THE EQUATIONS OF STOCHASTIC NONLINEAR OSCILLATOR DRIVEN BY FRACTIONAL BROWNIAN MOTION

Jana Šnupárková

Abstract: Existence of a weak solution to the n-dimensional equation of stochastic nonlinear oscillator driven by a fractional Brownian motion with Hurst parameter $H \in (0,1) \setminus \{\frac{1}{2}\}$ has been shown if a diffusion matrix is time-dependent but state-independent and a drift may be singular but has to satisfy conditions of Girsanov Theorem.

Keywords: Fractional Brownian Motion, Girsanov Theorem, Weak Solution, Stochastic Nonlinear Oscillator.

1 Introduction

In this paper, we prove an existence of a weak solution to the n-dimensional equation of stochastic oscillator

$$\frac{d^2}{dt^2}x_t + F(t, x_t, \frac{d}{dt}x_t) = \bar{\sigma}(t)\frac{d}{dt}B_t^H \ ,$$

where $B^H = \left((B^H)^i\right)_{i=1}^n$ is an n-dimensional **fractional Brownian motion** with Hurst parameter $H \in (0,1)$, i.e. B^H is a centered Gaussian process with covariance function

$$\mathbb{E}\left[(B_s^H)^i (B_t^H)^j\right] = \frac{1}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right)\delta_{ij}, \quad s,t \geq 0,\ i,j = 1,\ldots,n \ .$$

Process B^H is a standard Wiener process for $H = \frac{1}{2}$. For $H \neq \frac{1}{2}$ B^H has a version with Hölder continuous trajectories of order γ for any $0 < \gamma < H$ and has stationary increments. Nevertheless, B^H is neither Markov process nor a semimartingale, hence standard methods of integration are not applicable. This example of a stochastic oscillator and other results about existence of weak solutions with full proofs will appear in [11]. There are many papers devoted to equations driven by a fractional Brownian motion, e.g. in [2], [8] the strong existence and uniqueness of solutions to one-dimensional SDE's for any $H \in (0,1)$ is established. In the case $H > \frac{1}{2}$ the existence and uniqueness of solutions is also proved in [4], [5] (by the rough path approach, using Young type integrals and the concept on p-variation) and in [10]. An existence of weak solutions in one-dimensional case is studied in [8], [9], [7] and [1] using Girsanov Theorem for fractional Brownian motion. Stochastic equations in Hilbert spaces driven by a fractional Brownian motion are studied in [3], [6].

2 Preliminaries

Consider an n-dimensional stochastic differential equation

$$X_t = \tilde{x} + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s)\, dB_s^H \ , \tag{2.1}$$

where $\tilde{x} \in \mathbb{R}^n$ is deterministic initial condition, $b : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ is a drift which can be divided into a regular part and a singular part and $\sigma : [0,T] \to \mathcal{L}(\mathbb{R}^n)$ is a diffusion, which is time-dependent but state-independent. $\mathcal{L}(\mathbb{R}^n)$ is a space of all linear bounded operators from $\mathbb{R}^n$ to $\mathbb{R}^n$. Furthermore, consider another simplier equation

$$Y_t = \tilde{x} + \int_0^t b_1(s, Y_s)\, ds + \int_0^t \sigma(s)\, dB_s^H \ . \tag{2.2}$$

Definition 2.1. *An adapted process with continuous trajectories is a* ***solution*** *to the equation (2.2) if* $\{Y_t,\, t \in [0,T]\}$ *satisfies the equation (2.2). The solution to the equation (2.2) is* ***pathwise unique*** *if*

$$\mathbb{P}\{Y_t = \tilde{Y}_t \quad \forall\, t \in [0,T]\} = 1$$

holds for any two solutions $\{Y_t,\, t \in [0,T]\}$, $\{\tilde{Y}_t,\, t \in [0,T]\}$.
By a ***weak solution*** *to the equation (2.1) we mean a couple of adapted processes* (B^H, X) *with continuous trajectories on a complete probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *such that* B^H *is an* n*-dimensional fractional Brownian motion on the interval* $[0,T]$ *and* X *and* B^H *satisfy (2.1).*

The next theorem is a main result in [11]. Remark that $\mathcal{C}^\delta([0,T]; \mathcal{L}(\mathbb{R}^n))$ is a space of all Hölder continuous mappings of order δ, $0 < \delta < 1$, from the interval $[0,T]$ to the space $\mathcal{L}(\mathbb{R}^n)$.

Theorem 2.2. *Let* $b_1, b_2 : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0,T] \to \mathcal{L}(\mathbb{R}^n)$ *be Borel mappings such that* $b = b_1 + b_2$ *on* $[0,T] \times \mathbb{R}^n$ *and assume that there exists a Borel measurable inverse* σ^{-1} *of* σ. *Suppose that*

$$\exists K_b > 0\ \forall t \in [0,T]\ \forall x \in \mathbb{R}^n \quad \|b_1(t,x)\| \le K_b(1 + \|x\|) \tag{2.3}$$

and there exists a solution Y *to the equation (2.2). Set* $u(t) = \sigma^{-1}(t) b_2(t, Y_t)$, $t \in [0,T]$. *Assume that* $u \in L^\infty([0,T]; \mathbb{R}^n)$ $\mathbb{P}$*-almost surely and either*
$H < \frac{1}{2}$, $\sigma \in \mathcal{C}^{\delta^*}([0,T]; \mathcal{L}(\mathbb{R}^n))$ *for some* $\delta^* \in (\frac{1}{2} - H, 1)$ *and* $\exists K > 0$
$\forall t \in [0,T]\ \forall x \in \mathbb{R}^n$

$$\|\sigma^{-1}(t)\, b_2(t,x)\| \le K(1 + \|x\|) \ ,$$

or
$H > \frac{1}{2}$, $\sigma \in L^\infty([0,T]; \mathcal{L}(\mathbb{R}^n))$ *and* $\exists \alpha \in (1 - \frac{1}{2H}, 1)$ $\exists \beta \in (H - \frac{1}{2}, 1)$
$\exists C > 0\ \forall s,t \in [0,T]\ \forall x,y \in \mathbb{R}^n$

$$\|\sigma^{-1}(t)\, b_2(t,x) - \sigma^{-1}(s)\, b_2(s,y)\| \le C(\|x - y\|^\alpha + |t - s|^\beta) \ .$$

Then there exists a weak solution to the equation (2.1).

Sketch of the proof (cf. [11] for details) First we show that there exists a version of the stochastic integral $\{\int_0^t \sigma(s)\, dB_s^H,\ t \in [0,T]\}$ with Hölder continuous trajectories of order $\gamma,\ 0<\gamma<H$, using Kolmogorov-Chentsov Theorem. As H increases, conditions on integrand σ are less restrictive and computations in the proof of Hölder continuity are easier for $H > \frac{1}{2}$ then for $H < \frac{1}{2}$. Next we prove that if we have solution Y to the equation (2.2), where b_1 is a Borel function satisfying condition (2.3), then there exists a version of $\{Y_t,\ t \in [0,T]\}$ with Hölder continuous trajectories of order $\gamma,\ 0<\gamma<H$. The next aim is to use Girsanov Theorem which for a fractional Brownian motion takes the following form

Theorem 2.3. *Let $B^H = \{B_t^H,\ t \in [0,T]\}$ be an n-dimensional fractional Brownian motion with Hurst parameter H on the interval $[0,T]$. Consider an adapted n-dimensional process $u = \{u_t,\ t \in [0,T]\}$ with integrable trajectories. Set*

$$v(s) = K_H^{-1}\left(\int_0^{\cdot} u_r\, dr\right)(s)\,, \quad s \in [0,T]\ ,$$

$$\xi_T = \exp\left\{\int_0^T v_s^{\mathrm{T}}\, dW_s - \frac{1}{2}\int_0^T \|v_s\|^2\, ds\right\}$$

and assume that

$$(i) \quad \int_0^{\cdot} u_s\, ds \in I_{0+}^{H+\frac{1}{2}}\big(L^2([0,T];\mathbb{R}^n)\big) \quad \mathbb{P}-a.s.\ ,$$

$$(ii) \quad \mathbb{E}\,(\xi_T) = 1\ .$$

Then $\{B_t^H - \int_0^t u_s\, ds,\ t \in [0,T]\}$ is an n-dimensional fractional Brownian motion with Hurst parameter H on the interval $[0,T]$ under the probability $\tilde{\mathbb{P}}$ defined by the density $\xi_T = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$ with respect to $\mathbb{P}$.

Note that $\{W_t,\ t \geq 0\}$ is an n-dimensional Wiener process defined by

$$W_t = \int_0^T \big(\mathcal{K}_H^*\big)^{-1}(I_{[0,t]}\, id)(s)\, dB_s^H, \quad t \in [0,T]\ ,$$

where $\mathcal{K}_H^*$ is an isometry between the space $(\mathcal{H}, \|\,.\,\|_{\mathcal{H}})$ of all deterministic time-dependent integrable functions with respect to the fractional Brownian motion and a space $L^2(\Omega, \mathcal{L}(\mathbb{R}^n))$ (for precise definitions see [11]). The space $I_{0+}^{H+\frac{1}{2}}\big(L^2([0,T];\mathbb{R}^n)\big)$ is an image of $L^2([0,T];\mathbb{R}^n))$ under the operator $I_{0+}^{H+\frac{1}{2}}$ defined by the formula

$$\big(I_{0+}^{H+\frac{1}{2}}\varphi\big)(t) := \frac{1}{\Gamma(H+\frac{1}{2})}\int_0^t (t-s)^{H-\frac{1}{2}}\varphi(s)\, ds$$

for $\varphi \in L^1([0,T];\mathbb{R}^n)$. The above integral is well-defined for a.e. $t \in [0,T]$ and Γ denotes Gamma function.

The operator $K_H^{-1} : I_{0+}^{H+\frac{1}{2}}\big(L^2([0,T];\mathbb{R}^n)\big) \longrightarrow L^2([0,T];\mathbb{R}^n)$ is the inverse of the linear operator $K_H : L^2([0,T];\mathbb{R}^n) \longrightarrow I_{0+}^{H+\frac{1}{2}}\big(L^2([0,T];\mathbb{R}^n)\big)$ defined by

$$\big(K_H\varphi\big)(t) := \int_0^t K_H(t,s)\,\varphi(s)\,ds\ ,$$

where $K_H(t,s)$ is an integral kernel having a form

$$K_H(t,s) = \begin{cases} \frac{C_H}{H^*} s^{-H^*}\Big[(t.(t-s))^{H^*} - H^*.I_H\Big] & ,\quad s < t\ , \\ 0 & ,\quad s \geq t\ , \end{cases}$$

where $H^* = H - \frac{1}{2}$, $I_H = \int_s^t u^{H^*-1}(u-s)^{H^*}\,du$ and

$$C_H = \sqrt{\frac{H.H^*}{2B(2-2H,\,H^*)}}\ ,$$

and B denotes Beta function.
To verify conditions $(i),(ii)$ of Girsanov Theorem it is sufficient to show that

$$\mathbb{E}\ \exp\left\{\int_{t_1}^{t_2} \|v_s\|^2\,ds\right\} < +\infty$$

for any $0 \leq t_1 < t_2 \leq T$ enough small. This can be shown using Fernique theorem

Theorem 2.4. *Let $(V, \|\,.\,\|, \mathcal{B}(V))$ be a separable Banach space with a Borel σ-field $\mathcal{B}(V)$. Suppose that G is a V-valued zero-mean Gaussian random variable. Then there exists $\zeta > 0$ such that*

$$\mathbb{E}\ \exp\{\zeta\|G\|_V^2\} < +\infty\ .$$

Contrary to the regularity of trajectories of the process $\{\int_0^t \sigma(s)\,dB_s^H,\ t \in [0,T]\}$, conditions on the product $\sigma^{-1}b_2$ are more restrictive and computations are more difficult in the case $H > \frac{1}{2}$ than in the case $H < \frac{1}{2}$. Finally we show that the couple $(\{B_t^H - \int_0^t u_s\,ds,\ t \in [0,T]\}, Y)$ is a weak solution to the equation (2.1) on the probability space $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$, where $\tilde{\mathbb{P}}$ is a changed probability measure defined by a density ξ_T with respect to probability measure $\mathbb{P}$.

△

3 Equation of stochastic oscillator

Consider the formal equation

$$\frac{d^2}{dt^2}x_t + F(t, x_t, \frac{d}{dt}x_t) = \bar{\sigma}(t)\frac{d}{dt}\bar{B}_t^H\ , \tag{3.1}$$

the weak solution of which is defined as the weak solution to

$$X_t = y_0 + \int_0^t \big(b_1(X_s) + b_2(s, X_s)\big)\, ds + \int_0^t \sigma(s)\, dB_s^H \ , \tag{3.2}$$

where $\{B_t^H, t \in [0,T]\}$ is a $2n$-dimensional fractional Brownian motion whose first n components are components of $\bar{B}^H$ and where for $t \in [0,T]$ and $y = (x,v)^{\mathrm{T}} \in \mathbb{R}^{2n}$

$$\begin{aligned} X_t &:= \begin{pmatrix} x_t \\ v_t \end{pmatrix}, \\ b_1(y) &:= \begin{pmatrix} v \\ 0 \end{pmatrix}, \\ b_2(t,y) &:= \begin{pmatrix} 0 \\ -F(t,y) \end{pmatrix}, \\ y_0 &:= \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \end{aligned}$$

and

$$\sigma(t) := \left(\begin{array}{cc} 0 & 0 \\ 0 & \bar{\sigma}(t) \end{array} \right) \ ,$$

$\sigma(t)$ being a $2n \times 2n$-dimensional matrix.
Moreover, consider the linear equation

$$Y_t = y_0 + \int_0^t b_1(Y_s)\, ds + \int_0^t \sigma(s)\, dB_s^H \ . \tag{3.3}$$

Proposition 3.1. *Suppose that* $\sigma : [0,T] \to \mathcal{L}(\mathbb{R}^n)$ *is a Borel mapping satisfying either* $H < \frac{1}{2}$ *and* $\sigma \in \mathcal{C}^{\delta^*}([0,T]\,;\mathcal{L}(\mathbb{R}^n))$ *for some* $\delta^* \in (\frac{1}{2} - H, 1)$ *or* $H > \frac{1}{2}$ *and* $\sigma \in L^{\infty}([0,T];\mathcal{L}(\mathbb{R}^n))$. *Further, let* $b_1 : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ *be a Borel function satisfying the following conditions:*

$$\forall N \in \mathbb{N}\ \exists K_N > 0\ \forall t \in [0,T]\ \forall x, y \in \mathbb{R}^n\ \ \|x\| + \|y\| \le N$$

$$\|b_1(t,x) - b_1(t,y)\| \le K_N \|x - y\|,$$

and

$$\exists K_b > 0\ \forall t \in [0,T]\ \forall x \in \mathbb{R}^n \quad \|b_1(t,x)\| \le K_b(1 + \|x\|)\ .$$

Then there exists the pathwise unique solution to the equation (2.2).

Proof. Cf. [11].

Q.E.D.

Suppose that matrix $\bar{\sigma}$ is regular for all $t \in [0,T]$. Let

$$\Sigma(t) = \left(\begin{array}{cc} 0 & 0 \\ 0 & \bar{\sigma}^{-1}(t) \end{array} \right), \quad t \in [0,T]\ ,$$

be a $2n \times 2n$-dimensional matrix. It is easy to see from the proof that the statement of Theorem 2.2 holds if we replace σ^{-1} in this theorem by Σ. Suppose that $\bar{\sigma}$ is a Borel function satisfying either

(A1) $H < \frac{1}{2}$ and $\bar{\sigma} \in \mathcal{C}^{\delta^*}([0,T];\mathcal{L}(\mathbb{R}^n))$ for some $\delta^* \in (\frac{1}{2} - H, 1)$,

or

(A2) $H > \frac{1}{2}$ and $\bar{\sigma} \in L^\infty([0,T];\mathcal{L}(\mathbb{R}^n))$.

Function $b_1 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$; $y = (x,v)^{\mathrm{T}} \mapsto (v,0)^{\mathrm{T}}$ is Lipschitz (consequently b_1 satisfies condition (2.3)). Then there exists the pathwise unique solution $\{Y_t,\, t \in [0,T]\}$ to the equation (3.3) (cf. Proposition 3.1).
Assume that trajectories of the process $\{\bar{\sigma}^{-1}(t)F(t,Y_t),\, t \in [0,T]\}$ are in $L^\infty([0,T];\mathbb{R}^n)$ and suppose moreover either

(B1) $H < \frac{1}{2}$ and $\exists K > 0\ \forall t \in [0,T]\ \forall y \in \mathbb{R}^{2n}$

$$\|\bar{\sigma}^{-1}(t)F(t,y)\| \le K(1 + \|y\|) ,$$

or

(B2) $H > \frac{1}{2}$ and $\exists\, \alpha \in (1 - \frac{1}{2H}, 1)\ \exists \beta \in (H - \frac{1}{2}, 1)\ \exists C > 0\ \forall\, s,t \in [0,T]$ $\forall\, y_1, y_2 \in \mathbb{R}^{2n}$

$$\|\bar{\sigma}^{-1}(t)F(t,y_1) - \bar{\sigma}^{-1}(s)F(t,y_2)\| \le C(\|y_1 - y_2\|^\alpha + |t-s|^\beta) .$$

Then assumptions of Theorem 2.2 on a map $(t,y) \mapsto \Sigma(t)\, b_2(t,y)$, $t \in [0,T]$, $y \in \mathbb{R}^{2n}$, are satisfied because

$$\Sigma(t)\, b_2(t,y) = \begin{pmatrix} 0 \\ -\bar{\sigma}(t)F(t,y) \end{pmatrix} ,$$

hence equations (3.2) and thereby (3.1) have weak solutions.

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Address: Institute of Information Theory and Automation of the ASCR,
Pod Vodárenskou věží 4, 182 08 Prague 8
Czech Republic

E-mail: `snupark@utia.cas.cz`