# STUDY OF THE TELEGRAPHER EQUATION BY USING CELLULAR AUTOMATA

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## Introduction

There are many problems of physics, which are modelled by the familiar diffusion equation, namely

$$\frac{\partial F}{\partial t} - \kappa \, \Delta F = 0 \, , \tag{1}$$

where $\Delta$ is the Laplace operator. This equation describes not only the diffusion processes but also many other phenomena such as heat conduction and others.

From a pure physical point of view, however, this equation is a little problematic because it allows an infinite speed of propagation. This problem – well-known especially in connection with the heat conduction – comes from the fact that the equation (1) is a parabolic one. Thus from the physical point of view the propagation should be rather described by hyperbolic equation, say

$$\frac{\partial F}{\partial t} + \tau \frac{\partial^2 F}{\partial t^2} - \kappa \, \Delta F = 0 \, , \tag{2}$$

where the correction $\tau \, \partial^2 F / \partial t^2$, however small, changes essentially the character of solution. This equation is usually called the telegrapher equation [1].

The correction leading to the telegrapher equation, however, completely violates an easy physical explanation of the diffusion equation (1). Namely the equation (1) can be straightforwardly derived from a simple statistical model based on a "random walk" in space – this is, in fact, the essence of diffusion. The key problem of a physical derivation of the telegrapher equation (2) is a lack of a clear statistical interpretation of the process described by (2).

In this paper we will present the famous work of M. Kac [2], in which statistical interpretation of (2) is given. By using cellular automata [3] we will directly model the Kac's stochastic process and show thus the essential features of solutions of the telegrapher equation (2).

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## Cellular automata

Cellular automata can be characterised by the following fundamental properties [4]: They consist of a regular discrete lattice of cells. Each cell is characterised by the state, which is chosen from a finite set of states. The state of all cells are updated in discrete time steps simultaneously and independent of one another. Each cell evolves according to the same rule, which depends only on the state of the cell and the states of a finite number of neighbouring cells. The neighbourhood is local and uniform.

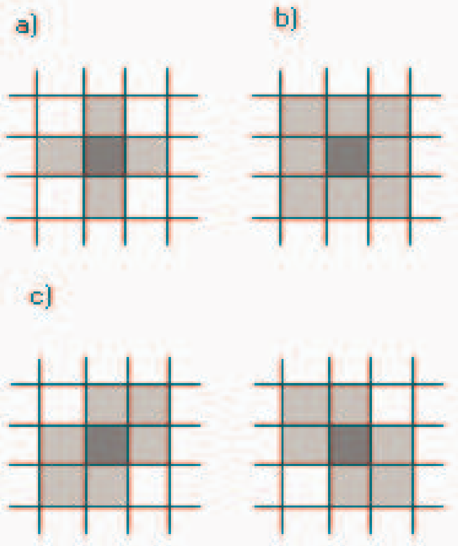


**Figure 1**: Definition of typical neighbourhoods which can be used in CA-simulations (a) von Neumann, (b) Moore, (c) hexagonal [5].

There are many choices of the lattice definition. In two-dimensions, there are three regular lattices, namely, square, triangular and hexagonal. There is no general restriction concerning the size and the type of neighbourhood. The only restriction is that the size of neighbourhood must be much smaller than the size of lattice. Let us define some formulas of typical neighbourhoods in square lattice. For the nearest neighbourhood, the von Neumann neighbourhood is described (see Figure 1a) by

$$N_{i,j} = \{(k,l) \in L;\ |k-i| + |l-j| \leq 1\}. \tag{3}$$

Another common neighbourhood is the Moore neighbourhood composed from first and second nearest neighbours (see Figure 1b)

$$N_{i,j} = \{(k,l) \in L;\ |k-i| \leq 1\ \&\ |l-j| \leq 1\}. \tag{4}$$

The hexagonal neighbourhood is describe (see Figure 1c) by

$$\begin{aligned} &1)\ N_{i,j} = \{(k,l) \in L;\ |k-i| \leq 1\ \&\ |l-j| \leq 1\ \&\ (k-i)(l-j) < 1\}, \\ &2)\ N_{i,j} = \{(k,l) \in L;\ |k-i| \leq 1\ \&\ |l-j| \leq 1\ \&\ (k-i)(l-j) > -1\}, \end{aligned} \tag{5}$$

One of these neighbourhoods is used in odd times and the other is used in even times or vice versa.

**A random walk model leading to the telegrapher equation**

Following the work [2] we will consider a very simple stochastic model, a random walk, which has very interesting features and leads not to a diffusion equation but to a hyperbolic one. The problem is following: Suppose you have a lattice of points. We mean discrete, equidistant points as in Figure2.

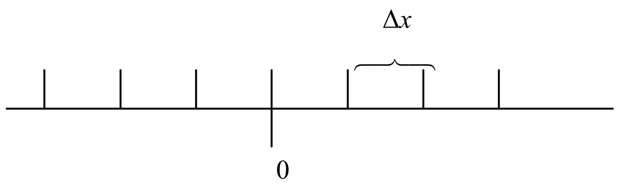


**Figure 2:** The spacing of the problem

Now we start a particle from the original $x = 0$ and the particle always moves with speed $v$. It can move either in a positive direction or in the negative direction. We flip a coin, let's say, to determine which. Each step is of duration $\Delta t$ and covers a distance $\Delta x$. So we have $\Delta x = v\Delta t$. Each time you arrive at a lattice point there is a probability of reversal of direction. We assume that $a\Delta t$ is to be this probability. Then, of course, $1 - a\Delta t$ is the probability that the direction of motion will be maintained. What is wanted is the probability that after certain time $t$ the particle is at a certain interval.

Let $x$ now stand only for abscissas of discrete points, the lattice points. And let us call the displacement after time $n\Delta t$. Now we will take a function $\varphi(x)$, an "arbitrary" function. And we will ask for the average $\langle\varphi(x + S_n)\rangle$. This will really give us all we want – for example $\varphi(x)$ could be the characteristic function of an interval. In that case this average will simply be probability of finding the particle in that interval after $n$ steps if it started at the point $x$. But instead of talking such a special function we will take more general one.

Now let us analyse the problem by introducing the following random variable:

$$\in = \begin{cases} 1 \ \text{ with probability } \ 1 - a\Delta t, \\ -1 \ \text{ with probability } \quad a\Delta t, \end{cases} \qquad (6)$$

and we consider a sequence of such independent random variables $\in_1, \in_2, \ldots, \in_{n-1}$. Each of them has the distribution (6) and they are all independent. In other words, we have a coin, an extremely biased coin and the $\in$'s are now the result of $n$ independent tosses. Now we can very easily write out the displacement. If we start in the positive direction from the origin then it will be

$$S_n = v\Delta t\,(1 + \in_1 + \in_1\in_2 + \in_1\in_2 \ldots \in_{n-1}). \qquad (7)$$

Indeed, the first step will certainly take us a distance $v\Delta t$ in the positive direction. Now we must toss our coin and find what will happen to velocity. It will change from $v$ into $\in_1 v$, i.e., it will be maintained or reverse according to the outcome of the toss. So in the next step we will move an additional distance $\in_1 v\,\Delta t$. And so it goes on, and you see how (7) comes about. If we had started in the negative direction then the displacement would have been

$$S_n^{|} = -v\Delta t\,(1 + \in_1 + \in_1\in_2 + \in_1\in_2 \ldots \in_{n-1}) = -S_n. \tag{8}$$

Now, let us consider the two functions

$$F_n^{+}(x) = \langle \varphi(x + S_n) \rangle. \tag{9}$$

$$F_n^{-}(x) = \langle \varphi(x - S_n) \rangle. \tag{10}$$

and write a recursion formula. First of all, let us write

$$F_n^{+}(x) = \langle \varphi[x + v\Delta t + v\Delta t \in_1(1+\in_2+\in_2\in_3+\ldots+\in_2\in_3\ldots\in_{n-1})] \rangle. \tag{11}$$

Let us notice we have factored out $\in_1$. Now the averaging is really just a weighted sum over all possible sequences of $\in$'s. The weight are dictated by the probability distribution. But we can perform the averaging in two different steps. We can first perform the average on $\in_1$, and then on all the remaining $\in$'s. So let us first of all average on $\in_1$. This variable can assume the value –1 with probability $a\Delta t$; and it can assume the value +1 with probability 1- $a\Delta t$. So we can simply write

$$F_n^{+}(x) = a\Delta t\,\langle \varphi[x + v\Delta t - v\Delta t\,(1+\in_2+\in_2\in_3+\ldots)] \rangle + (1- a\Delta t)\,\langle \varphi[x + v\Delta t + v\Delta t\,(1+\in_2+\in_2\in_3+\ldots)] \rangle. \tag{12}$$

The averages have exactly the same form as before – except that $x$ is replaced by $x + v\Delta t$ and $n$ is replaced by $n$-1. This gives us the formula

$$F_n^{+}(x) = a\Delta t\, F^{-}_{n-1}(x+ v\Delta t) + (1 - a\Delta t)\, F^{+}_{n-1}(x+ v\Delta t). \tag{13}$$

In exactly the same way we can obtain another relation using $F_n^{-}$. It is

$$F_n^{-}(x) = a\Delta t\, F^{+}_{n-1}(x- v\Delta t) + (1 - a\Delta t)\, F^{-}_{n-1}(x- v\Delta t). \tag{14}$$

So now we have a system of recursion relations.

Now the standard way is to pass from these difference equations to a differential equation in the limit $\Delta t \to 0$. In order to pass from the discrete to continuous, notice first of all that $n$ measures time. Actually, $n$ is the number of steps and $n\Delta t$ is the equal to our time $t$. Now let us rewrite relation (13):

$$\frac{F_n^{+}(x) - F_{n-1}^{+}(x)}{\Delta t} = \frac{F_{n-1}^{+}(x + v\Delta t) - F_{n-1}^{+}(x)}{\Delta t} - aF_{n-1}^{+}(x + v\Delta t) - aF_{n-1}^{-}(x + v\Delta t). \tag{15}$$

And now we can pass to the limit to get

$$\frac{\partial F^{+}}{\partial t} = v\frac{\partial F^{+}}{\partial x} - aF^{+} + aF^{-}. \tag{16}$$

There is no $n$ anymore, because we went to the limit. From the other relation, (14), we get in a similar way

$$\frac{\partial F^-}{\partial t} = -v\frac{\partial F^-}{\partial x} + aF^+ - aF^- . \tag{17}$$

Now, these two linear equations of first order can be combined into a hyperbolic equation. For this purpose we will introduce two new functions:

$$F = \tfrac{1}{2}(F^+ + F^-) \qquad \text{and} \qquad G = \tfrac{1}{2}(F^+ - F^-) . \tag{18}$$

Now, add up equations (16) and (17). Then we get, in this notation,

$$\frac{\partial F}{\partial t} = v\frac{\partial G}{\partial x} . \tag{19}$$

Now subtract (17) from (16) to get

$$\frac{\partial G}{\partial t} = v\frac{\partial F}{\partial x} - 2aG . \tag{20}$$

Now the problem is to eliminate G. To do this, differentiate (19) with respect to $t$ and (20) with respect to $x$. Everything then becomes obvious, and we obtain

$$\frac{1}{v}\frac{\partial^2 F}{\partial t^2} = v\frac{\partial^2 F}{\partial x^2} - \frac{2a}{v}\frac{\partial F}{\partial t} . \tag{21}$$

This is a nothing but the telegrapher equation (2) in one dimension. A generalisation for more dimensional space can be done in a straightforward way – we obtain the Laplace operator instead of the second derivative in (21).

Let us discuss a few points. First of all, there is one limiting case, which is extremely easy. That is when $a = 0$. Then, of course, the probability of reversing direction is zero. If you start moving in one direction, you never stop. What would F($x$, $t$) be? There are no reversals of direction and no random variables. So from (9) we see that $F_n^+(x) = \varphi(x + nv\Delta t)$ and from (10) that $F_n^-(x) = \varphi(x - nv\Delta t)$. So it follows that

$$F(x,t) = \frac{\varphi(x+vt) + \varphi(x-vt)}{2} . \tag{22}$$

And that, of course, is a well-known classical case of the vibrating string.

We can get something different if we send $a \to \infty$ and $v \to \infty$ in such a way that $2a/v^2$ remains constant, say 1/D. This can always be done, and we are allowed to choose D anyway we want to. This limiting case of equation (21) then becomes the diffusion equation (1) in one dimension,

$$\frac{1}{D}\frac{\partial F}{\partial t} = v\frac{\partial^2 F}{\partial x^2} . \tag{23}$$

The reason why we must let $a$ and $v$ go to infinity is as follows: The diffusion can be looked on as a random walk. But in the standard model the probability of a move to the right or left is one-half. Now we see the probabilities in our model are either extremely small or extremely large. The only way they can be brought to where they will be one-half and one-half is to let $a$ approach infinity as $\Delta t$ goes to zero. If $a$ does not go to infinity, there will always be a drift. We know, also, from the random walk model that the velocity of a particle is infinite in limit. So we have to let $v$ also go to infinity.

**The cellular automata model**

We use the square lattice for our model. The neighbourhood is created by two neighbour cells: left and right. Following properties of model were used for observation the largest number of particles: Each of particles moves only in two directions, left and right. The motion of separate particles is independent. Therefore we can understand one column of the lattice as a cell. We choose 64000 particles to start walking from original cell ($x = 0$) in two directions - left and right. The probability of motion in both direction is one half. It is our initial condition. Let us choose the time step

$$\Delta t=1. \tag{24}$$

It means that at each following time step any particle can reverse direction with probability $a$ (see (6)), i.e. 1 - $a$ is the probability that the direction of motion will be maintained. At each time we observe the position of all particles. A distribution of particles then gives an approximate solution of the telegrapher's equation at time $t$. We model this simulation for several different $a$. Results are presented for $a$ = 1/2, 1/7, 1/10, 1/15, 1/20,1/50. The choice $a$ = 1/2 leads to diffusion behaviour because the time step equals "one" and no continuum limit $\Delta t \to 0$ is done. When we compare the dependence obtained by using various $a$ we can see clearly the change of the probability distribution (see Figure 3).

At the Figure 3 we can see six graphs for various probabilities of inverting the velocity direction. At each graph there are ten curves modelling the distribution of particles after ten time steps. That is the first curve corresponds to the distribution of particles after ten time steps and the last one corresponds to that after one hundred time steps.

As seen from the graphs the distribution of particles during the time evolution is strongly influenced by the probability of the velocity inversion, $a$. If this probability equals one half the distribution of particles gives the Gaussian distribution. With decreasing value of the probability $a$ more and more particles move at the maximal distance which can be reached by the previous steps (the probability of the velocity inverting is very small). This is nothing but a wave moving with the velocity $\pm 1$ (in our "one-one" space-time units). Thus we easily see the wave behaviour "hidden" in the stochastic process (random walk) (6). The fact that the continuum limit of this process gives the telegrapher equation is now clearer: a hyperbolic equation does not allow an infinite speed of propagation of a pulse – any pulse cannot exceed the velocity

$$v = \sqrt{\frac{\kappa}{\tau}}\ . \tag{25}$$

In our model this maximal velocity equals 1. In fact, the solution of the telegrapher equation is a combination of a wave motion and the diffusion "smearing out". The combination of the both processes can be also seen at graphs with an intermediate parameter $a$.

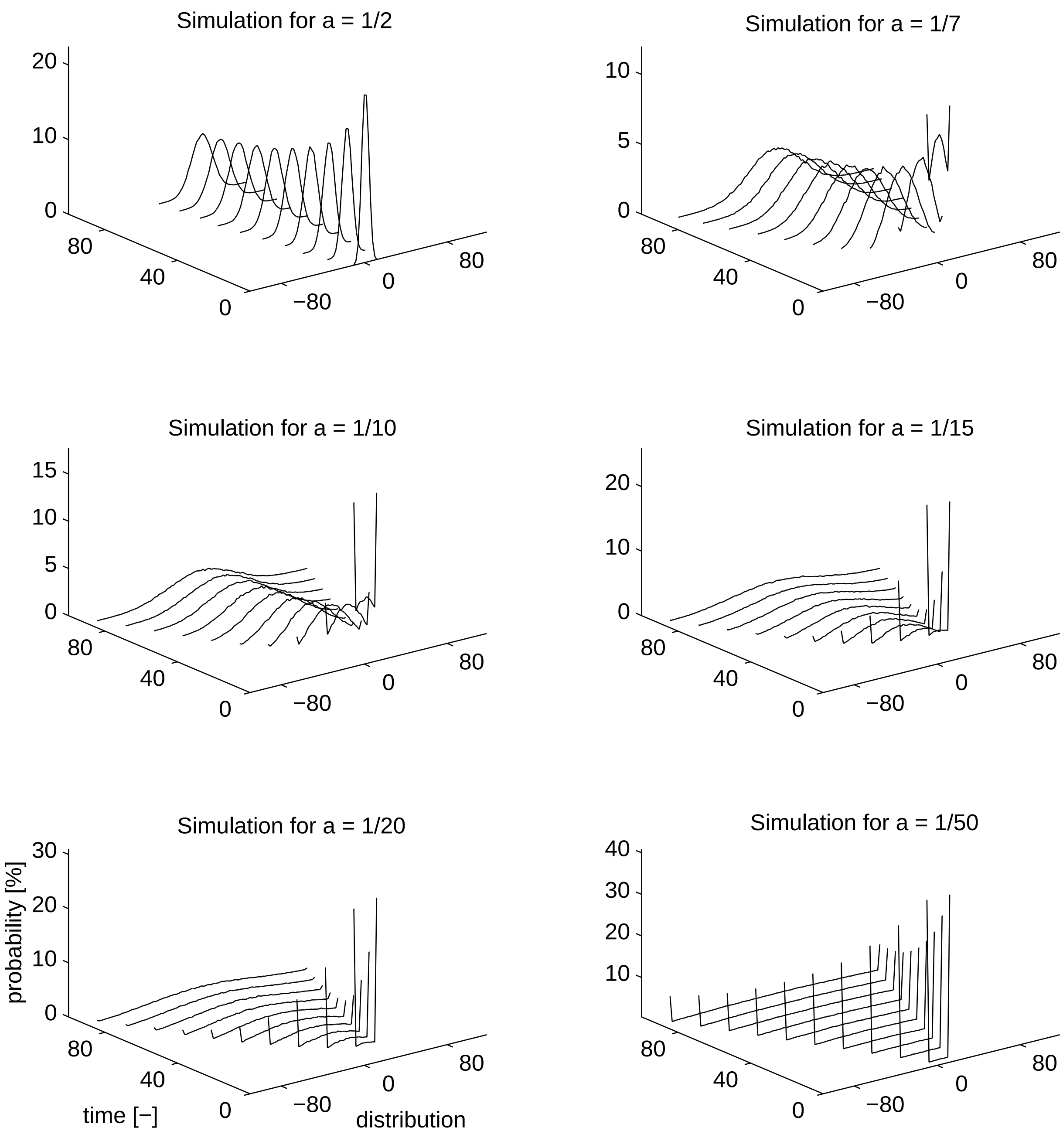


**Figure 3.** Comparison of behaviour of model for various probability of reverse direction.

## Conclusion

In the paper we model a special one-dimensional random walk problem by using the cellular automata. The "walk" corresponds to a process in which the direction of velocity can immediately invert into the opposite one. As known, this process leads in the continuum limit into the telegrapher equation. The use of the cellular automata, however, allows us to study directly the stochastic process alone without a limit sending time and space steps into zero.

The results show that by using 64000 particles we obtain perfectly stochastic behaviour. The tendencies obtained by the cellular automata modelling correspond exactly with those gained by a solution of the telegrapher equation. Moreover, the results are transparent and easily explainable. It supports the idea being presented in many studies concerning the cellular automata and other discrete methods that the continuum approach, regardless its enormous power and applicability, is maybe more complicated than a discrete modelling because Nature seems to be more discrete than continuous.

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**References**

[1] Jou D., Casas-Vazquez J., Lebon G.: Extended Irreversible Thermodynamics. Springer-Verlang, Berlin, Heidelberg 1993.

[2] Kac M.: A stochastic model related to the telegrapher's equation. Rocky Mountain Journal of mathematics,Volume 4, Number 3, Summer 1974.

[3] Eckart D.J.: A Cellular Automata Simulation System. Cellular/Cellang, http://www.csrunet.edu/~dana/.

[4] Kroc J.: An Introduction into the Cellular Automata Theory with Applications. Proceedings of West Bohemia University, Pilsen, Vol. 3 (1999) 99-110.

[5] Hesselbarth H.W. & Göbel I.R.: Simulation of recrystallization by cellular automata. Acta Matallurgica at Materialia, 39(9): 2135-2143.Great Britain 1991.

**Resume**

The work deals with modeling a random walk procedure simulating the solution of the telegrapher equation. The problem is formulated and solved by using the cellular automata.