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OSCILLATIONS AND CONCENTRATIONS GENERATED BY $\mathcal{A}$-FREE MAPPINGS AND WEAK LOWER SEMICONTINUITY OF INTEGRAL FUNCTIONALS

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Abstract. DiPerna's and Majda's generalization of Young measures is used to describe oscillations and concentrations in sequences of maps $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$ satisfying a linear differential constraint $\mathcal{A}u_k = 0$. Applications to sequential weak lower semicontinuity of integral functionals on $\mathcal{A}$-free sequences and to weak continuity of determinants are given. In particular, we state necessary and sufficient conditions for weak* convergence of $\det\nabla\varphi_k \overset{*}{\rightharpoonup} \det\nabla\varphi$ in measures on the closure of $\Omega \subset \mathbb{R}^n$ if $\varphi_k \rightharpoonup \varphi$ in $W^{1,n}(\Omega;\mathbb{R}^n)$. This convergence holds, for example, under Dirichlet boundary conditions. Further, we formulate a Biting-like lemma precisely stating which subsets $\Omega_j \subset \Omega$ must be removed to obtain weak lower semicontinuity of $u \mapsto \int_{\Omega\setminus\Omega_j} v(u(x))\,\mathrm{d}x$ along $\{u_k\} \subset L^p(\Omega;\mathbb{R}^m) \cap \ker\mathcal{A}$. Specifically, Ω_j are arbitrarily thin "boundary layers".

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1. Introduction

Oscillations and concentrations appear naturally in many problems in the calculus of variations, partial differential equations, and optimal control theory. While Young measures [39] successfully capture oscillatory behavior of sequences, they completely miss concentration effects. These may be dealt with appropriate generalizations of Young measures, as in DiPerna's and Majda's treatment of concentrations [9], following Alibert's and Bouchitté's approach [1] (see also [13,25,26]), etc. Detailed overviews of this subject may be found in [33,36].

We are interested in the interplay of oscillation and concentration effects generated by sequences $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$ which satisfy a linear differential constraint $\mathcal{A}u_k = 0$, or $\mathcal{A}u_k \to 0$ in $W^{-1,p}(\Omega;\mathbb{R}^d)$, $1 < p < +\infty$, where $\mathcal{A}$ is a first-order linear differential operator. An explicit characterization of Young measures generated by sequences fulfilling $\mathcal{A}u_k = 0$ ($\mathcal{A}$-free sequences) was completely given in [15], following earlier works by Kinderlehrer and Pedregal [19,21] in the special case $\mathcal{A} :=$ curl, the so-called gradient Young measures (see also [30,32]). The complete study of oscillations and concentrations when $\mathcal{A} =$ curl can be found in [16]

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(see also [18] for a more general setting). Another particularly interesting situation, that will be a corollary of the theory developed in this paper, is $\mathcal{A} := \operatorname{div}$ which is relevant in the theory of micromagnetics [8,31,32].

Here we will use DiPerna's and Majda's generalization of Young measures, the so-called DiPerna-Majda measures [9,33], to address oscillations and concentrations features in sequences $\{gv(u_k)\}$ where v agrees at infinity with a positively p-homogeneous function and $g \in C(\bar{\Omega})$.

The main results may be found in Section 2. First, we will state necessary and sufficient conditions for a DiPerna-Majda measure to be generated by an $\mathcal{A}$-free sequence that admits an $\mathcal{A}$-free p-equiintegrable extension, see Theorem 2.1. Secondly, we formulate necessary conditions for a DiPerna-Majda measure to be generated by a general $\mathcal{A}$-free sequence, see Theorem 2.2. New sequential weak lower semicontinuity theorems issue from this analysis (*cf*. Thms. 2.3 and 2.4). We further state a *necessary and sufficient* condition ensuring weak L^1 convergence of $\{\det\nabla\varphi_k\}_{k\in\mathbb{N}}$ if $\{\varphi_k\} \subset W^{1,n}(\Omega;\mathbb{R}^n)$ and $\det \nabla\varphi_k \geq 0$ for all $k \in \mathbb{N}$, see Proposition 2.6. In the absence of the sign assumption, the same condition is equivalent to the weak* convergence $\det\nabla\varphi_k \overset{*}{\rightharpoonup} \det\nabla\varphi$ in measures supported on the *closure* of $\bar{\Omega}$, *cf*. Proposition 2.8. In particular, this holds if $\varphi_k = \varphi$ on $\partial\Omega$ for some $\varphi \in W^{1,n}(\Omega;\mathbb{R}^m)$. Finally, we formulate a Biting-like Lemma for $\mathcal{A}$-quasiconvex functions, see Lemma 2.10, showing that sets which must be bitten to recover weak lower semicontinuity are *only* arbitrarily thin "boundary layers".

1.1. **Preliminaries and Young measures**

We recall some measure theory results and set the notation [10]. Let X be a topological space. We denote by $C(X)$ the space of real-valued continuous functions in X. If X is a locally compact space then $C_0(X)$ denotes the closure of the subspace of $C(X)$ of functions with the compact support. By the Riesz Representation Theorem, the dual space to $C_0(X)$, $C_0(X)'$, is isometrically isomorphic with $\mathcal{M}(X)$, the linear space of finite Radon measures supported on X, normed by the total variation. Moreover, if X is compact then the dual space to $C(X)$, $C(X)'$, is isometrically isomorphic with $\mathcal{M}(X)$. A positive Radon measure $\mu \in \mathcal{M}(X)$ with $\mu(X) = 1$ is called a probability measure, and the set of all probability measures is denoted $\mathcal{P}(X)$.

If not said otherwise, we will work with a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ equipped with the Euclidean topology and the n-dimensional Lebesgue measure $\mathcal{L}^n$. By $L^p(\Omega,\mu)$, $1 \leq p \leq +\infty$, we denote the space of p-integrable functions with respect to the measure $\mu \in \mathcal{M}(\Omega)$. Further, $W^{1,p}(\Omega;\mathbb{R}^m)$, $1 \leq p < +\infty$, stands for the usual space of measurable mappings, which together with their first (distributional) derivatives, are integrable with the p-th power. The closer of $C_0(\Omega;\mathbb{R}^m)$ in $W^{1,p}(\Omega;\mathbb{R}^m)$ is denoted $W^{1,p}_0(\Omega;\mathbb{R}^m)$. If $1 < p < +\infty$ then $W^{-1,p}(\Omega;\mathbb{R}^m)$ denotes the dual space to $W^{1,p'}_0(\Omega;\mathbb{R}^m)$, where $p'^{-1} + p^{-1} = 1$. If $\mu \in \mathcal{M}(\Omega)$ then $L^1(\Omega,\mu;C_0(X))'$ may be identified with $L^\infty_{\mathrm{w}}(\Omega,\mu;\mathcal{M}(X))$, the space of weakly* μ-measurable mappings $\eta : \Omega \to \mathcal{M}(X)$. We recall $\eta : \Omega \to \mathcal{M}(X)$ is weakly* μ-measurable if, for all $v \in C_0(X)$, the mapping $x \in \Omega \mapsto \langle\eta(x), v\rangle$ is μ-measurable. If X is compact then $L^1(\bar{\Omega},\mu;C(X))'$ may be identified with $L^\infty_{\mathrm{w}}(\bar{\Omega},\mu;\mathcal{M}(X))$. We drop the reference to μ in this notation if $\mu := \mathcal{L}^n \llcorner \Omega$.

The support of a measure $\mu \in \mathcal{M}(\Omega)$ is the smallest closed set S such that $\mu(A) = 0$ if $S \cap A = \emptyset$. Finally, if $\mu \in \mathcal{M}(\Omega)$ we write μ_s and d_μ for, respectively, the singular part and the density of μ with respect to the Lebesgue measure, *i.e.*, using the Radon-Nikodým theorem [12]

$$\mu = \frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^n}\mathcal{L}^n \llcorner \Omega + \mu_s \text{ and } d_\mu := \frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^n}\mathcal{L}^n.$$

For $p \geq 0$ we define

$$C_p(\mathbb{R}^m) := \{v \in C(\mathbb{R}^m)\colon v(s) = o(|s|^p) \text{ for } |s| \to \infty\}\cdot$$

The *Young measures* in a domain $\Omega \subset \mathbb{R}^n$ with values in $\mathcal{P}(\mathbb{R}^m)$ are the weakly* measurable mappings $\nu : \Omega \to \mathcal{P}(\mathbb{R}^m)$. In what follows, and when there is no possibility of confusion, we write ν_x in place of $\nu(x)$ and abbreviate $\nu := \{\nu_x\}_{x\in\Omega}$. We denote the set of all such Young measures by $\mathcal{Y}(\Omega;\mathbb{R}^m)$. Obviously, $\mathcal{Y}(\Omega;\mathbb{R}^m)$ is a convex subset of $L^\infty_{\mathrm{w}}(\Omega;\mathcal{M}(\mathbb{R}^m))$. A classical result [13,35,38,39] is that, for every sequence $\{y_k\}_{k\in\mathbb{N}}$ bounded in $L^\infty(\Omega;\mathbb{R}^m)$, there exists a subsequence (not relabeled) and a Young measure $\nu = \{\nu_x\}_{x\in\Omega} \in \mathcal{Y}(\Omega;\mathbb{R}^m)$ such that for all $v \in C(\mathbb{R}^m)$

$$\lim_{k\to\infty} v \circ y_k = v_\nu \qquad \text{weakly* in } L^\infty(\Omega), \tag{1.1}$$

$$v_\nu(x) := \int_{\mathbb{R}^m} v(s) d\nu_x(s) \text{ for a.e. } x \in \Omega. \tag{1.2}$$

We say that $\{y_k\}$ *generates* ν if (1.2) holds. We denote by $\mathcal{Y}^\infty(\Omega;\mathbb{R}^m)$ the set of all Young measures generated in this way, *i.e.*, all Young measures attained by bounded sequences in $L^\infty(\Omega;\mathbb{R}^m)$.

A generalization of this result was formulated by Schonbek [34] for the case $1 \le p < +\infty$ (*cf.* [2] where further results in this direction have been obtained; see also [23]): If $\{y_k\}_{k\in\mathbb{N}}$ is bounded in $L^p(\Omega;\mathbb{R}^m)$ then there exists a subsequence (not relabeled) and a Young measure $\nu := \{\nu_x\}_{x\in\Omega} \in \mathcal{Y}(\Omega;\mathbb{R}^m)$ such that for all $v \in C_p(\mathbb{R}^m)$

$$\lim_{k\to\infty} v \circ y_k = v_\nu \qquad \text{weakly in } L^1(\Omega). \tag{1.3}$$

As before, we say that $\{y_k\}$ *generates* ν if (1.3) holds. We denote by $\mathcal{Y}^p(\Omega;\mathbb{R}^m)$ the set of all Young measures which are generated in this way.

1.2. The operator $\mathcal{A}$ and $\mathcal{A}$-quasiconvexity

Following [5,15], we consider linear operators $A^{(i)} : \mathbb{R}^m \to \mathbb{R}^d$, $i = 1,\dots,n$, and define $\mathcal{A} : L^p(\Omega;\mathbb{R}^m) \to W^{-1,p}(\Omega;\mathbb{R}^d)$ by

$$\mathcal{A}u := \sum_{i=1}^n A^{(i)} \frac{\partial u}{\partial x_i}, \text{ where } u : \Omega \to \mathbb{R}^m,$$

i.e., for all $w \in W_0^{1,p'}(\Omega;\mathbb{R}^d)$

$$\langle \mathcal{A}u, w\rangle = -\sum_{i=1}^n \int_\Omega A^{(i)} u(x) \cdot \frac{\partial w(x)}{\partial x_i}\, \mathrm{d}x.$$

For $w \in \mathbb{R}^n$ we define the linear map

$$\mathbb{A}(w) := \sum_{i=1}^n w_i A^{(i)} \; : \; \mathbb{R}^m \to \mathbb{R}^d,$$

and assume that there is $r \in \mathbb{N} \cup \{0\}$ such that

$$\text{rank } \mathbb{A}(w) = r \text{ for all } w \in \mathbb{R}^n, \; |w| = 1,$$

i.e., $\mathcal{A}$ has the so-called *constant-rank property.*

Let Q be the unit cube $(-1/2, 1/2)^n$ in $\mathbb{R}^n$. We say that $u : \mathbb{R}^n \to \mathbb{R}^m$ is *Q-periodic* if for all $x \in \mathbb{R}^n$ and all $z \in \mathbb{Z}$

$$u(x+z) = u(x).$$

If $u \in L^p(\mathbb{R}^n;\mathbb{R}^m)$ then we say that $u \in \ker \mathcal{A}$ when for all open bounded sets $\Omega \subset \mathbb{R}^n$, $\mathcal{A}u = 0$ in $W^{-1,p}(\Omega;\mathbb{R}^d)$, *i.e.*,

$$\ker \mathcal{A} := \{u \in L^p(\Omega;\mathbb{R}^m)\colon \langle \mathcal{A}u, w\rangle = 0 \text{ for all } w \in W_0^{1,p'}(\Omega;\mathbb{R}^d)\}.$$

Although the definition of $\mathcal{A}$ depends on the domain Ω we will omit specifying it whenever it is obvious from the context. Let us finally define

$$L^p_{\#}(\mathbb{R}^n;\mathbb{R}^m) := \{u \in L^p_{\rm loc}(\mathbb{R}^n;\mathbb{R}^m)\colon u \text{ is } Q\text{-periodic}\}\cdot$$

We will use the following lemmas proved in [15], Lemmas 2.14, and [15], Lemma 2.15, respectively.

Lemma 1.1. *If $\mathcal{A}$ has the constant rank property then there is a linear bounded operator $\mathbb{T} : L^p_{\#}(\mathbb{R}^n;\mathbb{R}^m) \to L^p_{\#}(\mathbb{R}^n;\mathbb{R}^m)$ that vanishes on constant mappings, $\mathbb{T}(\mathbb{T}u) = \mathbb{T}u$ for all $u \in L^p_{\#}(\mathbb{R}^n;\mathbb{R}^m)$, and $\mathbb{T}u \in \ker \mathcal{A}$. Moreover, for all $u \in L^p_{\#}(\mathbb{R}^n;\mathbb{R}^m)$ with $\int_Q u(x)\,\mathrm{d}x = 0$ it holds that*

$$\|u - \mathbb{T}u\|_{L^p_{\#}(\mathbb{R}^n;\mathbb{R}^m)} \le C\|\mathcal{A}u\|_{W^{-1,p}(\mathbb{R}^n;\mathbb{R}^d)},$$

where $C > 0$ is a constant independent of u.

Lemma 1.2 (decomposition lemma)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded and open, $1 < p < +\infty$, and let $\{u_k\} \subset L^p(\Omega;\mathbb{R}^m)$ be bounded and such that $\mathcal{A}u_k \to 0$ in $W^{-1,p}(\Omega;\mathbb{R}^d)$ strongly, $u_k \rightharpoonup u$ in $L^p(\Omega;\mathbb{R}^m)$ weakly, and assume that $\{u_k\}$ generates $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$. Then there is a sequence $\{z_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m) \cap \ker \mathcal{A}$, $\{|z_k|^p\}$ is equiintegrable in $L^1(\Omega)$, $\{z_k\}$ generates the Young measure ν, and $u_k - z_k \to 0$ in measure in Ω.*

Definition 1.3 (see [15], Defs. 3.1 and 3.2)**.** We say that a continuous function $v : \mathbb{R}^m \to \mathbb{R}$, $|v| \le C(1 + |\cdot|^p)$ for some $C > 0$, is $\mathcal{A}$-*quasiconvex* if for all $s_0 \in \mathbb{R}^m$ and all $\varphi \in L^p(Q;\mathbb{R}^m) \cap \ker \mathcal{A}$ with $\int_Q \varphi(x)\,\mathrm{d}x = 0$ it holds

$$v(s_0) \le \int_Q v(s_0 + \varphi(x))\,\mathrm{d}x.$$

The $\mathcal{A}$-quasiconvex of v we define its $\mathcal{A}$-*quasiconvex envelope* as

$$Q_{\mathcal{A}}v(s_0) := \inf\left\{\int_Q v(s_0 + \varphi(x))\,\mathrm{d}x :\ \varphi \in L^p(Q;\mathbb{R}^m) \cap \ker \mathcal{A} \text{ and } \int_Q \varphi(x)\,\mathrm{d}x = 0\right\} \text{for all } s_0 \in \mathbb{R}^m.$$

If v is $\mathcal{A}$-quasiconvex then $v = Q_{\mathcal{A}}v$.

Definition 1.4. Let $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m) \cap \ker \mathcal{A}$. We say that $\{u_k\}$ has an $\mathcal{A}$-free p-equiintegrable extension if for every domain $\tilde\Omega \subset \mathbb{R}^n$ such that $\Omega \subset \tilde\Omega$, there is a sequence $\{\tilde u_k\}_{k\in\mathbb{N}} \subset L^p(\tilde\Omega;\mathbb{R}^m) \cap \ker \mathcal{A}$ such that

(i) $\tilde u_k = u_k$ a.e. in Ω for all $k \in \mathbb{N}$;
(ii) $\{|\tilde u_k|^p\}_{k\in\mathbb{N}}$ is equiintegrable on $\tilde\Omega \setminus \Omega$; and
(iii) there is $C > 0$ such that $\|\tilde u_k\|_{L^p(\tilde\Omega;\mathbb{R}^m)} \le C\|u_k\|_{L^p(\Omega;\mathbb{R}^m)}$ for all $k \in \mathbb{N}$.

Example 1.5. If $\mathcal{A} :=$ curl and $\{\varphi_k\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$, $\varphi_k \rightharpoonup \varphi$ weakly in $W^{1,p}(\Omega;\mathbb{R}^m)$ then $\{u_k\} := \{\nabla\varphi_k\}$ has a curl free p-equiintegrable extension if $\{\varphi_k - \varphi\} \subset W^{1,p}_0(\Omega;\mathbb{R}^m)$.

Other examples of $\mathcal{A}$-free mappings include solenoidal fields where $\mathcal{A} =$ div, higher-order gradients where $\mathcal{A}u = 0$ if and only if $u = \nabla^{(s)}\varphi$ for some $\varphi \in W^{s,p}(\Omega;\mathbb{R}^\ell)$, and some $s \in \mathbb{N}$, or symmetrized gradients where $\mathcal{A}u = 0$ if and only if $u = (\nabla\varphi + (\nabla\varphi)^\top)/2$ for some $\varphi \in W^{1,p}(\Omega;\mathbb{R}^\ell)$.

1.3. DiPerna-Majda measures

Consider a complete (*i.e.* containing constants, separating points from closed subsets and closed with respect to the supremum norm), separable (*i.e.* containing a dense countable subset) ring $\mathcal{R}$ of continuous bounded functions from $\mathbb{R}^m$ into $\mathbb{R}$. Such ring always contains $C_0(\mathbb{R}^m)$. It is known that there is a one-to-one correspondence $\mathcal{R} \mapsto \beta_{\mathcal{R}}\mathbb{R}^m$ between such rings and metrizable compactifications of $\mathbb{R}^m$ [11]; by a compactification we mean here a compact set, denoted by $\beta_{\mathcal{R}}\mathbb{R}^m$, into which $\mathbb{R}^m$ is embedded homeomorphically and densely. For simplicity, we will not distinguish between $\mathbb{R}^m$ and its image in $\beta_{\mathcal{R}}\mathbb{R}^m$. We set

$$\Upsilon^p_{\mathcal{R}} := \{v := v_0(1+|\cdot|^p) : \ v_0 \in \mathcal{R}\}\cdot$$

Let $\pi \in \mathcal{M}(\bar\Omega)$ be a finite positive Radon measure, and let $\lambda \in L^\infty_{\rm w}(\bar\Omega,\pi;\mathcal{M}(\beta_{\mathcal{R}}\mathbb{R}^m))$, $\lambda_x := \lambda(x) \in \mathcal{P}(\beta_{\mathcal{R}}\mathbb{R}^m)$, *i.e.* the parameterized measure $\lambda := \{\lambda_x\}_{x\in\bar\Omega}$ is a Young measure on $\bar\Omega$ equipped with π see [39], and also [2,33,35,37,38]). DiPerna and Majda [9] proved the following theorem:

Theorem 1.6. *Let Ω be an open domain in $\mathbb{R}^n$ with $\mathcal{L}^n(\partial\Omega)=0$, and let $\{y_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$, with $1 \le p < +\infty$, be bounded. Then there exists a subsequence (not relabeled), a positive Radon measure $\pi \in \mathcal{M}(\bar\Omega)$ and a mapping $\lambda \in L^\infty_{\rm w}(\bar\Omega,\pi;\mathcal{M}(\beta_{\mathcal{R}}\mathbb{R}^m))$, $\lambda_x \in \mathcal{P}(\beta_{\mathcal{R}}\mathbb{R}^m)$ for π-a.e. $x\in\bar\Omega$, such that for all $g\in C(\bar\Omega)$ and all $v \in \Upsilon^p_{\mathcal{R}}$*

$$\lim_{k\to\infty}\int_\Omega g(x)v(y_k(x))\mathrm{d}x \ = \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m} g(x)v_0(s)d\lambda_x(s)d\pi(x). \tag{1.4}$$

Take $v_0 := 1$ in (1.4) (recall that constants are elements of $\mathcal{R}$) to get

$$\lim_{k\to\infty}(1+|y_k|^p)\mathcal{L}^n \llcorner \Omega \ = \ \pi \qquad \text{weakly* in } \mathcal{M}(\bar\Omega). \tag{1.5}$$

If (1.4) holds then we say that $\{y_k\}_{\in\mathbb{N}}$ *generates* (π,λ), and we denote by $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ the set of all such pairs $(\pi,\lambda)\in\mathcal{M}(\bar\Omega)\times L^\infty_{\rm w}(\bar\Omega,\pi;\mathcal{M}(\beta_{\mathcal{R}}\mathbb{R}^m))$, $\lambda_x\in\mathcal{P}(\beta_{\mathcal{R}}\mathbb{R}^m)$ for π-a.e. $x\in\bar\Omega$. Note that, taking $v_0 := 1$ and $g := 1$ in (1.4), generating sequences must be necessarily bounded in $L^p(\Omega;\mathbb{R}^m)$. We say that $(\pi,\lambda)\in\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ is *homogeneous* if $x\mapsto\lambda_x$ is constant. In this case, the density of π with respect to the Lebesgue measure is constant (see formula (A.1) below).

1.3.1. *Compactification of $\mathbb{R}^m$ by the sphere*

In what follows we will work mostly with a particular compactification of $\mathbb{R}^m$, namely, with the compactification by the sphere. We will consider the following ring $\mathcal{R}$ of continuous bounded functions

$$\begin{aligned}\mathcal{S} \ \ := \ \ \Bigg\{ & v_0\in C(\mathbb{R}^m)\text{: there exist } c\in\mathbb{R},\ v_{0,0}\in C_0(\mathbb{R}^m),\ \text{and } v_{0,1}\in C(S^{m-1}) \text{ s.t.}\\ & v_0(s) = c + v_{0,0}(s) + v_{0,1}\left(\frac{s}{|s|}\right)\frac{|s|^p}{1+|s|^p} \text{ if } s\neq 0 \text{ and } v_0(0) = c+v_{0,0}(0)\Bigg\},\end{aligned} \tag{1.6}$$

where S^{m-1} denotes the $(m-1)$-dimensional unit sphere in $\mathbb{R}^m$. Then $\beta_{\mathcal{S}}\mathbb{R}^m$ is homeomorphic to the unit ball $\overline{B(0,1)}\subset\mathbb{R}^m$ *via* the mapping $f:\mathbb{R}^m\to B(0,1)$, $f(s) := s/(1+|s|)$ for all $s\in\mathbb{R}^m$. Note that $f(\mathbb{R}^m)$ is dense in $\overline{B(0,1)}$.

For any $v\in\Upsilon^p_{\mathcal{S}}$ there exists a continuous and positively p-homogeneous function $v_\infty : \mathbb{R}^m\to\mathbb{R}$, *i.e.*, $v_\infty(ts) = t^p v_\infty(s)$ for all $t\ge 0$ and $s\in\mathbb{R}^m$, such that

$$\lim_{|s|\to\infty}\frac{v(s)-v_\infty(s)}{|s|^p} = 0. \tag{1.7}$$

Indeed, if v_0 is as in (1.6) and $v = v_0(1+|\cdot|^p)$ then set

$$v_\infty(s) := \begin{cases} \left(c + v_{0,1}\left(\frac{s}{|s|}\right)\right)|s|^p & \text{if } s\neq 0,\\ 0 & \text{if } s = 0.\end{cases}$$

By continuity we define $v_\infty(0) := 0$. It is easy to see that v_∞ satisfies (1.7). Such v_∞ is called the *recession function* of v.

Remark 1.7. Notice that $\mathcal{S}$ contains all functions $v_0 := v_{0,0} + v_\infty/(1+|\cdot|^p)$ where $v_{0,0} \in C_0(\mathbb{R}^m)$ and $v_\infty : \mathbb{R}^m \to \mathbb{R}$ is continuous and positively p-homogeneous.

2. Characterization of the set $\mathcal{ADM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$ and weak lower semicontinuity

In what follows we will denote by $\mathcal{ADM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$ the set of DiPerna-Majda measures from $\mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$ which are generated by $\mathcal{A}$-free mappings. We restrict ourselves to the compactification of $\mathbb{R}^m$ by the sphere, although our results can be straightforwardly generalized to finer metrizable compactifications if the following two conditions are satisfied:

(i) two (sub)sequences whose difference tends to zero in $L^p(\Omega;\mathbb{R}^m)$ generate the same DiPerna-Majda measure;

(ii) $\mathcal{A}$-quasiconvex functions in $\Upsilon^p_{\mathcal{R}}$ are separately convex. If this is the case, and if $v \in \Upsilon^p_{\mathcal{R}}$ and $Q_{\mathcal{A}}v > -\infty$ then $|Q_{\mathcal{A}}v| \le C(1+|\cdot|^p)$ for some $C>0$; *cf.* [22] and, moreover, $Q_{\mathcal{A}}v$ is p-Lipschitz, see *e.g.* [27] or [7]. However, in general $\mathcal{A}$-quasiconvex functions do not need to be even continuous; *cf.* [15].

Let Ω be an open bounded Lipschitz domain and $1 < p < +\infty$. While the case $p = +\infty$ does not allow for concentrations and was fully resolved in [15], the case $p = 1$ is much more complicated due to non-reflexivity of $L^1(\Omega;\mathbb{R}^m)$.

Theorem 2.1. *Let $(\pi,\lambda) \in \mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$. Then there exists $\{u_k\} \subset L^p(\Omega;\mathbb{R}^m) \cap \ker \mathcal{A}$, having an $\mathcal{A}$-free p-equiintegrable extension, and generating (π,λ) if and only if the following three conditions hold:*
(i) there exists $u \in L^p(\mathbb{R}^n;\mathbb{R}^m) \cap \ker \mathcal{A}$ such that for a.e. $x \in \Omega$

$$u(x) = d_\pi(x) \int_{\mathbb{R}^m} \frac{s}{1+|s|^p} d\lambda_x(s);$$

(ii) for $\mathcal{L}^n$-almost every $x \in \Omega$ and for all $v \in \Upsilon^p_{\mathcal{S}}$

$$Q_{\mathcal{A}}v(u(x)) \le d_\pi(x) \int_{\beta_{\mathcal{S}}\mathbb{R}^m} \frac{v(s)}{1+|s|^p} d\lambda_x(s); \tag{2.1}$$

(iii) for π-almost every $x \in \bar\Omega$ and all positively p-homogeneous $v \in \Upsilon^p_{\mathcal{S}}$ with $Q_{\mathcal{A}}v(0) = 0$ it holds that

$$0 \le \int_{\beta_{\mathcal{S}}\mathbb{R}^m \setminus \mathbb{R}^m} \frac{v(s)}{1+|s|^p} \mathrm{d}\lambda_x(s). \tag{2.2}$$

The next theorem characterizes DiPerna-Majda measures generated by an arbitrary sequence of $\mathcal{A}$-free mappings, *i.e.*, there may not exist a generating sequence with an $\mathcal{A}$-free p-equiintegrable extension. Then inequality (2.2) does not have to hold on $\partial\Omega$.

Theorem 2.2. *Let $(\pi,\lambda) \in \mathcal{ADM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$ be generated by $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m) \cap \ker \mathcal{A}$. Then (i) and (ii) of Theorem 2.1 are satisfied but (2.2) in (iii) may hold only for π-a.e. $x \in \Omega$.*

The proof of the necessary conditions in Theorems 2.1 and 2.2 is the subject of Section 3 (see Prop. 3.5). Section 4 establishes the sufficient conditions (see Prop. 4.6).

The following two sequential weak lower semicontinuity theorems follow from Theorem 2.2. Their proofs may be found in Section 5.

Theorem 2.3. *Let $0 \le g \in C(\bar\Omega)$, let $v \in \Upsilon^p_{\mathcal{S}}(\mathbb{R}^m)$ be $\mathcal{A}$-quasiconvex, and let $1 < p < +\infty$. Let $\{u_k\} \subset L^p(\Omega;\mathbb{R}^m) \cap \ker \mathcal{A}$, $u_k \rightharpoonup u$ weakly, and assume that at least one of the following conditions is satisfied:*

(i) for any subsequence of $\{u_k\}$ (not relabeled) such that $|u_k|^p \mathcal{L}^n \llcorner \Omega \rightharpoonup \pi$ weakly in $\mathcal{M}(\bar\Omega)$, it holds $\pi(\partial\Omega) = 0$;*

(ii) $\lim_{|s|\to\infty}\frac{v^-(s)}{1+|s|^p}=0$ *where* $v^-:=\max\{0,-v\}$;
(iii) $\{u_k\}$ *has an* $\mathcal{A}$*-free* p*-equiintegrable extension;*
(iv) $g\in C_0(\Omega)$.

Then $I(u)\le\liminf_{k\to\infty}I(u_k)$, *where*

$$I(u):=\int_\Omega g(x)v(u(x))\,\mathrm{d}x. \tag{2.3}$$

Theorem 2.4. *Let* $0\le g\in C(\bar\Omega)$, *let* $v\in\Upsilon^p_{\mathcal{S}}(\mathbb{R}^m)$ *be* $\mathcal{A}$*-quasiconvex, and let* $1<p<+\infty$. *Then* I *is sequentially weakly lower semicontinuous in* $L^p(\Omega;\mathbb{R}^m)\cap\ker\mathcal{A}$ *if and only if for any bounded sequence* $\{u_k\}\subset L^p(\Omega;\mathbb{R}^m)\cap\ker\mathcal{A}$ *such that* $u_k\to0$ *in measure*

$$\liminf_{k\to\infty}I(u_k)\ge I(0).$$

2.1. **Weak/in measure continuity of determinants**

As an application of our results, we give necessary and sufficient conditions for weak sequential continuity of $\varphi\in W^{1,n}(\Omega;\mathbb{R}^n)\mapsto\det\nabla\varphi\in L^1(\Omega)$. Here $n=p$, $d=n^2$,

$$\mathcal{A}u=0\text{ if and only if }\operatorname{curl}u=0,$$

and the notion of $\mathcal{A}$-quasiconvexity reduces to the well-known notion of quasiconvexity, see [3,28]. We recall (see [7,13]) that a Borel measurable function $v:\mathbb{R}^{n\times n}\to\mathbb{R}$ is quasiconvex if for all $s\in\mathbb{R}^{m\times n}$ and all $\phi\in W^{1,\infty}_0(Q;\mathbb{R}^n)$ it holds that

$$v(s)\le\int_Q v(s+\nabla\phi(x))\,\mathrm{d}x. \tag{2.4}$$

If $|v|\le C(1+|\cdot|^n)$, a simple density argument shows that (2.4) remains valid if we take $\phi\in W^{1,n}_{Q-\mathrm{per}}(\mathbb{R}^n;\mathbb{R}^n)$, see [3].

In particular, $v(s):=\pm\det s$ is quasiconvex (see *e.g.* [7]) and, since it is n-homogeneous, $\pm\det/(1+|\cdot|^n)\in\mathcal{S}$ in view of Remark 1.7. Indeed, $\det(\alpha s)=\alpha^n\det s$ if $\alpha\ge0$ and $s\in\mathbb{R}^{n\times n}$. Consider $\{\varphi_k\}_{k\in\mathbb{N}}\subset W^{1,n}(\Omega;\mathbb{R}^n)$ such that w-$\lim_{k\to\infty}\varphi_k=\varphi$ in $W^{1,n}(\Omega;\mathbb{R}^n)$. We extract a further subsequence, if necessary, such that $\{\nabla\varphi\}_k$ generates $\nu\in\mathcal{Y}^n(\Omega;\mathbb{R}^{n\times n})$ and $(\pi,\lambda)\in\mathcal{DM}^n_{\mathcal{S}}(\Omega;\mathbb{R}^{n\times n})$, and so that (A.7) holds for $v:=\det$ and $y_k:=\nabla\varphi_k$, *i.e.*, if $g\in C(\bar\Omega)$ then

$$\lim_{k\to\infty}\int_\Omega g(x)\det\nabla\varphi_k(x)\mathrm{d}x=\int_\Omega\int_{\mathbb{R}^{n\times n}}\det s\;d\nu_x(s)g(x)\,\mathrm{d}x+\int_{\bar\Omega}\int_{\beta_{\mathcal{S}}\mathbb{R}^{n\times n}\setminus\mathbb{R}^{n\times n}}\frac{\det s}{1+|s|^n}\mathrm{d}\lambda_x(s)g(x)d\pi(x). \tag{2.5}$$

It is known that (see [21,30])

$$\int_\Omega\int_{\mathbb{R}^{n\times n}}\det s\;d\nu_x(s)\,\mathrm{d}x=\int_\Omega\det\nabla\varphi(x)\,\mathrm{d}x,$$

and, due to (2.2) applied to $v:=\pm\det$

$$\int_{\beta_{\mathcal{S}}\mathbb{R}^{n\times n}\setminus\mathbb{R}^{n\times n}}\frac{\det s}{1+|s|^n}\mathrm{d}\lambda_x(s)=0 \tag{2.6}$$

for π-almost all $x \in \Omega$. Therefore, we can rewrite (2.5) as

$$\lim_{k\to\infty}\int_\Omega g(x)\det\,\nabla\varphi_k(x)\mathrm{d}x = \int_\Omega \det\nabla\varphi(x)g(x)\,\mathrm{d}x + \int_{\partial\Omega}\int_{\beta_{\mathcal{S}}\mathbb{R}^{n\times n}\setminus\mathbb{R}^{n\times n}} \frac{\det\, s}{1+|s|^n}\mathrm{d}\lambda_x(s)g(x)d\pi(x), \tag{2.7}$$

and, in particular, we have

$$\lim_{k\to\infty}\int_\Omega g(x)\det\,\nabla\varphi_k(x)\mathrm{d}x = \int_\Omega g(x)\det\nabla\varphi(x)\,\mathrm{d}x \tag{2.8}$$

for all $g \in C_0(\Omega)$, *i.e.*, $\det\nabla\varphi_k \stackrel{*}{\rightharpoonup} \det\nabla\varphi$ in the sense of measures [3]. Moreover, if

$$\int_{\beta_{\mathcal{S}}\mathbb{R}^{n\times n}\setminus\mathbb{R}^{n\times n}} \frac{\det\, s}{1+|s|^n}\mathrm{d}\lambda_x(s) = 0$$

for π-almost all $x \in \partial\Omega$, then (2.8) holds for all $g \in C(\bar\Omega)$.

We will need the following lemma.

Lemma 2.5. *Let $0 \le v_0 \in \mathcal{R}$ and let $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$ generate $(\pi,\lambda) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$. Let $v := v_0(1+|\cdot|^p)$. Then $\{v(u_k)\}_{k\in\mathbb{N}}$ is weakly relatively compact in $L^1(\Omega)$ if and only if*

$$\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m\setminus\mathbb{R}^m} v_0(s)d\lambda_x(s)d\pi(x) = 0. \tag{2.9}$$

Proof. We follow the proof of [33], Lemma 3.2.14(i). Suppose first that (2.9) holds. For $\varrho \ge 0$ define the function $\xi^\varrho : \mathbb{R}^m \to \mathbb{R}$

$$\xi^\varrho(s) := \begin{cases} 0 & \text{if } |s| \le \varrho, \\ |s|-\varrho & \text{if } \varrho \le |s| \le \varrho+1, \\ 1 & \text{if } |s| \ge \varrho+1. \end{cases}$$

Note that always $\xi^\varrho \in \mathcal{R}$, hence $\xi^\varrho v_0 \in \mathcal{R}$ because $\mathcal{R}$ is closed under multiplication. We have due to the Lebesgue Dominated Convergence Theorem

$$\lim_{\varrho\to\infty}\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m\setminus B(0,\varrho)} v_0(s)d\lambda_x(s)d\pi(x) = \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m\setminus\mathbb{R}^m} v_0(s)d\lambda_x(s)d\pi(x) = 0.$$

Let $\varepsilon > 0$ and ϱ be large enough so that

$$\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m} \xi^\varrho(s)v_0(s)d\lambda_x(s)d\pi(x) \le \int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m\setminus B(0,\varrho)} v_0(s)\lambda_x(s)d\pi(x) \le \frac{\varepsilon}{2},$$

and choose $k_\varrho \in \mathbb{N}$ such that, if $k \ge k_\varrho$, then

$$\left|\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m} \xi^\varrho(s)v_0(s)d\lambda_x(s)d\pi(x) - \int_\Omega \xi^\varrho_0(u_k(x))v(u_k(x))\,\mathrm{d}x\right| \le \frac{\varepsilon}{2}.$$

Therefore, if $k \ge k_\varrho$ then $\int_\Omega \xi^\varrho_0(u_k(x))v(u_k(x))\,\mathrm{d}x \le \varepsilon$, and so

$$\int_{\{x\in\Omega:\ |u_k(x)|\ge\varrho+1\}} v(u_k(x))\,\mathrm{d}x \le \int_\Omega \xi^\varrho_0(u_k(x))v(u_k(x))\,\mathrm{d}x \le \varepsilon.$$

As $0 \le v \le C(1+|\cdot|^p)$ for some $C>0$, we get for $K \ge C(1+(\varrho+1)^p)$ that

$$\int_{\{x\in\Omega:\ |v(u_k(x))|\ge K\}} v(u_k(x))\,\mathrm{d}x \le \int_{\{x\in\Omega:\ |u_k(x)|\ge\varrho+1\}} v(u_k(x))\,\mathrm{d}x \le \varepsilon.$$

Clearly, the finite set $\{v(u_k)\}_{k=1}^{k_\varrho}$ is weakly relatively compact in $L^1(\Omega)$, which means that for $K_0>0$ sufficiently large and $1\le k\le k_\varrho$

$$\int_{\{x\in\Omega:\ |v(u_k(x))|\ge K_0\}} v(u_k(x))\,\mathrm{d}x \le \varepsilon.$$

Hence,

$$\sup_{k\in\mathbb{N}} \int_{\{x\in\Omega:\ |v(u_k(x))|\ge \max(K_0,K)\}} v(u_k(x))\,\mathrm{d}x \le \varepsilon,$$

and $\{v(u_k)\}$ is relatively weakly compact in $L^1(\Omega)$ by the Dunford-Pettis criterion. Consequently, if $\{v(u_k)\}$ is relatively weakly compact in $L^1(\Omega)$, then the limit of a (sub)sequence can be fully described by the Young measure generated by $\{u_k\}$, see *e.g.* [2,30,32]. Formula (2.9) then follows from (A.7). □

Suppose now that $\det\nabla\varphi_k \ge 0$ for all $k\in\mathbb{N}$. Then Lemma 2.5 applied to $v := |\det|$, together with (2.6), implies that (notice that $|\det\nabla\varphi_k| = \det\nabla\varphi_k$, $k\in\mathbb{N}$) if

$$\int_{\partial\Omega}\int_{\beta_{\mathcal{S}}\mathbb{R}^{n\times n}\setminus\mathbb{R}^{n\times n}} \frac{\det\ s}{1+|s|^n}\mathrm{d}\lambda_x(s)\mathrm{d}\pi(x) = 0 \tag{2.10}$$

then $\text{w-}\lim_{k\to\infty}\det\nabla\varphi_k = \det\nabla\varphi$ in $L^1(\Omega)$. On the other hand, if $\text{w-}\lim_{k\to\infty}\det\nabla\varphi_k = \det\nabla\varphi$ in $L^1(\Omega)$ then (2.7) yields (2.10). We proved the following proposition, which is a generalization of Müller's result [29]; *cf.* also [17,20].

Proposition 2.6. *Let $\{\varphi_k\}_{k\in\mathbb{N}} \subset W^{1,n}(\Omega;\mathbb{R}^n)$ be such that w-$\lim_{k\to\infty}\varphi_k = \varphi$ in $W^{1,n}(\Omega;\mathbb{R}^n)$, $\det\nabla\varphi_k \ge 0$ a.e. in Ω for all $k\in\mathbb{N}$, and $\{\nabla\varphi_k\}_{k\in\mathbb{N}}$ generates $(\pi,\lambda)\in\mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$. Then w-$\lim_{k\to\infty}\det\nabla\varphi_k = \det\nabla\varphi$ in $L^1(\Omega)$ if and only if (2.10) holds.*

Condition (2.10) can be ensured, for instance, if $\varphi_k=\varphi$ on $\partial\Omega$ in the sense of traces [18]. The fact that $\text{w-}\lim_{k\to\infty}\det\nabla\varphi_k = \det\nabla\varphi$ in $L^1(\Omega)$ if $\det\nabla\varphi_k\ge 0$ and $\varphi_k=\varphi$ on $\partial\Omega$ was already mentioned in [20], Theorem 4.1. However, (2.10) also holds if $\{\varphi_k\}$ has an extension to $\tilde\Omega\supset\Omega$ such that $\{|\nabla\varphi_k|^p|_{\tilde\Omega\setminus\Omega}\}$ is weakly relatively compact in $L^1(\Omega)$, see *(iii)* in Theorem 2.1.

Corollary 2.7. *Let $\{\varphi_k\}_{k\in\mathbb{N}}\subset W^{1,n}(\Omega;\mathbb{R}^n)$ be such that w-$\lim_{k\to\infty}\varphi_k=\varphi$ in $W^{1,n}(\Omega;\mathbb{R}^n)$, $\varphi_k\in\varphi+W^{1,n}_0(\Omega;\mathbb{R}^n)$, and $\det\nabla\varphi_k(x)\ge 0$ for all $k\in\mathbb{N}$ and a.e. $x\in\Omega$. Then w-$\lim_{k\to\infty}\det\nabla\varphi_k=\det\nabla\varphi$ in $L^1(\Omega)$.*

Removing the assumption $\det\nabla\varphi_k\ge 0$ from Proposition 2.6 substantially weakens the assertion. Its proof follows again from (2.7).

Proposition 2.8. *Let $\{\varphi_k\}_{k\in\mathbb{N}}\subset W^{1,n}(\Omega;\mathbb{R}^n)$ be such that w-$\lim_{k\to\infty}\varphi_k=\varphi$ in $W^{1,n}(\Omega;\mathbb{R}^n)$, and $\{\nabla\varphi_k\}_{k\in\mathbb{N}}$ generates $(\pi,\lambda)\in\mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$. Then w^*-$\lim_{k\to\infty}\det\nabla\varphi_k=\det\nabla\varphi$ in the sense of measures on $\bar\Omega$ if and only if (2.10) holds.*

Remark 2.9. Analogous variants of Propositions 2.6 and 2.8 clearly hold for $\mathcal{A}$-quasiaffine functions, *i.e.*, if v and $-v$ are both $\mathcal{A}$-quasiconvex.

2.2. Biting lemma for $\mathcal{A}$-quasiconvex functions

The next proposition can be seen as a version of the Biting Lemma [6] for $\mathcal{A}$-quasiconvex functions. It generalizes a result from [4]. It is known that if $v\in\Upsilon^p_{\mathcal{S}}$ is $\mathcal{A}$-quasiconvex then the functional I given in (2.3) does not have to be sequentially weakly lower semicontinuous in $L^p(\Omega;\mathbb{R}^m)\cap\ker\mathcal{A}$; *cf.* [3] for a particular example with the determinant. Our next lemma asserts that the weak lower semicontinuity is preserved if we remove (bite) an arbitrarily thin "boundary layer" of Ω.

Lemma 2.10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and such that $0 \in \Omega$. Let $u_k \rightharpoonup u$ weakly in $L^p(\Omega;\mathbb{R}^m) \cap \ker \mathcal{A}$, $1 < p < +\infty$. Let further $0 \le g \in C(\bar\Omega)$ and let $v \in \Upsilon^p_{\mathcal{S}}$ be $\mathcal{A}$-quasiconvex. Then there exists a subsequence of $\{u_k\}$ (not relabeled) and $\{\varepsilon_\ell\}_{\ell\in\mathbb{N}} \subset (0,1]$ such that*

$$\liminf_{k\to\infty} \int_{\varepsilon\Omega} g(x)v(u_k(x))\,\mathrm{d}x \ge \int_{\varepsilon\Omega} g(x)v(u(x))\,\mathrm{d}x, \tag{2.11}$$

if $\varepsilon \notin \{\varepsilon_\ell\}_{\ell\in\mathbb{N}}$ and $\varepsilon\Omega := \{\varepsilon y : \ y \in \Omega\}$.

The proofs of Theorems 2.3, 2.4, and of Lemma 2.10 can be found in Section 5. The next two sections will be devoted to proving Theorems 2.1 and 2.2.

3. Theorems 2.1 and 2.2: Necessary conditions

The following result can be found in [18], Lemma 3.2. It follows by the approximation of the characteristic function by continuous ones.

Lemma 3.1. *Let $(\pi,\lambda) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ and let $\omega \subseteq \Omega$ be an open set such that $\pi(\partial\omega) = 0$. Let $\{u_k\}_{k\in\mathbb{N}}$ generate (π,λ) in the sense of (1.4). Then for all $v_0 \in \mathcal{R}$ and all $g \in C(\bar\Omega)$*

$$\lim_{k\to\infty} \int_\omega v(u_k)g(x)\,\mathrm{d}x = \int_\omega \int_{\beta_{\mathcal{R}}\mathbb{R}^m} v_0(s)d\lambda_x(s)g(x)\,d\pi(x). \tag{3.1}$$

Proposition 3.2. *Let $1 < p < +\infty$, let $\{u_k\} \subset L^p(\Omega;\mathbb{R}^m)$ be such that $\mathcal{A}u_k \to 0$ in $W^{-1,p}(\Omega;\mathbb{R}^d)$, $u_k \rightharpoonup u$ weakly in $L^p(\Omega;\mathbb{R}^m)$. If $\{u_k\}$ generates a DiPerna-Majda measure $(\pi,\lambda) \in \mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$ with π absolutely continuous with respect to the Lebesgue measure, then there is $\{w_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m) \cap \ker \mathcal{A}$ that also generates (π,λ). Moreover, $\int_\Omega (w_k(x) - u(x))\,\mathrm{d}x = 0$ for all $k \in \mathbb{N}$.*

Proof. We follow the proof of [15], Lemma 2.15. After an affine rescaling, we may assume that $\Omega \subset Q$. Clearly $\mathcal{A}u = 0$, and by linearity and Lemma A.5 we may suppose that $u = 0$.

For any $\eta \in C_0^\infty(\Omega)$, $0 \le \eta \le 1$, it follows

$$\mathcal{A}(\eta u_k) = \eta\mathcal{A}u_k + \sum_{i=1}^n u_k A^{(i)} \frac{\partial\eta}{\partial x_i} \to 0 \text{ in } W^{-1,p}(\Omega;\mathbb{R}^d)$$

because $L^p(\Omega;\mathbb{R}^m)$ is compactly embedded into $W^{-1,p}(\Omega;\mathbb{R}^m)$. Take $\eta_k \in C_0^\infty(\Omega)$, $0 \le \eta_k \le 1$ for any $k \in \mathbb{N}$, $\eta_k \to \chi_\Omega$ pointwise everywhere. Define $w_{jk} := \eta_j u_k$, $j,k \in \mathbb{N}$. By Lemma A.7 extract a subsequence of $\{w_{jk}\}_{j,k}$, denoted by $\{w_k\}_{k\in\mathbb{N}}$, that generates (π,λ) with $u_k \rightharpoonup 0$ weakly in $L^p(\Omega;\mathbb{R}^m)$ and $\mathcal{A}w_k \to 0$ in $W^{-1,p}(\Omega;\mathbb{R}^d)$. We extend w_k by zero to $Q \setminus \Omega$, and then periodically to the whole $\mathbb{R}^n$. Define

$$\tilde w_k := \mathbb{T}\left(w_k - \int_Q w_k(x)\,\mathrm{d}x\right).$$

By Lemma 1.1 $\{\tilde w_k\} \subset L^p(\Omega;\mathbb{R}^m) \cap \ker \mathcal{A}$ and we have due to the fact that $\int_Q w_k(x)\,\mathrm{d}x \to 0$ as $k \to \infty$

$$\begin{aligned}\lim_{k\to\infty} \|w_k - \tilde w_k\|_{L^p(\Omega;\mathbb{R}^m)} &= \lim_{k\to\infty} \left\| w_k - \int_Q w_k\,\mathrm{d}x - \mathbb{T}\left(w_k - \int_Q w_k(x)\,\mathrm{d}x\right)\right\|_{L^p_\#(\mathbb{R}^n;\mathbb{R}^m)} \\ &\le \lim_{k\to\infty} C\|\mathcal{A}w_k\|_{W^{-1,p}(Q;\mathbb{R}^d)} = 0.\end{aligned}$$

Therefore, by Lemma A.6 $\{\tilde{u}_k\}_{k\in\mathbb{N}}$ generates the same DiPerna-Majda measure as $\{u_k\}$. Finally, we set $w_k := \tilde{w}_k - \mathcal{L}^n(\Omega)^{-1}\int_\Omega \tilde{w}_k(x)\,\mathrm{d}x$ for any k. □

Proposition 3.3. *Let $1 < p < +\infty$, let $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$ be such that $\mathcal{A}u_k \to 0$ in $W^{-1,p}(\Omega;\mathbb{R}^d)$ as $k\to\infty$, and let $\{u_k\}$ generate $(\pi,\lambda) \in \mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$. Let further $u_k \rightharpoonup u$ weakly in $L^p(\Omega;\mathbb{R}^m)$. Then for almost every $a\in\Omega$ $(\lambda_a, d_\pi(a)\mathcal{L}^n\llcorner\Omega)$ is a DiPerna-Majda measure. Moreover, $(\lambda_a, d_\pi(a)\mathcal{L}^n\llcorner\Omega)$ is generated by a sequence in $L^p_\#(\mathbb{R}^n;\mathbb{R}^m)\cap\ker\mathcal{A}$.*

Proof. We remark that

$$d_\pi(a) = \left(\int_{\mathbb{R}^m} \frac{d\lambda_a(s)}{1+|s|^p}\right)^{-1} \tag{3.2}$$

as follows from (A.1). Define $\gamma := d_\pi(a)\mathcal{L}^n\llcorner\Omega$ and $\mu_x := \lambda_a$ for a.e. $x\in\Omega$. Notice that $(\gamma,\mu)\in\mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$ by Proposition A.1. We proceed as in [32], Theorem 7.2, and apply Lemma 3.1 to any $\omega := a+\varrho Q$ with ϱ small enough and such that $\pi(\partial(a+\varrho Q)) = 0$. Define $\bar{V}_\ell(y) := d_\pi(y)\int_{\beta_{\mathcal{S}}\mathbb{R}^m} v_0^\ell(s)d\lambda_y(s)$ where $\{v_0^\ell\}_{\ell\in\mathbb{N}}$ is a dense subset of $\mathcal{S}$. Consider $a\in\Omega$ a common Lebesgue point of u, d_π, $\bar{V}_\ell$, for any $\ell\in\mathbb{N}$, and such that $\pi_s(\{a\}) = 0$. The set of such points has full Lebesgue measure.

We recall that $\mathrm{w}^*-\lim_{k\to\infty}(1+|u_k|^p) = \pi$, *i.e.*, for any $\xi\in C(\bar{\Omega})$

$$\lim_{k\to\infty}\int_\Omega \xi(x)(1+|u_k(x)|^p)\,\mathrm{d}x = \int_{\bar{\Omega}}\xi(x)\,d\pi(x).$$

Let $\xi_{a,\varrho}\in C_0(\Omega)$ be such that

$$0 \le \chi_{a+\varrho Q}(x) \le \xi_{a,\varrho}(x) \le \chi_{a+2\varrho Q}(x),\;\; x\in\Omega.$$

Then

$$\begin{aligned}
\limsup_{\varrho\to 0}\limsup_{k\to\infty}\varrho^{-n}\int_\Omega(1+|u_k(x)|^p)\chi_{a+\varrho Q}(x)\,\mathrm{d}x &\le \limsup_{\varrho\to 0}\limsup_{k\to\infty}\varrho^{-n}\int_\Omega(1+|u_k(x)|^p)\xi_{a,\varrho}(x)\,\mathrm{d}x\\
&= \limsup_{\varrho\to 0}\varrho^{-n}\int_\Omega \xi_{a,\varrho}(x)\,d\pi(x)\\
&\le \limsup_{\varrho\to 0}\varrho^{-n}\int_\Omega \chi_{a+2\varrho Q}(x)\,d\pi(x) \le C d_\pi(a).
\end{aligned}$$

Hence,

$$\limsup_{\varrho\to 0}\limsup_{k\to\infty}\varrho^{-n}\int_\Omega |u_k(x)|^p\chi_{a+\varrho Q}(x)\,\mathrm{d}x = \limsup_{\varrho\to 0}\limsup_{k\to\infty}\int_\Omega |u_k(a+\varrho x)|^p\,\mathrm{d}x < +\infty.$$

Define

$$u^a_{k,\varrho}(x) := u_k(a+\varrho x),\;\; x\in Q,\;\; \varrho>0.$$

Taking $v\in\Upsilon^p_{\mathcal{S}}$ and $g\in C(\bar{Q})$, we have

$$\int_Q v(u^a_{k,\varrho}(x))g(x)\,\mathrm{d}x = \int_Q v(u_k(a+\varrho x))g(x)\,\mathrm{d}x = \varrho^{-n}\int_\Omega v(u_k(y))\chi_{a+\varrho Q}(y)g\left(\frac{y-a}{\varrho}\right)\mathrm{d}y.$$

Using Lemma 3.1, we get for all $v^\ell := v_0^\ell(1+|\cdot|^p)$ and all $g \in C(\bar{Q})$ that

$$\lim_{k\to\infty}\int_Q v^\ell(u^a_{k,\varrho}(x))g(x)\,\mathrm{d}x = \varrho^{-n}\int_\Omega \bar{V}_\ell(y)\chi_{a+\varrho Q}(y)g\left(\frac{y-a}{\varrho}\right)\,\mathrm{d}y \\ + \varrho^{-n}\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m} v_0^\ell(s)d\lambda_y(s)\chi_{a+\varrho Q}(y)g\left(\frac{y-a}{\varrho}\right)d\pi_s(y). \tag{3.3}$$

Since $\pi_s(\{a\})=0$, we have

$$\limsup_{\varrho\to 0}\varrho^{-n}\int_{\bar\Omega}\int_{\beta_{\mathcal{R}}\mathbb{R}^m}\left|v_0^\ell(s)\,d\lambda_y(s)\chi_{a+\varrho Q}(y)g\left(\frac{y-a}{\varrho}\right)\right|d\pi_s(y) \le \lim_{\varrho\to 0} C\varrho^{-n}\int_{a+\varrho Q} d\pi_s(y) = 0.$$

Thus,

$$\lim_{\varrho\to 0}\lim_{k\to\infty}\int_Q v^\ell(u^a_{k,\varrho}(x))g(x)\,\mathrm{d}x = \lim_{\varrho\to 0}\int_Q \bar{V}_\ell(a+\varrho x)g(x)\,\mathrm{d}x = \bar{V}_\ell(a)\int_Q g(x)\,\mathrm{d}x \\ = \int_Q\int_{\beta_{\mathcal{S}}\mathbb{R}^m} v_0^\ell(s)d\lambda_a(s)g(x)d_\pi(a)\,\mathrm{d}x = \int_Q\int_{\beta_{\mathcal{S}}\mathbb{R}^m} v_0^\ell(s)d\mu_x(s)g(x)\,d\gamma(x).$$

As $\mathcal{S}$ and $C(\bar{Q})$ are separable, we use a diagonalization procedure to find $\{u^a_k\}_{k\in\mathbb{N}}$ such that for any $v \in \Upsilon^p_{\mathcal{S}}$ and any $g \in C(\bar{Q})$

$$\lim_{k\to\infty}\int_Q v(u^a_k(x))g(x)\,\mathrm{d}x = \int_{\bar{Q}}\int_{\beta_{\mathcal{S}}\mathbb{R}^m} v_0(s)d\mu_x(s)g(x)\,d\gamma(x).$$

To modify the sequence such that it belongs to $L^p_\#(\mathbb{R}^n;\mathbb{R}^m)\cap\ker\mathcal{A}$ we follow the proof of Proposition 3.2. $\square$

Lemma 3.4. *Let $(\pi,\lambda)\in\mathcal{ADM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$, $1<p<+\infty$. Then for π-almost every $x\in\Omega$*

$$\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m}\frac{v(s)}{1+|s|^p}\mathrm{d}\lambda_x(s)\ge 0 \tag{3.4}$$

for all positively p-homogeneous $v\in\Upsilon^p_{\mathcal{S}}$ with $Q_{\mathcal{A}}v(0)=0$.

Proof. By rescaling, we can assume that $\Omega\subset Q$. Fix v, $x_0\in\Omega$, a π-Lebesgue point of $x\mapsto\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m}\frac{v(s)}{1+|s|^p}\mathrm{d}\lambda_x(s)$, and $r>0$ such that $B(x_0,r):=\{x\in\Omega\colon |x-x_0|<r\}\subset\Omega$ and $\pi(\partial B(x_0,r))=0$. Suppose further that $\{u_k\}\subset L^p(\Omega;\mathbb{R}^m)\cap\ker\mathcal{A}$ generates $(\pi,\lambda)\in\mathcal{ADM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$ and decompose $u_k=z_k+w_k$ using Lemma 1.2 with $z_k\in L^p(\Omega;\mathbb{R}^m)\cap\ker\mathcal{A}$ and $w_k\to 0$ in measure. By Lemma 3.1 and by (A.7)

$$\lim_{k\to\infty}\int_{B(x_0,r)} v(w_k(x))g(x)\,\mathrm{d}x = \int_{B(x_0,r)}\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m} v_0(s)\mathrm{d}\lambda_x(s)g(x)d\pi(x) \tag{3.5}$$

for all $v\in\Upsilon^p_{\mathcal{S}}$ positively p-homogeneous and all $g\in C(\bar\Omega)$. As in the proof of Lemma A.7, we find a sequence $\{\eta_k\}_{k\in\mathbb{N}}\subset C_0^\infty(B(x_0,r))$, $\eta_k\to\chi_{B(x_0,r)}$, $\eta_k\in[0,1]$ for all $x\in B(x_0,r)$, such that $\{\hat{w}_k\}:=\{\eta_k w_k\}$ still satisfies (3.5). Moreover, by the compact embedding of $L^p(B(x_0,r);\mathbb{R}^m)$ into $W^{-1,p}(B(x_0,r);\mathbb{R}^m)$ and the assumption that $\mathcal{A}w_k=0$, we have that $\mathcal{A}\hat{w}_k\to 0$ in $W^{-1,p}(B(x_0,r);\mathbb{R}^m)$. We extend $\hat{w}_k$ by zero to $Q\setminus B(x_0,r)$ and then periodically to the whole $\mathbb{R}^n$. The extension is still denoted by $\hat{w}_k\in L^p_\#(\mathbb{R}^n;\mathbb{R}^m)$. We define

$$\tilde{w}_k := \mathbb{T}\left(\hat{w}_k - \int_Q \hat{w}_k\right).$$

By Lemma 1.1 $\{\tilde w_k\}\subset L^p_{\#}(\mathbb{R}^n;\mathbb{R}^m)\cap\ker\mathcal{A}$ and we have, due to the fact that $\int_Q \hat w_k\,\mathrm{d}x\to 0$ as $k\to\infty$,

$$\begin{aligned}\lim_{k\to\infty}\|\hat w_k-\tilde w_k\|\|_{L^p_{\#}(\mathbb{R}^n;\mathbb{R}^m)} &= \lim_{k\to\infty}\left\|\hat w_k-\int_Q\hat w_k\,\mathrm{d}x-\mathbb{T}\left(\hat w_k-\int_Q\hat w_k\,\mathrm{d}x\right)\right\|_{L^p_{\#}(\mathbb{R}^n;\mathbb{R}^m)}\\ &\le \lim_{k\to\infty}C\|\mathcal{A}\hat w_k\|_{W^{-1,p}(Q;\mathbb{R}^d)}=0.\end{aligned}$$

Hence, for all $v\in\Upsilon^p_{\mathcal{S}}$, positively p-homogeneous and all $g\in C(\bar\Omega)$ it holds that

$$\lim_{k\to\infty}\int_Q v(\tilde w_k(x))g(x)\,\mathrm{d}x=\int_Q\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m}\frac{v(s)}{1+|s|^p}\,\mathrm{d}\lambda_x(s)g(x)d\pi(x).$$

Suppose that $v\in\Upsilon^p_{\mathcal{S}}$, positively p-homogeneous is such that $Q_{\mathcal{A}}v(0)=0$. By the definition of $\mathcal{A}$-quasiconvexity

$$0\le\lim_{k\to\infty}\int_Q v(\tilde w_k(x))\,\mathrm{d}x=\lim_{k\to\infty}\int_{B(x_0,r)}v(\tilde w_k(x))\,\mathrm{d}x=\int_{B(x_0,r)}\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m}\frac{v(s)}{1+|s|^p}\mathrm{d}\lambda_x(s)d\pi(x),$$

and so

$$0\le\lim_{r\to 0}\frac{1}{\pi(B(x_0,r))}\int_{B(x_0,r)}\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m}\frac{v(s)}{1+|s|^p}\mathrm{d}\lambda_x(s)d\pi(x)=\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m}\frac{v(s)}{1+|s|^p}\,d\lambda_{x_0}(s).$$

Proceeding as in [16], the previous calculation yields the existence of a π-null set $E_v\subset\Omega$ such that

$$0\le\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m}\frac{v(s)}{1+|s|^p}\mathrm{d}\lambda_x(s)$$

if $x\notin E_v$. Let $\{v_0^k\}_{k\in\mathbb{N}}$ be a dense subset of $\mathcal{S}$, so that $\{v^k\}_{k\in\mathbb{N}}=\{v_0^k(1+|\cdot|^p)\}_{k\in\mathbb{N}}\subset\Upsilon^p_{\mathcal{S}}$. We define

$$E:=\bigcup_k\ \bigcup_{\{j\in\mathbb{N};\ Q_{\mathcal{A}}(v_0^k+1/j)(1+|\cdot|^p)(0)=0\}}E_{(v_0^k+1/j)(1+|\cdot|^p)}.$$

Clearly $\pi(E)=0$. Fix $x\in(\Omega\setminus E)$, a positively p-homogeneous $v\in\Upsilon^p_{\mathcal{S}}$ such that $Q_{\mathcal{A}}v(0)=0$, and choose a subsequence (not relabeled) $\{v_0^k\}_{k\in\mathbb{N}}$ such that

$$v_0^k\to v_0\text{ in }C(\beta_{\mathcal{S}}\mathbb{R}^m)\text{ and }\|v_0^k-v_0\|_{C(\beta_{\mathcal{S}}\mathbb{R}^m)}<\frac{1}{k},$$

where $k\to\infty$ if $k\to\infty$. Denote $\hat v^k:=v^k+\frac{1}{k}(1+|\cdot|^p)$. We have

$$\begin{aligned}\hat v^k(s)&\ge v^k(s)+(1+|s|^p)\|v_0^k-v_0\|_{C(\beta_{\mathcal{S}}\mathbb{R}^m)}\\ &\ge v^k(s)+|v_0^k(s)-v_0(s)|(1+|s|^p)\ge v(s).\end{aligned}$$

Finally, as $x\notin E$ then $x\notin E_{(v_0^k+1/k)(1+|\cdot|^p)}$ and

$$0\le\lim_{k\to\infty}\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m}\frac{\hat v^k(s)}{1+|s|^p}\mathrm{d}\lambda_x(s)=\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m}\frac{v(s)}{1+|s|^p}\mathrm{d}\lambda_x(s).\qquad\square$$

Proposition 3.5. *Let $(\pi,\lambda)\in\mathcal{ADM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$, $1<p<+\infty$, be generated by $\{u_k\}_{k\in\mathbb{N}}\subset L^p(\Omega;\mathbb{R}^m)\cap\ker\mathcal{A}$. Then the following conditions are satisfied:*
(i) there exists $u\in L^p(\Omega;\mathbb{R}^m)\cap\ker\mathcal{A}$ such that $u_k\rightharpoonup u$ and for a.e. $x\in\Omega$

$$u(x)=d_\pi(x)\int_{\beta_{\mathcal{S}}\mathbb{R}^m}\frac{s}{1+|s|^p}\mathrm{d}\lambda_x(s); \tag{3.6}$$

and for all $v\in\Upsilon^p_{\mathcal{S}}$

$$Q_{\mathcal{A}}v(u(x))\le d_\pi(x)\int_{\beta_{\mathcal{S}}\mathbb{R}^m}\frac{v(s)}{1+|s|^p}\mathrm{d}\lambda_x(s), \tag{3.7}$$

for almost all $x\in\Omega$;
(ii) for all $v\in\Upsilon^p_{\mathcal{S}}$ such that $Q_{\mathcal{A}}v(0)=0$

$$0\le\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m}\frac{v(s)}{1+|s|^p}\mathrm{d}\lambda_x(s) \tag{3.8}$$

for π-a.e. $x\in\Omega$. Moreover, if $\{u_k\}$ has an $\mathcal{A}$-free p-equiintegrable extension then (3.8) holds for π-a.e. $x\in\bar\Omega$ and $u\in L^p(\mathbb{R}^n;\mathbb{R}^m)\cap\ker\mathcal{A}$.

Proof. Using (1.4) with $v_0(s)=s_i/(1+|s|^p)$ for $i=1,\dots,m$ and $g\in C(\bar\Omega)$ shows that (3.6) is the expression of the weak limit of $\{u_k\}$, u, in terms of DiPerna-Majda measures. Clearly, $\mathcal{A}u=0$ because $u_k\rightharpoonup u$ and $u_k\in\ker\mathcal{A}$. In order to prove (3.7) we use Lemma 3.3 and consider for almost all $a\in\Omega$ a sequence $\{u^a_k\}_{x\in\Omega}\subset L^p_\#(\mathbb{R}^n;\mathbb{R}^m)\cap\ker\mathcal{A}$ generating $(d_\pi(a)\mathrm{d}x,\lambda_a)\in\mathcal{ADM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$, and converging weakly to $u(a)$. We define for all $k\in\mathbb{N}$

$$\tilde u^a_k(x):=u^a_k(x)+\int_Q(u(a)-u^a_k(x))\,\mathrm{d}x.$$

Notice that $\int_Q\tilde u^a_k(x)\,\mathrm{d}x=u(a)$ and that $\|u^a_k-\tilde u^a_k\|_{L^p_\#(\mathbb{R}^n;\mathbb{R}^m)}\to0$ as $k\to\infty$, and therefore $\{\tilde u^a_k\}_{k\in\mathbb{N}}$ also generates $(d_\pi(a)\mathcal{L}^n\llcorner\Omega,\lambda_a)$. Then we have by (1.4) and the definition of $\mathcal{A}$-quasiconvexity for any $v\in\Upsilon^p_{\mathcal{S}}$

$$Q_{\mathcal{A}}v(u(a))\le\lim_{k\to\infty}\int_Q v(\tilde u^a_k(x))\,\mathrm{d}x=d_\pi(a)\int_{\beta_{\mathcal{S}}\mathbb{R}^m}\frac{v(s)}{1+|s|^p}\,d\lambda_a(s),$$

which proves (3.7). Finally, (3.8) follows from Lemma 3.4.

Assume now that $\{u_k\}$ has an $\mathcal{A}$-free p-equiintegrable extension $\{\tilde u_k\}_{k\in\mathbb{N}}$ with $\tilde u_k\rightharpoonup\tilde u$ weakly in $L^p(\mathbb{R}^n;\mathbb{R}^m)$, $\tilde u\in\ker\mathcal{A}$, and $\tilde u=u$ a.e. in Ω.

Let $\tilde\Omega$ be an arbitrary bounded domain such that $\overline{\Omega}\subset\tilde\Omega$, and consider $v\in\Upsilon^p_{\mathcal{S}}$ and $g\in C(\overline{\tilde\Omega})$, write

$$\int_{\tilde\Omega}v(\tilde u_k(x))g(x)\mathrm{d}x=\int_{\tilde\Omega\setminus\Omega}v(\tilde u(x))g(x)\mathrm{d}x+\int_\Omega v(u_k(x))g(x)\mathrm{d}x.$$

Suppose that $\{\tilde u_k\}_{k\in\mathbb{N}}$ restricted to $\tilde\Omega\setminus\bar\Omega$ generates a DiPerna-Majda measure $(\gamma,\mu)\in\mathcal{DM}^p_{\mathcal{S}}(\tilde\Omega\setminus\bar\Omega;\mathbb{R}^m)$. Since $\{|\tilde u_k|^p\}$ is weakly relatively compact in $L^1(\tilde\Omega\setminus\bar\Omega)$ we have that $\gamma(\partial\tilde\Omega\cup\partial\Omega)=0$, see Lemma 2.5. Altogether, $\{\tilde u_k\}$ generates a DiPerna-Majda measure $(\tilde\pi,\tilde\lambda)$ on $\tilde\Omega$ such that

$$\tilde\pi=\begin{cases}\gamma & \text{in}\quad \tilde\Omega\setminus\overline{\Omega}\\ \pi & \text{in}\quad \overline{\Omega},\end{cases}\qquad \tilde\lambda_x=\begin{cases}\mu_x & \text{if}\quad x\in\tilde\Omega\setminus\overline{\Omega}\\ \lambda_x & \text{if}\quad x\in\bar\Omega.\end{cases}$$

Using Lemma 3.4 applied to $(\tilde\pi,\tilde\lambda)$ that (3.4) holds true for $\tilde\pi$-almost all $x\in\tilde\Omega$. In particular, it holds true for π-almost every $x\in\overline{\Omega}$. $\square$

4. Theorems 2.1 and 2.2: Sufficient conditions

We will follow [15]. Let us take $\lambda \in \mathcal{P}(\beta_{\mathcal{S}}\mathbb{R}^m)$ such that $\lambda(\mathbb{R}^m) > 0$, and

$$0 = \int_{\beta_{\mathcal{S}}\mathbb{R}^m} \frac{s}{1+|s|^p} d\lambda(s). \tag{4.1}$$

Define

$$d_\pi := \left(\int_{\beta_{\mathcal{S}}\mathbb{R}^m} \frac{d\lambda(s)}{1+|s|^p} \right)^{-1}. \tag{4.2}$$

Consider a set of DiPerna-Majda measures $\eta \cong (\pi, \lambda)$ defined for all $g \in C(\overline{\Omega})$ and all $v_0 \in \mathcal{S}$ by

$$\langle \eta, g \otimes v_0 \rangle := \int_{\bar{\Omega}\times\beta_{\mathcal{S}}\mathbb{R}^m} v_0(s) g(x) d\lambda(s)\, d\pi(x), \tag{4.3}$$

where π is absolutely continuous with respect to the Lebesgue measure with the density d_π. Here we used the fact that the linear hull of $\{g \otimes v_0;\ v_0 \in \mathcal{S},\ g \in C(\bar{\Omega})\}$ is dense in $C(\bar{\Omega} \times \beta_{\mathcal{S}}\mathbb{R}^m)$. We denote by $\mathbb{H}$ the set of DiPerna-Majda measures of the form (4.3) with the first moment zero, *i.e.* (4.1) holds, and generated by p-equiintegrable sequences in $L^p_\#(\mathbb{R}^n;\mathbb{R}^m) \cap \ker \mathcal{A}$.

There is an obvious one-to-one mapping from $\mathbb{H}$ to the Young measures in $\mathcal{Y}^p(Q;\mathbb{R}^m)$ generated by p-integrable sequences in $L^p_\#(\mathbb{R}^n;\mathbb{R}^m) \cap \ker \mathcal{A}$; *cf.* (A.6). This is clear because any such sequence generates both a DiPerna-Majda measure as well as a Young measure. Let us denote by $\mathbb{Y}$ the set of homogeneous Young measures from $\mathcal{Y}^p(Q;\mathbb{R}^m)$ generated by p-integrable sequences in $L^p_\#(\mathbb{R}^n;\mathbb{R}^m) \cap \ker \mathcal{A}$, and define

$$E_p := \left\{ v \in C(\mathbb{R}^m)\colon \lim_{|s|\to\infty} \frac{v(s)}{1+|s|^p} \in \mathbb{R} \right\}.$$

It is well-known that E_p is a separable ring corresponding to a one-point compactification of $\mathbb{R}^m$. The dual space of E_p, E_p', can thus be identified with $\mathcal{M}(\beta_{E_p}\mathbb{R}^m)$.

Lemma 4.1. $\mathbb{H}$ *is convex.*

Proof. We first show that $\mathbb{H}$ is convex. We follow [15], Proof of Proposition 4.2. Let $\{u_k\}, \{\tilde{u}_k\} \subset L^p_\#(\mathbb{R}^n;\mathbb{R}^m) \cap \ker \mathcal{A}$ be p-equiintegrable and generating $\eta, \tilde{\eta} \in \mathbb{H}$ and Young measures $\nu, \tilde{\nu} \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$, respectively. There is a one-to-one correspondence between η and ν and $\tilde{\eta}$ and $\tilde{\nu}$; *cf.* (A.6).

By mollification we may suppose that $\{u_k\}, \{\tilde{u}_k\} \subset C^\infty(Q;\mathbb{R}^m)$, and because $\{u_k\}, \{\tilde{u}_k\}$ converge weakly to 0 we may suppose that $\int_Q u_k(x)\,\mathrm{d}x = \int_Q \tilde{u}_k(x)\,\mathrm{d}x = 0$. Fix $\theta \in (0,1)$. As $\{u_k\}$ and $\{\tilde{u}_k\}$ converge strongly to zero in $W^{-1,p}(Q;\mathbb{R}^m)$ we have for every $\xi \in C_0^\infty((0,\theta) \times Q_{n-1})$ with $Q_{n-1} := (-1/2, 1/2)^{n-1}$ that

$$\|\mathcal{A}(\xi(u_k - \tilde{u}_k))\|_{W^{-1,p}(Q;\mathbb{R}^d)} = \left\| \sum_{i=1}^n \frac{\partial \xi}{\partial x_i} A^{(i)}(u_k - \tilde{u}_k) \right\|_{W^{-1,p}(Q;\mathbb{R}^d)} \to 0.$$

Hence, we may find a sequence $\{\varphi_k\} \subset C_0^\infty((0,\theta) \times Q_{n-1})$, $\varphi_k \to \chi_{(0,\theta)\times Q_{n-1}}$ pointwise, such that

$$\|\mathcal{A}(\varphi_k(u_k - \tilde{u}_k))\|_{W^{-1,p}(Q;\mathbb{R}^d)} \to 0.$$

We define

$$w_k =: u_k + \mathbb{T}\left(\varphi_k(\tilde{u}_k - u_k) - \int_Q \varphi_k(x)(\tilde{u}_k(x) - u_k(x)) \right) \mathrm{d}x.$$

Then $\{w_k\}\subset L^p_\#(\mathbb{R}^n;\mathbb{R}^m)\cap\ker\mathcal{A}$, $\int_Q \varphi_k(x)(\tilde u_k(x)-u_k(x))\,\mathrm{d}x\to 0$, and by properties of $\mathbb{T}$ it holds

$$w_k = u_k + \varphi_k(\tilde u_k - u_k) + h_k,$$

where $h_k\to 0$ in $L^p_\#(\mathbb{R}^n;\mathbb{R}^m)$. In particular, $\{w_k\}$ is p-equiintegrable and generates a Young measure $\{\mu_x\}_{x\in Q}$ such that $\mu_x=\nu_x$ if $x_1\in(0,\theta)$ and $\mu_x=\tilde\nu_x$ if $x_1\in(\theta,1)$. Finally, we set $\bar w_{k,j}:=w_k(jx)$ for $j\in\mathbb{N}$. Then $\{\bar w_{k,j}\}\subset C^\infty(Q;\mathbb{R}^m)\cap\ker\mathcal{A}$, $\{\bar w_{k,j}\}$ is bounded in $L^p_\#(\mathbb{R}^n;\mathbb{R}^m)$, and it is equiintegrable for every $j\in\mathbb{N}$. Hence for any $v\in\Upsilon^p_{\mathcal{S}}$ and any $g\in C(\overline{Q})$

$$\begin{aligned}\lim_{k\to\infty}\lim_{j\to\infty}\int_Q g(x)v(\bar w_{k,j}(x))\,\mathrm{d}x &= \lim_{k\to\infty}\int_Q g(x)\left(\int_Q v(w_k(y))\,\mathrm{d}y\right)\mathrm{d}x\\ &= \int_Q g(x)\,\mathrm{d}x\left(\theta\int_{\mathbb{R}^m} v(s)d\tilde\nu(s)+(1-\theta)\int_{\mathbb{R}^m} d\nu(s)\right)\\ &= \theta\,\langle\tilde\eta,g\otimes v\rangle+(1-\theta)\,\langle\eta,g\otimes v\rangle\cdot\end{aligned}$$

As $\mathcal{S}$ and $C((\overline{Q})$ are separable we diagonalize to find a sequence $\{\bar w_{k,j(k)}\}\subset C^\infty(Q;\mathbb{R}^m)\cap\ker\mathcal{A}$ generating $\theta\,\langle\tilde\eta,g\otimes v\rangle+(1-\theta)\,\langle\eta,g\otimes v\rangle$, *i.e.*, $\mathbb{H}$ is convex. □

Lemma 4.2. $\mathbb{H}$ *is closed.*

Proof. We follow [15], p. 1385. We show that $\mathbb{Y}$ is closed in the weak* topology of E'_p. Suppose that $\nu\in\overline{\mathbb{Y}}$. Let $\{f_i\}_{i\in\mathbb{N}}\subset C^\infty(Q)$ be dense in $L^1(Q)$ and $\{g_j\}_{j\in\mathbb{N}}$ be dense in $C_0(\mathbb{R}^m)$. Moreover, we take $f=1$ and $g_0(s)=|s|^p$ for any $s\in\mathbb{R}^m$. By the definition of the weak* topology in E^*_p there is $\nu_k\in\mathbb{Y}$ such that

$$|\,\langle\nu_k-\nu,g_j\rangle\,|\le\frac{1}{2k},\ j=0,\dots,k;$$

hence by the Fundamental Theorem of Young measures [2] we can find $w_k\in L^p_\#(\mathbb{R}^n;\mathbb{R}^m)$ such that

$$\left|\langle\nu,g_j\rangle\int_Q f_i(x)\,\mathrm{d}x-\int_Q f_i(x)g_j(w_k(x))\,\mathrm{d}x\right|<\frac{1}{k},\quad 0\le i,j\le k.$$

Taking $i=j=0$ in the above formula we get that $\{w_k\}$ is bounded in $L^p_\#(\mathbb{R}^n;\mathbb{R}^m)$ and it generates a Young measure in $\mathcal{Y}^p(Q;\mathbb{R}^m)$. Clearly, this Young measure is ν. Again, setting $i=j=0$ yields

$$\|w_k\|^p_{L^p_\#(\mathbb{R}^n;\mathbb{R}^m)}\to\langle\nu,|\cdot|^p\rangle\,,$$

as $k\to\infty$. Hence, $\{w_k\}$ is p-equiintegrable. Therefore, $\nu\in\mathbb{Y}$. Correspondingly, $\mathbb{H}$ is closed. □

Take $u\in L^p_\#(\mathbb{R}^n;\mathbb{R}^m)\cap\ker\mathcal{A}$, $\int_Q u(x)\,\mathrm{d}x=0$. It is well-known [3] that the sequence $\{u_k\}_{k\in\mathbb{N}}$ with $u_k(x)=u(kx)$, $x\in Q$, $k\in\mathbb{N}$, generates the homogeneous Young measure $\overline{\delta_u}$ given, for any $v\in C_p(\mathbb{R}^m)$, by

$$\int_{\mathbb{R}^m} v(s)d\overline{\delta_u}(s):=\int_Q v(u(x))\,\mathrm{d}x.$$

We can embed $\overline{\delta_u}$ in $\mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$ as follows. Define $\eta_u\cong(\pi,\vartheta)\in\mathbb{H}$ where for any $v\in\Upsilon^p_{\mathcal{S}}$

$$\int_{\beta_{\mathcal{S}}\mathbb{R}^m} v_0(s)\,d\vartheta(s):=d_\pi^{-1}\int_Q v(u(x))\,\mathrm{d}x \tag{4.4}$$

and

$$d_\pi := \int_Q (1+|u(x)|^p)\,\mathrm{d}x, \tag{4.5}$$

where d_π is the density with respect to the Lebesgue measure of the absolutely continuous measure $\pi \in \mathcal{M}(\bar\Omega)$.

Lemma 4.3. *Let $1<p<+\infty$ and let $(\pi,\lambda)\in\mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$ be such that λ is homogeneous,* i.e., *$\lambda_x=\lambda_y$ for all $x,y\in\Omega$, and π is absolutely continuous with respect to the Lebesgue measure with the constant density*

$$d_\pi = \left(\int_{\mathbb{R}^m}\frac{d\lambda(s)}{1+|s|^p}\right)^{-1},$$

such that

$$\int_{\beta_{\mathcal{S}}\mathbb{R}^m}\frac{s}{1+|s|^p}d\lambda(s)=0$$

and for any $v\in\Upsilon^p_{\mathcal{S}}$

$$Q_{\mathcal{A}}v(0)\le d_\pi\int_{\beta_{\mathcal{S}}\mathbb{R}^m}\frac{v(s)}{1+|s|^p}d\lambda(s). \tag{4.6}$$

Then $(\pi,\lambda)\in\mathcal{ADM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$.

Proof. We define $\xi\in\mathcal{M}(\beta_{\mathcal{S}}\mathbb{R}^m)$ by

$$\langle\xi,v_0\rangle := d_\pi\int_{\beta_{\mathcal{S}}\mathbb{R}^m}v_0(s)d\lambda(s),$$

where $v_0\in\mathcal{S}$. By (4.6)

$$\langle\xi,v_0\rangle\ge Q_{\mathcal{A}}v(0). \tag{4.7}$$

We will use the Hahn-Banach Theorem to prove that ξ cannot be separated from $\mathbb{H}$ in the weak* topology by an element of $C(\beta_{\mathcal{S}}\mathbb{R}^m)$. Suppose that ξ does not belong to $\mathbb{H}$. Then it does not belong to $\overline{\mathrm{co}(\mathbb{H})}$ by Lemmas 4.1 and 4.2 and there is $v_0\in\mathcal{S}$ and $\alpha\in\mathbb{R}$ such that $\langle\mu,v_0\rangle\ge\alpha$ for all $\mu\in\mathbb{H}$ and $\langle\xi,v_0\rangle<\alpha$, *i.e.*, by (4.7) $Q_{\mathcal{A}}v(0)<\alpha$. Consider $u\in L^p_\#(\mathbb{R}^n;\mathbb{R}^m)\cap\ker\mathcal{A}$ and η_u defined as in (4.4) and (4.5). Then we have that $\langle\eta_u,1\otimes v_0\rangle=\int_Q v(u(x))\,\mathrm{d}x\ge\alpha$, hence $Q_{\mathcal{A}}v(0)\ge\alpha$, and we reached a contradiction. Therefore, $\xi\in\mathbb{H}$. □

Lemma 4.4 (see [32], Lem. 7.9 for a more general case)**.** *Let $\Omega\subset\mathbb{R}^n$ be an open domain with $|\partial\Omega|=0$, and let $N\subset\Omega$ be of the zero Lebesgue measure. For $r_k:\Omega\setminus N\to(0,+\infty)$ and $\{f_k\}_{k\in\mathbb{N}}\subset L^1(\Omega)$ there exists a set of points $\{a_{ik}\}\subset\Omega\setminus N$ and positive numbers $\{\epsilon_{ik}\}$, $\epsilon_{ik}\le r_k(a_{ik})$ such that $\{a_{ik}+\epsilon_{ik}\bar\Omega\}$ are pairwise disjoint for each $k\in\mathbb{N}$, $\bar\Omega=\cup_i\{a_{ik}+\epsilon_{ik}\bar\Omega\}\cup N_k$ with $\mathcal{L}^n(N_k)=0$, and for any $j\in\mathbb{N}$ and any $g\in L^\infty(\Omega)$*

$$\lim_{k\to\infty}\sum_i f_j(a_{ik})\int_{a_{ik}+\epsilon_{ik}\Omega}g(x)\,\mathrm{d}x=\int_\Omega f_j(x)g(x)\,\mathrm{d}x.$$

Proposition 4.5. *Let $(\pi,\lambda)\in\mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$, $1<p<+\infty$, be such that π is absolutely continuous with respect to the Lebesgue measure and let d_π be its density. Set for almost every $x\in\Omega$*

$$u(x):=d_\pi(x)\int_{\mathbb{R}^m}\frac{s}{1+|s|^p}d\lambda_x(s). \tag{4.8}$$

If $u \in L^p(\mathbb{R}^n;\mathbb{R}^m)\cap\ker\mathcal{A}$ and if for all $v\in\Upsilon^p_{\mathcal{S}}$ and for almost every $x\in\Omega$

$$Q_{\mathcal{A}}v(u(x)) \le d_\pi(x)\int_{\beta_{\mathcal{S}}\mathbb{R}^m}\frac{v(s)}{1+|s|^p}d\lambda_x(s), \tag{4.9}$$

then $(\pi,\lambda)\in\mathcal{ADM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$. Moreover, its generating sequence can be chosen to be $\mathcal{A}$-free with a p-equiintegrable extension.

Proof. Using a rescaling argument, we may assume that $\Omega\subset Q$.

(i) Suppose first that u in (4.8) is zero. We are looking for a sequence $\{u_k\}_{k\in\mathbb{N}}\subset L^p(\Omega;\mathbb{R}^m)\cap\ker\mathcal{A}$ satisfying

$$\lim_{k\to\infty}\int_\Omega v(u_k(x))g(x)\,\mathrm{d}x = \int_{\bar\Omega}\int_{\beta_{\mathcal{S}}\mathbb{R}^m}\frac{v(s)}{1+|s|^p}\mathrm{d}\lambda_x(s)g(x)d\pi(x)$$

for all $g\in\Gamma$ and all $v=v_0(1+|\cdot|^p)$, $v_0\in\Sigma$, where Γ and Σ are countable dense subsets of $C(\bar\Omega)$ and $\mathcal{S}$, respectively.

Take $r_k:=1/k$ and, using Lemma 4.4, find $a_{ik}\in\Omega\setminus N$, $\epsilon_{ik}\le 1/k$, such that for $v_0\in\Sigma$ and $g\in C(\bar\Omega)$

$$\lim_{k\to\infty}\sum_i\bar V(a_{ik})\int_{a_{ik}+\epsilon_{ik}\Omega}g(x)\,\mathrm{d}x = \int_\Omega\bar V(x)g(x)\,\mathrm{d}x, \tag{4.10}$$

and

$$\lim_{k\to\infty}\sum_i|\bar V(a_{ik})|\int_{a_{ik}+\epsilon_{ik}\Omega}g(x)\,\mathrm{d}x = \int_\Omega|\bar V(x)|g(x)\,\mathrm{d}x, \tag{4.11}$$

where

$$\bar V(x) := d_\pi(x)\int_{\beta_{\mathcal{S}}\mathbb{R}^m}v_0(s)\mathrm{d}\lambda_x(s).$$

Notice that $\bar\Omega=\cup_i\{a_{ik}+\epsilon_{ik}\bar\Omega\}\cup N_k$ with $\mathcal{L}^n(N_k)=0$. By (4.9) and by Lemma 4.3, we can assume that $(d_\pi(a_{ik})\mathcal{L}^n\llcorner\Omega,\lambda_{a_{ik}})$ is a homogeneous $\mathcal{A}$-free DiPerna-Majda measure in $\mathcal{ADM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$. Let $\{u^{ik}_j\}_{j\in\mathbb{N}}\subset L^p_{\#}(\mathbb{R}^n;\mathbb{R}^m)\cap\ker\mathcal{A}$ be a generating sequence. Recall that $u=0$, so $\text{w-}\lim_{j\to\infty}u^{ik}_j=0$ in $L^p_{\#}(\mathbb{R}^n;\mathbb{R}^m)$, and for all $g\in C(\bar\Omega)$ and all $v\in\Upsilon^p_{\mathcal{S}}$,

$$\lim_{j\to\infty}\int_\Omega v(u^{ik}_j(x))g(x)\,\mathrm{d}x = \bar V(a_{ik})\int_\Omega g(x)\,\mathrm{d}x. \tag{4.12}$$

We define a sequence of smooth cut-off functions $\{\eta_\ell\}_{\ell\in\mathbb{N}}\subset C^\infty_0(\Omega)$, $0\le\eta_\ell\le1$, such that $\eta_\ell(x)=1$ if $x\in\Omega_\ell:=\{x\in\Omega\colon \operatorname{dist}(x,\partial\Omega)>\ell^{-1}\}$ and $|\nabla\eta_\ell|\le C\ell$ for some $C>0$. Define

$$u^\ell_k(x) := \begin{cases}\eta_\ell\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right)u^{ik}_j\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right) & \text{if } x\in a_{ik}+\epsilon_{ik}\Omega,\\ 0 & \text{otherwise.}\end{cases}$$

Let $\Gamma\times\Sigma=\cup_k E_k$, with $E_k\subset E_{k+1}$, finite sets. For k,i,ℓ fixed, take $j=j(k,i,\ell)$ so large that for all $(g,v_0)\in E_k$

$$\left|\epsilon^n_{ik}\int_\Omega g(a_{ik}+\epsilon_{ik}y)v(u^{ik}_j(y))\,\mathrm{d}y - \bar V(a_{ik})\int_{a_{ik}+\epsilon_{ik}\Omega}g(x)\,\mathrm{d}x\right| \le \frac{1}{2^ik} \tag{4.13}$$

and

$$\left|\epsilon^n_{ik}\int_{\Omega\setminus\Omega_\ell} g(a_{ik}+\epsilon_{ik}y)v(u^{ik}_j(y))\,\mathrm{d}y - \epsilon^n_{ik}\bar V(a_{ik})\int_{\Omega\setminus\Omega_\ell}g(a_{ik}+\epsilon_{ik}y)\,\mathrm{d}y\right| \le \frac{1}{2^ik}\cdot \tag{4.14}$$

Here we used (4.12) written for $\tilde{g}(z) := g(a_{ik} + \epsilon_{ik}z)$ instead of g. Using this estimate and (4.10), we get for any $(g, v_0) \in \Gamma \times \Sigma$

$$\int_\Omega g(x)v(u_k^\ell(x))\,\mathrm{d}x = \sum_i \epsilon_{ik}^n \int_\Omega g(a_{ik}+\epsilon_{ik}y)v(u_j^{ik}(y))\,\mathrm{d}y - \sum_i \epsilon_{ik}^n \int_{\Omega\setminus\Omega_\ell} g(a_{ik}+\epsilon_{ik}y)v(u_j^{ik}(y))\,\mathrm{d}y$$
$$+ \sum_i \epsilon_{ik}^n \int_{\Omega\setminus\Omega_\ell} g(a_{ik}+\epsilon_{ik}y)v(u_k^\ell(a_{ik}+\epsilon_{ik}y))\,\mathrm{d}y =: T_{k\ell}^1 - T_{k\ell}^2 + T_{k\ell}^3.$$

As $T_{k\ell}^1$ is independent of ℓ, (4.13) yields

$$\begin{aligned}\lim_{\ell\to\infty}\lim_{k\to\infty} T_{k\ell}^1 &= \lim_{k\to\infty}\sum_i \bar{V}(a_{ik})\int_{a_{ik}+\epsilon_{ik}\Omega} g(x)\,\mathrm{d}x = \int_\Omega \bar{V}(x)g(x)\,\mathrm{d}x \\ &= \int_\Omega\int_{\beta_{\mathcal{S}}\mathbb{R}^m} v_0(s)\mathrm{d}\lambda_x(s)g(x)\,d\pi(x).\end{aligned}$$

Applying (4.11) with $g = 1$, yields

$$\lim_{k\to\infty}\sum_i |\bar{V}(a_{ik})|\epsilon_{ik}^n\mathcal{L}^n(\Omega) = \int_\Omega |\bar{V}(x)|\,\mathrm{d}x.$$

Therefore, we have due to (4.14)

$$\begin{aligned}\lim_{\ell\to\infty}\lim_{k\to\infty}|T_{k\ell}^2| &= \lim_{\ell\to\infty}\lim_{k\to\infty}\left|\sum_i \epsilon_{ik}^n\bar{V}(a_{ik})\int_{\Omega\setminus\Omega_\ell} g(a_{ik}+\epsilon_{ik}y)\,\mathrm{d}y\right| \\ &\le \lim_{\ell\to\infty}\lim_{k\to\infty}\|g\|_{C(\bar{\Omega})}\frac{\mathcal{L}^n(\Omega\setminus\Omega_\ell)}{\mathcal{L}^n(\Omega)}\sum_i \epsilon_{ik}^n\mathcal{L}^n(\Omega)|\bar{V}(a_{ik})| \\ &= \lim_{\ell\to\infty}\frac{\mathcal{L}^n(\Omega\setminus\Omega_\ell)}{\mathcal{L}^n(\Omega)}\|g\|_{C(\bar{\Omega})}\int_\Omega |\bar{V}(x)|\,\mathrm{d}x = 0\end{aligned} \tag{4.15}$$

because $\mathcal{L}^n(\Omega\setminus\Omega_\ell) \to 0$. We show that also $\lim_{\ell\to\infty}\lim_{k\to\infty} T_{k\ell}^3 = 0$. Indeed,

$$\begin{aligned}\left|\sum_i \epsilon_{ik}^n\int_{\Omega\setminus\Omega_\ell} g(a_{ik}+\epsilon_{ik}y)v(u_k^\ell(a_{ik}+\epsilon_{ik}y))\,\mathrm{d}y\right| &\le C\sum_i \epsilon_{ik}^n\int_{\Omega\setminus\Omega_\ell}(1+|\eta_\ell u_j^{ik}(y)|^p)\,\mathrm{d}y \\ &\le C\sum_i \epsilon_{ik}^n\int_{\Omega\setminus\Omega_\ell}(1+|u_j^{ik}(y)|^p)\,\mathrm{d}y =: J_{kl}.\end{aligned}$$

But $\lim_{\ell\to\infty}\lim_{k\to\infty} J_{kl} = 0$ because it is (4.15) written for $v_0 = 1$. Altogether, we have

$$\lim_{\ell\to\infty}\lim_{k\to\infty}\int_\Omega g(x)v(u_k^\ell(x))\,\mathrm{d}x = \int_\Omega\int_{\beta_{\mathcal{S}}\mathbb{R}^m} v_0(s)\mathrm{d}\lambda_x(s)g(x)\,d\pi(x). \tag{4.16}$$

Further, for $\phi \in W_0^{1,p'}(\Omega;\mathbb{R}^d)$, $\|\nabla\phi\|_{L^{p'}(\Omega;\mathbb{R}^{d\times n})} \le 1$, we write

$$\phi_{ik}(y) := \epsilon_{ik}^{n-1}\phi(a_{ik}+\epsilon_{ik}y) - |\Omega|^{-1}\int_\Omega \epsilon_{ik}^{n-1}\phi(a_{ik}+\epsilon_{ik}y)\,\mathrm{d}y.$$

In view of the Poincaré inequality $\{\phi_{ik}\}_{i,k}$ is uniformly bounded in $W^{1,p'}(\Omega;\mathbb{R}^m)$. Notice that

$$\|\nabla\phi_{ik}\|_{L^{p'}(\Omega;\mathbb{R}^{d\times n})} = \left(\int_{a_{ik}+\epsilon_{ik}\Omega} |\nabla\phi(x)|^{p'}\,\mathrm{d}x\right)^{1/p'} \leq 1.$$

Hence,

$$\begin{aligned}
\int_\Omega \sum_{l=1}^n A^{(l)} u_k^\ell(x)\frac{\partial\phi}{\partial x_l}\,\mathrm{d}x &= \sum_i \int_{a_{ik}+\epsilon_{ik}\Omega} \sum_{l=1}^n A^{(l)}\frac{\partial\phi(x)}{\partial x_l}\eta_\ell\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right) u_j^{ik}\left(\frac{x-a_{ik}}{\epsilon_{ik}}\right)\mathrm{d}x\\
&= \sum_i \epsilon_{ik}^n \int_\Omega \eta_\ell(y)u_k^\ell(y)\sum_{l=1}^n A^{(l)}\frac{\partial\phi(a_{ik}+\epsilon_{ik}y)}{\partial y_l}\,\mathrm{d}y\\
&= \sum_i \int_\Omega \eta_\ell(y)u_j^{ik}(y)\sum_{l=1}^n A^{(l)}\frac{\partial\phi_{ik}(y)}{\partial y_l}\,\mathrm{d}y \qquad (4.17)\\
&= \sum_i \int_\Omega u_j^{ik}(y)\sum_{l=1}^n A^{(l)}\frac{\partial(\phi_{ik}(y)\eta_\ell(y))}{\partial y_l}\,\mathrm{d}y\\
&\quad - \sum_i \int_\Omega u_j^{ik}(y)\sum_{l=1}^n A^{(l)}\phi_{ik}(y)\frac{\partial(\eta_\ell(y))}{\partial y_l}\,\mathrm{d}y.
\end{aligned}$$

On the other hand, $\int_\Omega u_j^{ik}(y)\sum_{l=1}^n A^{(l)}\frac{\partial(\phi_{ik}(y)\eta_\ell(y))}{\partial y_l}\,\mathrm{d}y = 0$ for all ℓ,i,k,j because $u_j^{ik}\in\ker\mathcal{A}$. Moreover, $u_j^{ik}\sum_{l=1}^n A^{(l)}\frac{\partial(\eta_\ell(y))}{\partial y_l} \rightharpoonup 0$ weakly in $L^p(\Omega;\mathbb{R}^m)$ (and strongly in $W^{-1,p}(\Omega;\mathbb{R}^m)$) as $j\to\infty$. Thus, for j large enough

$$\left|\int_\Omega u_j^{ik}(y)\sum_{l=1}^n A^{(l)}\phi_{ik}(y)\frac{\partial(\eta_\ell(y))}{\partial y_l}\,\mathrm{d}y\right| \leq \frac{1}{2^i k},$$

so that

$$\left|\sum_i\int_\Omega u_j^{ik}(y)\sum_{l=1}^n A^{(l)}\phi_{ik}(y)\frac{\partial(\eta_\ell(y))}{\partial y_l}\,\mathrm{d}y\right| \leq \frac{1}{k}.$$

Relying on the separability of $\mathcal{S}$ and $C(\bar\Omega)$, and taking into account (4.16), we can choose a subsequence of $\{u_{k(\ell)}^\ell\}_{\ell\in\mathbb{N}}$, denoted by $\{u_k\}_{k\in\mathbb{N}}$, such that

$$\lim_{k\to\infty}\int_\Omega g(x)v(u_k(x))\,\mathrm{d}x = \int_\Omega\int_{\beta_{\mathcal{S}}\mathbb{R}^m} v_0(s)\mathrm{d}\lambda_x(s)g(x)\,d\pi(x)$$

and

$$\lim_{k\to\infty}\|\mathcal{A}u_k\|_{W^{-1,p}(\Omega;\mathbb{R}^d)} = 0.$$

If we extend u_k by zero on $Q\setminus\Omega$ and set for all $k\in\mathbb{N}$

$$\tilde u_k := \mathbb{T}\left(u_k - \int_Q u_k(x)\,\mathrm{d}x\right)$$

we have $\{\tilde u_k\}\subset\ker\mathcal{A}$ and $\lim_{k\to\infty}\|u_k-\tilde u_k\|_{L^p(\Omega;\mathbb{R}^m)} = 0$ and therefore by Lemma A.6 $\{\tilde u_k\}_{k\in\mathbb{N}}$ generates (π,λ).

It remains to show that the generating sequence has an $\mathcal{A}$-free p-equiintegrable extension. We take a Lipschitz domain $\hat\Omega\subset\mathbb{R}^n$ such that $\Omega\subset\hat\Omega\subset Q$ and extend (π,λ) to $\hat\Omega$ by $(\mathcal{L}^n\mathsf{L}(\hat\Omega\setminus\Omega),\delta_0)$. This extended DiPerna-Majda measure satisfies (4.8) and (4.9) for almost every $x\in\hat\Omega$ and we denote it $(\hat\pi,\hat\lambda)$. Hence, by our previous result, there is $\{\hat{u}_k\}_{k\in\mathbb{N}} \subset L^p(\hat{\Omega};\mathbb{R}^m) \cap \ker \mathcal{A}$ generating it. Due to Lemma 3.1, $\{\hat{u}_k|_{\hat{\Omega}\setminus\Omega}\}$ generates $(\mathcal{L}^n \mathsf{L}(\hat{\Omega}\setminus\Omega), \delta_0)$, so it must be p-equiintegrable.

(ii) Suppose now that $u \neq 0$ with $\mathcal{A}u = 0$. We rewrite (4.8) using the Young measure $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$ corresponding to $(\pi,\lambda) \in \mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$. A generating sequence of (π,λ), $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$, can be decomposed as $u_k = z_k + w_k$ by decomposition Lemma 1.2 applied for $\mathcal{A} := 0$. Then $\{z_k\}$ is p-equiintegrable. In view of Lemma A.5

$$\begin{aligned} Q_{\mathcal{A}}v(u(x)) &\leq \int_{\mathbb{R}^m} v(s)d\nu_x(s) + d_\pi(x)\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m} \frac{v(s)}{1+|s|^p}d\lambda_x(s) \\ &= \int_{\mathbb{R}^m} v(s)d\nu_x(s) + d_\pi(x)\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m} \frac{v_\infty(s)}{1+|s|^p}d\lambda_x(s). \end{aligned} \tag{4.18}$$

For $x \in \Omega$ and all $s \in \mathbb{R}^m$ define $f(s) := v(s+u(x))$. By Lemma A.4, $f_0 := f/(1+|\cdot|^p) \in \mathcal{S}$, and (see the proof of Lem. A.4) $f_\infty = v_\infty$. In particular, f_∞ does not depend on the choice of $x \in \Omega$. Therefore, we write (4.18) in the form

$$\begin{aligned} Q_{\mathcal{A}}f(0) &\leq \int_{\mathbb{R}^m} f(s-u(x))d\nu_x(s) + d_\pi(x)\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m} \frac{f_\infty(s)}{1+|s|^p}d\lambda_x(s) \\ &= \int_{\mathbb{R}^m} f(s)d\mu_x(s) + d_\pi(x)\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m} \frac{f_\infty(s)}{1+|s|^p}d\lambda_x(s), \end{aligned} \tag{4.19}$$

where we used formula (A.12). This defines the Young measure $\mu := \{\mu_x\}_{x\in\Omega} \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$ which is by Lemma A.5 generated by the p-equiintegrable sequence $\{z_k - u\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$. Altogether, using (A.10) for $\{z_k - u\}$ instead of $\{z_k\}$ we have for all $g \in C(\bar{\Omega})$ and all $v \in \Upsilon^p_{\mathcal{S}}$

$$\begin{aligned} \lim_{k\to\infty}\int_\Omega g(x)v(z_k(x)-u(x)+w_k)\,dx &= \lim_{k\to\infty}\int_\Omega g(x)v(z_k(x)-u(x))\,dx + \lim_{k\to\infty}\int_\Omega g(x)v(w_k(x))\,dx \\ &= \int_\Omega\int_{\mathbb{R}^m} v(s)d\mu_x(s)g(x)\,dx + \int_{\bar{\Omega}}\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m} \frac{v_\infty(s)}{1+|s|^p}d\lambda_x(s)g(x)d_\pi(x)\,dx \\ &= \int_{\bar{\Omega}}\int_{\beta_{\mathcal{S}}\mathbb{R}^m} \frac{v(s)}{1+|s|^p}d\alpha_x(s)g(x)d\kappa(x) \\ &= \int_{\Omega}\int_{\beta_{\mathcal{S}}\mathbb{R}^m} \frac{v(s)}{1+|s|^p}d\alpha_x(s)g(x)d_\kappa(x)\,dx, \end{aligned}$$

where $(\kappa,\alpha) \in \mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$ is generated by $\{z_k - u + w_k\}$. As $g \in C(\bar{\Omega})$ is arbitrary, we get for a.e. $x \in \Omega$ and all $v \in \Upsilon^p_{\mathcal{S}}$ that

$$\int_{\mathbb{R}^m} v(s)d\mu_x(s) + d_\pi(x)\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m} \frac{v_\infty(s)}{1+|s|^p}d\lambda_x(s) = d_\kappa(x)\int_{\beta_{\mathcal{S}}\mathbb{R}^m} \frac{v(s)}{1+|s|^p}d\alpha_x(s). \tag{4.20}$$

By (4.20), (4.19) now reads

$$Q_{\mathcal{A}}f(0) \leq d_\kappa(x)\int_{\beta_{\mathcal{S}}\mathbb{R}^m} \frac{f(s)}{1+|s|^p}d\alpha_x(s),$$

and therefore, by (i) (κ,α) is generated by an $\mathcal{A}$-free sequence $\{\tilde{u}_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m) \cap \ker \mathcal{A}$, $\tilde{u}_k \rightharpoonup 0$. Clearly, $\{\tilde{u}_k + u\}$ generates (π,λ). □

Finally, we prove the general result with π possibly having also a singular part.

Proposition 4.6. *Let $1 < p < +\infty$ and let $(\pi,\lambda) \in \mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$ be such that the following three conditions hold:*
(i) $u \in L^p(\Omega;\mathbb{R}^m)\cap\ker\mathcal{A}$ where for almost every $x\in\Omega$

$$u(x) = d_\pi(x)\int_{\mathbb{R}^m}\frac{s}{1+|s|^p}d\lambda_x(s); \tag{4.21}$$

(ii) for almost every $x\in\Omega$ and for any $v\in\Upsilon^p_{\mathcal{S}}$

$$Q_{\mathcal{A}}v(u(x)) \le d_\pi(x)\int_{\beta_{\mathcal{S}}\mathbb{R}^m}\frac{v(s)}{1+|s|^p}d\lambda_x(s); \tag{4.22}$$

(iii) for π-almost all $x\in\bar\Omega$ and all positively p-homogeneous $v\in\Upsilon^p_{\mathcal{S}}$ with $Q_{\mathcal{A}}v(0)=0$, it holds that

$$0 \le \int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m}\frac{v(s)}{1+|s|^p}\mathrm{d}\lambda_x(s). \tag{4.23}$$

Then $(\pi,\lambda)\in\mathcal{ADM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$. Moreover, a generating sequence $\{u_k\}_{k\in\mathbb{N}}$ can be chosen so that it has an $\mathcal{A}$-free p-equiintegrable extension.

Proof. If the singular part of π vanishes, then the assertion follows from Proposition 4.5. Hence, we suppose that $\pi_s\neq 0$. The proof is divided into two steps.

(i) We assume first that the singular part of π, π_s, consists of a finite sum of atoms, *i.e.*, $\pi_s=\sum_{i=1}^n a_i\delta_{x_i}$, where $a_i>0$ and $x_i\in\Omega$, $1\le i\le N$.

Note that by Lemma A.2 $\lambda_{x_i}(\mathbb{R}^m)=0$ for $1\le i\le N$. Choose $r>0$ sufficiently small and balls $B(x_i,r)\subset\Omega$, such that $B(x_i,r)\cap B(x_j,r)=\emptyset$ if $i\neq j$. We define, for $i=1,\dots,N$,

$$\alpha_i(r) := \frac{1}{a_i}\int_{B(x_i,r)}(1+|u(x)|^p)\,\mathrm{d}x.$$

As $\lim_{r\to 0}\alpha_i(r)=0$ we will only consider $r<r_0$ for $r_0>0$ so small that $0<\alpha_i(r)<1$ for all $i=1,\dots,N$.

Further, for a.e. $x\in\Omega$ we define

$$\lambda^r_x := \begin{cases} \lambda_x & \text{if } x\in\bar\Omega\setminus\cup_{i=1}^n B(x_i,r),\\ \alpha_i(r)\delta_{u(x)}+(1-\alpha_i(r))\lambda_{x_i} & \text{if } x\in B(x_i,r) \text{ for some } 1\le i\le N\end{cases}$$

and introduce the measure $\pi_r := d_{\pi_r}\mathcal{L}^n\llcorner\Omega$ defined through its density d_{π_r} as

$$d_{\pi_r}(x) := \begin{cases} d_\pi(x) & \text{if } x\in\bar\Omega\setminus\cup_{i=1}^n B(x_i,r),\\ \frac{1+|u(x)|^p}{\alpha_i(r)} & \text{if } x\in B(x_i,r) \text{ for some } 1\le i\le N.\end{cases}$$

We claim that $(\pi_r,\lambda^r)\in\mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$. For almost all $x\in\Omega$

$$u(x) = d_{\pi_r}(x)\int_{\mathbb{R}^m}\frac{s}{1+|s|^p}d\lambda^r_x(s).$$

Indeed, if $x\in B(x_i,r)$, then we get

$$d_{\pi_r}(x)\int_{\mathbb{R}^m}\frac{s}{1+|s|^p}d\lambda^r_x(s) = u(x)+\frac{(1-\alpha_i(r))(1+|u(x)|^p)}{\alpha_i(r)}\int_{\mathbb{R}^m}\frac{s}{1+|s|^p}\,\mathrm{d}\lambda_{x_i}(s) = u(x)$$

and due to (4.23), for almost all $x\in B(x_i,r)$

$$Q\mathcal{A}v(u(x))\le v(u(x))+\frac{(1-\alpha_i(r))(1+|u(x)|^p)}{\alpha_i(r)}\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m}\frac{v(s)}{1+|s|^p}\,\mathrm{d}\lambda_{x_i}(s).$$

Altogether we have for any $v\in\Upsilon^p_{\mathcal{S}}$ with $Q_{\mathcal{A}}v>-\infty$

$$Q_{\mathcal{A}}v(u(x))\le d_{\pi_r}(x)\int_{\beta_{\mathcal{S}}\mathbb{R}^m}\frac{v(s)}{1+|s|^p}d\lambda^r_x(s),$$

and by Proposition 4.5 there is $\{u^r_k\}\in L^p(\Omega;\mathbb{R}^m)\cap\ker\mathcal{A}$ such that $\{u^r_k\}_{k\in\mathbb{N}}$ generates $(\pi_r,\lambda^r)\in\mathcal{ADM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$.
We calculate for fixed $v_0\in\mathcal{S}$ and $g\in C(\bar\Omega)$

$$\begin{aligned}\lim_{r\to0}\int_{\bar\Omega}\int_{\beta_{\mathcal{S}}\mathbb{R}^m}v_0(s)d\lambda^r_x(s)g(x)\,d\pi_r(x)&=\lim_{r\to0}\int_{\bar\Omega\setminus\cup_{i=1}^nB(x_i,r)}\int_{\beta_{\mathcal{S}}\mathbb{R}^m}v_0(s)\mathrm{d}\lambda_x(s)g(x)d_\pi(x)\,\mathrm{d}x\\&+\lim_{r\to0}\sum_{i=1}^n\int_{B(x_i,r)}v(u(x))g(x)\,\mathrm{d}x+\lim_{r\to0}\sum_{i=1}^n\frac{1-\alpha_i(r)}{\alpha_i(r)}\\&\times\int_{B(x_i,r)}g(x)(1+|u(x)|^p)\,\mathrm{d}x\int_{\beta_{\mathcal{S}}\mathbb{R}^m}v_0(s)d\lambda_{x_i}(s)=:I+II+III.\end{aligned}$$

Obviously, $I+II=\int_{\bar\Omega}\int_{\beta_{\mathcal{S}}\mathbb{R}^m}v_0(s)\mathrm{d}\lambda_x(s)g(x)d_\pi(x)\,\mathrm{d}x$, while

$$\begin{aligned}III&=\lim_{r\to0}\sum_{i=1}^n\frac{1}{\alpha_i(r)}\int_{B(x_i,r)}g(x)(1+|u(x)|^p)\,\mathrm{d}x\int_{\beta_{\mathcal{S}}\mathbb{R}^m}v_0(s)d\lambda_{x_i}(s)\\&=\sum_{i=1}^na_i\left(\int_{\beta_{\mathcal{S}}\mathbb{R}^m}v_0(s)d\lambda_{x_i}(s)\right)\lim_{r\to0}\frac{1}{\int_{B(x_i,r)}(1+|u(x)|^p)\,\mathrm{d}x}\int_{B(x_i,r)}g(x)(1+|u(x)|^p)\,\mathrm{d}x\\&=\sum_{i=1}^na_ig(x_i)\int_{\beta_{\mathcal{S}}\mathbb{R}^m}v_0(s)d\lambda_{x_i}(s)=\int_{\bar\Omega}\int_{\beta_{\mathcal{S}}\mathbb{R}^m}v_0(s)\mathrm{d}\lambda_x(s)g(x)\,d\pi_s(x).\end{aligned}$$

We conclude that

$$\lim_{r\to0}\lim_{k\to\infty}\int_\Omega v(u^r_k(x))g(x)\,\mathrm{d}x=\int_{\bar\Omega}\int_{\beta_{\mathcal{S}}\mathbb{R}^m}v_0(s)\mathrm{d}\lambda_x(s)g(x)\,d\pi(x).$$

A suitable diagonalization yields the existence of a bounded sequence $\{u_k\}_{k\in\mathbb{N}}\subset L^p(\Omega;\mathbb{R}^m)\cap\ker\mathcal{A}$ such that

$$\lim_{k\to\infty}\int_\Omega v(u_k(x))g(x)\,\mathrm{d}x=\int_{\bar\Omega}\int_{\beta_{\mathcal{S}}\mathbb{R}^m}v_0(s)\mathrm{d}\lambda_x(s)g(x)\,d\pi(x),$$

whenever $v\in\Upsilon^p_{\mathcal{S}}$ and $g\in C(\bar\Omega)$.

(ii) Now we prove the general case. Take $l\in\mathbb{N}$. There exists a finite partition $P_l:=\{\Omega^l_j\}_{j=1}^{J(l)}$ of $\bar\Omega$ such that $\Omega^l_{j_1}\bigcap\Omega_{j_2}=\emptyset,\ 1\le j_1<j_2\le J(l)$ and all Ω^l_j are measurable with $\mathrm{diam}(\Omega^l_j)<1/l$. We suppose that, for any $l\in\mathbb{N}$, the partition P_{l+1} is a refinement of P_l and that $\mathrm{int}(\Omega^l_j)\ne\emptyset$ for all j. Set $a^l_i:=\pi_s(\Omega^l_i)$, and

$$N(l):=\{1\le j\le J(l);\ a^l_j\ne0\}\cdot$$

If $i\in N(l)$ then fix $x_i\in \mathrm{int}(\Omega_i^l)$ and define a measure (π^l,λ^l) with $\pi^l:=d_\pi\mathcal{L}^n\llcorner\Omega+\sum_{i\in N(l)}a_i^l\delta_{x_i}$ and

$$\lambda_x^l:=\begin{cases}\lambda_x & \text{if } x\neq x_i\\ \lambda_{x_i}^l & \text{if } x=x_i,\end{cases}$$

and for any $v_0\in\mathcal{S}$

$$\int_{\beta_{\mathcal{S}}\mathbb{R}^m} v_0(s)d\lambda_{x_i}^l(s):=\frac{1}{\pi_s(\Omega_i^l)}\int_{\Omega_i^l}\int_{\beta_{\mathcal{R}}\mathbb{R}^m} v_0(s)\mathrm{d}\lambda_x(s)\,d\pi_s(x). \tag{4.24}$$

By Lemma A.2 and because supp $\lambda_{x_i}^l\subset\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m$ we can rewrite (4.24) as

$$\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m} v_0(s)\lambda_{x_i}^l(\mathrm{d}s)=\frac{1}{\pi_s(\Omega_i^l)}\int_{\Omega_i^l}\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m} v_0(s)\mathrm{d}\lambda_x(s)\,d\pi_s(x). \tag{4.25}$$

Part (i) implies $(\pi^l,\lambda^l)\in\mathcal{ADM}_{\mathcal{S}}^p(\Omega;\mathbb{R}^m)$. Indeed, Proposition A.1 ensures that $(\pi^l,\lambda^l)\in\mathcal{DM}_{\mathcal{S}}^p(\Omega;\mathbb{R}^m)$ Moreover, an easy verification shows that (4.21), (4.22), and (4.23) are also satisfied for (π^l,λ^l), and (4.21) holds with the same function u.

Let $\{u_k^l\}_{k\in\mathbb{N}}\subset L^p(\Omega;\mathbb{R}^m)\cap\ker\mathcal{A}$ be such that $\{u_k^l\}_{k\in\mathbb{N}}$ generates (π^l,λ^l) and, in addition, it has an $\mathcal{A}$-free p-equiintegrable extension. We have for any $l\in\mathbb{N}$

$$\lim_{k\to\infty}\int_\Omega(1+|u_k^l(x)|^p)\,\mathrm{d}x=\pi^l(\bar\Omega)=\pi(\bar\Omega),$$

and for any $v_0\in\mathcal{S}$ and any $g\in C(\bar\Omega)$

$$\begin{aligned}
&\lim_{l\to\infty}\left|\int_{\bar\Omega}\int_{\beta_{\mathcal{S}}\mathbb{R}^m} v_0(s)d\lambda_x^l(s)g(x)\,d\pi^l(x)-\int_{\bar\Omega}\int_{\beta_{\mathcal{S}}\mathbb{R}^m} v_0(s)d\lambda_x(s)g(x)\,d\pi(x)\right|\\
&=\lim_{l\to\infty}\left|\sum_{i\in N(l)}g(x_i)\pi_s(\Omega_i^l)\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m} v_0(s)d\lambda_{x_i}^l(s)-\int_{\bar\Omega}\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m} v_0(s)d\lambda_x(s)g(x)\,d\pi_s(x)\right|\\
&=\lim_{l\to\infty}\left|\sum_{i\in N(l)}\left(\int_{\Omega_i^l}\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m} v_0(s)\mathrm{d}\lambda_x(s)g(x_i)\,d\pi_s(x)-\int_{\Omega_i^l}\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m} v_0(s)d\lambda_x(s)g(x)\,d\pi_s(x)\right)\right|\\
&\leq\lim_{l\to\infty}\sum_{i\in N(l)}\int_{\Omega_i^l}\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m}|v_0(s)|\mathrm{d}\lambda_x(s)|g(x)-g(x_i)|\,d\pi_s(x)\leq\|v_0\|_{C(\mathbb{R}^m)}\pi_s(\bar\Omega)\lim_{l\to\infty}M_g\left(\frac{1}{l}\right)=0,
\end{aligned}$$

where M_g is the modulus of continuity of the uniformly continuous $g\in C(\bar\Omega)$. Hence, for any $v\in\Upsilon_{\mathcal{S}}^p$ and any $g\in C(\bar\Omega)$ we obtain

$$\lim_{l\to\infty}\lim_{k\to\infty}\int_\Omega v(u_k^l(x))g(x)\,\mathrm{d}x=\int_{\bar\Omega}\int_{\beta_{\mathcal{S}}\mathbb{R}^m} v_0(s)d\lambda_x(s)g(x)\,d\pi(x)$$

and we complete the proof using a diagonalization argument. □

5. Proof of Theorems 2.3, 2.4, and Lemma 2.10

Proof of Theorem 2.3. It follows from Theorem 2.1 that each of conditions (i)–(iv) ensures that $\int_{\partial\Omega}\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m}\frac{v(s)}{1+|s|^p}\mathrm{d}\lambda_x(s)g(x)\,d\pi(x)\geq 0$ for v and g as in the statement of the theorem. Thus, it suffices to integrate (2.1) and (2.2) with respect to π over $\bar\Omega$ and use (1.4). □

Proof of Theorem 2.4. Let us first prove the "only if part". We have from the assumption that $w_k \rightharpoonup 0$ in $L^p(\Omega;\mathbb{R}^m)$. By sequential weak lower semicontinuity of I we have $\liminf_{k\to\infty} I(w_k) \geq I(0)$.

Now we are going to prove the "if part". Let us take any bounded $\{u_k\} \subset L^p(\Omega;\mathbb{R}^m) \cap \ker \mathcal{A}$ such that w-$\lim_{k\to\infty} u_k = u$. Suppose that a subsequence of $\{u_k\}$ (not relabeled) generates $(\pi,\lambda) \in \mathcal{ADM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$. Using Lemma 1.2, we decompose $u_k = z_k + w_k$ for any $k \in \mathbb{N}$. Then (A.8) and the assumption $\liminf_{k\to\infty} I(w_k) \geq I(0)$ imply that

$$\int_{\bar{\Omega}} \int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m} \frac{v(s)}{1+|s|^p} \mathrm{d}\lambda_x(s) g(x)\,\mathrm{d}\pi(x) \geq 0 \tag{5.1}$$

for any subsequence of $\{w_k\}$ (not relabeled) such that $\{I(w_k)\}$ converges. Let $\{u_k\}_{k\in\mathbb{N}} \cap \ker \mathcal{A}$ generate a Young measure $\nu = \{\nu_x\}_{x\in\Omega} \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$. We have using (1.4) and Lemma A.5

$$\begin{aligned}\lim_{k\to\infty} \int_\Omega g(x) v(u_k(x))\mathrm{d}x &= \int_\Omega \int_{\mathbb{R}^m} g(x) v(s) d\nu_x(s)\,\mathrm{d}x \\ &\quad + \int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}}\mathbb{R}^m\setminus\mathbb{R}^m} g(x) \frac{v(s)}{1+|s|^p} \mathrm{d}\lambda_x(s)\, d\pi(x) \geq \int_\Omega g(x) v(u(x))\,\mathrm{d}x.\end{aligned}$$

The last inequality follows from (5.1) and from the characterization of $\mathcal{A}$-free Young measures given in [15]. The theorem is proved. □

Proof of Lemma 2.10. Without loss of generality, we will assume that $\{u_k\}$ generates $(\pi,\lambda) \in \mathcal{ADM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$. Since, $\pi(\partial(\varepsilon\Omega)) > 0$ only for at most countably many values of ε, which we denote ε_ℓ, $\ell \in \mathbb{N}$. Thus we take $\varepsilon > 0$ such that $\pi(\partial(\varepsilon\Omega)) = 0$. Then using Lemma 3.1 we have that the restriction of $\{u_k\}$ on $\varepsilon\Omega$ has the property that $\{u_k|_{\varepsilon\Omega}\}$ generates $(\pi,\lambda)|_{\varepsilon\Omega}$, and now (2.11) follows from Theorem 2.3(i). □

A. Appendix

A.1. **Characterization of DiPerna-Majda measures**

The explicit description of the elements from $\mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$, called *DiPerna-Majda measures*, for unconstrained sequences was obtained in [24], Theorem 2.

Proposition A.1 (see [24])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain, let $(\pi,\lambda) \in \mathcal{M}(\bar{\Omega}) \times L^\infty_{\mathrm{w}}(\bar{\Omega},\pi;\mathcal{M}(\beta_{\mathcal{R}}\mathbb{R}^m))$, and let $1 \leq p < +\infty$. Then the following two statements are equivalent:*

(i) $(\pi,\lambda) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$;

(ii) The following properties hold:

(1) *π is positive;*

(2) *$\pi_\lambda \in \mathcal{M}(\bar{\Omega})$, defined for all $\psi \in C_0(\mathbb{R}^m)$ by $\int_{\bar{\Omega}} \psi(x) d\pi_\lambda(x) := \int_{\bar{\Omega}} \psi(x) \lambda_x(\mathbb{R}^m) d\pi(x)$, is absolutely continuous with respect to the Lebesgue measure (d_{π_λ} will denote its density);*

(3) *for a.e. $x \in \Omega$ it holds*

$$\lambda_x(\mathbb{R}^m) > 0, \qquad d_{\pi_\lambda}(x) = \left(\int_{\mathbb{R}^m} \frac{d\lambda_x(s)}{1+|s|^p}\right)^{-1} \lambda_x(\mathbb{R}^m);$$

(4) *$\lambda_x \in \mathcal{P}(\beta_{\mathcal{R}}\mathbb{R}^m)$ for π-a.e. $x \in \bar{\Omega}$.*

We will also use the following result, whose proof can be found in various contexts (see [24], Lem. 1, Thm. 1,2, [33], Prop. 3.2.17), [1], Proposition 4.1, part (iii).

Lemma A.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $(\pi, \lambda) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega; \mathbb{R}^m)$. Then for $\mathcal{L}^n$-a.e. $x \in \Omega$*

$$d_\pi(x) = \left(\int_{\mathbb{R}^m} \frac{\mathrm{d}\lambda_x(s)}{1 + |s|^p} \right)^{-1} \tag{A.1}$$

and for π_s-almost all $x \in \bar{\Omega}$ we have

$$\lambda_x(\mathbb{R}^m) = 0.$$

Proof. Setting $v_0 := (1 + |\cdot|^p)^{-1}$ in (1.4) we get for all $g \in C(\bar{\Omega})$

$$\int_\Omega g(x)\,\mathrm{d}x = \int_\Omega g(x) \left(\int_{\mathbb{R}^m} \frac{\mathrm{d}\lambda_x(s)}{1 + |s|^p} \right) d_\pi(x)\,\mathrm{d}x + \int_{\bar{\Omega}} g(x) \left(\int_{\mathbb{R}^m} \frac{\mathrm{d}\lambda_x(s)}{1 + |s|^p} \right) \mathrm{d}\pi_s(x). \tag{A.2}$$

Here we used the fact that $v_0 = 0$ on $\beta_{\mathcal{R}}\mathbb{R}^m \setminus \mathbb{R}^m$. Hence, it follows from (A.2) that $d_\pi(x) \left(\int_{\mathbb{R}^m} \frac{\mathrm{d}\lambda_x(s)}{1+|s|^p} \right) = 1$ a.e. in Ω and $\lambda_x(\mathbb{R}^m) = 0$ for π_s-a.a. $x \in \bar{\Omega}$. □

A.2. DiPerna-Majda measures on the sphere compactification

We start with an easy lemma from [16].

Lemma A.3. *Let $v \in C(\mathbb{R}^m)$ be Lipschitz continuous on the unit sphere S^{m-1} and p-homogeneous, $p \geq 1$. Then v is p-Lipschitz,* i.e., *there is a constant $\alpha > 0$ such that for any $s_1, s_2 \in \mathbb{R}^m$ it holds*

$$|v(s_1) - v(s_2)| \leq \alpha(|s_1|^{p-1} + |s_2|^{p-1})|s_1 - s_2|. \tag{A.3}$$

Lemma A.4. *Let $v_0 \in \mathcal{S}$, $s_0 \in \mathbb{R}^m$, and $v(s) := v_0(s)(1 + |s|^p)$ for all $s \in \mathbb{R}^m$. Then $s \mapsto v_0(s) := \frac{v(s+s_0)}{1+|s|^p}$ also belongs to $\mathcal{S}$.*

Proof. Since v_∞ is continuous on S^{m-1}, using the Stone-Weierstrass theorem, we can uniformly approximate $v_\infty|_{S^{m-1}}$ by Lipschitz functions. Take a sequence $\{\psi_j\}_{j\in\mathbb{N}}$ such that $\psi_j : S^{m-1} \to \mathbb{R}$ is Lipschitz continuous for all $j \in \mathbb{N}$ and identify ψ_j with its positively p-homogeneous extension to the whole $\mathbb{R}^m$. We assume that for all $j \in \mathbb{N}$

$$\|\psi_j - v_\infty\|_{C(S^{m-1})} := \max_{s \in S^{m-1}} |\psi_j(s) - v_\infty(s)| \leq \frac{1}{j}\cdot \tag{A.4}$$

Then

$$\lim_{|s|\to\infty} \frac{|v_\infty(s+s_0) - v_\infty(s)|}{|s|^p} \leq \lim_{|s|\to\infty} \frac{|\psi_j(s+s_0) - \psi_j(s)|}{|s|^p} + \limsup_{|s|\to\infty} \frac{|v_\infty(s+s_0) - \psi_j(s+s_0)|}{|s|^p}$$
$$+ \limsup_{|s|\to\infty} \frac{|\psi_j(s) - v_\infty(s)|}{|s|^p}\cdot$$

The first term on the right-hand side is zero due to Lemma A.3. By (A.4) and using the p-homogeneity, we further estimate the remaining two terms

$$\lim_{|s|\to\infty} \frac{|v_\infty(s+s_0) - v_\infty(s)|}{|s|^p} \leq \limsup_{|s|\to\infty} \left| v_\infty\left(\frac{s+s_0}{|s+s_0|}\right) - \psi_j\left(\frac{s+s_0}{|s+s_0|}\right) \right| \frac{|s+s_0|^p}{|s|^p}$$
$$+ \limsup_{|s|\to\infty} \left| \psi_j\left(\frac{s}{|s|}\right) - v_\infty\left(\frac{s}{|s|}\right) \right| \leq \frac{2}{j}\cdot$$

As $j \in \mathbb{N}$ is arbitrary we deduce that

$$\lim_{|s|\to\infty} \frac{|v_\infty(s+s_0) - v_\infty(s)|}{|s|^p} = 0. \tag{A.5}$$

Hence, we have in view of (A.3)

$$\lim_{|s|\to\infty} \frac{|v(s+s_0) - v_\infty(s)|}{|s|^p} \leq \lim_{|s|\to\infty} \frac{|v(s+s_0) - v_\infty(s+s_0)|}{|s|^p} + \lim_{|s|\to\infty} \frac{|v_\infty(s+s_0) - v_\infty(s)|}{|s|^p} = 0,$$

which means that $v(\cdot + s_0)$ has the recession function v_∞. Denote $\tilde{v}_0 := v(\cdot + s_0)/(1 + |\cdot|^p)$ and write

$$\tilde{v}_0(s) = \frac{v(s+s_0) - v_\infty(s)}{1+|s|^p} + \frac{v_\infty(s)}{1+|s|^p}.$$

The first term on the right-hand side belongs to $C_0(\mathbb{R}^m)$ and v_∞ is positively p-homogeneous. Hence, $\tilde{v}_0 \in \mathcal{S}$ in view of Remark 1.7. □

Given a bounded sequence in $L^p(\Omega;\mathbb{R}^m)$ that generates a DiPerna-Majda measure $(\pi,\lambda) \in \mathcal{DM}^p_{\mathcal{R}}(\Omega;\mathbb{R}^m)$ and that also generates an L^p-Young measure $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$ we have for all $g \in C(\bar{\Omega})$ and all $v \in C_p(\mathbb{R}^m)$ (*i.e.* $v = 0$ on $\beta_{\mathcal{R}}\mathbb{R}^m \setminus \mathbb{R}^m$)

$$\int_\Omega \int_{\mathbb{R}^m} g(x)v(s)d\nu_x(s)\,\mathrm{d}x = \int_{\bar{\Omega}} \int_{\mathbb{R}^m} g(x)\frac{v(s)}{1+|s|^p} d\lambda_x(s)d\pi(x). \tag{A.6}$$

Observe that (A.6) holds in fact for all $v \in \Upsilon^p_{\mathcal{R}}$ and all $g \in C(\bar{\Omega})$. Indeed, for any $j \in N$ let $a_j \in C_0(\mathbb{R}^m)$ be such that $0 \leq a_j \leq 1$, $a_j(s) = 1$ if $|s| \leq j$. Then $va_j \in C_p(\mathbb{R}^m)$ is admissible for (A.6) and the Lebesgue Dominated Convergence Theorem finishes the argument. Therefore, for all $g \in C(\bar{\Omega})$ and all $v \in \Upsilon^p_{\mathcal{R}}$ we have

$$\lim_{k\to\infty} \int_\Omega g(x)v(y_k(x))\mathrm{d}x = \int_\Omega \int_{\mathbb{R}^m} v(s)d\nu_x(s)g(x)\,\mathrm{d}x + \int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}}\mathbb{R}^m\setminus\mathbb{R}^m} v_0(s)\mathrm{d}\lambda_x(s)g(x)d\pi(x). \tag{A.7}$$

We now show that oscillations and concentration effects, generated by a sequence bounded in $L^p(\Omega;\mathbb{R}^m)$ and encoded in $(\pi,\lambda) \in \mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$, can be separated. Suppose that $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$, $1 < p < +\infty$, is a bounded sequence generating $(\pi,\lambda) \in \mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$, $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$, and converging weakly to zero in $L^p(\Omega;\mathbb{R}^m)$. Notice that for all $v \in \Upsilon^p_{\mathcal{S}}$ and all $g \in C(\bar{\Omega})$ we have

$$\lim_{k\to\infty} \int_\Omega g(x)v(u_k(x))\,\mathrm{d}x = \lim_{k\to\infty} \int_\Omega g(x)(v - v_\infty)(u_k(x))\,\mathrm{d}x + \lim_{k\to\infty} \int_\Omega g(x)v_\infty(u_k(x))\,\mathrm{d}x. \tag{A.8}$$

As $(v - v_\infty) \in C_p(\mathbb{R}^m)$ the first term on the right-hand side of (A.8) can be represented by the Young measure ν. The second term on the right-hand side of (A.8) carries all concentrations and is described by (π,λ). Applying Lemma 1.2 with $\mathcal{A} := 0$ to the sequence $\{u_k\}$ we may decompose $u_k = z_k + w_k$ where $\{z_k\}_{k\in\mathbb{N}}, \{w_k\}_{k\in\mathbb{N}} \subset L^p(\Omega;\mathbb{R}^m)$ are bounded, $\{z_k\}_{k\in\mathbb{N}}$ is p-equiintegrable and $w_k \to 0$ in measure. Moreover, $\{u_k\}$ and $\{z_k\}$ generate the same Young measure $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$, see also [14], Corollary 8.8, and setting

$$\Omega_k := \{x \in \Omega\colon\, w_k(x) \neq 0\},$$

we have that $\mathcal{L}^n(\Omega_k) \to 0$ as $k \to \infty$. Thus, (A.8) can be written as

$$\lim_{k\to\infty} \int_\Omega g(x)v(u_k(x))\,\mathrm{d}x = \lim_{k\to\infty} \int_\Omega g(x)(v - v_\infty)(z_k(x))\,\mathrm{d}x + \lim_{k\to\infty} \int_\Omega g(x)v_\infty(u_k(x))\,\mathrm{d}x. \tag{A.9}$$

Take the sequence $\{\psi_j\}_{j\in\mathbb{N}}$ of Lipschitz functions on S^{m-1} as in the proof of Lemma A.4 to get for all $g \in C(\bar\Omega)$ and all $j \in \mathbb{N}$

$$\begin{aligned}
&\lim_{k\to\infty}\left|\int_\Omega g(x)(v_\infty(w_k(x)) - (v_\infty(u_k(x)) - v_\infty(z_k(x))))\,\mathrm{d}x\right| \\
&\quad\le \lim_{k\to\infty}\left|\int_\Omega g(x)(\psi_j(w_k(x)) - (\psi_j(u_k(x)) - \psi_j(z_k(x))))\,\mathrm{d}x\right| \\
&\quad+ \limsup_{k\to\infty}\|g\|_{C(\bar\Omega)}\int_\Omega\left|v_\infty\left(\frac{w_k(x)}{|w_k(x)|}\right) - \psi_j\left(\frac{w_k(x)}{|w_k(x)|}\right)\right||w_k(x)|^p\,\mathrm{d}x \\
&\quad+ \limsup_{k\to\infty}\|g\|_{C(\bar\Omega)}\int_\Omega\left|\psi_j\left(\frac{u_k(x)}{|u_k(x)|}\right) - v_\infty\left(\frac{u_k(x)}{|u_k(x)|}\right)\right||u_k(x)|^p\,\mathrm{d}x \\
&\quad+ \limsup_{k\to\infty}\|g\|_{C(\bar\Omega)}\int_\Omega\left|v_\infty\left(\frac{z_k(x)}{|z_k(x)|}\right) - \psi_j\left(\frac{z_k(x)}{|z_k(x)|}\right)\right||z_k(x)|^p\,\mathrm{d}x \le \frac{C}{j} \overset{j\to\infty}{\to} 0,
\end{aligned}$$

as $C > 0$ depends only on g and L^p bounds of $\{z_k\}$ and $\{w_k\}$. Altogether we see that

$$\lim_{k\to\infty}\int_\Omega g(x)v(u_k(x))\,\mathrm{d}x = \lim_{k\to\infty}\int_\Omega g(x)v(z_k(x))\,\mathrm{d}x + \lim_{k\to\infty}\int_\Omega g(x)v_\infty(w_k(x))\,\mathrm{d}x \tag{A.10}$$

holds for all $g \in C(\Omega)$ and all $v \in \Upsilon^p_{\mathcal{S}}$.

If $u \in L^p(\Omega;\mathbb{R}^m)$ then we have $u_k - u = (z_k - u) + w_k$. Again $\{z_k - u\}_{k\in\mathbb{N}}$ is p-equiintegrable, so we get by (A.10)

$$\lim_{k\to\infty}\int_\Omega g(x)v(u_k(x) - u(x))\,\mathrm{d}x = \lim_{k\to\infty}\int_\Omega g(x)v(z_k(x) - u(x))\,\mathrm{d}x + \lim_{k\to\infty}\int_\Omega g(x)v_\infty(w_k(x))\,\mathrm{d}x. \tag{A.11}$$

Note that $\{z_k - u\}$ generates the Young measure $\mu \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$ given for almost all $x \in \Omega$ and all $v \in C_p(\mathbb{R}^m)$ by the formula (see [15], Prop. 24)

$$\int_{\mathbb{R}^m} v(s)d\mu_x(s) := \int_{\mathbb{R}^m} v(s - u(x))\mathrm{d}\nu_x(s). \tag{A.12}$$

Comparing (A.10) with (A.11) we see that $\{u_k\}$ and $\{u_k - u\}$ generate the same concentration effects, namely those related to $\{w_k\}$. The shift by u is recorded only in the first terms in the right-hand sides of (A.10) and (A.11) which generates only oscillations but no concentrations. It will be occasionally convenient to assign to a generating sequence a Young measure-DiPerna-Majda measure pair $[\nu,(\pi,\lambda)] \in \mathcal{Y}^p(\Omega;\mathbb{R}^m) \times \mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$. We have the following result.

Lemma A.5. *Let $\{u_k\} \subset L^p(\Omega;\mathbb{R}^m)$, $1 \le p < +\infty$, generate a DiPerna-Majda measure $(\pi,\lambda) \in \mathcal{DM}^p_{\mathcal{S}}(\Omega;\mathbb{R}^m)$ and a Young measure $\nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^m)$, and let $u \in L^p(\Omega;\mathbb{R}^m)$. Then for all $g \in C(\bar\Omega)$ and all $v \in \Upsilon^p_{\mathcal{S}}$ it holds*

$$\begin{aligned}
&\lim_{k\to\infty}\int_\Omega v(u_k(x) - u(x))g(x)\,\mathrm{d}x \\
&= \int_\Omega\int_{\mathbb{R}^m} v(s - u(x))\mathrm{d}\nu_x(s)g(x)\,\mathrm{d}x \;+\; \int_{\bar\Omega}\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m}\frac{v_\infty(s)}{1+|s|^p}\mathrm{d}\lambda_x(s)g(x)d\pi(x) \\
&= \int_\Omega\int_{\mathbb{R}^m} v(s - u(x))\mathrm{d}\nu_x(s)g(x)\,\mathrm{d}x \;+\; \int_{\bar\Omega}\int_{\beta_{\mathcal{S}}\mathbb{R}^m\setminus\mathbb{R}^m}\frac{v(s)}{1+|s|^p}\mathrm{d}\lambda_x(s)g(x)d\pi(x).
\end{aligned}$$

Proof. We decompose $u_k = z_k + w_k$ using Lemma 1.2. In view of (A.11) and (A.12) and of the fact that $\{u_k\}$ and $\{z_k\}$ generate the same Young measure, we have by (1.4) for all $g \in C(\bar\Omega)$ and all $v_\infty : \mathbb{R}^m \to \mathbb{R}$ positively

p-homogeneous and continuous

$$\lim_{k\to\infty}\int_\Omega g(x)v(w_k(x))\,\mathrm{d}x=\int_{\bar\Omega}\int_{\beta_{\mathcal S}\mathbb{R}^m\setminus\mathbb{R}^m}\frac{v_\infty(s)}{1+|s|^p}d\lambda_x(s)g(x)\,d\pi(x). \tag{A.13}$$

Finally, it remains to prove that

$$\int_{\bar\Omega}\int_{\beta_{\mathcal S}\mathbb{R}^m\setminus\mathbb{R}^m}\frac{v_\infty(s)}{1+|s|^p}\mathrm{d}\lambda_x(s)g(x)d\pi(x)=\int_{\bar\Omega}\int_{\beta_{\mathcal S}\mathbb{R}^m\setminus\mathbb{R}^m}\frac{v(s)}{1+|s|^p}\mathrm{d}\lambda_x(s)g(x)d\pi(x). \tag{A.14}$$

Let $\varepsilon>0$. By (1.7), there is $\varrho>0$ such that $|v(s)-v_\infty(s)|/(1+|s|^p)<\varepsilon$ whenever $|s|>\varrho$. Thus, for π-a.e. $x\in\bar\Omega$

$$\int_{\beta_{\mathcal R}\mathbb{R}^m\setminus B(0,\varrho)}\frac{|v(s)-v_\infty(s)|}{1+|s|^p}d\lambda_x(s)<\varepsilon$$

and we obtain (A.14). □

Lemma A.6. *Let $\{u_k\},\{w_k\}\subset L^p(\Omega;\mathbb{R}^m)$ be bounded sequences such that $\lim_{k\to\infty}\|u_k-w_k\|_{L^p(\Omega;\mathbb{R}^m)}=0$ and $\{u_k\}$ generates $(\pi,\lambda)\in\mathcal{DM}^p_{\mathcal S}(\Omega;\mathbb{R}^m)$. Then $\{w_k\}$ also generates (π,λ).*

Proof. Suppose that $v\in\Upsilon^p_{\mathcal S}$ is such that v_∞ is Lipschitz on S^{m-1}. By (A.3)

$$\begin{aligned}\left|\int_\Omega g(x)v_\infty(u_k(x))\,\mathrm{d}x-\int_\Omega g(x)v_\infty(w_k(x))\,\mathrm{d}x\right|&\le\|g\|_{C(\bar\Omega)}\int_\Omega|v_\infty(u_k(x))-v_\infty(w_k(x))|\,\mathrm{d}x\\&\le C\|g\|_{C(\bar\Omega)}\int_\Omega(|u_k(x)|^{p-1}+|w_k(x)|^{p-1})|u_k(x)-w_k(x)|\,\mathrm{d}x\\&\le C\|g\|_{C(\bar\Omega)}(\|u_k\|^{p-1}_{L^p(\Omega;\mathbb{R}^m)}\\&\quad+\ \||w_k\|^{p-1}_{L^p(\Omega;\mathbb{R}^m)})\|u_k-w_k\|_{L^p(\Omega;\mathbb{R}^m)}\to0\end{aligned}$$

as $k\to\infty$. By density, the result extends to any continuous v_∞. Hence, the second term on the right-hand side of (A.8) is the same for both sequences $\{u_k\}$ and $\{w_k\}$. As $\lim_{k\to\infty}\|u_k-w_k\|_{L^p(\Omega;\mathbb{R}^m)}=0$ then both sequences generate the same Young measure $\nu\in\mathcal{Y}^p(\Omega;\mathbb{R}^m)$, thus the first term on the right-hand side of (A.8) is also the same for both sequences. □

Lemma A.7. *Let $\{u_k\}_{k\in\mathbb{N}}\subset L^p(\Omega;\mathbb{R}^m)$, $1\le p<+\infty$, generate $(\pi,\lambda)\in\mathcal{DM}^p_{\mathcal S}(\Omega;\mathbb{R}^m)$ with $\pi(\partial\Omega)=0$. Let $\{\eta_j\}_{j\in\mathbb{N}}\subset C_0(\Omega)$, $0\le\eta_j\le1$, $j\in\mathbb{N}$, be such that $\eta_j(x)\to\chi_\Omega$ everywhere in Ω. Then there is a subsequence of $\{u_{k(j)}\eta_j\}_{j\in\mathbb{N}}$ generating (π,λ).*

Proof. If $v\in\Upsilon^p_{\mathcal S}$, and if v_∞ is Lipschitz on S^{m-1}, then

$$\begin{aligned}\left|\int_\Omega g(x)v_\infty(u_k(x))\,\mathrm{d}x-\int_\Omega g(x)v_\infty(u_k(x)\eta_j(x))\,\mathrm{d}x\right|&\le\|g\|_{C(\bar\Omega)}\int_\Omega|v_\infty(u_k(x))-v_\infty(u_k(x)\eta_j(x))|\,\mathrm{d}x\\&\le C\|g\|_{C(\bar\Omega)}\int_\Omega|u_k(x)|^p(1+\eta_j(x)^{p-1})(1-\eta_j(x))\,\mathrm{d}x.\end{aligned}$$

Further, as $\pi(\partial\Omega)=0$ we get

$$\begin{aligned}&\lim_{j\to\infty}\lim_{k\to\infty}\int_\Omega|u_k(x)|^p(1+\eta_j(x)^{p-1})(1-\eta_j(x))\,\mathrm{d}x=\\&\qquad\lim_{j\to\infty}\int_\Omega\int_{\beta_{\mathcal S}\mathbb{R}^m}\frac{|s|^p}{1+|s|^p}\mathrm{d}\lambda_x(s)(1+\eta_j(x)^{p-1})(1-\eta_j(x))d\pi(x)=0\end{aligned}$$

by the Lebesgue Dominated Convergence Theorem. Therefore

$$\lim_{j\to\infty}\lim_{k\to\infty}\int_\Omega g(x)v_\infty(u_k(x)\eta_j(x))\,\mathrm{d}x = \int_\Omega\int_{\beta_{\mathcal{S}}\mathbb{R}^m}\frac{v_\infty(s)}{1+|s|^p}\mathrm{d}\lambda_x(s)g(x)d\pi(x). \tag{A.15}$$

By density (A.15) holds for all continuous v_∞. As $\mathcal{S}$ and $C(\bar\Omega)$ are separable, we conclude by using a diagonalization argument. Similarly, the chosen subsequence generates the same Young measure as $\{u_k\}$. Therefore, the constructed sequence generates the same DiPerna-Majda measure as $\{u_k\}$ by (A.8). □

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