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GENERALIZED ZERO RANGE PROCESS AS A TRAFFIC MODEL

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Abstract: In the paper, a model of conservative particle system, which generalises a well known zero range process, is studied. The generalisation consists in allowing jumps of more than one particle in one moment. We describe what this generalisation means in the context of modeling a traffic flow.

Abstrakt: Příspěvek je věnován konzervativnímu částicovému systému, který zobecňuje známý částicový systém zvaný Zero-range proces. V tomto zobecnění povolíme přeskok více než jedné částice v jednom kroku. Proces použijeme pro modelování dopravního toku, kde má takové zobecnění velmi dobrý význam.

1 Introduction

We are interested in particle systems with zero range or exclusion interactions [1, 8]. A particle system is a large system of indistinguishable particles which occupy sites of a graph, typically lattice $\mathbb{Z}^d$ or for example binary trees [4]. The evolution of such a system in time is given by movement of particles between sites and this movement is influenced by interactions among particles. In case of zero range and exclusion interactions, the whole system is conservative, i.e. there are no births or deaths. Exclusion models allow at most one particle per site and interactions between two particles arise if one particle attempts to jump to the site occupied by the second one. Such jumps are excluded. Zero range models allow arbitrary number of particles per site and interactions arise just among particles at the same site, the particles are queueing. These systems are models for queuing systems, networks or traffic. From the theoretical point of view they are Markov processes with continuous time with a state space given by allowable particle configurations.

Particularly, the one dimensional ($d = 1$) exclusion process can be used as a simple model for a traffic flow. Each particle (a car) can move forward only if the site ahead is unoccupied. If zero range models are employed we gain wider possibilities how to describe different speeds of columns of various lengths. Both models were used as models for a traffic flow in [7].

In this contribution, we focus on a generalisation of zero range processes with allowance of multiple jumps. This generalisation would permit to model, in the traffic context, splitting of columns. Our aim is to study stationary states of this process and also a closely connected problem of the existence of a coupling process for the given dynamics.

Previously, stationary states for zero range dynamics generalised by multiple jumps were studied by Evans [2, 3] in the framework of finite volume systems, i.e. particles live only on a finite interval $\{1, ..., L\} \subset \mathbb{Z}$.

In a special case, when jumps of particles have a constant rate - independent of the number of particles at sites and also independent of the amount of jumping particles, a multiple jump model was treated in [9]. The model from paper [9], called *stick process* was studied in the context with Ulam's problem, i.e. evaluation of a limit theorem for the longest increasing subsequence of a random permutation of n symbols.

2 Definition of model

Let us consider a particle system living on one dimensional lattice $\mathbb{Z}$, each site $i \in \mathbb{Z}$ is occupied by an arbitrary number $\eta(i)$ of particles and the whole particle configuration $\eta = (\eta(i) : i \in \mathbb{Z})$ is a state of our process $(\eta_t)_{t\geq 0}$. We suppose that the particle system is evolving in time in the sense given by series of independent Poisson processes, where each of them rules movement of particles at just one site i, and exactly events of these Poisson processes are only possible moments for jumps of particles from one site to another. We will consider here the totally asymmetric case, it means that jumps are possible only between neighbours on $\mathbb{Z}$ and only in one direction (let us say from the left to the right). In classical zero range processes, the jump of at most one particle in one moment is possible. In generalised - *multiple jumps* - zero range processes (MJ-ZRP), the following jumps may occur in time t of a Poisson event at i:

> k-many particles from total amount $\eta_{t_-}(i)$ leaves site i and moves to site $i+1$.

The rate of this jump is equal to

$$g(k, \eta_{t_-}(i)) \tag{1}$$

where g is a nonnegative function on $\mathbb{N} \times \mathbb{N}$, $g(k, \alpha) = 0$ if $k > \alpha$.

The classical zero range process (ZRP) is then a special case with

$$g(k, \alpha) = \mathrm{I}_{[k=1]} r(\alpha) \tag{2}$$

for a nonnegative function r on $\mathbb{N}$, $r(0) = 0$. So only one particle can jump in time t of a Poisson event at i and a rate of the jump is $r(\eta_{t_-}(i))$. A meaning of the rate is following: looking at site i in time s we will wait for a jump of one particle from i an exponential time with mean $1/r(\eta_s(i))$. We can imagine an exponential clock at each site instead of the Poisson process. Note that always when a jump happens some of exponential clocks have to change their means according to new configuration η.

A totally asymmetric *multiple jumps - zero range process* on $\mathbb{Z}$ can be defined as a Markov process $(\eta_t)_{t\geq 0}$ with a state space $\mathrm{X} \subset \mathbb{N}^{\mathbb{Z}}$ given by infinitesimal generator

$$(\text{Ł}f)(\eta) = \sum_{i\in\mathbb{Z}} \sum_{k\in\mathbb{N}^+} g\left(k, \eta\left(i\right)\right) \; \left(f\left(\eta^{k,i}\right) - f\left(\eta\right)\right) \tag{3}$$

for every $\eta \in \mathrm{X}$ and every cylinder function $f : X \to \mathbb{R}$.

Notation and remark on definition:
1. We denoted by $\eta^{k,i}$ a changed configuration after a multiple jump

$$\eta^{k,i}(j) = \begin{cases} \eta(i) - k & \text{if } j = i \\ \eta(i+1) + k & \text{if } j = i+1 \\ \eta(j) & \text{otherwise.} \end{cases}$$

2. A function $f : \mathbb{N}^{\mathbb{Z}} \to \mathbb{R}$ is called *cylinder function*, if there exists a finite $K \subset \mathbb{N}$ such that $f(\eta)$ depends only on $\eta_{\upharpoonright K} = (\eta(j) : j \in K)$ for each $\eta \in \mathbb{N}^{\mathbb{Z}}$.
3. The reason, why we can not consider the whole product space $\mathbb{N}^{\mathbb{Z}}$ as a state space, is that even if we assume bounded function g there are particle configurations that could induce arrival of infinitely many particles to the same site in a finite time. This is the case of configurations with too many particles laying on the left halfline near $-\infty$. Hence we allow only following set of configuration $\mathrm{X} = \{\eta \in \mathbb{N}^{\mathbb{Z}} : \lim_{n\to-\infty} \frac{1}{n^2} \sum_{i=n}^{-1} \eta(i) = 0\}$, see [9].

Let us state here for completeness a generator of the classical zero range process (totally asymmetric):

$$(\text{Ł}_c f)(\eta) = \sum_{i\in\mathbb{Z}} r\left(\eta\left(i\right)\right) \; \left(f\left(\eta^{1,i}\right) - f\left(\eta\right)\right) \tag{4}$$

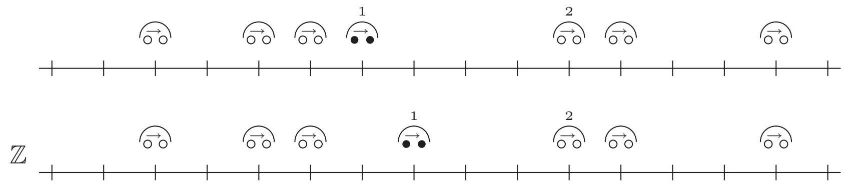
for every $\eta \in \mathbb{N}^{\mathbb{Z}}$ and every cylinder function $f : \mathbb{N}^{\mathbb{Z}} \to \mathbb{R}$. No restriction on the state space are needed if we assume r bounded. For a case of unbounded r, see [1].

3 Traffic interpretation

In the framework of *finite volume*, zero range process was used as a model for traffic flow in work [7]. Particle systems in finite volume consider only finite configurations of particles living on an interval $[1, ..., L]$ instead of $\mathbb{Z}$, usually with periodic boundary condition (site 1 is assumed as the neighbour on the right from site L). In this paper, we are interested in models in infinite volume, defined in (3).

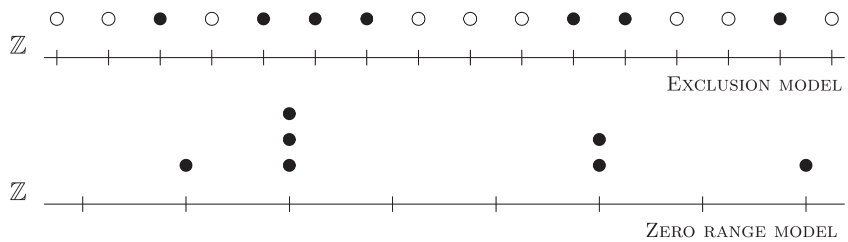
Let us interpret particles as cars and one dimensional lattice $\mathbb{Z}$ as a road on which cars move in one direction (to the right, by our definition). So each $i \in \mathbb{Z}$ is either occupied by a car or is empty. The car at site i can move to site $i+1$ only if site $i+1$ is empty. If there is an interval $[i, i+1, ..., j]$ of sites which are occupied by cars we will say that there is a column (convoy) of cars. This simple model can be described by a totally asymmetric exclusion process, which refers to evolution of a configuration $(\eta(i) : i \in \mathbb{Z})$, where

$\eta(i) = 1$ if site i is occupied or $\eta(i) = 0$ if it is empty. A jump of a particle from site i to $i+1$ has rate $\mathrm{I}_{[\eta(i)=1,\eta(i+1)=0]}$, i.e. a constant rate of jumps from an occupied site to its empty right neighbour.



Therefore, if cars have empty space ahead of them they move fluently with constant rate 1 (car 1 in the picture above). If, on the other hand, the traffic is heavy then next site could be occupied and then the car has to wait (car 2 in the picture above).

But we would like if our traffic model covers the fact that the speed of a single car, a short column and a long columns differs. From this reason we use instead of the simple exclusion process the classical zero range process (4). Note that the models can be transformed one by one in a simple way demonstrated on a folllowing picture.



Now the first car of column i moves with a rate (speed) $r(\eta(i))$ where $\eta(i)$ is a number of cars of i-th column. In this context, it is natural to assume that r is nonincreasing. Since ZRP with decreasing rate function r is not attractive (attractiveness is defined in the next section), a lot of works on ZRP excluded this case. ZRP with decreasing r was studied in details in [6].

The generalisation of zero range process defined by (3) allows to consider situations that k particles of $\eta(i)$ move together in one moment, which means that a column of length k breaks away from i-th columns with a rate $g(k, \eta(i))$.

4 Coupling process

In this section, we investigate the problem of attractiveness of MJ-ZRP (3). Since the attractiveness is closely related to ordering of particle configurations, let us start with some notation.

Let us consider the following partial order on state space $\mathrm{X} \subset \mathbb{N}^{\mathbb{Z}}$: $\eta \leq \zeta$ if and only if $\eta(i) \leq \zeta(i)$ for every $i \in \mathbb{Z}$, $\eta, \zeta \in \mathrm{X}$. We say that a function f on X is *monotone* if $f(\eta) \leq f(\zeta)$ whenever $\eta \leq \zeta$.

We say that a particle system is *attractive* if for every bounded, monotone continuous function f and every time $t > 0$, function $S_t f$ is again bounded, monotone continuous function, where $(S_t f)(\xi)$ for each $\xi \in \mathrm{X}$ is the expected value of $f(\eta_t)$ under assumption that the process $(\eta_t)_{t\geq 0}$ have started from configuration ξ at time 0. In another words, an attractive system preserves ordering in the following sense. If we imagine two initial configurations η, ζ which are ordered, $\eta \leq \zeta$, and then we let them evolve by the same dynamics and control both of them by the same set of Poisson processes (same realisation of exponential clocks), the configurations of these systems in arbitrary time will be again ordered, $\eta_t \leq \zeta_t$.

The technique, we have just now used to explain what attractiveness means, is called *coupling technique*. It is very usual tool and, specially, the construction of a *coupling process* (i.e. particle system (η_t^1, η_t^2) on $\mathrm{X}\times\mathrm{X}$ where each its marginal η_t^i is the original process) which preserves ordering of its marginals (i.e. $\eta_0^1 \leq \eta_0^2$ implies $\eta_t^1 \leq \eta_t^2$ for every $t > 0$) acts as a proof of attractiveness. However it is not the aim of this paper to present results about coupling process. We focus only on conditions on rate function g under which the process is or is not attractive. A thorough characterisation of coupling rates for a big class of conservative process, MJ-ZRP among them, can be found in [5].

The attractiveness is an important property and a lot of results on particle systems were proved only using the fact that a process is attractive. On the other hand, if a process is not attractive then it could have some special properties. At the end of the previous section, we said that using ZRP as a traffic model it is natural to assume nonincreasing speed function r. And it is well known fact, see e.g. [1], that in this case (except the constant speed function) ZRP is not attractive. In [6], cases of decreasing speed function r are studied for which so called *condensation* occurs. They showed that there exists a critical density of particles and if we start with a configuration with a supercritical density, a mass of particles will concentrate at one single site. If we interpret this condensation phenomena in the traffic context we obtain a critical density of traffic flow beyond which traffic jam occurs.

In the following proposition we examine single cases of rate function g of MJ-ZRP and state conditions on g so that the process is attractive.

Proposition 4.1. Let us consider MJ-ZRP given by generator (3). We distinguish following cases:

$$g(k,\alpha) = h(k)\,\mathrm{I}_{[k\leq\alpha]} \tag{5}$$
$$g(k,\alpha) = r(\alpha)\,\mathrm{I}_{[k\leq\alpha]} \tag{6}$$
$$g(k,\alpha) = r^*(\alpha)r^*(\alpha-1)...r^*(\alpha-k+1)\,\mathrm{I}_{[k\leq\alpha]}. \tag{7}$$

for some functions h, r and r^*.

The process is attractive if and only if following conditions on function h, r and r^*, respectively, are satisfied:

- $h(k)$ nonincreasing for $k \geq 1$, nonnegative,
- $r(\alpha)$ nonincreasing & $\alpha r(\alpha)$ nondecreasing for $\alpha \geq 1$, nonnegative, and
- $0 < r^*(\alpha) \leq 1$ & $\sum_{i=1}^{k} r^*(k) \dots r^*(i) \leq \frac{r^*(\alpha+1)}{1-r^*(\alpha+1)}$ for every $1 \leq k \leq \alpha$,

respectively.

This result is a simple consequence of a general theorem presented in [5] where necessary and sufficient conditions for a general class of conservative particle systems are proved. Note that a sufficient condition on r^* to have the process attractive is to assume r^* positive, bounded by 1 and nondecreasing.

An interesting problem now could be to study the condensation phenomena for generalised zero range process (3) with such rate function g for which we have obtained by Proposition 4.1 the process is not attractive. To study this problem we need to know how invariant measures in mentioned cases of the rate function look like.

5 Invariant measures

This part is devoted to the study of invariant measures of MJ-ZRP (3). Invariant measures are stationary states of a Markov process which means: if the distribution of the initial particle configuration is just a stationary distribution then looking at the system in arbitrary fixed time we can observe again the same distribution as at the start.

A measure μ on $\mathrm{X} \subset \mathbb{N}^Z$ is called invariant measure for the process with generator Ł if

$$\int Łf \; \mathrm{d}\mu = 0 \tag{8}$$

for every cylinder function $f : \mathrm{X} \to \mathbb{R}$.

Note that (8) is other, more useful formulation of a standard formula for invariant measures

$$\int (S_t f)(\xi) \; \mu(\mathrm{d}\xi) = \int f(\xi) \; \mu(\mathrm{d}\xi), \text{ for every continuous, bounded } f,$$

since generator Ł is given explicitly for a majority of particle systems. In case of particle systems there is usually large set of invariant measures indexed by a real parameter. An exhaustive description of all invariant measures for given dynamics is then to set exactly how $\mathcal{I}_e$, the set of extremal invariant measures, looks like. Since the problem of finding $\mathcal{I}_e$ is nontrivial, a complete description of $\mathcal{I}_e$ is known only for basic models. Sometimes, a simpler task is to look for invariant measures which are in addition translation invariant,

$$\text{i.e. } \mu(\{\eta : \eta(i) \in A_i,\, i \in I\}) = \mu(\{\eta : \eta(i+j) \in A_i,\, i \in I\}) \text{ for each } j \in \mathbb{Z}.$$

A result on invariant measures for MJ-ZRP we want to present is following.

Theorem 5.1. Let us consider MJ-ZRP given by generator (3). If

$$g(k,\alpha) = \frac{h(k)r(\alpha)}{r(\alpha-k)} \text{ for every } 1 \le k \le \alpha,\ \alpha \ge 1, \tag{9}$$

for some h nonnegative, r positive function, then product measures ν^φ on X,

$$\nu^\varphi(\{\eta : \eta(i) = n\}) = Z_\varphi \frac{\varphi^n}{r(n)} \text{ for every } i \in \mathbb{Z}, n \in \mathbb{N}, \tag{10}$$

with a normalising constant Z_φ, are invariant for (3) for every $\varphi \in (0, c)$.

Let us discuss now the particular cases which we studied in the previous section. We can see that cases (5) and (7) are covered by Theorem 5.1 and product invariant measures ν^φ have the following form:

$$\begin{aligned} \nu^\varphi(\eta(i) = n) &= Z_\varphi \varphi^n / (r^*(n) r^*(n-1) ... r^*(1)), && \text{in case (7)}, \\ \nu^\varphi(\eta(i) = n) &= \varphi^n(1-\varphi), && \text{in case (5)}. \end{aligned}$$

Hence invariant measures ν^φ in case (5) do not depend on the exact form of function h. Note that the same measures are also invariant for a classical (totally asymmetric) ZRP (4) with a constant rate function $r(\alpha) = \mathrm{I}_{[a>0]}$.

The situation in case (6) is no more analogical to classical ZRP (4) which has product invariant measures, $\nu^\varphi(\eta(i) = n) = Z_\varphi \frac{\varphi^n}{r(n) r(n-1) ... \cdot r(1)}$, see e.g. [1]. In case (6) of MJ-ZRP, invariant and translation invariant measures are no more product ones.

Theorem 5.2. MJ-ZRP with rate function $g(k,\alpha) = r(\alpha)$, case (6), has no product translation invariant measures which are invariant with respect to (4), except a trivial case when r is constant.

A similar problem was treated in [3] in the framework of finite volume particle systems. They proved that finite volume MJ-ZRP has product invariant measures if and only if rate function g can be formulated as in (9). So Theorem 5.1 generalises one implication of this result for infinite volume MJ-ZRP and Theorem 5.2 generalises the second implication for one particular case of infinite volume MJ-ZRP.

Sketch of proof of Theorem 5.1. One can show that following condition on rate function g:

$$g(k, k+l) \frac{\mu(\eta(i) = k+l)}{\mu(\eta(i) = l)} = c(k), \text{ for each } k \ge 1,$$

where c is a function independent of $l \ge 0, i \in \mathbb{Z}$, is sufficient for (8) to be satisfied for a product measure μ on X. The proof is done, if we verify the above conditions with g as given in (9) and $\mu = \nu^\varphi$ defined in (10).

Sketch of proof of Theorem 5.2. We find necessary conditions on a product, translation invariant measure μ on X for (8) to be satisfied. Let us denote $\pi(n) = \mu(\eta(i) = n)$. Then for arbitrary $n \in \mathbb{N}$, $(l_1, ..., l_n) \in \mathbb{N}^n$

$$\begin{aligned} 0 \quad = \quad & \sum_{i=2}^{n} \sum_{k=1}^{l_i} \left(r(l_{i-1}+k) \frac{\pi(l_{i-1}+k)}{\pi(l_{i-1})} \frac{\pi(l_i-k)}{\pi(l_i)} - r(l_i) \right) + \\ & + \sum_{k=1}^{l_1} \sum_{m \geq 0} \left(r(m+k) \frac{\pi(m+k)}{\pi(m)} \frac{\pi(l_1-k)}{\pi(l_1)} - r(l_1) \right) \pi(m) + \\ & + \sum_{k \geq 1} \left(r(l_n+k) \frac{\pi(l_n+k)}{\pi(l_n)} - \sum_{m \geq 0} r(m+k) \pi(m+k) \right) \end{aligned}$$

are desired necessary conditions. One can show that these conditions does not hold if r is not a constant function.

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