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MULTIPLES OF HYPERCYCLIC OPERATORS

by

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Abstract. — We give a negative answer to a question of Prajitura by showing that there exists an invertible bilateral weighted shift T on $\ell_2(\mathbb{Z})$ such that T and $3T$ are hypercyclic but $2T$ is not. Moreover, any G_δ set $M \subseteq (0, \infty)$ which is bounded and bounded away from zero can be realized as $M = \{t > 0 \,;\, tT \text{ is hypercyclic}\}$ for some invertible operator T acting on a Hilbert space.

1. Introduction

This note is devoted to the study of multiples of hypercyclic operators acting on a real or complex separable Banach space X. An operator $T \in \mathcal{B}(X)$ is said to be *hypercyclic* if there exists a vector $x \in X$ which has a dense orbit, i.e. the set $\{T^n x \,;\, n \geq 0\}$ is dense in X. Hypercyclic operators have been the subject of active investigation in the past twenty years, and we refer the reader to the book [**1**] for a thorough survey of this area. The first examples of hypercyclic operators were given by Rolewicz in 1969: if B is the backward shift on $\ell_p(\mathbb{N})$, $1 \leq p < +\infty$, or $c_0(\mathbb{N})$, with the canonical basis $(e_n)_{n\geq 0}$, defined by $Be_0 = 0$ and $Be_n = e_{n-1}$ for $n \geq 1$, then λB is hypercyclic for any complex number λ such that $|\lambda| > 1$. This can be seen very easily using the Hypercyclicity Criterion, which is the most useful tool for proving that a given operator is hypercyclic. We recall it here in the version of Bès and Peris [**2**]:

Hypercyclicity Criterion. — *Suppose that there exist a strictly increasing sequence (n_k) of positive integers, two dense subsets V and W of X and a sequence (S_k) of maps (not necessarily linear nor continuous) $S_k : W \to W$ such that:*

1. *for every $x \in V$, $T^{n_k} x \to 0$*
2. *for every $x \in W$, $S_k x \to 0$*

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3. *for every* $x \in W$, $T^{n_k} S_k x \to x$.

Then the operator T *is hypercyclic.*

Despite its somewhat involved aspect, the Hypercyclicity Criterion follows directly from a simple Baire Category argument, using the fact that T is hypercyclic if and only if it is *topologically transitive* (i.e. for every pair (U, V) of non empty open subsets of X there exists an integer n such that $T^{-n}(U) \cap V \neq \emptyset$). The "legitimity" of the Hypercyclicity Criterion comes from the fact [**2**] that $T \in \mathcal{B}(X)$ satisfies the Hypercyclicity Criterion if and only if $T \oplus T$ is hypercyclic on $X \oplus X$. Until very recently it was unknown whether every hypercyclic operator satisfied the Hypercyclicity Criterion or not: the answer is no, see [**3**].

Let $T \in B(X)$ satisfy the Hypercyclicity Criterion. Note that for any $0 < t < 1$ the operator tT satisfies condition (1) (for the same sequence (n_k) and set V). Similarly, the operator tT for $t > 1$ satisfies conditions (2) and (3) (for the same set W and for the mappings $t^{-n_k} S_k$). Therefore in many concrete examples the set $\{t > 0 \ ; \ tT \text{ is hypercyclic}\}$ is convex. This motivates the following question of Prajitura [**6**], see also [**5**] about multiples of hypercyclic operators:

Question 1.1. — *Let* T *be a bounded operator on* X. *Suppose that there exist two positive numbers* t_1 *and* t_2, $0 < t_1 < t_2$, *such that* $t_1 T$ *and* $t_2 T$ *are hypercyclic. Is it true that* tT *is hypercyclic for every* $t \in [t_1, t_2]$*?*

We give a negative answer to this question, and prove the following stronger result:

Theorem 1.2. — *Let* M *be a subset of* $(0, +\infty)$. *The following assertions are equivalent:*

(1) M *is a* G_δ *subset of* $(0, +\infty)$ *which is bounded and bounded away from zero;*

(2) *there exists an invertible operator* T *acting on a Hilbert space such that*

$$M = \{t > 0 \ ; \ tT \text{ is hypercyclic}\}.$$

Remark that as soon as M coincides with the set of positive t's such that tT is hypercyclic, M must be bounded away from zero, since tT is a contraction for small enough t. As a corollary we obtain for instance:

Corollary 1.3. — *There exists an operator* T *acting on a Hilbert space such that* T *and* $3T$ *are hypercyclic but* $2T$ *is not.*

Note that by [**4**], if T is a hypercyclic operator in a complex Banach space and $\theta \in \mathbb{R}$, then $e^{i\theta} T$ is hypercyclic (with the same set of hypercyclic vectors as T). Thus the set $M = \{\lambda \in \mathbb{C} \ ; \ \lambda T \text{ is hypercyclic}\}$ is *circularly symmetric* (if λ belongs to M, $e^{i\theta}\lambda$ belongs to M for any $e^{i\theta}$ in the unit circle). We thus obtain the following variant of Theorem 1.2:

Theorem 1.4. — *Let* M *be a subset of the complex plane* $\mathbb{C}$. *The following assertions are equivalent:*

(i) *there exists an invertible operator* T *acting on a Hilbert space such that*

$$M = \{\lambda \in \mathbb{C} \ ; \ \lambda T \text{ is hypercyclic}\};$$

(ii) *M is circularly symmetric and $M \cap (0,+\infty)$ is a G_δ subset of $(0,+\infty)$ which is bounded and bounded away from zero.*

The operators constructed in Theorem 1.2 are bilateral weighted shifts on the space $\ell_2(\mathbb{Z})$, and for these shifts the Hypercyclicity Criterion takes a particularly simple form (see [**7**] for a necessary and sufficient condition for a general bilateral weighted shift to be hypercyclic):

Fact 1.5. — *Let T be an invertible bilateral weighted shift on the space $\ell_2(\mathbb{Z})$ endowed with its canonical basis $(e_n)_{n\in\mathbb{Z}}$. Then T is hypercyclic if and only if there exists a strictly increasing sequence $(n_k)_{k\geq 0}$ of positive integers such that $||T^{n_k}e_0||$ and $||T^{-n_k}e_0||$ tend to zero as k goes to infinity.*

Multiples of the shifts constructed in the proof of Theorem 1.2 are not mixing (recall that T is said to be *mixing* if for every pair (U,V) of non empty open subsets of X there exists an integer N such that $T^{-n}(U)\cap V \neq \emptyset$ for every $n \geq N$): this is coherent with the next result, which implies that the answer to Question 1.1 is affirmative for a large class of operators.

Theorem 1.6. — *Let $T \in \mathcal{B}(X)$ be such that for some $0 < t_1 < t_2$, $t_1T \oplus t_2T$ is hypercyclic. Then tT is hypercyclic for every $t \in [t_1, t_2]$. This holds true in particular if either t_1T or t_2T is mixing.*

2. Proofs of Theorems 1.2 and 1.6

The proof of the implication (2) ⇒ (1) in Theorem 1.2 is quite standard: suppose that $T \in B(X)$ is invertible. Let $M = \{t > 0 \text{ ; } tT \text{ is hypercyclic}\}$, and we can suppose that M is non empty. As it was previously mentioned, $\|tT\| \leq 1$ for $0 < t \leq \|T\|^{-1}$ and so tT is not hypercyclic in this case. Hence M is bounded away from zero. Since T is invertible, the same argument applied to T^{-1} shows that M must be bounded above. Let $(U_j)_{j\geq 1}$ be a countable basis of open subsets of X (which is separable). Clearly

$$M = \{t > 0 \text{ ; } tT \text{ is hypercyclic}\} = \bigcap_{i\geq 1}\bigcap_{j\geq 1}\bigcup_{n\geq 0}\{t > 0 \text{ ; } (tT)^n U_i \cap U_j \neq \emptyset\},$$

which is a G_δ set.

The first step in the proof of the reverse implication (1) ⇒ (2) of Theorem 1.2 is the following proposition, which proves the result when M is an open set. One of its interests is that it shows the existence of *common* subsets V and W in the Hypercyclicity Criterion for *all* operators tT with t belonging to this open set.

Proposition 2.1. — *Let G be an open subset of an interval of the form (K^{-1}, K) for some $K > 1$. Then*

(i) *there exists an invertible bilateral weighted shift on $\ell_2(\mathbb{Z})$ such that $\|T\| \leq K^3$ and $G = \{t > 0 \text{ ; } tT \text{ is hypercyclic}\}$;*

(ii) *write G as a (finite or countable) union*

$$G = \bigcup_{\lambda \in \Lambda} (a_\lambda, b_\lambda)$$

of open intervals. For each $\lambda \in \Lambda$ let A_λ be an infinite subset of $\mathbb{N}$. Then for each $\lambda \in \Lambda$ there exists an increasing sequence $(m_{\lambda,k})_{k\geq 1}$ of integers belonging to A_λ such that for every $t \in (a_\lambda, b_\lambda)$, $\|(tT)^{m_{\lambda,k}} e_0\|$ and $\|(tT)^{-m_{\lambda,k}} e_0\|$ tend to zero as k tends to infinity (where $\{e_n \ ; n \in \mathbb{Z}\}$ is the standard orthonormal basis in $\ell^2(\mathbb{Z})$).

Proof. — The statement is trivial if G is empty, so suppose that G is non empty. Order the intervals (a_λ, b_λ) into a sequence (a_k, b_k) in which every interval (a_λ, b_λ) appears infinitely many times. Then fix a function $f : \mathbb{N} \to \Lambda$ such that $(a_k, b_k) = (a_{f(k)}, b_{f(k)})$ and for each $\lambda \in \Lambda$, $f(k) = \lambda$ for infinitely many k's.

Set formally $n_0 = 1$ and choose inductively a sequence $(n_k)_{k\geq 1}$ such that $n_k \in A_{f(k)}$ and $n_k \geq 4n_{k-1}$ for each $k \geq 1$.

The operator T will be the weighted bilateral shift defined on $\ell_2(\mathbb{Z})$ by

$$Te_i = c_{i+1} e_{i+1} \ \text{ and } \ T^{-1} e_{-i} = \widetilde{c}_{i+1} e_{-i-1} \text{ for } i \geq 0,$$

i.e. $Te_i = (1/\widetilde{c}_{-i}) e_{i+1}$ for $i < 0$. The weights c_i and $\widetilde{c}_i$ are defined for $i \geq 1$ in the following way:

- $c_1 = c_2 = \widetilde{c}_1 = \widetilde{c}_2 = K$;
- for $k \in \mathbb{N}$ and $2n_{k-1} < j \leq n_k$,

$$c_j = \left(\frac{1}{K^{2n_{k-1}} b_k^{n_k}} \right)^{\frac{1}{n_k - 2n_{k-1}}} \quad \text{and} \quad \widetilde{c}_j = \left(\frac{a_k^{n_k}}{K^{2n_{k-1}}} \right)^{\frac{1}{n_k - 2n_{k-1}}};$$

- for $k \in \mathbb{N}$ and $n_k < j \leq 2n_k$,

$$c_j = K^2 b_k \ \text{ and } \ \widetilde{c}_j = \frac{K^2}{a_k} \cdot$$

For $n \in \mathbb{N}$ write the products of the n first coefficients c_i or $\widetilde{c}_i$ as $w_n = \prod_{i=1}^n c_i$ and $\widetilde{w}_n = \prod_{i=1}^n \widetilde{c}_i$. It is easy to show by induction that for every $k \in \mathbb{N}$,

$$w_{2n_k} = \widetilde{w}_{2n_k} = K^{2n_k}, \ w_{n_k} = b_k^{-n_k} \ \text{ and } \ \widetilde{w}_{n_k} = a_k^{n_k}.$$

Since $1/K < a_k < b_k < K$ for every k, we have for every k and every j such that $n_k < j \leq 2n_k$,

$$K \leq c_j \leq K^3 \ \text{ and } \ K \leq \widetilde{c}_j \leq K^3.$$

Then since $n_k \geq 4n_{k-1}$, we have for $2n_{k-1} < j \leq n_k$

$$\frac{1}{c_j} = \left(K^{2n_{k-1}} b_k^{n_k}\right)^{\frac{1}{n_k - 2n_{k-1}}} \leq K^{\frac{2n_{k-1} + n_k}{n_k - 2n_{k-1}}} \leq K^3,$$

$$\frac{1}{\widetilde{c}_j} = \left(\frac{K^{2n_{k-1}}}{a_k^{n_k}} \right)^{\frac{1}{n_k - 2n_{k-1}}} \leq K^{\frac{2n_{k-1} + n_k}{n_k - 2n_{k-1}}} \leq K^3,$$

$$\widetilde{c}_j \leq K^{\frac{n_k - 2n_{k-1}}{n_k - 2n_{k-1}}} \leq K$$

and similarly, $c_j \leq K$. Hence $K \leq c_j \leq K^3$ and $K \leq \widetilde{c}_j \leq K^3$ for every j, and this proves that T is bounded and invertible with $\|T\| \leq K^3$ and $||T^{-1}|| \leq K^3$. Note that for $t \in (a_k, b_k)$ we have

$$\|(tT)^{n_k} e_0\| = t^{n_k} b_k^{-n_k} = (t/b_k)^{n_k} \quad \text{and} \quad \|(tT)^{-n_k} e_0\| = t^{-n_k} a_k^{n_k} = (a_k/t)^{n_k},$$

where $t/b_k < 1$ and $a_k/t < 1$.

Let now $\lambda \in \Lambda$. Since the interval (a_λ, b_λ) appears in the sequence (a_k, b_k) infinitely many times, let $(m_{\lambda,i})_{i\geq 1}$ be the increasing sequence consisting of the integers of the set $\{n_k\}$ for which $f(k) = \lambda$. Then each $m_{\lambda,i}$ belongs to A_λ since $n_k \in A_{f(k)}$ for every k.

Let t belong to the interval (a_λ, b_λ). Then by the computation above $||(tT)^{m_{\lambda,i}} e_0||$ and $||(tT)^{-m_{\lambda,i}} e_0||$ tend to zero as i tends to infinity, and, by Fact 1.5, tT is hypercyclic. Since this is true for every $\lambda \in \Lambda$ this shows that $G \subseteq \{t > 0 \ ; \ tT \text{ is hypercyclic}\}$.

Conversely, suppose that t does not belong to G. In order to show that tT is not hypercyclic, it suffices to prove that for each $j \in \mathbb{N}$, $\max\{\|(tT)^j e_0\|, \|(tT)^{-j} e_0\|\} \geq 1$. Let $2n_{k-1} < j \leq 2n_k$ for some $k \geq 1$. Since $t \notin G$, either $t \leq a_k$ or $t \geq b_k$.

- If $n_k < j \leq 2n_k$ and $t \geq b_k$ then

$$\|(tT)^j e_0\| = t^j \|T^j e_0\| \geq b_k^j \|T^{n_k} e_0\| \cdot (K^2 b_k)^{j-n_k} = b_k^{j-n_k} (K^2 b_k)^{j-n_k} = (K b_k)^{2(j-n_k)} \geq 1.$$

- if $n_k < j \leq 2n_k$ and $t \leq a_k$, then

$$\|(tT)^{-j} e_0\| \geq a_k^{-j} \|T^{-n_k} e_0\| \cdot \left(\frac{K^2}{a_k}\right)^{j-n_k} = a_k^{-(j-n_k)} \left(\frac{K^2}{a_k}\right)^{j-n_k} = \left(\frac{K}{a_k}\right)^{2(j-n_k)} \geq 1.$$

- if $2n_{k-1} < j \leq n_k$ for some $k \geq 1$, and $t \geq b_k$, then

$$\begin{aligned} \|(tT)^j e_0\| &\geq b_k^j \|T^j e_0\| = b_k^j \|T^{n_k} e_0\| \cdot (K^{2n_{k-1}} b_k^{n_k})^{\frac{n_k - j}{n_k - 2n_{k-1}}} \\ &= b_k^{j-n_k} (K^{2n_{k-1}} b_k^{n_k})^{\frac{n_k - j}{n_k - 2n_{k-1}}} = (K^{2n_{k-1}} b_k^{2n_{k-1}})^{\frac{n_k - j}{n_k - 2n_{k-1}}} \geq 1 \end{aligned}$$

since $K b_k \geq 1$.

- Finally if $2n_{k-1} < j \leq n_k$ and $t \leq a_k$ then

$$\begin{aligned} \|(tT)^{-j} e_0\| &\geq a_k^{-j} \|T^{-j} e_0\| = a_k^{-j} \|T^{-n_k} e_0\| \cdot \left(\frac{K^{2n_{k-1}}}{a_k^{n_k}}\right)^{\frac{n_k - j}{n_k - 2n_{k-1}}} \\ &= a_k^{n_k - j} \left(\frac{K^{2n_{k-1}}}{a_k^{n_k}}\right)^{\frac{n_k - j}{n_k - 2n_{k-1}}} = \left(\frac{K^{2n_{k-1}}}{a_k^{2n_{k-1}}}\right)^{\frac{n_k - j}{n_k - 2n_{k-1}}} \geq 1 \end{aligned}$$

since $K/a_k \geq 1$ this time.

Hence $\max\{\|(tT)^j e_0\|, \|(tT)^{-j} e_0\|\} \geq 1$ for all j, and consequently, tT is not hypercyclic for $t \notin G$. This shows that $G = \{t > 0 \ ; \ tT \text{ is hypercyclic}\}$ and finishes the proof of Proposition 2.1. □

We are now ready for the proof of Theorem 1.2.

Proof of Theorem 1.2. — Let $K > 1$ be such that $M \subseteq (1/K, K)$. Write $M = \bigcap_{j\geq 1} G_j$ where $(G_j)_{j\geq 1}$ is a decreasing sequence of non empty open sets. Then each G_j can be decomposed as a disjoint union $G_j = \bigcup_{\lambda\in\Lambda_j}(a_\lambda, b_\lambda)$ of open intervals, where Λ_j are suitable finite or infinite sets. By Proposition 2.1, there exists a bilateral weighted shift T_1 such that $\|T\| \leq K^3$ and $G_1 = \{t > 0 \; ; \; tT_1 \text{ is hypercylic}\}$. Moreover, for each $\lambda \in \Lambda_1$ there is an increasing sequence $(m^{(1)}_{\lambda,i})_{i\geq 1}$ such that tT_1 satisfies the Hypercyclicity Criterion with respect to this sequence for each $t \in (a_\lambda, b_\lambda)$.

We then define a sequence of weighted bilateral shifts T_j, $j \geq 2$, in the following way. For each $j \geq 2$ define a (uniquely determined) function $g_j : \Lambda_j \to \Lambda_{j-1}$ such that $(a_\lambda, b_\lambda) \subseteq (a_{g_j(\lambda)}, b_{g_j(\lambda)})$ for every $\lambda \in \Lambda_j$. By Proposition 2.1 we can define inductively weighted bilateral shifts T_j such that

- $\|T_j\| \leq K^3$;
- $G_j = \{t > 0 \; ; \; tT_j \text{ is hypercyclic}\}$;
- for each $\lambda \in \Lambda_j$ there is an increasing sequence $(m^{(k)}_{\lambda,i})_{i\geq 1}$ of integers such that tT_j satisfies the Hypercyclicity Criterion with respect to this sequence for each $t \in (a_\lambda, b_\lambda)$, $\lambda \in \Lambda_j$. Moreover, we may assume that

$$\{m^{(j)}_{\lambda,i} \; ; \; i \geq 1\} \subseteq \{m^{(j-1)}_{g_j(\lambda),i} \; ; \; i \geq 1\}.$$

Consider now the direct sum $T = \bigoplus_{j=1}^{\infty} T_j$ acting on $\bigoplus_{j=1}^{\infty} \ell_2(\mathbb{Z})$. Clearly $\|T\| \leq K^3$. Suppose that tT is hypercyclic for some $t > 0$. Then tT_j is hypercyclic for each $j \geq 1$ and thus $t \in G_j$ for every $j \geq 1$. Hence t belongs to M.

Conversely, let t belong to M. For each j choose the (uniquely determined) element $\lambda^{(j)}$ of Λ_j such that $t \in (a_{\lambda^{(j)}}, b_{\lambda^{(j)}})$. Consider then the sequence $m_k = m^{(k)}_{\lambda^{(k)},k}$, $k \geq 1$. Then it is easy to check that tT satisfies the Hypercylicity Criterion with respect to the sequence $(m_k)_{k\geq 1}$, and Theorem 1.2 is proved. □

The proof of Theorem 1.6 is a straightforward application of the Hypercyclicity Criterion:

Proof of Theorem 1.6. — Let $t \in (t_1, t_2)$. In order to show that tT satisfies the Hypercyclicity Criterion, it suffices to prove that for all nonempty open subsets U, V of X and for any open neighborhood W of 0 there exists an $n \in \mathbb{N}$ such that $T^n(W) \cap V$ and $T^n(U) \cap W$ are non empty. Let $\varepsilon > 0$ be such that the open ball of radius ε is contained in W. Since $t_1T \oplus t_2T$ is hypercyclic, there exists a vector $x \oplus y$ with $\|x\| < \varepsilon$ and $y \in U$ which is hypercyclic for $t_1T \oplus t_2T$. Thus there exists an $n \in \mathbb{N}$ such that $(t_1T)^n x \in V$ and $\|(t_2T)^n y\| < \varepsilon$. Then $\|t_1^n t^{-n} x\| \leq \|x\| < \varepsilon$, so $t_1^n t^{-n} x \in W$, and $(tT)^n t_1^n t^{-n} x = (t_1T)^n x \in V$. Hence $(tT)^n(W) \cap V \neq \emptyset$. Furthermore, $\|(tT)^n y\| \leq \|(t_2T)^n y\| < \varepsilon$, and so $(tT)^n(U) \cap W \neq \emptyset$. Hence tT is hypercyclic. □

In view of Theorem 1.6, one may wonder whether the condition $t_1T \oplus t_2T$ hypercyclic is necessary for tT to be hypercyclic whenever t belongs to $[t_1, t_2]$. This is not the case, as shown by the following example:

Example 2.2. — There exists a bilateral weighted shift T on $\ell_2(\mathbb{Z})$ such that tT is hypercyclic for every $t \in (1, 4)$ but $2T \oplus 3T$ is not hypercyclic.

Proof. — We define T using the notation of the proof of Proposition 2.1 with $M = (a_1, b_1) \cup (a_2, b_2)$, where $\Lambda = \{1, 2\}$, $(a_1, b_1) = (1, 3)$ and $(a_2, b_2) = (2, 4)$. Then we define the function f as $f(k) = 1$ if k is odd and $f(k) = 2$ if k is even. Let $K = 5$ and construct a sequence (n_k) and the operator T as in Proposition 2.1. The proof of Proposition 2.1 shows that tT is hypercyclic if and only if $t \in (1, 3) \cup (2, 4) = (1, 4)$. Furthermore, it is easy to check that $2T \oplus 3T$ is not hypercyclic. Indeed if k is odd, then:

• if $2n_{k-1} < j \le n_k$,

$$w_j = 5^{2n_{k-1}} \left(\frac{1}{5^{2n_{k-1}} 3^{n_k}} \right)^{\frac{j-2n_{k-1}}{n_k - 2n_{k-1}}}.$$

Hence $||(3T)^j e_0|| = 3^j w_j = (15)^{2n_{k-1}} \left(\frac{1}{15^{2n_{k-1}}} \right)^{\frac{j-2n_{k-1}}{n_k-2n_{k-1}}} = 15^{\frac{2n_{k-1}(n_k - j)}{n_k - 2n_{k-1}}} \ge 1$.

• if $n_k < j \le 2n_k$, $||(3T)^j e_0|| = 3^j w_j = 15^{2(j-n_k)} \ge 1$.

If k is even, then

• if $2n_{k-1} < j \le n_k$,

$$||(2T)^{-j} e_0|| = 2^{-j} \widetilde{w}_j = (5/2)^{2n_{k-1}} \left(\frac{1}{(2/5)^{2n_{k-1}}} \right)^{\frac{j-2n_{k-1}}{n_k-2n_{k-1}}} \ge 1.$$

• if $n_k < j \le 2n_k$, $||(2T)^{-j} e_0|| = 2^{-j} \widetilde{w}_j = (5/2)^{2(j-n_k)} \ge 1$.

Hence there is no sequence (m_j) such that both $||(2T)^{m_j} e_0 \oplus (3T)^{m_j} e_0||$ and $||(2T)^{-m_j} e_0 \oplus (3T)^{-m_j} e_0||$ tend to zero as j tends to infinity, and $2T \oplus 3T$ is not hypercyclic. □

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