in J. That is straightforward, as the change of speculative money in the stock market will have an immediate impact on the rate of change of the stock index, while even large change in the J will take perhaps weeks or months to change F due to research involved. Consequently, the flow lines will be nearly vertical almost everywhere. In other words, fixing C and F will cause J to rapidly seek a stable equilibrium, even if the starting point of the system is not on a surface as the dynamics will quickly find the equilibrium. As we will not test this relationship in this paper, we will not define the nonlinear model which would describe it more precisely. Interested readers are advised to consult the original Zeeman paper.

Also, when the number of chartists is small, J is a continuous monotonic increasing function of F; as the market is dominated by well-informed investors, the demand for buying and selling is equal, and the only changes in J are caused by excess demand or supply. On the contrary, a large number of chartists introduces instability into the market. That is also a reason why they are on the bifurcation side of the control space. The larger C is, the less stable J is.

Another important observation is that C has the same sign as J since chartists follow the trend and are attracted by the bull market. Fundamentalists, on the other hand, exit the market after large rises, even though it may continue to rise, and enter the market after short falls. The model is illustrated in Fig. 2, and interested readers are encouraged to consult Zeeman (1974) where it is suggested that all Zeeman's hypotheses about the stock market behavior may be addressed by the catastrophe model.

3.2. Data description

We primarily test the model on the set of daily returns data which contains the most discussed stock market crash of October 19, 1987, known as Black Monday. The crash was the greatest single-day loss that Wall Street had ever suffered in continuous trading, 20.5%. The reasons for Black Monday have been widely discussed among professional investors and academics in many books and research papers. From many, Waldrop (1987) was the first who essayed to explain the causes of the crash, Bates (1991) provided rigorous analysis, Gennotte and Leland (1990) proposed an explanation of the 1987 crash along the lines of catastrophe theory and finally Carlson (2007) gave a very comprehensive review of the causes and events surrounding the crash. However, not even today is there a consensus on the cause; the most discussed potential causes are computer trading, overvaluation, and problems with liquidity or market psychology. For comparison, we use another large crash, that of September 11, 2001. Our hypothesis is that while in 1987 the crash was caused by internal forces, the 2001 crash happened due to external forces, namely the terrorist attack on the Twin Towers. Therefore the catastrophe model should fit the data of 1987 well, as bifurcations leading to instability should be present. However, this model should not perform better than linear regression in the case of the 2001 data. Except for different forces which may drive the stock market to crash, there is one more important difference in the two tested datasets: diverse distribution of stock market participants. Naturally, the stock market changes over time. Accordingly, in the years of 1987–2001 changes occurred internally in the stock market and behavior of investors, which may have led to different results.

According to Zeeman's interpretation we are modeling the rate of change of the stock index, which is the first difference of its prices. The data thus represent the daily returns of S&P 500[10] in the years 1987–1988 and 2001–2002 as the crashes took place within these intervals. With the transformation of prices into the returns we also gain stationarity. Augmented Dickey–Fuller statistics exceed the critical values on 1% level of significance; thus we can reject the null hypothesis of unit root presence in the returns data. Tables 1 and 2 show[11] the descriptive statistics of the data. It can be seen that the data are leptokurtic. For illustration of bimodality, we use kernel densityestimation[12] – see Figs. 4a and b.

Kernel density of the two-year returns of 1987 and 1988 indicates bimodality, and so does the kernel density of the second set of the data, i.e. years 2001 and 2002. These periods of multimodality are our candidates for the testing of bifurcations.

As control variables, we have chosen the daily change of total trading volume, ratio of advancing stocks volume and declining stocks volume, OEX put/call ratio, Dow Jones Composite Bond Index, and a one-day lag of S&P 500 returns. The

[10] Standard and Poor's 500 index represents good approximation of the U.S. stock exchange.

[11] See Appendix.

[12] We use Epanechnikov kernel, which is of the following form: $K(u) = \frac{3}{4}(1 - u^2)(|u| \leq 1)$. A smoother bandwidth was chosen so the bimodality can be seen.

Table 1
Descriptive statistics for 1987 data.

	Volume	Put/call	S&P500
Mean	0.37082	0.02871	0.00036
Standard error	0.06927	0.01433	0.00087
Median	0.01277	0.00000	0.00102
Mode	Multi	0	Multi
St. dev.	1.39928	0.28943	0.01764
Sample variance	1.95798	0.08377	0.00031
Kurtosis	50.54401	58.09584	47.26796
Skewness	5.41975	5.35765	−3.92644
Minimum	−0.93786	−0.74005	−0.20457
Maximum	16.70808	3.54255	0.09099

Table 2
Descriptive statistics for 2001 data.

	Volume	Put/call	S&P 500
Mean	0.24162	0.06894	−0.00108
Standard error	0.06161	0.01973	0.00072
Median	0.02315	0.01867	−0.00147
Mode	Multi	0	Multi
St. dev.	1.24446	0.39850	0.01458
Sample variance	1.54868	0.15880	0.00021
Kurtosis	153.42761	2.36763	1.31866
Skewness	10.19473	1.09433	0.24263
Minimum	−0.95221	−0.78571	−0.04922
Maximum	19.88924	2.05063	0.05728

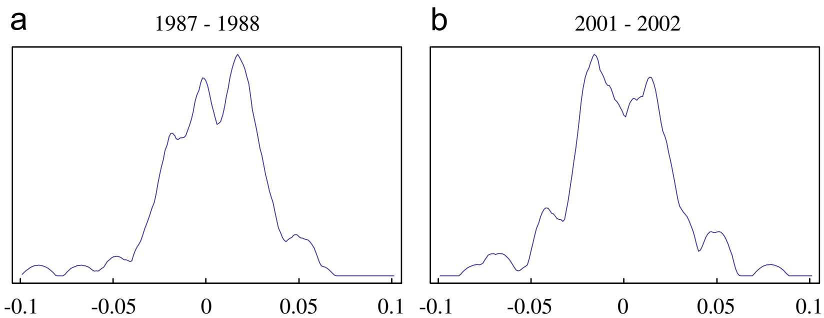

Fig. 4. (a) Kernel density estimate of S&P 500 returns for years 1987–1988. (b) Kernel density estimate of S&P 500 returns for years 2001–2002.

trading volume represents a good measure of the fundament, as it correlates with the volatility, and more importantly good measure of what the large funds, representing fundamental investors, are doing. Thus we suspect the total volume indicator and the ratio of advancers and decliners volume to be on the asymmetry side of the model. The Dow Jones Composite Index should also have an impact on fundamentals, thus we expect its contribution to be on the asymmetry side. The lagged index returns are also added to see whether it might contribute to the model. Finally, OEX put/call ratio represents a very good measure of speculative money. It is a ratio of daily put volume divided by daily call volume of the options with the underlying Standard and Poor's 100 index. As financial options provide the most popular vehicle for speculation, they represent the data of speculative money, while extraordinarily biased volume or premium suggests excessive fear or greed in the stock market. These should be internal forces which cause the bifurcation. Overall, we expect this OEX put/call ratio and advancing stocks volume over declining stock volume to have greater impact on the bifurcation and asymmetry side, respectively. In the next section we will discuss the results of our analysis.

3.3. Results

The first test we consider is Hartelman's test for multimodality, which is similar to Silverman (1986) and can be used to test the presence of bifurcation points in the data. It applies to two-dimensional cross-section data, control and behavioral

Table 3
Catastrophe analysis of '87 data.

Model	α_0	α_1	α_2	β_0	β_1	β_2	λ	σ	LL	par	R^2	AIC	BIC
1	1.60	0.00	0.00	3.79	0.00	0.00	−0.09	0.04	−516	4	0.32	1039	1055
2	1.60	0.00	0.00	3.99	0.00	−1.50	−0.09	0.04	−501	5	0.36	1011	1031
3	2.19	0.00	0.00	3.88	0.54	0.00	−0.09	0.04	−352	5	0.66	713	733
4	2.29	0.00	0.00	3.92	0.53	−0.70	−0.09	0.04	−348	6	0.67	708	732
5	1.75	0.00	−0.43	3.71	0.00	0.00	−0.09	0.04	−514	5	0.33	1038	1058
6	4.81	0.00	6.81	1.85	0.00	−5.73	−0.09	0.04	−483	6	0.39	977	1001
7	2.20	0.00	−0.09	3.88	0.54	0.00	−0.09	0.04	−352	6	0.66	715	739
8	3.30	0.00	4.73	3.91	0.57	−2.99	−0.09	0.04	−332	7	0.68	679	707
9	0.53	1.83	0.00	4.85	0.00	0.00	−0.09	0.04	−317	5	0.71	644	664
10	0.53	1.83	0.00	4.98	0.00	−0.66	−0.09	0.04	−314	6	0.72	639	663
11	0.60	1.83	0.00	4.86	0.07	0.00	−0.09	0.04	−315	6	0.72	641	665
12	0.62	1.83	0.00	4.90	0.06	−0.59	−0.09	0.03	−312	7	0.72	638	666
13	0.51	1.83	0.15	4.85	0.00	0.00	−0.09	0.04	−317	6	0.71	646	670
14	1.03	1.83	2.12	4.71	0.00	−1.68	−0.09	0.03	−309	7	0.72	631	659
15	0.58	1.83	0.22	4.85	0.07	0.00	−0.09	0.04	−315	7	0.72	643	671
16	1.03	1.83	2.12	4.88	0.08	−1.61	−0.09	0.03	−307	8	0.73	629	661
Linear									−663	4	0.35	1334	1350
Logistic									−410	5	0.56	830	850

There are 16 cusp models, two unconstrained linear and logistic models, α_1 and β_1 are parameters for x_1 – volume, and α_2 and β_2 are parameters for x_2 – put/call option ratio, LL – log likelihood, AIC, BIC – Akaike and Bayes's criterion.

variable, respectively. If present, a bifurcation point is assumed to occur with respect to the behavioral variable. This test cannot be used for rigorous hypothesis testing with respect to complexity of the catastrophe model, but it serves us as an indication of the presence of bifurcations. We are particularly interested in testing the S&P 500 returns for bimodality, as the previous figures suggest that the returns might be bimodal. We have found that there is 75% probability that the 1987–1988 data contain at least one bifurcation point, and 26% probability that the 2001–2002 data contain at least one bifurcation point. These results are not so statistically strong, but suggest that the first crisis was drawn by internal market forces (cf. the presence of the bifurcations in the data), whereas the 2001 crash was caused mainly due to external forces, the 9/11 attack (the presence of the bifurcations in the data is very weak).

Encouraged by the knowledge that the bifurcations might be present in our datasets we can now move to cusp fitting. As we mentioned before, we use Hartelman's cuspfit software[13] for this purpose. In all experiments, the linear, the nonlinear (logistic) and the cusp catastrophe models have been fitted to the data. Then we have tested whether the cusp catastrophe model fits the data better than the other two models by the procedure described at previous sections. We look at log likelihood, Akaike and Bayesian information criteria and we also use a simple chi-square test to compare the models. Let us have a look at this analysis.

3.3.1. 1987 crash

First of all we will study the stock market crash of October 19, 1987, known as Black Monday. We begin the analysis using only two variables which we suspect to have the greatest impact on the stock market returns, x_1 – advancing stocks volume over declining stock volume, and x_2 – OEX put/call ratio. Thus we have control variables $\alpha_x = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2$ and $\beta_x = \beta_0 + \beta_1 x_1 + \beta_2 x_2$ and we expect the put/call ratio to have more effect on bifurcation side, and volume ratio on asymmetry side. Thus $\alpha_2 = \beta_1 = 0$ should be the best model. To perform rigorous analysis we fix the parameters to zero one by one to test their significance and the contribution of the variables. Since two different control variables were manipulated, 16 different cusp models are possible. Table 3 shows the results. Most of the cusp models have performed better than the alternative linear and logistic models. Among the cusp models, the best performing is unconstrained model Cusp16. However, the unconstrained model is only slightly better than models Cusp14, Cusp12 and Cusp10, so we can see that both variables also have an impact on both sides of control space, but this impact is deniable. Fixing parameters to zero results in worse fits. We can conclude that our expectation that volume ratio drives the market more from the asymmetry side and options data have more impact on the bifurcation side proved to be right.

We follow our analysis by adding other variables to see if they help to explain the data better. First of all we consider an unconstrained model, meaning that all five variables we have chosen as control variables enter on both bifurcation and asymmetry sides. The variables $x_1, \ldots, x_5$ in $\alpha_x = \alpha_0 + \alpha_1 x_1 + \cdots + \alpha_5 x_5$ and $\beta_x = \beta_0 + \beta_1 x_1 + \cdots + \beta_5 x_5$ are lagged returns of S&P500, advancers/decliners volume, change of total volume, OEX put/call ratio and Dow Jones Bond Index, respectively. Todetermine the best model we would need to fix the parameters $\alpha_0, \ldots, \alpha_5, \beta_1, \ldots, \beta_5$ to zero subsequently, so 1023 cusp models would be possible. Instead of this exhausting analysis, we present results of the models where only parameters of

[13] Applications are available at Han van der Maas's Website (http://users.fmg.uva.nl/hvandermaas/).

Table 4
Catastrophe analysis of '87 data using additional variables.

Model	17	18	19	20	21	22	23	24	25
fixed par.	α_1	β_1	α_1,β_1	α_2	β_2	α_2, β_2	α_3	β_3	α_3, β_3
R^2	0.76	0.76	0.75	0.72	0.75	0.39	0.76	0.75	0.75
LL	−282	−288	−296	−300	−289	−488	−282	−285	−288
No. of par.	13	13	12	13	13	12	13	13	12
AIC	590	601	615	300	603	1000	589	597	599
BIC	642	653	664	677	655	1012	641	649	647
Model	26	27	28	29	30	31	32	Lin	Log
Fixed par.	α_4	β_4	α_4,β_4	α_5	β_5	α_5,β_5			
R^2	0.76	0.76	0.76	0.76	0.76	0.76	0.76	0.44	0.58
LL	−282	−282	−285	−285	−286	−286	−283	−691	−399
No. of par.	13	13	12	13	13	12	14	7	8
AIC	590	591	594	596	598	597	594	1395	813
BIC	642	643	642	648	650	645	650	1423	845

Results for fixed parameters of x_1 – returns of S&P 500 with 1-day lag, x_2 – up volume/down volume, x_3 – change of total volume, x_4 – OEX put/call ratio, x_5 – Dow Jones Bond Index.

one variable are fixed, i.e. $\alpha_1 = 0, \beta_1 = 0$ or $\alpha_1 = \beta_1 = 0$ to test the contribution of the variable. In Table 4 we have the results of all considered models.

We can see that most of the models perform better than a linear or logistic model. The unconstrained model is model 32; from the results we can see that the largest contribution has variable 2, advancers/decliners volume. When we fix its parameters to zero, $\alpha_2 = \beta_2 = 0$, the model deteriorates (log likelihood of the model is substantially lower). Most of the results represent a small improvement to the previous simpler cusp model with two variables. As all the models are nested, we can use the χ^2 test to verify whether the constrained model is significantly better than unconstrained one. The test shows that the models are significantly different meaning that also other variables may play a role in the model. We can conclude that cusp models explained the 1987 stock market crash significantly better than alternative linear and logistic models, as all the cusp models performed much better than the linear and logistic models.

3.3.2. 2001 crash

In this section we will apply the methodology from the previous section to the data from September 11, 2001. We again begin with two variables which we suspect to have the greatest impact on the stock market returns, x_1 – advancing stocks volume over declining stock volume, and x_2 – OEX put/call ratio. Thus we have control variables $\alpha_x = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2$ and $\beta_x = \beta_0 + \beta_1 x_1 + \beta_2 x_2$ and we expect the put/call ratio to have greater effect on the bifurcation side, and volume ratio on the asymmetry side. Thus $\alpha_2 = \beta_1 = 0$ should be the best model. We again fix the parameters to zero one by one in order to test their significance and the contribution of the variables. Since two different control variables were manipulated, 16 different cusp models are possible. Table 5 shows the results. In this case the logistic model performed significantly better than all 16 cusp models according to the log likelihood ratio test. Among the cusp models, models 11, 12, 15 and 16 do not have significantly different results, and all others are significantly worse according to the chi-squared test.

As in the previous case we follow our analysis by adding more variables to the model. The variables $x_1, \ldots, x_5$ in $\alpha_x = \alpha_0 + \alpha_1 x_1 + \cdots + \alpha_5 x_5$ and $\beta_x = \beta_0 + \beta_1 x_1 + \cdots + \beta_5 x_5$ are lagged returns of S&P500, advancers/decliners volume, change of total volume, OEX put/call ratio and Dow Jones Bond Index, respectively. To determine the best model we would need to subsequently fix the parameters $\alpha_0, \ldots, \alpha_5, \beta_1, \ldots, \beta_5$ to zero, so 1023 cusp models would again be possible. Instead of this exhausting analysis, wepresent models with only parameters of one variable fixed, i.e., $\alpha_1 = 0, \beta_1 = 0$ and $\alpha_1 = 0, \beta_1 = 0$ to test the contribution of the variable. In Table 6 we have the results of all considered models.

We can see that none of the models performs better than the logistic model, but they all are better than the linear model. Fixing the parameters of variables to zero gives us very similar results as in the case of 1987 data, but it does not make an important contribution to the model; in other words, it does not lead to a better model than the logistic model.

The fact that the 2001 data are better fitted by an alternative logistic model leads us to the conclusion that this market was not in the bifurcation area. The logistic model describes the data better, thus the market was outside of the cusp area and there were no internal bifurcations which could lead to the market crash in 2001.

4. Conclusions

Uncertain behavior of stock markets has always been at the leading edge of research. Using Cobb and Zacks (1985), Hartelman (1997), Hartelman et al. (1998) and Wagenmakers et al. (2005) stochastic methods we have managed to test cusp catastrophe theory on financial data, and we have arrived at very interesting results which may help to advance the

Table 5
Catastrophe analysis of '01 data.

Model	α_0	α_1	α_2	β_0	β_1	β_2	λ	σ	LL	Par	R^2	AIC	BIC
1	−5.00	0.00	0.00	−5.00	0.00	0.00	0.03	0.04	−575	4	0.00	1158	1174
2	−5.00	0.00	0.00	−5.07	0.00	0.94	0.03	0.04	−572	5	0.01	1154	1174
3	−5.00	0.00	0.00	−5.02	2.98	0.00	0.03	0.04	−572	5	0.01	1152	1179
4	1.90	0.00	0.00	3.21	1.32	−0.45	−0.07	0.03	−373	6	0.67	758	782
5	−4.95	0.00	−0.74	−5.00	0.00	0.00	0.03	0.04	−573	5	0.01	1155	1175
6	−5.02	0.00	0.33	−5.09	0.00	1.29	0.03	0.04	−572	6	0.01	1156	1180
7	2.23	0.00	−0.78	3.19	1.32	0.00	−0.07	0.03	−373	6	0.67	759	783
8	2.29	0.00	−0.46	3.20	1.33	−0.24	−0.07	0.03	−373	7	0.67	760	788
9	−4.95	−0.74	0.00	−5.00	0.00	0.00	0.03	0.04	−573	5	0.01	1155	1175
10	0.51	3.27	0.00	−3.36	0.00	−0.66	−0.02	0.02	−318	6	0.80	648	672
11	−8.18	2.32	0.00	−0.86	−3.01	0.00	0.02	0.02	−302	6	0.80	617	641
12	−8.21	2.34	0.00	−0.96	−2.97	0.62	0.02	0.02	−300	7	0.80	615	643
13	0.56	3.27	−0.61	−3.54	0.00	0.00	−0.02	0.02	−317	6	0.80	647	671
14	0.55	3.27	−0.48	−3.50	0.00	−0.20	−0.02	0.02	−317	7	0.80	649	677
15	−8.17	2.34	−0.70	−0.91	−2.98	0.00	0.02	0.02	−301	7	0.80	615	643
16	−8.20	2.34	−0.29	−0.95	−2.97	0.41	0.02	0.02	−300	8	0.80	617	649
Linear									−761	4	0.60	1531	1547
Logistic									−245	5	0.80	499	519

There are 16 cusp models, two unconstrained linear and logistic models, α_1 and β_1 are parameters for x_1 – volume, and α_2 and β_2 are parameters for x_2 – put/call option ratio, LL – log likelihood, AIC, BIC – Akaike and Bayes's criterion.

Table 6
Catastrophe analysis of '01 data using additional variables.

Model	17	18	19	20	21	22	23	24	25
Fixed parameters	α_1	β_1	α_1,β_1	α_2	β_2	α_2,β_2	α_3	β_3	α_3,β_3
R^2	0.82	0.81	0.82	0.7	0.82	0.05	0.81	0.81	0.80
LL	−289	−285	−291	−363	−288	−553	−289	−291	−292
no. of par.	13	13	12	13	13	12	13	13	12
AIC	603	595	605	753	602	1129	605	609	609
BIC	655	647	653	805	654	1177	657	661	657
Model	26	27	28	29	30	31	32	Lin	Log
Fixed parameters	α_4	β_4	α_4,β_4	α_5	β_5	α_5,β_5			
R^2	0.81	0.81	0.81	0.8077	0.81	0.81	0.81	0.60	0.81
LL	−279	−278	−279	−278	−279	−279	−278	−762	−240
No. of par.	13	13	12	13	13	12	14	7	8
AIC	583	583	581	583	583	583	585	1538	495
BIC	635	635	630	635	635	631	641	1566	527

Results for fixed parameters of x_1 – returns of S&P 500 with 1-lag, x_2 – up volume/down volume, x_3 – change of total volume, x_4 – OEX put/call ratio, x_5 – Dow Jones Bond Index.

frontier of understanding stock market crashes. We may thus confirm that the stochastic catastrophe model explains the stock market crash much better than alternative linear regression models, or a nonlinear logistic model. We have fitted the data of the two stock market crashes, the first being the crash of October 19, 1987, and the second September 11, 2001. We have used the sentiment measures to model the proportion of technical and fundamental players in the market. We have chosen the daily change of total trading volume, the ratio of advancing stocks volume and declining stocks volume, the OEX put/call ratio, the Dow Jones Composite Bond Index, and one lag of S&P 500 returns. We expected OEX put/call ratio to be a very good measure of the technical players, as it represents speculative money in the market and the trading volume to be the measure of fundamental players as it represents the excess demand.

Our most important result is that the data from the year 1987 contained bifurcation points. We have identified the bimodality of the returns by the test for multimodality, which confirms that there is a 75% probability that there is at least one bifurcation point in the data. More important, the cusp catastrophe models fit these data much better than other models that have been used. Hence we conclude that the internal processes of the first dataset led to the crash in 1987. On the other hand, the crash of September 11, 2001, can be better explained by the alternative logistic model. We have also found only a 26% probability that there is at least one bifurcation point in these data, which is also in line with our second hypothesis: that due to the fact that this crash was caused by external forces, the presence of bifurcations in the data is much weaker.

Our findings may advance the frontier of research, as it is the first attempt to quantitatively explain stock market crashes by stochastic cusp catastrophe theory. We have also managed to show that not just the price information is important for stock market analysis; other measures, such as the measures of sentiment of the stock markets, may also have a crucial impact.

Finally, it is necessary to mention that the testing has been conducted only on the restricted datasets. Thus, further work is needed to test on different data, in which the changes in speculative money in the stock market may lead to a crash. The main significant question, whether cusp catastrophe theory may help as an early indication of stock market crashes, still remains to be answered.

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