A SCALE DEPENDENT MODEL OF THE URETHRAL TISSUE

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1 Introduction

A very important problem of biomechanics consists in finding reasonable mechanical models of studied tissues. These models should be not only realistic but appropriately simple and robust so that the number of "free" parameters (usually very problematic to be determined experimentally) be minimal. A typical tissue whose realistic mechanical model is very needed is the muscle tissue [1], [2].

In this paper, we present a very simple mechanical model of muscular work within the urethral tissue being able to close the urethra when stimulated and to open it when tending to a passive state. The model is based on the so called *scale dependent continuum approach* [3], [4], [5] in which the continual quantities are supposed to depend on the scale at which they are studied. The approach allows us to formulate the continuum theory simultaneously at least at two scales – a 'micro' one at which individual contractile units are simply modelled by using just two real parameters λ_i and the standard (macroscopic) scale at which we study the macroscopic deformation of the urethral pipe. We model 'strain' energy of these contractile elements by using the idea of *simple* thermodynamic systems which have, in the simplest case, the only one volume variable [6]. To define the change of the volume of individual particles we introduce not only the standard deformation gradient but also that defined at the lower scale. Though the tissue may be supposed to be incompressible from the macroscopic point of view we shall show that a compressibility at the micro-scale plays the crucial role. Namely it enables the living tissue to *relax*, i.e. find out a "low" minimum of energy at the microstructural level. This relaxation mechanism is very sensitive on a change of microstructural parameters (such as changes of protein conformations within contractile units) and, as a result, it allows to control effectively the stiffness of tissue at the macroscopic level. When "switching off" this mechanism ("dead" tissue) a striking change of mechanical properties happens.

We formulate the model and propose a simple transformation of the Young modulus measured on a dead tissue [7] to a crucial energetic parameter of the model of living tissue. Then we find out a simple (two dimensional) model of the urethral wall enabling us to explain two typical states of the urethra (the closed and open one) as just a result of some microstructural changes within contractile units. The model gives correct values of pressure

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being measured at those states without using *any* additional "free" parameter than that above mentioned one, which is fully fixed by the measurement.

2 A simple scale dependent model

The continuum description identifies some parts of a deformable body – usually called the *particles* – with material points. These points form a continuous structure modelling the body so that an (actual) configuration of the body can be described by a vector function $\mathbf{x} : \Omega \to R^n$, i.e. $\mathbf{x}(\mathbf{X})$. In our approach each particle is understood to be a *simple thermodynamic system* [6]. It means that it can exchange energy with its surrounding and has own state variables (in standard continuum description the energy and state is connected only with a relative position of particles). We ignore thermal effects and thus we can use the simplest thermodynamic description in which the energy of the system is a function of its volume, namely $E(V)$. We will study only elastic deformations and the energy E is understood to be the *strain* energy stored in the particle as a result of its deformation. Since we work in continuum description we introduce the relative change of volume, $v \equiv V/V_r$, where V_r is the volume of the particle in the reference configuration, and, similarly, we will work with the *specific strain energy* per unit mass, e, instead of E. Both quantities are scalar fields, i.e. $v(\mathbf{X})$ and $e(\mathbf{X})$.

In continuum mechanics, the relative change of local volume is defined as $v_0 = |\det \mathbf{F}|$. However, understanding the particle as an infinitesimal thermodynamic system, v_0 cannot be a relative change of the volume of *this* system. We need more such systems (particles) 'surrounding' that to determine v_0 ($\mathbf{F}$ describes a relative change of *positions* of particles) [8]. Generally, the field v can be different from v_0. The crucial step of our approach is based on the idea of the so called *scale dependent continuum description*: We suppose that the deformation gradient is scale dependent, i.e. $\mathbf{F}$ can be understood as a function $\mathbf{F}(l)$, where l is a characteristic length – scale. Moreover we assume that there is a scale l_m at which the deformation gradient $\mathbf{F}(l_m) \equiv \mathbf{F_m}$ describes inner deformations of individual particles so that the relative change of the particle's volume v is defined as

$$v = |\det \mathbf{F_m}|. \tag{1}$$

In what follows, the symbol $\mathbf{F}$ denotes the deformation gradient at some fixed scale l_0, $l_0 > l_m$, at which the macroscopic deformation of the body is studied.

The total specific strain energy is given by the sum of the deformation energy of individual particles, $e(v)$, and the strain energy coming from mutual interactions of individual particles. The second one depends not only on the deformation gradient $\mathbf{F}$ because there is also an influence of own deformations of particles. Namely the energy connected with couplings of particles can be increased or decreased only by changing their own deformations – see Fig. 1. This influence may be estimated by a correction $\sim \mathbf{M} \cdot (\mathbf{F_m} - \mathbf{F})$ to the standard deformation gradient, where the tensor $\mathbf{M}$ describes a measure of influence of the shape of individual particles and will be called the *microstructural tensor* in what follows. Thus we express the specific strain energy w as a function

$$w = W(\mathbf{F_{ef}}, v), \tag{2}$$

where

$$\mathbf{F_{ef}} = \mathbf{F} + \mathbf{M} \cdot (\mathbf{F_m} - \mathbf{F}) \tag{3}$$

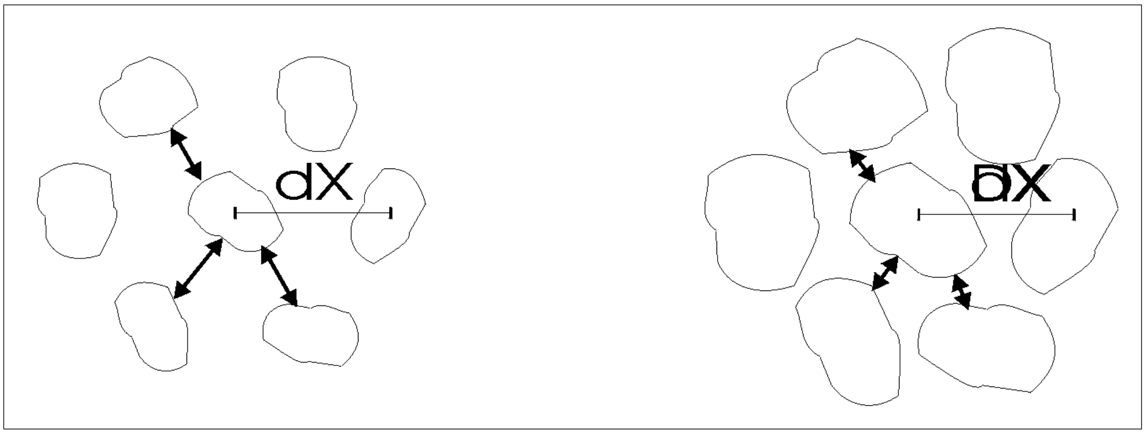

Figure 1: The interaction energy of particles can be changed by changing the size of particles while the relative position of particles (described by the field $\mathbf{F}$) is fixed.

and v is given by (1). Let us notice that if there are *no scale effects*, i.e. $\mathbf{F} = \mathbf{F_m}$, we obtain the strain energy in a form $w = W(\mathbf{F}, \det \mathbf{F})$, which corresponds with some useful constitutive laws of standard continuum mechanics [9].

Let us restrict our discussion to elastic materials whose possible configuration minimizes the integral

$$I = \int_\Omega w(\mathbf{X})\mathbf{d^n X} \tag{4}$$

within a suitable class of functions $\mathbf{x}(\mathbf{X})$ respecting boundary conditions. When solving the problem of minimizing the integral (4) we can obtain two different solutions:
1. The microstructural tensor is fixed and we obtain a solution of (4) explicitly dependent on microstructure.
2. We suppose that the structure at lower scale l_m *relaxes* so that it minimizes the local stored energy at any point $\mathbf{X}$ while $\mathbf{F}$ has a fixed value (we look for the shape of particle minimizing the strain energy while positions of surrounding particles are fixed). In other words, finding the minimal value over all possible tensors $\mathbf{F_m}$ (with fixed $\mathbf{F}$) we obtain an effective strain energy w_0 defined at the scale l_0, i.e.

$$w_0(\mathbf{F}, \mathbf{X}) = \min_{\mathbf{F_m}} \mathbf{W}(\mathbf{F_{ef}}, \det \mathbf{F_m}). \tag{5}$$

Then the energy w_0 instead of w is used in the minimizing problem (4).

3 Deformation of elastic, isotropic, uniform pipes

Let us study radial deformation of a pipe whose deformation is supposed to be the same along the pipe. The pipe with radii R_a (inner) and R_b (outer) is exposed to the pressures p_a (internal) and p_b (external) so that $\Delta p \equiv p_a - p_b > 0$. Since the deformation is supposed to be uniform (the same along the pipe) and radial we can describe it by the function $r(R)$ where R is the point in reference configuration. The principal values of the (macroscopic) strain tensor $\mathbf{F}$ are r' ($\equiv dr/dR$) and r/R. Concerning the microstructure we suppose that, in the cylindrical coordinate system, the tensor is diagonal too, i.e. we have

$$\mathbf{F} = \begin{pmatrix} r' & 0 \\ 0 & r/R \end{pmatrix}, \qquad \mathbf{F_m} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}. \tag{6}$$

The microstructural tensor is supposed to be of the form

$$\mathbf{M} = \begin{pmatrix} -\frac{\lambda_1}{1-\lambda_1} & 0 \\ 0 & -\frac{\lambda_2}{1-\lambda_2} \end{pmatrix}. \tag{7}$$

Taking into account that the strain energy has its minimum at the reference configuration

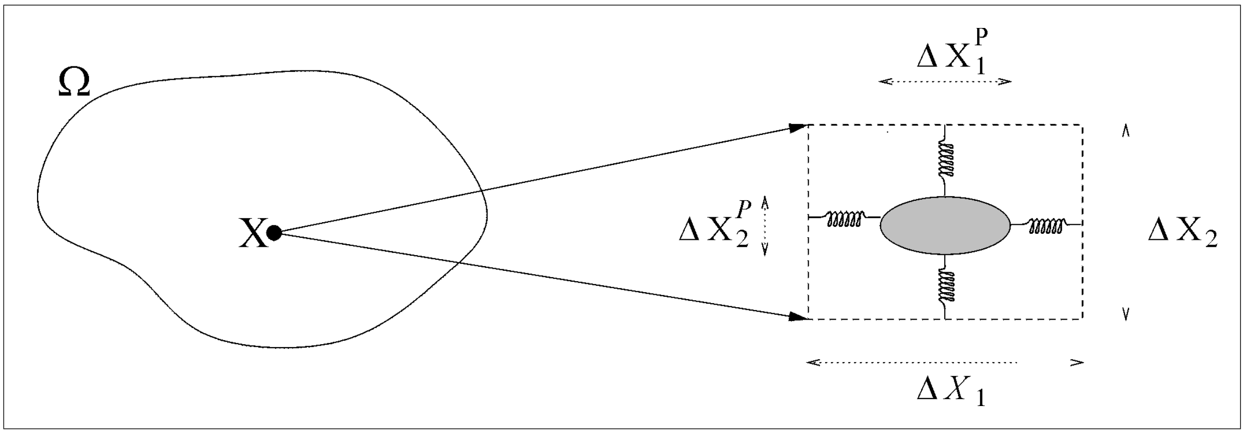

Figure 2: A simple spring model with deformable particles

we can choose its quadratic approximation as follows

$$w(\lambda_i, \alpha_i, E, K) \equiv \frac{E}{2}\left[\left(\frac{r' - \alpha_1\lambda_1}{1-\lambda_1} - 1\right)^2 + \left(\frac{r/R - \alpha_2\lambda_2}{1-\lambda_2} - 1\right)^2\right] + K\left(\alpha_1\alpha_2 - 1\right)^2. \tag{8}$$

where E and K are physical parameters, which should be determined by an experiment. In fact, this model describes a simple model consisting of deformable particles interconnected by linear springs so that $\lambda_i = \Delta X_i^p / \Delta X_i$ – see Fig. 2.

Living tissues may be supposed to be incompressible [1]. It means that there is no volume change at the (macro) scale l_0, i.e.

$$r' \frac{r}{R} = 1. \tag{9}$$

If we study only the deformations with $r \geq R$ the condition (9) has the solutions

$$r(R) = \sqrt{R^2 + C}, \tag{10}$$

where the constant C depends on the pressure difference Δp and the solution of the problem reduces to specification of this dependence. As known ([5]), the minimizing problem leads to the integral relation

$$\Delta p = \int_{R_a}^{R_b} \frac{1}{r}\left(\psi_2 - \left(\frac{R}{r}\right)^2 \psi_1\right) dR, \tag{11}$$

where ψ_1, ψ_2 are partial derivatives (with respect to r' and r/R respectively) of the strain energy function. That can be defined by (8) or, by accepting the relaxation of microstructure, it is defined by (5) at each point R, i.e.

$$\psi = \min_{\alpha_1, \alpha_2} w(\lambda_i, \alpha_i, E, K). \tag{12}$$

4 An identification of coefficients

To apply the model to a urethra we have to explain in which way we understand the model so that the coefficients and parameters were correctly identified. In fact, the energy w depends not only on some material coefficients, E and K, but also on the microstructural parameters λ_i. As explained in [12] a change of these parameters can lead to an extremal change of the energy (12) yielding a broad spectrum of mechanical properties of the studied material. This is exactly what muscle tissue does. Thus we expect a deeper connection between these parameters and some parameters describing muscle stimulation. It implies that the passive state of muscle is defined by such λ's giving the softest tissue.

Nevertheless, what happens if a piece of a tissue is not involved in living processes but it is removed from the tissue and used as a measuring sample. May we identify its state as a state having muscle tissue in a passive state? This question is extremely important because we, in fact, measure mechanical properties on such a "dead" tissue. A simple calculation reveals that we *cannot* identify dead tissue with that in a passive state: A measurement of the Young modulus of urethral tissue has been done [7] with the result $E_Y \approx 0.25$MPa (it is in a good agreement with other measurements given in various works). Calculating a small deformation of an incompressible pipe with the Young modulus E_Y exposed by the pressure difference Δp we obtain the displacement

$$u(R) = \frac{3}{2} E_Y^{-1} (R_a^{-2} - R_b^{-2})^{-1} R^{-1} \Delta p. \tag{13}$$

Taking into account characteristic values found in urethra ($R_a \sim 1$mm, $R_b \sim 3$mm, $u(R_a) \sim 2$mm) we see that the pressure must be in order of $\sim 10^5$Pa. However, pressure measurements [13] show that $\Delta p \sim 5 \cdot 10^3$Pa! We see that living tissue seems to be softer that a "dead" one on which measurements are performed.

It motivates us to accept a hypothesis as follows: The "living" tissue has a big ability to *relax*, that means to "find" the parameters α_i solving the minimizing problem (12). The "dead" one, in the contrary, has fixed fibers (living processes being stopped) and it cannot relax and thus it seems to be much stiffer than the living one. This hypothesis allows us to identify the parameters of our model. Let us calculate the deformation of a pipe having the energy (8). The parameters α_i are supposed to be near one (in fact they, when being fixed, do not play an important role) and we obtain the displacement for small deformations

$$u(R) = 2E^{-1}((1-\lambda_1)^2 + (1-\lambda_2)^2)^{-1} (R_a^{-2} - R_b^{-2})^{-1} R^{-1} \Delta p. \tag{14}$$

Taking an average value over all λ's ($0 < \lambda_i < 1$) we obtain after comparing with (13)

$$E \approx \frac{2}{3} \left(1 - \frac{\pi}{4}\right) E_Y, \tag{15}$$

where E_Y is the measured Young modulus, $E_Y = 0.25$MPa, i.e. $E \approx 0.036$MPa.

5 Numerical results

We solved numerically the minimizing problem (12) with E defining by (15) for various λ_1 and λ_2. The coefficient K describing some micro-compressibility and must be high enough but it is not determined. Nevertheless, numerical calculations show that for $K \geq 1$MPa the results are almost independent on K. Geometrical parameters of the urethra cross section at the closing state and the state of full opening during micturition were defined as follows: $r_a(\text{closed}) = 1.5\text{mm}$, $r_a(\text{open}) = 2.81\text{mm}$ as explained at Fig. 3. The only problem was to choose a suitable reference configuration – we estimated $R_a = 0.9\text{mm}$. Since an effective thickness of the

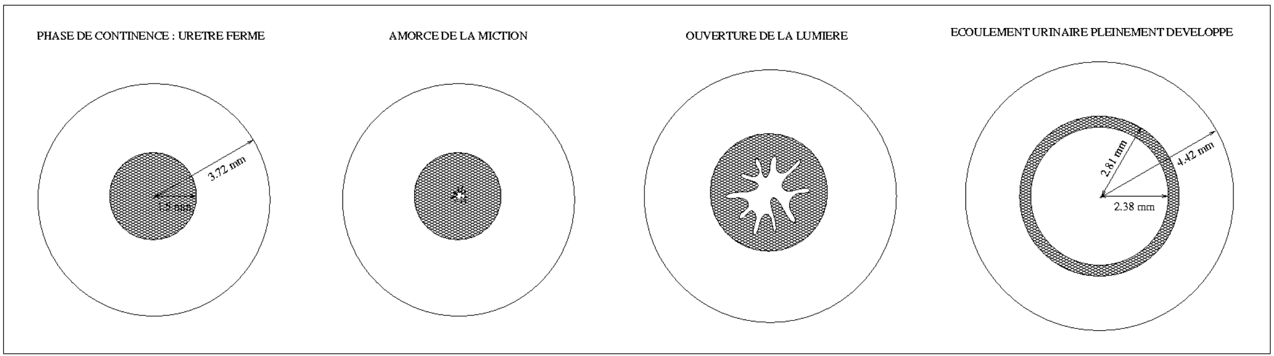

Figure 3: The innermost part of the urethra (painted in grey) is a soft, "forming" matter whose inner energy seems to be negligible with that of muscular tissue. Thus it only "transports the pressure" but its concrete shape may be neglect (it behaves like an incompressible fluid – in open state it forms a thin layer).

urethral wall being important for its deformation (a white annulus at Fig. 3) is about 2.4mm the parameter C is defined at the both states: $C(\text{open}) = 7.09\text{mm}^2$, $C(\text{closed}) = 1.44\text{mm}^2$. Calculating numerically the minimizing problem (12) and then finding the pressure difference

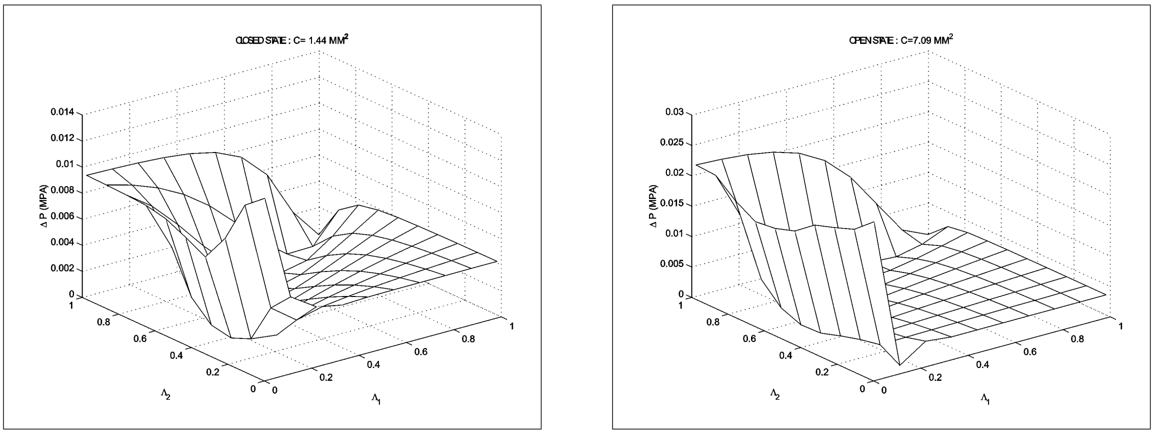

Figure 4: Pressure difference in dependence on microstructural parameters. Roughly speaking, the "flat parts" correspond to passive states, the rapidly increasing areas to stimulated ones. An averaged value of Δp over passive parts is around 5kPa.

(11) we obtain the "landscape" of various micro-states – Fig. 4. We see that an average value of pressure is around 7kPa, which corresponds to pressures measured at these two states: $\Delta p(\text{open}) = 5\text{kPa}$, $\Delta p(\text{closed}) = 9\text{kPa}$ [13]. We can find precise values of micro-structural parameters λ_i which gives the correct values of pressure at these states. Supposing λ_1 is fixed (because it describes the microstructural change at radial direction) we can estimate it to be $\lambda_1 \sim 0.1$ because around this value we can find λ_2's corresponding to the both states.

6 Conclusion

The model presented in this paper is based on a quadratic approximation of the strain energy. In spite of that it gives a nontrivial non-linear model. The reason consists in the fact that the energy is defined at some microscopic level and its macroscopic value is determined by "relaxing" microscopic parameters to obtain a local minimum at each point of (macroscopic) continua. However, this relaxation is not possible for any material – it seems as to be typical for living tissue. By accepting this hypothesis we obtain correct results when modelling the pressure dependence on deformation of urethra. We use the Young modulus measured on "dead" samples of the tissue and by interpreting it in the above way we obtain the model giving results in a good agreement with experiments and enabling us to model simulations within muscle tissue. We would like to emphasis that this model has no free parameter, which (being chosen "properly") could adjust the model so that it gives "only" correct results (the only problematic parameter is the reference radius R_a, nevertheless computer simulations show that its choice influences the results only very small). That is why be believe that it might lead to a better understanding of behaviour of tissue from the mechanical point of view and be useful in formulating some more precise models taking into account a layered structure of tissue and so on.

Acknowledgements

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Resume

A scale-dependent model of the urethral tissue is proposed. The model uses the deformation gradient defined at two scales. A relaxation at micro-scales leads to an essential decreasing of an effective Young modulus, which seems to be a typical feature of living tissues. It is confirmed by our model of the urethral pipe in which that relaxation enables as to interpret correctly the measured data of the tissue's Young modulus and give a realistic model of opening and closing the urethra in a good agreement with experiments.