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Embeddings and the growth envelope of Besov spaces involving only slowly varying smoothness

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Abstract

We characterize local embeddings of Besov spaces $B^{0,b}_{p,r}$ involving only a slowly varying smoothness b into classical Lorentz spaces. These results are applied to establish sharp local embeddings of Besov spaces in question into Lorentz-Karamata spaces. As consequence of these results, we are able to determine growth envelopes of spaces $B^{0,b}_{p,r}$ and to show that we cannot describe all local embeddings of Besov spaces $B^{0,b}_{p,r}$ into Lorentz-Karamata spaces in terms of growth envelopes.

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1 Introduction

There are two aims of this paper:

First, to find *easily verifiable* conditions which are necessary and sufficient for the validity of embeddings of Besov spaces $B^{0,b}_{p,r} = B^{0,b}_{p,r}(\mathbb{R}^n)$, $1 \le p < \infty$, $1 \le r \le \infty$, (involving the zero classical smoothness and a slowly varying smoothness b) into classical Lorentz spaces $\Lambda^{loc}_q(\omega)$, $0 < q \le \infty$ (ω is a nonnegative measurable function on the interval $(0,1)$).

Second, to determine growth envelopes of spaces $B^{0,b}_{p,r}$, the notion introduced in [14] and [22].

To achieve our first goal, we use Kolyada's inequality (see [17]) and its converse form (see [3, Proposition 3.5]) to characterize the given local embedding by means of a reverse Hardy inequality restricted to the cone of non-increasing functions (see Theorem 3.1 below). Then we apply results of [9] and [10] (together with Theorem 5.3 below), to solve such an inequality completely and to obtain the desired characterization of embeddings of Besov spaces $B^{0,b}_{p,r}$ into classical Lorentz spaces $\Lambda^{loc}_q(\omega)$ (cf. Theorem 3.2 below).

Note that Theorem 3.1 below also follows from Theorem 1 and item 5 of Remarks to Theorem 1 in [19], where the rearrangement invariant hull of the Besov space in question is described. However, in [19] proofs are mainly sketched, details are often omitted. Moreover, our proof of Theorem 3.1 mentioned below is completely different.

Although Section 4 of [19] concerns embeddings of Besov spaces into Lorentz spaces, our Theorem 3.2 cannot be found there. The point is that in [19] Lorentz spaces are defined in terms of f^{**} (the maximal function of the non-increasing rearrangement f^* of a function f) rather than in terms of f^* and that the range of the parameter corresponding to q is restricted to the interval $[1,\infty]$. Moreover, in [19] a condition characterizing the embedding in question is much more involved and, in fact, the original problem is transfered to another one (cf. [19, Theorem 2, part a)].

On the other hand, the author of [19] investigates embeddings of Besov spaces which are more general than those considered here.

To achieve our second goal, first we apply Theorem 3.2 to the particular case when the target space of the given embedding is a Lorentz-Karamata space $L^{loc}_{p,q;\tilde{b}}$ (here $\tilde{b}$ is another slowly varying function) to establish sharp local embeddings of Besov spaces in question into Lorentz-Karamata spaces (cf. Theorems 3.3 and 3.4 below). Then, as consequence of these results, we are able to determine the growth envelope of the space $B^{0,b}_{p,r}$ (see Theorem 3.5 below) and to show that we cannot describe all local embeddings of Besov spaces $B^{0,b}_{p,r}$ into Lorentz-Karamata spaces in terms of growth envelopes (see Remark 3.6 below).

The paper is a direct continuation of [3], where sharp embeddings of Besov spaces $B^{0,b}_{p,r}$ into Lorentz-Karamata spaces $L^{loc}_{p,q;\tilde{b}}$ were established and the growth envelope of the space $B^{0,b}_{p,r}$ was determined in the particular case when $b(t) = \ell^{\beta}(t)$ and $\tilde{b} = \ell^{\gamma}(t)$ with $\ell(t) := 1 + |\ln t|$, $t > 0$, and $\gamma \in \mathbb{R}$. Note also that our approach in [3] was more complicated. First, we have converted the given embedding to a weighted inequality, which was more involved than that of Theorem 3.1 below (cf. [3, Proposition 3.6]). Then we have discretized the weighted inequality to find sufficient conditions for the validity of the embedding in question. Finally, convenient test functions have been used to prove that these conditions are also necessary.

Note that Theorems 3.3, 3.4(i) and 3.5 were also proved in [4] by a method slightly different from that used here.

Embeddings of Besov spaces into rearrangement invariant spaces were also considered in [11] and [12]. The authors of these papers used different methods and considered a more general setting. However, the methods used there do not allow to consider the full range of parameters. For example, after a careful checking, one can see that the restriction $1 < p \le r$ appears in the relevant result of [11] (cf. [11, Theorem 3]).

The paper is organized as follows. Section 2 contains notation, basic definitions and preliminary assertions. In Section 3 we present main results (Theorems 3.1-3.5.) Section 4 is devoted to the proof of Theorem 3.1. Theorem 3.2 is proved in Section 5. The proof of Theorem 3.3 is given in Section 6 while Theorem 3.4 is proved in Section 7. Finally, the proof of Theorem 3.5 is given in Section 8.

2 Notation, basic definitions and preliminaries

For two non-negative expressions (i.e. functions or functionals) $\mathcal{A}$ and $\mathcal{B}$, the symbol $\mathcal{A} \precsim \mathcal{B}$ (or $\mathcal{A} \succsim \mathcal{B}$) means that $\mathcal{A} \le c\,\mathcal{B}$ (or $c\,\mathcal{A} \ge \mathcal{B}$), where c is a positive constant independent of appropriate quantities involved in $\mathcal{A}$ and $\mathcal{B}$. If $\mathcal{A} \precsim \mathcal{B}$ and $\mathcal{A} \succsim \mathcal{B}$, we write $\mathcal{A} \approx \mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are *equivalent*. Throughout the paper we use the abbreviation LHS($*$) (RHS($*$)) for the left- (right-) hand side of the relation ($*$). Furthermore, we adopt the convention that $\frac{a}{0} = \infty$, $\frac{\infty}{a} = \infty$ if $0 < a < \infty$, $\frac{0}{0} = 0$, $\frac{\infty}{\infty} = 0$ and $0 \cdot \infty = 0$.

Given a set A, its characteristic function is denoted by χ_A. By $A \Delta B$ we mean the symmetric difference of sets A and B. For $a \in \mathbb{R}^n$ and $r \ge 0$, the notation $B(a, r)$ stands for the closed ball in $\mathbb{R}^n$ centered at a with the radius r. The volume of $B(0, 1)$ in $\mathbb{R}^n$ is denoted by V_n though, in general, we use the notation $|\cdot|_n$ for Lebesgue measure in $\mathbb{R}^n$.

Let Ω be a Borel subset of $\mathbb{R}^n$. The symbol $\mathcal{M}_0(\Omega)$ is used to denote the family of all complex-valued or extended real-valued (Lebesgue-)measurable functions defined and finite a.e. on Ω. By $\mathcal{M}_0^+(\Omega)$ we mean the subset of $\mathcal{M}_0(\Omega)$ consisting of those functions which are non-negative a.e. on Ω. If $\Omega = (a, b) \subset \mathbb{R}$, we write simply $\mathcal{M}_0(a, b)$ and $\mathcal{M}_0^+(a, b)$ instead of $\mathcal{M}_0((a, b))$ and $\mathcal{M}_0^+((a, b))$, respectively. By $\mathcal{M}_0^+(a, b; \downarrow)$ or $\mathcal{M}_0^+(a, b; \uparrow)$ we mean the collection of all $f \in \mathcal{M}_0^+(a, b)$ which are non-increasing or non-decreasing on (a, b), respectively. [1] Furthermore, by $AC(a, b)$ we denote the family of all functions which are locally absolutely continuous on (a, b) (that is, absolutely continuous on any closed subinterval of (a, b)). Finally, we put

$$S := \{f \in \mathcal{M}_0(\mathbb{R}^n) : \ |\operatorname{supp} f|_n \le 1\}.$$

For $f \in \mathcal{M}_0(\mathbb{R}^n)$, we define the *non-increasing rearrangement* f^* by

$$f^*(t) := \inf\{\lambda \ge 0 : |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|_n \le t\}, \quad t \ge 0.$$

[1]Consequently, functions from $\mathcal{M}_0^+(a, b; \downarrow)$ or $\mathcal{M}_0^+(a, b; \uparrow)$ are defined on the whole interval (a, b). On the other hand, functions from $\mathcal{M}_0^+(a, b)$ are defined only a. e. on (a, b).

The corresponding *maximal function* f^{**} is given by

$$f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)\,ds \tag{2.1}$$

and is also non-increasing on the interval $(0,\infty)$.

Given a Borel subset Ω of $\mathbb{R}^n$ and $0 < r \leq \infty$, $L_r(\Omega)$ is the usual *Lebesgue space* of measurable functions for which the quasi-norm

$$\|f\|_{r,\Omega} := \begin{cases} (\int_\Omega |f(t)|^r\,dt)^{1/r} & \text{if } 0 < r < \infty \\ \operatorname{ess\,sup}_{t\in\Omega}|f(t)| & \text{if } r = \infty \end{cases}$$

is finite. When $\Omega = \mathbb{R}^n$, we simplify $L_r(\Omega)$ to L_r and $\|\cdot\|_{r,\Omega}$ to $\|\cdot\|_r$.

Definition 2.1 *Let (α,β) be one of the intervals $(0,\infty)$, $(0,1)$ or $(1,\infty)$. A function $b \in \mathcal{M}_0^+(\alpha,\beta)$, $0 \not\equiv b \not\equiv \infty$, is said to be slowly varying on (α,β), notation $b \in SV(\alpha,\beta)$, if, for each $\varepsilon > 0$, there are functions $g_\varepsilon \in \mathcal{M}_0^+(\alpha,\beta;\uparrow)$ and $g_{-\varepsilon} \in \mathcal{M}_0^+(\alpha,\beta;\downarrow)$ such that*

$$t^\varepsilon b(t) \approx g_\varepsilon(t) \quad \text{and} \quad t^{-\varepsilon} b(t) \approx g_{-\varepsilon}(t) \quad \text{for all} \quad t \in (\alpha,\beta).$$

Here we follow the definition of $SV(0,+\infty)$ given in [8]; for other definitions see, for example, [1, 5, 6, 20]. The family of all slowly varying functions includes not only powers of iterated logarithms and the broken logarithmic functions of [7] but also such functions as $t \to \exp\left(|\log t|^a\right)$, $a \in (0,1)$. (The last mentioned function has the interesting property that it tends to infinity more quickly than any positive power of the logarithmic function.)

We shall need some properties of slowly varying functions.

Lemma 2.2 *Let $b \in SV(0,1)$.*

1. *Given $\alpha > 0$ and $\beta \in \mathbb{R}$, then the functions $t \mapsto b(t^\alpha)$ and $t \mapsto (b(t))^\beta$ are also in $SV(0,1)$; given $a \in SV(0,1)$, then $ab \in SV(0,1)$.*
2. *If $\varepsilon > 0$, then $t^\varepsilon b(t) \to 0$ as $t \to 0+$.*
3. *The extension of b by 1 outside of $(0,1)$ gives a function in $SV(0,\infty)$. (Such an extension will be assumed throughout this lemma, whenever b is considered in points outside of $(0,1)$.)*
4. *The functions b and b^{-1} are bounded in the interval $(\delta,1]$ for any $\delta \in (0,1)$.*
5. *Given $c > 0$, then $b(ct) \approx b(t)$ for all $t \in (0,1)$.*

6. *If* $\varepsilon > 0$ *and* $0 < r \le \infty$*, then*

$$\|t^{\varepsilon-1/r}b(t)\|_{r,(0,T)} \approx T^{\varepsilon}b(T) \quad \textit{and} \quad \|t^{-\varepsilon-1/r}b(t)\|_{r,(T,2)} \approx T^{-\varepsilon}b(T)$$

for all $T \in (0,1]$.

7. *If* $0 < r \le \infty$*, then the function* $B(t) := \|\tau^{-1/r}b(\tau)\|_{r,(t,2)}$, $t \in (0,1)$, *belongs to* $SV(0,1)$ *and the estimate* $b(t) \lesssim B(t)$ *holds for all* $t \in (0,1)$.

8. $\displaystyle\limsup_{t\to 0+} \frac{\int_t^1 s^{-1}b(s)\,ds}{b(t)} = \infty.$

Proof. We only prove assertion 8 here, as some of the others are easy consequences of Definition 2.1, and the proofs of the rest of them can be found, e.g., in [8, Proposition 2.2] and [13, Lemma 2.1].

Assume that assertion 8 does not hold. Then there exist $b \in SV(0,1)$, $c_1 > 0$ and $t_0 \in (0,1)$ such that $\int_t^1 s^{-1}b(s)\,ds \le c_1\, b(t)$ for all $t \in (0,t_0)$. Since $\int_t^1 s^{-1}b(s)\,ds \approx \int_t^2 s^{-1}b(s)\,ds$ for all $t \in (0,t_0)$,

$$\exists c_2 > 0: \quad f(t) := \int_t^2 s^{-1}b(s)\,ds \;\le\; c_2 b(t) \quad \forall t \in (0,t_0). \tag{2.2}$$

Consequently, given $\varepsilon \in (0, c_2^{-1})$, the function $t \mapsto t^{\varepsilon}f(t)$ (which belongs to $AC(0,t_0)$) is decreasing on $(0,t_0)$. Indeed, by (2.2), $(t^{\varepsilon}f(t))' = t^{\varepsilon-1}(\varepsilon f(t) - b(t)) < 0$ for all $t \in (0,t_0)$. However, by assertion 7, $f \in SV(0,1)$ and, by assertion 2, $\lim_{t\to 0+} t^{\varepsilon}f(t) = 0$. Thus, $f \equiv 0$ on $(0,t_0)$, which is a contradiction. Hence, assertion 8 holds. □

More properties and examples of slowly varying functions can be found in [23, Chapt. V, p. 186], [1], [5], [6], [18], [20] and [8].

Throughout the paper we put

$$\ell(t) := 1 + |\ln t|, \quad t \in (0,\infty)$$

(note that $\ell \in SV(0,\infty)$). We also adopt the following convention.

Convention 2.3 *If* $b \in SV(0,1)$*, then we assume that* b *is extended by 1 in the interval* $[1,\infty)$.

Given $q \in (0,\infty]$ and a non-negative measurable function ω on the interval $(0,1)$, the *classical Lorentz space* $\Lambda_q^{loc}(\omega)$ is defined to be the set of all measurable functions $f \in \mathbb{R}^n$ such that

$$\|\omega f^*\|_{q;(0,1)} < \infty.$$

In particular, putting $\omega(t) := t^{1/p-1/q}\, b(t)$, $t \in (0,1)$, where $b \in SV(0,1)$, we obtain the *Lorentz-Karamata space* $L_{p,q;b}^{loc}$. Note that Lorentz-Karamata spaces involve as particular cases the generalized Lorentz-Zygmund spaces, the Lorentz spaces, the Zygmund classes and Lebesgue spaces (cf., e.g., [5]).

Given $f \in L_p$, $1 \le p < \infty$, the *first difference operator* Δ_h of step $h \in \mathbb{R}^n$ transforms f in $\Delta_h f$ defined by

$$(\Delta_h f)(x) := f(x+h) - f(x), \quad x \in \mathbb{R}^n,$$

whereas the *modulus of continuity* of f is given by

$$\omega_1(f,t)_p := \sup_{\substack{h\in\mathbb{R}^n \\ |h|\le t}} \|\Delta_h f\|_p, \quad t>0.$$

Definition 2.4 *Let $1 \le p < \infty$, $1 \le r \le \infty$ and let $b \in SV(0,1)$ be such that*

$$\|t^{-1/r} b(t)\|_{r,(0,1)} = \infty. \tag{2.3}$$

The Besov space $B^{0,b}_{p,r} = B^{0,b}_{p,r}(\mathbb{R}^n)$ consists of those functions $f \in L_p$ for which the norm

$$\|f\|_{B^{0,b}_{p,r}} := \|f\|_p + \|t^{-1/r} b(t)\, \omega_1(f,t)_p\|_{r,(0,1)} \tag{2.4}$$

is finite.

Remark 2.5 (i) *Note that only the case when* (2.3) *holds is of interest. Indeed, otherwise $B^{0,b}_{p,r} \equiv L_p$ since*

$$\omega_1(f,t)_p \le 2\|f\|_p \quad \text{for all } t>0 \text{ and } f \in L_p. \tag{2.5}$$

(ii) *An equivalent norm results on $B^{0,b}_{p,r}(\mathbb{R}^n)$ if the modulus of continuity $\omega_1(f,\cdot)_p$ in* (2.4) *is replaced by the k-th order modulus of continuity $\omega_k(f,\cdot)_p$, where $k \in \{2,3,4,\ldots\}$. Indeed, this is a corollary of the Marchaud theorem* (cf. [2, Thm. 4.4, Chapt. 5]) *and the Hardy-type inequality from* Lemma 4.1 (*with $P=Q$, $b_1=b_2$*) *below.*

(iii) *Let the function $b \in SV(0,\infty)$ satisfy*

$$\|t^{-1/r} b(t)\|_{r,(1,\infty)} < \infty. \tag{2.6}$$

Then the functional

$$\|f\|_p + \|t^{-1/r} b(t)\, \omega_1(f,t)_p\|_{r,(0,\infty)} \tag{2.7}$$

is an equivalent norm on $B^{0,b}_{p,r}(\mathbb{R}^n)$. Indeed, this follows from (2.6) *and* (2.5).

Note also that assumption (2.6) *is natural. Otherwise the space of all functions on $\mathbb{R}^n$ for which norm* (2.7) *is finite is trivial* (*that is, it consists only of the zero element*)*. This is a consequence of the estimate*

$$\omega_1(f,1)_p \|t^{-1/r} b(t)\|_{r,(1,\infty)} \le \|t^{-1/r} b(t)\omega_1(f,t)_p\|_{r,(1,\infty)}.$$

In the next definition (we refer to [14] for details – see also [22, Chapt. II]) we need the notion of a Borel measure μ associated with a non-decreasing function $g : (a, b) \to \mathbb{R}$, where $-\infty \leq a < b \leq \infty$. We mean by this the unique (non-negative) measure μ on the Borel subsets of (a, b) such that $\mu([c, d]) = g(d+) - g(c-)$ for all $[c, d] \subset (a, b)$.

Definition 2.6 *Let* $(A, \|\cdot\|_A) \subset \mathcal{M}_0(\mathbb{R}^n)$ *be a quasi-normed space such that* $A \not\hookrightarrow L_\infty$*. A positive, non-increasing, continuous function* h *defined on some interval* $(0, \varepsilon]$, $\varepsilon \in (0, 1)$, *is called the* (local) growth envelope function *of the space* A *provided that*

$$h(t) \approx \sup_{\|f\|_A \leq 1} f^*(t) \quad \textit{for all } \; t \in (0, \varepsilon]. \tag{2.8}$$

Given a growth envelope function h *of the space* A (*determined up to equivalence near zero*) *and a number* $u \in (0, \infty]$, *we call the pair* (h, u) *the* (local) growth envelope *of the space* A *when the inequality*

$$\Big(\int_{(0,\varepsilon)} \Big(\frac{f^*(t)}{h(t)} \Big)^q \, d\mu_H(t) \Big)^{1/q} \lesssim \|f\|_A$$

(*with the usual modification when* $q = \infty$) *holds for all* $f \in A$ *if and only if the positive exponent* q *satisfies* $q \geq u$*. Here* μ_H *is the Borel measure associated with the non-decreasing function* $H(t) := -\ln h(t)$, $t \in (0, \varepsilon)$*. The component* u *in the growth envelope pair is called the* fine index.

3 Main Results

Theorem 3.1 *Let* $1 \leq p < \infty$, $1 \leq r \leq \infty$, $0 < q \leq \infty$ *and let* $b \in SV(0, 1)$ *satisfy* (2.3)*. Assume that* ω *is a non-negative measurable function on* $(0, 1)$*. Then*

$$\|\omega(t) f^*(t)\|_{q,(0,1)} \lesssim \|f\|_{B^{0,b}_{p,r}} \tag{3.1}$$

for all $f \in B^{0,b}_{p,r}$ *if and only if*

$$\|\omega(t) f^*(t)\|_{q,(0,1)} \lesssim \Big\| t^{-1/r} b(t^{1/n}) \Big(\int_0^t (f^*(u))^p \, du \Big)^{1/p} \Big\|_{r,(0,1)} \tag{3.2}$$

for all $f \in \mathcal{M}_0(\mathbb{R}^n)$.

Since we are able to characterize inequality (3.2) completely, we obtain the following assertion (which, in fact, describes embeddings of Besov spaces $B^{0,b}_{p,r}$ into classical Lorentz spaces $\Lambda^{loc}_q(\omega)$).

Theorem 3.2 *Let $1 \le p < \infty$, $1 \le r \le \infty$, $0 < q \le \infty$. Let $b \in SV(0,1)$ satisfy* (2.3) *and let b_r be defined by* (3.7). *Put $\rho = \infty$ if $p \le q$ and define ρ by $\frac{1}{\rho} = \frac{1}{q} - \frac{1}{p}$ if $q < p$. Assume that ω is a non-negative measurable function on $(0,1)$ and put*

$$\Omega_q(t) := \|\omega(s)\|_{q,(0,t)}, \quad t \in (0,1].$$

(i) *Let $1 \le r \le q \le \infty$. Then inequality* (3.1) *holds for all $f \in B^{0,b}_{p,r}$ if and only if*

$$\Omega_q(1) + \|s^{-\frac{1}{p}-\frac{1}{\rho}}\Omega_q(s)\|_{\rho,(t,1)} \lesssim b_r(t) \quad \textit{for all} \quad t \in (0,1). \tag{3.3}$$

(ii) *Let $0 < q < r < \infty$. Then inequality* (3.1) *holds for all $f \in B^{0,b}_{p,r}$ if and only if*

$$\Omega_q(1) + \int_0^1 \left(\|s^{-\frac{1}{p}-\frac{1}{\rho}}\Omega_q(s)\|_{\rho,(t,1)}\right)^{\frac{qr}{r-q}} b_r(t)^{\frac{r^2}{q-r}} b(t^{\frac{1}{n}})^r \,\frac{dt}{t} < \infty. \tag{3.4}$$

(iii) *Let $0 < q < r = \infty$. Put (cf.* (2.1)*)*

$$b^{**}_\infty(t) := t^{-1}\int_0^t b_\infty(\tau)\,d\tau, \quad t \in (0,1). \tag{3.5}$$

Then inequality (3.1) *holds for all $f \in B^{0,b}_{p,r}$ if and only if*

$$\Omega_q(1) + \int_{(0,1)} \left(\|s^{-\frac{1}{p}-\frac{1}{\rho}}\Omega_q(s)\|_{\rho,(t,1)}\right)^q d(b^{**}_\infty(t)^{-q}) < \infty. \tag{3.6}$$

Using Theorem 3.2, we are able to prove the following two assertions. The former describes the embedding of the Besov space $B^{0,b}_{p,r}$ into a Lorentz-Karamata space $L^{loc}_{p,q;\tilde{b}}$ while the latter concerns the sharpness of such an embedding.

Theorem 3.3 *Let $1 \le p < \infty$, $1 \le r \le \infty$, $0 < q \le \infty$ and let $b \in SV(0,1)$ satisfy* (2.3). *Define, for all $t \in (0,1)$,*

$$b_r(t) := \|s^{-1/r} b(s^{1/n})\|_{r,(t,2)} \quad {}^2 \tag{3.7}$$

and

$$\tilde{b}(t) := \begin{cases} b_r(t)^{1-r/q+r/\max\{p,q\}} b(t^{1/n})^{r/q-r/\max\{p,q\}} & \textit{if } r \ne \infty \\ b_\infty(t) & \textit{if } r = \infty \end{cases}. \tag{3.8}$$

Then the inequality

$$\|t^{1/p-1/q}\tilde{b}(t) f^*(t)\|_{q,(0,1)} \lesssim \|f\|_{B^{0,b}_{p,r}} \tag{3.9}$$

holds for all $f \in B^{0,b}_{p,r}$ if and only if $q \ge r$.

[2] Recall that throughout the paper we use Convention 2.3.

Theorem 3.4 *Let $1 \le p < \infty$, $1 \le r \le q \le \infty$ and let $b \in SV(0,1)$ satisfy* (2.3)*. Define b_r and $\tilde{b}$ by* (3.7) *and* (3.8)*.*

(i) *Let $\kappa \in \mathcal{M}_0^+(0,1;\downarrow)$. Then the inequality*

$$\|t^{1/p-1/q}\tilde{b}(t)\kappa(t)f^*(t)\|_{q,(0,1)} \lesssim \|f\|_{B^{0,b}_{p,r}} \tag{3.10}$$

holds for all $f \in B^{0,b}_{p,r}$ if and only if κ is bounded.

(ii) *Let $\kappa \in \mathcal{M}_0^+(0,1)$ and $q = \infty$. Then inequality* (3.10) *holds for all $f \in B^{0,b}_{p,r}$ if and only if $\|\kappa\|_{\infty,(0,1)} < \infty$.*

Theorems 3.3 and 3.4 enable us to determine the growth envelope of the Besov space $B^{0,b}_{p,r}$.

Theorem 3.5 *Let $1 \le p < \infty$, $1 \le r \le \infty$ and let $b \in SV(0,1)$ satisfy* (2.3)*. Define b_r by* (3.7)*. Then the growth envelope of $B^{0,b}_{p,r}$ is the pair*

$$(t^{-1/p}\, b_r(t)^{-1}, \max\{p,r\}).$$

Remark 3.6 (i) *Strictly speaking, $t^{-\frac{1}{p}} b_r(t)^{-1}$ might not have all the properties associated to a growth envelope function mentioned in* Definition 2.6 *but, with the help of part 6 of* Lemma 2.2*, it is possible to show that there is always an equivalent function defined on* $(0,1)$*, namely,*

$$h(t) := \int_t^2 s^{-1/p-1}\, b_r(s)^{-1}\, ds,$$

which does.

(ii) *Put $H(t) := -\ln h(t)$ for $t \in (0,\varepsilon)$, where $\varepsilon \in (0,1)$ is small enough. Since $H'(t) \approx \frac{1}{t}$ for a.e. $t \in (0,\varepsilon)$ (cf.* (8.4) *below), the measure μ_H associated with the function H satisfies $d\mu_H(t) \approx \frac{dt}{t}$. Thus, by* Definition 2.6*,* Theorem 3.5 *and part* (i) *of this remark,*

$$\|t^{1/p-1/q} b_r(t) f^*(t)\|_{q,(0,\varepsilon)} \lesssim \|f\|_{B^{0,b}_{p,r}} \quad \text{for all } f \in B^{0,b}_{p,r} \tag{3.11}$$

if and only if

$$q \ge \max\{p,r\}. \tag{3.12}$$

Hence, if (3.12) *holds, then inequality* (3.11) *gives the same result as inequality* (3.9) *of* Theorem 3.3 (*since* (3.12) *implies that $\tilde{b} = b_r$). However, if $r \le q < p$, then inequality* (3.11) *does not hold, while inequality* (3.9) *does. This means that the embeddings of Besov spaces $B^{0,b}_{p,r}$ given by* Theorem 3.3 *cannot be described in terms of growth envelopes when $1 \le r \le q < p < \infty$.*

4 Proof of Theorem 3.1

We shall need the following Hardy-type inequality, which is a consequence of [21, Thm. 6.2].

Lemma 4.1 *Let $1 \le P \le Q \le \infty$, $\nu \in \mathbb{R} \setminus \{0\}$ and let $b_1, b_2 \in SV(0,1)$. Then the inequality*

$$\Big\| t^{\nu-1/Q}\, b_2(t) \int_t^1 g(s)\, ds \Big\|_{Q,(0,1)} \lesssim \| t^{\nu+1-1/P} b_1(t) g(t) \|_{P,(0,1)}$$

holds for all $g \in \mathcal{M}_0^+(0,1)$ if and only if $\nu > 0$ and $b_2 \lesssim b_1$ on $(0,1)$.

We refer to [15, Thm. 2.4] for the next auxiliary result.

Lemma 4.2 *Let $0 < Q \le P \le 1$, $\Phi \in \mathcal{M}_0^+(\mathbb{R}_+ \times \mathbb{R}_+)$ and $v, w \in \mathcal{M}_0^+(0,\infty)$. Then the inequality*

$$\Big[\int_0^\infty \Big(\int_0^\infty \Phi(x,y) h(y)\, dy \Big)^P w(x)\, dx \Big]^{1/P} \lesssim \Big[\int_0^\infty h(x)^Q v(x)\, dx \Big]^{1/Q} \tag{4.1}$$

holds for every $h \in \mathcal{M}_0^+(0,\infty;\uparrow)$ if and only, for all $R > 0$,

$$\Big[\int_0^\infty \Big(\int_R^\infty \Phi(x,y)\, dy \Big)^P w(x)\, dx \Big]^{1/P} \lesssim \Big[\int_R^\infty v(x)\, dx \Big]^{1/Q}. \tag{4.2}$$

We shall also need the next assertion.

Lemma 4.3 (see [3, Proposition 4.2]) *Given $p > 0$ and a non-increasing function $g : (0,\infty) \to \mathbb{R}$, the function*

$$t \mapsto \int_0^t (g(s) - g(t))^p\, ds$$

is non-decreasing on $(0,\infty)$. In particular, if $f \in \mathcal{M}_0(\mathbb{R}^n)$, then the functions

$$t \to \int_0^t (f^*(s) - f^*(t))^p\, ds \tag{4.3}$$

and

$$t \to t(f^{**}(t) - f^*(t))$$

are non-decreasing on $(0,\infty)$.

To prove Theorem 3.1 we shall also make use of the following lemmas.

Lemma 4.4 *Let $1 \le p < \infty$, $1 \le r \le \infty$, and let $b \in SV(0,1)$. Then*

$$\begin{aligned} &\Big\| t^{1-1/r} b(t) \Big(\int_{t^n}^2 s^{-p/n} \int_0^s (f^*(u) - f^*(s))^p\, du\, \frac{ds}{s} \Big)^{1/p} \Big\|_{r,(0,1)} \\ &\approx \|f\|_p + \Big\| t^{-1/r} b(t^{1/n}) \Big(\int_0^t (f^*(u) - f^*(t))^p\, du \Big)^{1/p} \Big\|_{r,(0,1)} \end{aligned} \tag{4.4}$$

for all $f \in S$.

Proof. If $f \in S$, then function (4.3) is non-decreasing on $(0, \infty)$. Therefore, for all $t \in (0,1)$ and every $f \in S$,

$$\begin{aligned}
&\Big(\int_{t^n}^2 s^{-p/n} \int_0^s (f^*(u) - f^*(s))^p\, du\, \frac{ds}{s}\Big)^{1/p} \\
&\geq \Big(\int_0^{t^n} (f^*(u) - f^*(t^n))^p\, du\Big)^{1/p} \Big(\int_{t^n}^2 s^{-p/n}\, \frac{ds}{s}\Big)^{1/p} \\
&\approx t^{-1} \Big(\int_0^{t^n} (f^*(u) - f^*(t^n))^p\, du\Big)^{1/p}.
\end{aligned}$$

Together with the change of variables $t^n = \tau$, this implies that, for all $f \in S$,

$$\mathrm{LHS}(4.4) \gtrsim \Big\| \tau^{-1/r} b(\tau^{1/n}) \Big(\int_0^\tau (f^*(u) - f^*(\tau))^p\, d\tau\Big)^{1/p} \Big\|_{r,(0,1)}. \tag{4.5}$$

If $f \in S$, then $f^*(s) = 0$ for all $s \in [1, \infty)$. Thus, for all $t \in (0,1)$ and every $f \in S$,

$$\begin{aligned}
&\Big(\int_{t^n}^2 s^{-p/n} \int_0^s (f^*(u) - f^*(s))^p\, du\, \frac{ds}{s}\Big)^{1/p} \\
&\geq \Big(\int_1^2 s^{-p/n} \int_0^s f^*(u)^p\, du\, \frac{ds}{s}\Big)^{1/p} \\
&\geq \Big(\int_0^1 f^*(u)^p\, du\Big)^{1/p} \Big(\int_1^2 s^{-p/n}\, \frac{ds}{s}\Big)^{1/p} \\
&\approx \|f\|_p.
\end{aligned}$$

Consequently,

$$\mathrm{LHS}(4.4) \gtrsim \|f\|_p\ \|t^{1-1/r} b(t)\|_{r,(0,1)} \approx \|f\|_p \quad \text{for all } f \in S.$$

This estimate and (4.5) show that

$$\mathrm{LHS}(4.4) \gtrsim \mathrm{RHS}(4.4) \quad \text{for all } f \in S.$$

Now, we are going to prove the reverse estimate. Given $f \in S$, we put

$$h(s) = h_f(s) := \int_0^s (f^*(u) - f^*(s))^p\, du,\ \ s \in (0,2). \tag{4.6}$$

Then

LHS(4.4)

$$
\begin{aligned}
&\approx \Big\|\tau^{1/n-1/r} b(\tau^{1/n}) \Big(\int_\tau^2 s^{-p/n} \int_0^s (f^*(u)-f^*(s))^p \, du \, \frac{ds}{s} \Big)^{1/p} \Big\|_{r,(0,1)} \\
&\lesssim \Big\|\tau^{1/n-1/r} b(\tau^{1/n}) \Big(\int_\tau^1 s^{-p/n} \int_0^s (f^*(u)-f^*(s))^p \, du \, \frac{ds}{s} \Big)^{1/p} \Big\|_{r,(0,1)} \\
&+ \|\tau^{1/n-1/r} b(\tau^{1/n}) \Big(\int_1^2 s^{-p/n} \int_0^s (f^*(u)-f^*(s))^p \, du \, \frac{ds}{s} \Big)^{1/p} \Big\|_{r,(0,1)} \\
&\le \Big\|\tau^{1/n-1/r} b(\tau^{1/n}) \Big(\int_\tau^1 s^{-p/n} h(s) \, \frac{ds}{s} \Big)^{1/p} \Big\|_{r,(0,1)} \\
&+ \Big\|\tau^{1/n-1/r} b(\tau^{1/n}) \Big(\int_1^2 s^{-p/n} \int_0^s f^*(u)^p \, du \, \frac{ds}{s} \Big)^{1/p} \Big\|_{r,(0,1)} \\
&=: N_1 + N_2. \qquad (4.7)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
N_2 &\le \Big(\int_0^2 f^*(u)^p \, du \Big)^{1/p} \Big(\int_1^2 s^{-p/n} \frac{ds}{s} \Big)^{1/p} \|\tau^{1/n-1/r} b(\tau^{1/n})\|_{r,(0,1)} \\
&\approx \|f\|_p \quad \text{for all } f \in S. \qquad (4.8)
\end{aligned}
$$

To estimate N_1, we distinguish two cases.

(i) Assume that $r/p \in [1, +\infty]$. Then, using Lemma 4.1 (with $P = Q = r/p$, $\nu = p/n$, $b_2(t) = b_1(t) = b(t^{1/n})$, $g(s) = s^{-p/n-1} h(s)$), we obtain, for all $f \in S$,

$$
\begin{aligned}
N_1^p &= \Big\|\tau^{p/n-p/r} b(\tau^{1/n})^p \int_\tau^1 g(s) \, ds \Big\|_{r/p,(0,1)} \\
&\lesssim \|\tau^{p/n+1-p/r} b(\tau^{1/n})^p g(\tau)\|_{r/p,(0,1)} \\
&= \|\tau^{-p/r} b(\tau^{1/n})^p h(\tau)\|_{r/p,(0,1)} \\
&\approx \|\tau^{-1/r} b(\tau^{1/n}) h(\tau)^{1/p}\|_{r,(0,1)}^p \\
&= \Big\|\tau^{-1/r} b(\tau^{1/n}) \Big(\int_0^\tau (f^*(u)-f^*(\tau))^p \, du \Big)^{1/p} \Big\|_{r,(0,1)}^p. \qquad (4.9)
\end{aligned}
$$

Combining estimates (4.7)–(4.9), we see that

$$
\text{LHS}(4.4) \lesssim \text{RHS}(4.4) \quad \text{for all } f \in S.
$$

(ii) Assume that $r/p \in (0,1)$. First we prove that, for all $f \in S$,

$$
\begin{aligned}
N_1^p &= \Big\|\tau^{p/n-p/r} b(\tau^{1/n})^p \int_\tau^1 s^{-p/n-1} h(s) \, ds \Big\|_{r/p,(0,1)} \\
&\lesssim \|\tau^{-p/r} b(\tau^{1/n})^p h(\tau)\|_{r/p,(0,2)} =: N_3 \qquad (4.10)
\end{aligned}
$$

The function h given by (4.6) is non-decreasing on $(0,\infty)$. Thus, to verify (4.10), we apply Lemma 4.2. On putting $Q = P = r/p$ and

$$
\begin{aligned}
w(x) &= \chi_{(0,1)}(x)x^{r/n-1}b(x^{1/n})^r, \\
v(x) &= \chi_{(0,2)}(x)x^{-1}b(x^{1/n})^r, \\
\Phi(x,y) &= \chi_{(x,1)}(y)y^{-p/n-1}
\end{aligned}
$$

for all $x, y \in (0,\infty)$, we see that inequality (4.10) can be rewritten as (4.1). Consequently, by Lemma 4.2, inequality (4.10) holds for every $h \in \mathcal{M}_0^+(0,\infty;\uparrow\,)$ provided that condition (4.2) is satisfied.

Making use of Lemma 2.2, we obtain that, for all $R > 0$,

$$
\mathrm{LHS}(4.2) \lesssim \Big[b(R^{1/n})^p + \Big(\int_R^1 x^{-1}b(x^{1/n})^r\,dx\Big)^{p/r}\Big]\chi_{(0,1)}(R)
$$

and

$$
\mathrm{RHS}(4.2) \approx \Big[\int_R^2 x^{-1}b(x^{1/n})^r\,dx\Big]^{p/r}\chi_{(0,2)}(R).
$$

Therefore, condition (4.2) is satisfied, which means that inequality (4.10) holds.

To finish the proof, it is sufficient to show that

$$
N_3^{1/p} \lesssim \mathrm{RHS}(4.4) \quad \text{for all } f \in S.
$$

The definition of N_3 and (4.6) imply that, for all $f \in S$,

$$
\begin{aligned}
N_3^{1/p} &= \|\tau^{-1/r}b(\tau^{1/n})h(\tau)^{1/p}\|_{r,(0,2)} \\
&\approx \|\tau^{-1/r}b(\tau^{1/n})h(\tau)^{1/p}\|_{r,(0,1)} + \|\tau^{-1/r}b(\tau^{1/n})h(\tau)^{1/p}\|_{r,(1,2)} \\
&\approx \Big\|\tau^{-1/r}b(\tau^{1/n})\Big(\int_0^\tau (f^*(u) - f^*(\tau))^p\,du\Big)^{1/p}\Big\|_{r,(0,1)} \\
&\quad + \Big\|\tau^{-1/r}b(\tau^{1/n})\Big(\int_0^\tau (f^*(u) - f^*(\tau))^p\,du\Big)^{1/p}\Big\|_{r,(1,2)}.
\end{aligned}
$$

Comparing this estimate with RHS(4.4), we see that it is enough to verify that

$$
\Big\|\tau^{-1/r}b(\tau^{1/n})\Big(\int_0^\tau (f^*(u) - f^*(\tau))^p\,du\Big)^{1/p}\Big\|_{r,(1,2)} \lesssim \|f\|_p
$$

for all $f \in S$. However, such an estimate is an easy consequence of the facts that function (4.3) is non-decreasing on $(0,\infty)$, that $|\mathrm{supp}\, f|_n \le 1$, and that $\|\tau^{-1/r}b(\tau^{1/n})\|_{r,(1,2)} < \infty$. $\square$

Lemma 4.5 *Let $1 \le p < \infty$, $1 \le r \le \infty$ and let $b \in SV(0,\infty)$. Then*

$$\begin{aligned}
&\|f\|_p + \Big\| t^{-1/r} b(t) \Big(\int_0^t f^*(s)^p\, ds \Big)^{1/p} \Big\|_{r,(0,1)} \\
&\approx \Big\| t^{-1/r} b(t) \Big(\int_0^t f^*(s)^p\, ds \Big)^{1/p} \Big\|_{r,(0,1)} \\
&\approx \Big\| t^{-1/r} b(t) \Big(\int_0^t f^*(s)^p\, ds \Big)^{1/p} \Big\|_{r,(0,2)} \qquad (4.11)
\end{aligned}$$

for all $f \in S$.

Proof. Since, for all $f \in S$,

$$\begin{aligned}
&\Big\| t^{-1/r} b(t) \Big(\int_0^t f^*(s)^p\, ds \Big)^{1/p} \Big\|_{r,(1/2,1)} \\
&\ge \Big(\int_0^{1/2} f^*(s)^p\, ds \Big)^{1/p} \| t^{-1/r} b(t) \|_{r,(1/2,1)} \\
&\approx \Big(\int_0^{1/2} f^*(s)^p\, ds \Big)^{1/p} \ge \frac{1}{2^{1/p}} \Big(\int_0^1 f^*(s)^p\, ds \Big)^{1/p} = \frac{1}{2^{1/p}} \|f\|_p, \qquad (4.12)
\end{aligned}$$

the first estimate in (4.11) is clear. Furthermore, for all $f \in S$,

$$\begin{aligned}
\Big\| t^{-1/r} b(t) \Big(\int_0^t f^*(s)^p\, ds \Big)^{1/p} \Big\|_{r,(1,2)} &\le \Big(\int_0^2 f^*(s)^p\, ds \Big)^{1/p} \| t^{-1/r} b(t) \|_{r,(1,2)} \\
&\approx \Big(\int_0^2 f^*(s)^p\, ds \Big)^{1/p} = \|f\|_p. \qquad (4.13)
\end{aligned}$$

The second estimate in (4.11) is a consequence of (4.13) and (4.12). □

Lemma 4.6 *Let $1 \le p < \infty$, $1 \le r \le \infty$, and let $b \in SV(0,1)$. Then*

$$\begin{aligned}
&\|f\|_p + \Big\| t^{-1/r} b(t^{1/n}) \Big(\int_0^t (f^*(u) - f^*(t))^p\, du \Big)^{1/p} \Big\|_{r,(0,1)} \\
&\approx \Big\| t^{-1/r} b(t^{1/n}) \Big(\int_0^t f^*(u)^p\, du \Big)^{1/p} \Big\|_{r,(0,1)} \qquad (4.14)
\end{aligned}$$

for all $f \in S$.

Proof. The estimate LHS(4.14) $\le$ RHS(4.14) follows immediately from Lemma 4.5.

To prove the reverse estimate, first assume that $p = 1$. Since (see [3, (16)]), for all $t \in (0,1)$,

$$f^{**}(t) - f^{**}(1) = \int_t^1 \frac{f^{**}(s) - f^*(s)}{s}\, ds, \qquad (4.15)$$

Lemma 4.1 (with $P = Q = r$, $\nu = 1$, $b_2(t) = b_1(t) = b(t^{1/n})$, $g(s) = s^{-1}(f^{**}(s) - f^*(s)))$ and the identity

$$f^{**}(t) - f^*(t) = t^{-1} \int_0^t (f^*(u) - f^*(t))\, du, \tag{4.16}$$

yields that, for all $f \in S$,

$$\begin{aligned} \mathrm{RHS}(4.14) &\lesssim f^{**}(1) + \Big\| t^{1-1/r} b(t^{1/n}) \int_t^1 \frac{f^{**}(s) - f^*(s)}{s}\, ds \Big\|_{r,(0,1)} \\ &\lesssim \|f\|_1 + \Big\| t^{1-1/r} b(t^{1/n}) (f^{**}(t) - f^*(t)) \Big\|_{r,(0,1)} = \mathrm{LHS}(4.14). \end{aligned}$$

Assume now that $1 < p < \infty$. Since $f^* \le f^{**}$, (4.15) implies that, for all $f \in S$ and $t \in (0,1)$,

$$\begin{aligned} \mathrm{RHS}(4.14) &\le \Big\| t^{-1/r} b(t^{1/n}) \Big(\int_0^t f^{**}(u)^p\, du \Big)^{1/p} \Big\|_{r,(0,1)} \\ &\lesssim f^{**}(1) + \Big\| t^{-1/r} b(t^{1/n}) \Big(\int_0^t \Big(\int_u^1 \frac{f^{**}(s) - f^*(s)}{s}\, ds \Big)^p du \Big)^{1/p} \Big\|_{r,(0,1)} \\ &\lesssim \|f\|_p + \Big\| t^{-1/r} b(t^{1/n}) \Big(\int_0^t \Big(\int_u^t \frac{f^{**}(s) - f^*(s)}{s}\, ds \Big)^p du \Big)^{1/p} \Big\|_{r,(0,1)} \\ &\quad + \Big\| t^{1/p-1/r} b(t^{1/n}) \int_t^1 \frac{f^{**}(s) - f^*(s)}{s}\, ds \Big\|_{r,(0,1)} \\ &=: \|f\|_p + N_1 + N_2. \end{aligned} \tag{4.17}$$

By Lemma 4.1 (with $P = Q = p$, $\nu = 1/p$, $b_2 = b_1 \equiv 1$, $g(s) = g_t(s) = s^{-1}(f^{**}(s) - f^*(s))\chi_{(0,t)}(s))$,

$$\Big\| \int_u^t \frac{f^{**}(s) - f^*(s)}{s}\, ds \Big\|_{p,(0,1)} \lesssim \Big\| f^{**}(s) - f^*(s) \Big\|_{p,(0,t)} \quad \text{for all} \quad t \in (0,1).$$

As also (see [3, Proposition 4.5])

$$\Big(\int_0^t (f^{**}(s) - f^*(s))^p\, ds \Big)^{1/p} \lesssim \Big(\int_0^t (f^*(s) - f^*(t))^p\, ds \Big)^{1/p} \quad \text{for all } t \in (0.1),$$

we obtain

$$N_1 \lesssim \Big\| t^{-1/r} b(t^{1/n}) \Big(\int_0^t (f^*(s) - f^*(t))^p\, ds \Big)^{1/p} \Big\|_{r,(0,1)}. \tag{4.18}$$

By Lemma 4.1 (with $P = Q = r$, $\nu = 1/p$, $b_2(t) = b_1(t) = b(t^{1/n})$, $g(s) = s^{-1}(f^{**}(s) - f^*(s)))$,

$$N_2 \lesssim \Big\| t^{1/p-1/r} b(t^{1/n}) (f^{**}(t) - f^*(t)) \Big\|_{r,(0,1)}.$$

Since (4.16) and the Hölder inequality imply that

$$f^{**}(t) - f^*(t) \le t^{-1/p}\Big(\int_0^t (f^*(u) - f^*(t))^p\, du\Big)^{1/p},$$

we arrive at

$$N_2 \lesssim \mathrm{RHS}(4.18). \tag{4.19}$$

The desired estimate follows from (4.17), (4.18) and (4.19). □

The last result, which we need to prove Theorem 3.1, reads as follows.

Proposition 4.7 *Let* $1 \le p < \infty$, $1 \le r \le \infty$, $0 < q \le \infty$ *and let* $b \in SV(0,1)$ *satisfy* (2.3). *Assume that* ω *is a non-negative measurable function on* $(0,1)$. *Then*

$$\|\omega(t) f^*(t)\|_{q,(0,1)} \lesssim \|f\|_{B^{0,b}_{p,r}}$$

for all $f \in B^{0,b}_{p,r}$ *if and only if*

$$\|\omega(t) f^*(t)\|_{q,(0,1)} \lesssim \Big\| t^{1-1/r} b(t) \Big(\int_{t^n}^2 s^{-p/n} \int_0^s (f^*(u) - f^*(s))^p\, du \,\frac{ds}{s}\Big)^{1/p}\Big\|_{r,(0,1)}$$

for all $f \in S$.

Proof is analogous to that of [3, Proposition 3.6] (where the slowly varying function b was of logarithmic type). □

Proof of Theorem 3.1. The result follows from Proposition 4.7 and Lemmas 4.4 and 4.6. (Note that if inequality (3.2) holds for all $f \in \mathcal{M}_0(\mathbb{R}^n)$ with $|\mathrm{supp} f|_n \le 1$, then it also holds for all $f \in \mathcal{M}_0(\mathbb{R}^n)$).

□

5 Proof of Theorem 3.2

We shall start with the following two assertions. The first is a consequence of [9, Thm. 4.2] while the later of [10, Thm. 1.8]. [3]

Theorem 5.1 *Let* $0 < P, Q < \infty$, *let* v, w *be non-negative measurable functions on* $(0,\infty)$ *such that* $V_P(t) := \|v(s)\|_{P,(0,t)}$, $\mathcal{V}_P(t) := t\|\frac{v(s)}{s}\|_{P,(t,\infty)}$ *and* $W_Q(t) := \|w(s)\|_{Q,(0,t)}$ *are finite for all* $t > 0$. *Assume that*

$$\Big\|\frac{v(s)}{s}\Big\|_{P,(0,1)} = \|v(s)\|_{P,(1,\infty)} = \infty.$$

[3]Note that in [10, Thm. 1.8] it is assumed that $\int_0^t v(s)\, ds < \infty$, $t \in (0,\infty)$. However, one can check that this assumption is superfluous.

(i) *Let* $0 < P \le Q$ *and* $1 \le Q < \infty$*. Then the inequality*

$$\|wf^*\|_{Q,(0,\infty)} \lesssim \|vf^{**}\|_{P,(0,\infty)} \tag{5.1}$$

holds for all $f \in \mathcal{M}_0(\mathbb{R}^n)$ *if and only if*

$$\sup_{t\in(0,\infty)} \frac{W_Q(t)}{V_P(t) + \mathcal{V}_P(t)} < \infty. \tag{5.2}$$

(ii) *Let* $1 \le Q < P < \infty$ *and* $R = PQ/(P-Q)$*. Then inequality* (5.1) *holds for all* $f \in \mathcal{M}_0(\mathbb{R}^n)$ *if and only if*

$$\int_0^\infty \frac{\sup_{y\in(t,\infty)} y^{-R} W_Q(y)^R}{(V_P(t) + \mathcal{V}_P(t))^{R+2P}}\, V_P(t)^P \mathcal{V}_P(t)^P\, t^{R-1}\, dt < \infty.$$

(iii) *Let* $0 < P \le Q < 1$*. Then inequality* (5.1) *holds for all* $f \in \mathcal{M}_0(\mathbb{R}^n)$ *if and only if*

$$\sup_{t\in(0,\infty)} \frac{W_Q(t) + t\left(\int_t^\infty W_Q(s)^{\frac{Q^2}{1-Q}} w(s)^Q s^{-\frac{Q}{1-Q}}\, ds\right)^{\frac{1-Q}{Q}}}{V_P(t) + \mathcal{V}_P(t)} < \infty.$$

(iv) *Let* $0 < Q < 1$*,* $Q < P$ *and* $R = PQ/(P-Q)$*. Then inequality* (5.1) *holds for all* $f \in \mathcal{M}_0(\mathbb{R}^n)$ *if and only if*

$$\int_0^\infty \frac{\left[W_Q(t)^{\frac{Q}{1-Q}} + t^{\frac{Q}{1-Q}} \int_t^\infty W_Q(s)^{\frac{Q^2}{1-Q}} w(s)^Q s^{-\frac{Q}{1-Q}}\, ds\right]^{\frac{R(1-Q)}{Q}}}{(V_P(t) + \mathcal{V}_P(t))^{R+2P}} \times$$
$$\times V_P(t)^P \mathcal{V}_P(t)^P t^{-1}\, dt < \infty.$$

Theorem 5.2 *Let* $Q \in (0,\infty)$ *and let* v, w *be non-negative measurable functions on* $(0,\infty)$*. Assume that the function* $W_Q(t) := \|w(s)\|_{Q,(0,t)}$ *is finite for all* $t > 0$*. Define the quasi-concave function*

$$\phi(t) := \operatorname*{ess\,sup}_{s\in(0,t)} \left(s \operatorname*{ess\,sup}_{\tau\in(s,\infty)} \frac{v(\tau)}{\tau} \right), \quad t \in (0,\infty).^4 \tag{5.3}$$

Assume that ϕ *is non-degenerate, that is,*

$$\lim_{t\to0+} \phi(t) = \lim_{t\to\infty} \frac{1}{\phi(t)} = \lim_{t\to\infty} \frac{\phi(t)}{t} = \lim_{t\to0+} \frac{t}{\phi(t)} = 0. \tag{5.4}$$

Let ν *be a non-negative Borel measure on* $[0,\infty)$ *such that*

$$\frac{1}{\phi(t)^Q} \approx \int_{[0,\infty)} \frac{d\nu(s)}{s^Q + t^Q} \quad \textit{for all } t \in (0,\infty). \tag{5.5}$$

[4] Recall that ϕ is quasi-concave if ϕ is equivalent to a function from $\mathcal{M}_0^+(0,\infty;\uparrow)$ while $\phi(t)/t$ is equivalent to a function from $\mathcal{M}_0^+(0,\infty;\downarrow)$.

(i) *Let* $1 \le Q < \infty$. *Then inequality*

$$\|wf^*\|_{Q,(0,\infty)} \lesssim \|vf^{**}\|_{\infty,(0,\infty)} \tag{5.6}$$

holds for all $f \in \mathcal{M}_0(\mathbb{R}^n)$ *if and only if*

$$\int_{[0,\infty)} \left(\sup_{s\in(t,\infty)} \frac{W_Q(s)}{s} \right)^Q d\nu(t) < \infty.$$

(ii) *Let* $0 < Q < 1$. *Then inequality* (5.6) *holds for all* $f \in \mathcal{M}_0(\mathbb{R}^n)$ *if and only if*

$$\int_{[0,\infty)} \frac{\zeta(t)}{t^Q}\, d\nu(t) < \infty,$$

where

$$\zeta(t) = W_Q(t)^Q + t^Q \Big(\int_t^\infty W_Q(s)^{\frac{Q^2}{1-Q}} w(s)^Q s^{-\frac{Q}{1-Q}}\, ds \Big)^{1-Q}, \quad t \in (0,\infty).$$

To have a complete characterization of inequality (5.1), we shall also need the next result.

Theorem 5.3 *Let* $0 < P \le \infty$ *and let* v, w *be non-negative measurable functions on* $(0,\infty)$. *Put* $W_\infty(t) := \|w(s)\|_{\infty,(0,t)}$ *for all* $t > 0$. *Then the inequality*

$$\|wf^*\|_{\infty,(0,\infty)} \lesssim \|vf^{**}\|_{P,(0.\infty)} \tag{5.7}$$

holds for all $f \in \mathcal{M}_0(\mathbb{R}^n)$ *if and only if*

$$W_\infty(t) \lesssim \|v(s)\min\{1, t/s\}\|_{P,(0,\infty)} \quad \text{for all} \quad t \in (0,\infty). \tag{5.8}$$

Proof. (i) Necessity. Testing inequality (5.7) with $f = \chi_E$, where $E \subset \mathbb{R}^n$, $|E|_n = t > 0$, we arrive at (5.8).

(ii) Sufficiency. Together with the estimate $w(t) \le W_\infty(t)$ for a.e. $t \in (0,\infty)$ (cf. [21, Lemma 5.3]), (5.8) implies that [5]

$$\begin{aligned}
\text{LHS}(5.7) &\le \|W_\infty(t) f^*(t)\|_{\infty,(0,\infty)} \\
&\lesssim \|\|v(s)\min\{1,t/s\}\|_{P,(0,\infty)} f^{**}(t)\|_{\infty,(0,\infty)} \\
&\lesssim \|(\|v(s)\|_{P,(0,t)} + t\|v(s)/s\|_{P,(t,\infty)}) f^{**}(t)\|_{\infty,(0,\infty)} \\
&\lesssim \|\|v(s)f^{**}(s)\|_{P,(0,t)} + \|v(s)f^{**}(s)\|_{P,(t,\infty)}\|_{\infty,(0,\infty)} \\
&\lesssim \|vf^{**}\|_{P,(0,\infty)}.
\end{aligned}$$

□

[5] In the next estimate the first inequality can be replaced by the equality, that is, $\text{LHS}(5.7) = \|W_\infty(t)\, f^*(t)\|_{\infty,(0,\infty)}$.

Remark 5.4 *Define functions V_P and $\mathcal{V}_P$ as in* Theorem 5.1 *for all* $P \in (0,\infty]$.

(i) *Note that*

$$\mathrm{RHS(5.8)} \approx \mathrm{V_P(t)} + \mathcal{V}_\mathrm{P}(\mathrm{t}) \quad \textit{for all} \quad \mathrm{t} \in (0,\infty).$$

Consequently, condition (5.8) *corresponds to* (5.2).

(ii) *The function ϕ given by* (5.3) *satisfies*

$$\phi(t) \approx V_\infty(t) + \mathcal{V}_\infty(t) \approx \|v(s)\min\{1, t/s\}\|_{\infty,(0,\infty)} \quad \textit{for all} \quad t \in (0,\infty).$$

We shall also need the following two lemmas.

Lemma 5.5 *Let $1 \le p < \infty$ and let $b \in SV(0,1)$. Put*

$$v(t) := t\, b(t^{1/n})^p \chi_{(0,1)}(t) + \ell(t)\chi_{[1,\infty)}(t), \quad t \in (0,\infty).$$

If ϕ is given by (5.3) *and b_∞ defined by* (3.7)*, then*

$$\phi(t) \approx t\, b_\infty(t)^p \chi_{(0,1]}(t) + \ell(t)\chi_{(1,\infty)}(t) \quad \textit{for all } t \in (0,\infty). \tag{5.9}$$

Proof. Assume first that $t \in (0,1]$. Then, using assertions 4, 1, 7 and 6 of Lemma 2.2, we obtain

$$\begin{aligned}
\phi(t) &= \operatorname*{ess\,sup}_{s\in(0,t)} \Big(s \max\{ \operatorname*{ess\,sup}_{\tau\in(s,1)} b(\tau^{1/n})^p, \operatorname*{ess\,sup}_{\tau\in[1,\infty)} \frac{\ell(\tau)}{\tau}\}\Big)\\
&\approx \operatorname*{ess\,sup}_{s\in(0,t)} \big(s\, b_\infty(s)^p\big)\\
&= \|s^{1-1/\infty} b_\infty(s)^p\|_{\infty,(0,t)}\\
&\approx t\, b_\infty(t)^p \quad \text{for all } t \in (0,1].
\end{aligned}$$

Assume now that $t \in (1,\infty)$. Then

$$\begin{aligned}
\phi(t) &= \max\{ \operatorname*{ess\,sup}_{s\in(0,1)} \Big(s \operatorname*{ess\,sup}_{\tau\in(s,\infty)} \frac{v(\tau)}{\tau}\Big), \operatorname*{ess\,sup}_{s\in[1,t)} \Big(s \operatorname*{ess\,sup}_{\tau\in(s,\infty)} \frac{v(\tau)}{\tau}\Big)\}\\
&\approx \max\{\phi(1), \ell(t)\}\\
&\approx \ell(t) \quad \text{for all } t \in (1,\infty).
\end{aligned}$$

□

In the next lemma we consider the maximal function b_∞^{**} given by (3.5). By part 6 of Lemma 2.2,

$$b_\infty^{**} \approx b_\infty \quad \text{on} \quad (0,1]. \tag{5.10}$$

Moreover,

$$b_\infty^{**} \in AC(0,1). \tag{5.11}$$

Lemma 5.6 *Let p, b, b_∞, v and ϕ be the same as in* Lemma 5.5. Assume that (2.3) with $r = \infty$ holds. Let $0 < q < \infty$ and ν be the measure on $[0,\infty)$ which is absolutely continuous with respect to the Lebesgue measure on $[0,\infty)$ and satisfies

$$d\nu(t) = \begin{cases} -b_\infty^{**}(t)^{-q-1}(b_\infty^{**})'(t)\,dt & \text{if } 0 < t \le 1 \\ t^{q/p-1}\ell^{-q/p-1}(t)\,dt & \text{if } t > 1 \end{cases} . \tag{5.12}$$

Then

$$\frac{1}{\phi(t)^{q/p}} \approx \int_{[0,\infty)} \frac{d\nu(s)}{s^{q/p} + t^{q/p}} \quad \text{for all } t \in (0,\infty).$$

Proof. Since $(b_\infty^{**})'(t) = t^{-1}(b_\infty(t) - b_\infty^{**}(t)) \le 0$ *a.e.* on $(0,1)$, the measure ν is non-negative.

(i) Let $t \in (1,\infty)$. In view of (5.9), we need to show that

$$I = I(t) := \int_{[0,\infty)} \frac{d\nu(s)}{s^{q/p} + t^{q/p}} \approx \ell(t)^{-q/p} \quad \text{for all } t \in (1,\infty).$$

Split the integral in the following three terms:

$$\begin{aligned} I_1 &:= \int_{(0,1)} \frac{-b_\infty^{**}(s)^{-q-1}(b_\infty^{**})'(s)}{s^{q/p} + t^{q/p}}\,ds\,, \\ I_2 &:= \int_{(1,t)} \frac{s^{q/p-1}\ell^{-q/p-1}(s)}{s^{q/p} + t^{q/p}}\,ds\,, \\ I_3 &:= \int_{(t,\infty)} \frac{s^{q/p-1}\ell^{-q/p-1}(s)}{s^{q/p} + t^{q/p}}\,ds\,. \end{aligned}$$

Since $(b_\infty^{**}(s)^{-q})' = -q\,b_\infty^{**}(s)^{-q-1}(b_\infty^{**})'(s)$ for a.e. $s \in (0,1)$ and b_∞^{**-q} is non-decreasing on $[0,1]$,

$$\begin{aligned} I_1 &\le t^{-q/p}\int_0^1 -b_\infty^{**}(s)^{-q-1}(b_\infty^{**})'(s)\,ds \\ &\le \frac{1}{q}\,t^{-q/p}b_\infty^{**}(1)^{-q} \\ &\approx t^{-q/p} \le \ell(t)^{-q/p} \quad \text{for all } t \in (1,\infty). \end{aligned}$$

Furthermore, for all $t \in (1,\infty)$,

$$\begin{aligned} I_2 &\le t^{-q/p}\int_1^t s^{q/p-1}\ell^{-q/p-1}(s)\,ds \\ &\le t^{-q/p}\int_0^t s^{q/p-1}\ell^{-q/p-1}(s)\,ds \\ &\approx \ell(t)^{-q/p-1} \le \ell(t)^{-q/p} \end{aligned}$$

and

$$I_3 \;\le\; \int_t^\infty s^{-1}\ell^{-q/p-1}(s)\,ds \;\approx\; \ell(t)^{-q/p}.$$

So, we have got the estimate of I by $\ell(t)^{-q/p}$ from above. To prove the reverse estimate, note that

$$I_3 \;\ge\; \frac{1}{2}\int_t^\infty s^{-1}\ell^{-q/p-1}(s)\,ds \;\approx\; \ell(t)^{-q/p} \qquad \text{for all } \; t\in(1,\infty).$$

(ii) Consider now $t\in(0,1]$. By (5.9), we need to show that

$$J = J(t) := \int_{[0,\infty)} \frac{d\nu(s)}{s^{q/p}+t^{q/p}} \approx t^{-q/p} b_\infty(t)^{-q} \qquad \text{for all } \; t\in(0,1].$$

Again, we split the integral in three terms:

$$\begin{aligned}
J_1 \;&:=\; \int_{(0,t)} \frac{-b_\infty^{**}(s)^{-q-1}(b_\infty^{**})'(s)}{s^{q/p}+t^{q/p}}\,ds\,,\\
J_2 \;&:=\; \int_{[t,1]} \frac{-b_\infty^{**}(s)^{-q-1}(b_\infty^{**})'(s)}{s^{q/p}+t^{q/p}}\,ds\,,\\
J_3 \;&:=\; \int_{(1,\infty)} \frac{s^{q/p-1}\ell^{-q/p-1}(s)}{s^{q/p}+t^{q/p}}\,ds\,.
\end{aligned}$$

As before,

$$\begin{aligned}
J_1 \;&\le\; t^{-q/p}\int_0^t -b_\infty^{**}(s)^{-q-1}(b_\infty^{**})'(s)\,ds\\
&\lesssim\; t^{-q/p}b_\infty^{**}(t)^{-q} \;\approx\; t^{-q/p}b_\infty(t)^{-q} \qquad \text{for all } \; t\in(0,1].
\end{aligned}$$

By (5.11), the integration by parts, assertions 6 and 1 of Lemma 2.2 together with the definition of slowly varying functions, we obtain , for all $t\in(0,1]$,

$$\begin{aligned}
J_2 \;&\le\; \int_t^1 -s^{-q/p}b_\infty^{**}(s)^{-q-1}(b_\infty^{**})'(s)\,ds\\
&\lesssim\; b_\infty^{**}(1)^{-q} + \int_t^2 s^{-q/p-1}b_\infty^{**}(s)^{-q}\,ds\\
&\approx\; 1 + t^{-q/p}b_\infty^{**}(t)^{-q} \;\approx\; t^{-q/p}b_\infty(t)^{-q},
\end{aligned}$$

$$J_3 \;\le\; \int_1^\infty s^{-1}\ell^{-q/p-1}(s)\,ds \;\approx\; 1 \;\lesssim\; t^{-q/p}b_\infty(t)^{-q}.$$

So, we have got the estimate of J by $t^{-q/p}b_\infty(t)^{-q}$ from above. To prove the converse estimate, we apply (5.11) and hypothesis (2.3), to arrive at

$$\begin{aligned} J_1 &\geq \frac{1}{2}\, t^{-q/p} \int_0^t -b_\infty^{**}(s)^{-q-1}(b_\infty^{**})'(s)\, ds \\ &\approx t^{-q/p} b_\infty^{**}(t)^{-q} \approx t^{-q/p} b_\infty(t)^{-q} \quad \text{for all } t \in (0,1]. \end{aligned} \qquad \square$$

Proof of Theorem 3.2. If $\Omega_q(1) = \|\omega\|_{q,(0,1)} = \infty$, then inequality (3.1) does not hold for all $f \in B^{0,b}_{p,r}$. One can also show that none of conditions (3.3), (3.4) and (3.6) is satisfied in this case.

Assume that $\Omega_q(1) < \infty$. By Theorem 3.1, inequality (3.1) is equivalent to (3.2). Let $g \in \mathcal{M}_0(\mathbb{R}^n)$ and $f := |g|^{1/p}$. Then (3.2) yields

$$\|\omega^p(t) g^*(t)\|_{q/p,(0,1)} \lesssim \|t^{1-p/r} b(t^{1/n})^p g^{**}(t)\|_{r/p,(0,1)} \tag{5.13}$$

for all $g \in \mathcal{M}_0(\mathbb{R}^n)$ (or even for any measurable function g on $\mathbb{R}^n$). Inequality (5.13) is equivalent to

$$\|w g^*\|_{Q,(0,\infty)} \lesssim \|v g^{**}\|_{P,(0,\infty)}, \tag{5.14}$$

where $Q = q/p$, $P = r/p$,

$$w(t) := \begin{cases} \omega(t)^p & \text{for all } t \in (0,1) \\ 0 & \text{for all } t \in [1,\infty) \end{cases} \tag{5.15}$$

and

$$v(t) := t^{1-p/r} b(t^{1/n})^p \chi_{(0,1)}(t) + t^{-p/r}\chi_{[1,\infty)}(t), \quad t \in (0,\infty),$$

if $1 \leq r \leq q \leq \infty$ or $0 < q < r < \infty$, while

$$v(t) := t b(t^{1/n})^p \chi_{(0,1)}(t) + \ell(t)\chi_{[1,\infty)}(t), \quad t \in (0,\infty),$$

if $0 < q < r = \infty$.

Indeed, the implication (5.13) $\Rightarrow$ (5.14) is trivial. To prove the converse implication, take $g \in S$. Since $g^{**}(t) = g^{**}(1)/t$ for all $t \in (1,\infty)$, and

$$\Big\|\frac{v(t)}{t}\Big\|_{P,(1,\infty)} \approx 1 \approx \|v(t)\|_{P,(0,1)},$$

we get

$$\|v g^{**}\|_{P,(1,\infty)} \approx g^{**}(1) \lesssim \|v g^{**}\|_{P,(0,1)}.$$

Consequently, for all $g \in S$,

$$\text{RHS}(5.14) \approx \|v g^{**}\|_{P,(0,1)} = \text{RHS}(5.13).$$

Together with (5.14), this shows that (5.13) holds for all $g \in S$, and hence, (5.13) holds for all $g \in \mathcal{M}_0(\mathbb{R}^n)$.

To characterize inequality (5.14), we apply Theorems 5.1-5.3.

(i) Let $1 \le r \le q \le \infty$.

When $1 \le p \le q$, then the number ρ is from Theorem 3.2 satisfies $\rho = \infty$. By Theorem 5.1 (i) and Theorem 5.3, inequality (5.14) holds for all $g \in \mathcal{M}_0(\mathbb{R}^n)$ if and only if

$$\Omega_q(t) \lesssim t^{\frac{1}{p}} b_r(t) \quad \text{for all} \quad t \in (0,1). \tag{5.16}$$

Since $b_r \in \mathcal{M}_0^+(a,b;\downarrow)$, one can easily verify that (5.16) is equivalent to

$$\|s^{-\frac{1}{p}-\frac{1}{\rho}} \Omega_q(s)\|_{\rho,(t,1)} \lesssim b_r(t) \quad \text{for all} \quad t \in (0,1).$$

As $\Omega_q(1) \lesssim b_r(1) \le b_r(t)$ for all $t \in (0,1)$, the last estimate can be rewritten as (3.3).

When $q < p$, then $\rho = \frac{pq}{p-q}$. By Theorem 5.1 (iii), inequality (5.14) holds for all $g \in \mathcal{M}_0(\mathbb{R}^n)$ if and only if

$$\Omega_q(t) + t^{\frac{1}{p}} \Big(\int_t^1 \Omega_q(s)^{\frac{q^2}{p-q}} \omega(s)^q s^{\frac{q}{q-p}} \, ds \Big)^{\frac{p-q}{pq}} \lesssim t^{\frac{1}{p}} b_r(t) \ \text{ for all } \ t \in (0,1). \tag{5.17}$$

Using integration by parts, we obtain that, for all $t \in (0,1)$,

$$\begin{aligned}
&\Omega_q(1) + \bigg(\int_t^1 \Omega_q(s)^{\frac{pq}{p-q}} s^{\frac{q}{q-p}-1} \, ds \bigg)^{\frac{p-q}{pq}} \\
&\approx \quad \bigg(\frac{p-q}{q} \Omega_q(1)^{\frac{pq}{p-q}} + \int_t^1 \Omega_q(s)^{\frac{pq}{p-q}} s^{\frac{q}{q-p}-1} \, ds \bigg)^{\frac{p-q}{pq}} \\
&= \quad \bigg(\frac{p-q}{q} t^{\frac{q}{q-p}} \Omega_q(t)^{\frac{pq}{p-q}} + \frac{p}{q} \int_t^1 \Omega_q(s)^{\frac{q^2}{p-q}} \omega(s)^q s^{\frac{q}{q-p}} \, ds \bigg)^{\frac{p-q}{pq}} \\
&\approx \quad t^{-\frac{1}{p}} \, \text{LHS}(5.17).
\end{aligned}$$

Consequently,

$$\text{LHS}(5.17) \approx t^{\frac{1}{p}} \Omega_q(1) + t^{\frac{1}{p}} \|s^{-\frac{1}{p}-\frac{1}{\rho}} \Omega_q(s)\|_{\rho,(t,1)} \ \text{ for all } \ t \in (0,1), \tag{5.18}$$

which implies that (5.17) is equivalent to (3.3).

(ii) Let $0 < q < r < \infty$.

When $1 \le p \le q$, then by Theorem 5.1(ii), inequality (5.14) holds for all $g \in \mathcal{M}_0(\mathbb{R}^n)$ if and only if

$$\int_0^1 \Big[\sup_{t < y \le 1} y^{-\frac{1}{p}} \Omega_q(y) \Big]^{\frac{qr}{r-q}} b_r(t)^{\frac{r^2}{q-r}} b(t^{\frac{1}{n}})^r \, \frac{dt}{t} < \infty. \tag{5.19}$$

As $\rho = \infty$ in this case, (5.19) is equivalent to (3.4).

When $q < p$, then, by Theorem 5.1(iv), inequality (5.14) holds for all $g \in \mathcal{M}_0(\mathbb{R}^n)$ if and only if

$$\begin{aligned}\infty &> \int_0^1 \frac{\Big(\Omega_q(t)^{\frac{pq}{p-q}} + t^{\frac{q}{p-q}} \int_t^1 \Omega_q(s)^{\frac{q^2}{p-q}} \omega(s)^q s^{\frac{q}{q-p}}\, ds\Big)^{\frac{r(p-q)}{p(r-q)}}}{t^{\frac{rq}{p(r-q)}} b_r(t)^{\frac{r^2}{r-q}}} b(t^{\frac{1}{n}})^r \,\frac{dt}{t} \\ &\approx \int_0^1 \Big(\frac{\mathrm{LHS}(5.17)}{t^{\frac{1}{p}}}\Big)^{\frac{qr}{r-q}} b_r(t)^{\frac{r^2}{q-r}} b(t^{\frac{1}{n}})^r \,\frac{dt}{t}\end{aligned}$$

Using estimate (5.18), one can prove that the last condition is equivalent to (3.4).

(iii) Let $0 < q < r = \infty$. To characterize inequality (5.14), we apply Theorem 5.2. Define ϕ by (5.3). Lemma 5.5, part 2 of Lemma 2.2, (2.3) and (3.7) imply that (5.4) holds. Let ν be the measure given by (5.12). By Lemma 5.6, assumption (5.5) (with $Q = q/p$) is satisfied.

When $1 \le p \le q$, then Theorem 5.2 (i) shows that inequality (5.14) holds for all $g \in \mathcal{M}_0(\mathbb{R}^n)$ if and only if

$$\int_0^1 \Big[\sup_{t \le y \le 1} y^{-\frac{q}{p}} \Omega_q(y)^q\Big] b_\infty^{**}(t)^{-q-1} (-(b_\infty^{**})'(t))\, dt < \infty. \tag{5.20}$$

As $\rho = \infty$ and (5.11) holds, (5.20) is equivalent to (3.6).

When $q < p$, then Theorem 5.2 (ii) shows that inequality (5.14) holds for all $g \in \mathcal{M}_0(\mathbb{R}^n)$ if and only if

$$\begin{aligned}\infty &> \int_{[0,1)} \frac{\Omega_q(t)^q + t^{\frac{q}{p}} \Big(\int_t^1 \Omega_q(s)^{\frac{q^2}{p-q}} \omega(s)^q s^{\frac{q}{q-p}}\, ds\Big)^{\frac{p-q}{p}}}{t^{\frac{q}{p}}}\, d\nu(t) \\ &\approx \int_{[0,1)} \Big(\frac{\mathrm{LHS}(5.17)}{t^{\frac{1}{p}}}\Big)^q d\nu(t).\end{aligned}$$

Using estimate (5.18), one can prove that the last condition is equivalent to (3.6). □

6 Proof of Theorem 3.3

Proof of the sufficiency part. Assume that $q \ge r$. Put

$$\omega(t) := t^{1/p-1/q} \tilde{b}(t), \qquad t \in (0,1). \tag{6.1}$$

By Theorem 3.2, it is sufficient to verify that inequality (3.3) holds for $t \in (0,1)$. Since, by Lemma 2.2,

$$\Omega_q(t) \approx t^{\frac{1}{p}} \tilde{b}(t) \text{ for all } t \in (0,1], \tag{6.2}$$

we get

$$\|s^{-\frac{1}{p}-\frac{1}{\rho}}\Omega_q(s)\|_{\rho,(t,1)} \approx \|s^{-\frac{1}{\rho}}\tilde{b}(s)\|_{\rho,(t,1)} \text{ for all } t \in (0,1). \tag{6.3}$$

If $p \le q$, then $\rho = \infty$ and

$$\tilde{b}(t) = b_r(t), \quad t \in (0,1).$$

Thus,

$$\|s^{-\frac{1}{\rho}}\tilde{b}(s)\|_{\rho,(t,1)} = b_r(t), \quad t \in (0,1). \tag{6.4}$$

Combining (6.2), (6.3) and (6.4), we see that (3.3) is satisfied.

If $q < p$, then $\rho < \infty$,

$$\tilde{b}(t) = b_r(t)^{1-r/\rho} b(t^{1/n})^{r/\rho}, \quad t \in (0,1), \tag{6.5}$$

and

$$\|s^{-\frac{1}{\rho}}\tilde{b}(s)\|_{\rho,(t,1)} = \|s^{-\frac{1}{\rho}} b_r(s)^{1-\frac{r}{\rho}} b(s^{\frac{1}{n}})^{\frac{r}{\rho}}\|_{\rho,(t,1)} \approx (b_r(t)^\rho - b_r(1)^\rho)^{1/\rho} \tag{6.6}$$

for all $t \in (0,1)$. Together with (6.5), estimates (6.2), (6.3) and (6.6) imply that (3.3) holds.

Proof of the necessity part. Assume that $q < r$ and define ω by (6.1). By Theorem 3.2, it is sufficient to verify that inequalities (3.4) and (3.6) are not satisfied.

By (6.4), (6.6) and (2.3), there is $\varepsilon \in (0,1)$ such that

$$\|s^{-\frac{1}{\rho}}\tilde{b}(s)\|_{\rho,(t,1)} \approx b_r(t) \quad \text{for all} \quad t \in (0,\varepsilon). \tag{6.7}$$

Together with (6.3), this gives

$$\|s^{-\frac{1}{p}-\frac{1}{\rho}}\Omega_q(s)\|_{\rho,(t,1)} \approx b_r(t) \quad \text{for all} \quad t \in (0,\varepsilon).$$

Hence, if $r < \infty$, then

$$\text{LHS}(3.4) \gtrsim \int_0^\varepsilon b_r(s)^{-r} b(s^{\frac{1}{n}})^r \,\frac{ds}{s} = \int_{b_r(\varepsilon)^r}^\infty \frac{dx}{x} = \infty.$$

Thus, (3.4) does not hold. If $r = \infty$, then (cf. (5.10))

$$\text{LHS}(3.6) \gtrsim \int_{(0,\varepsilon)} b_\infty^{**}(s)^q \, d(b_\infty^{**}(s)^{-q}) = \infty.$$

Therefore, (3.6) does not hold. □

7 Proof of Theorem 3.4

In view of Theorem 3.3, the sufficiency of the condition that κ is essentially bounded is obvious. To prove that this condition is also necessary, suppose that (3.10) holds for all $f \in B^{0,b}_{p,r}$. Put $\omega(t) := t^{1/p-1/q}\tilde{b}(t)\kappa(t)$ for all $t \in (0,1)$. By Theorem 3.2(i), condition (3.3) is satisfied. In particular, $\Omega(1) < \infty$.

(i) Assume that $\kappa \in \mathcal{M}_0^+(0,1;\downarrow)$. Fix $y \in (0,1)$. Then

$$\Omega_q(s) \geq \|\tau^{\frac{1}{p}-\frac{1}{q}}\tilde{b}(\tau)\|_{q,(0,s)}\kappa(s) \approx s^{\frac{1}{p}}\tilde{b}(s)\kappa(s) \quad \text{for all} \quad s \in (0,1).$$

Thus,

$$\|s^{-\frac{1}{p}-\frac{1}{\rho}}\Omega_q(s)\|_{\rho,(t,1)} \gtrsim \kappa(y)\|s^{-\frac{1}{\rho}}\tilde{b}(s)\|_{\rho,(t,y)} \quad \text{for all} \quad t \in (0,y).$$

Put $\tau = 1$ if $\rho = \infty$ and $\tau = \rho$ otherwise. Then (2.3) implies that there is $\delta \in (0,y)$ such that

$$\begin{aligned}\|s^{-\frac{1}{\rho}}\tilde{b}(s)\|^{\tau}_{\rho,(t,y)} &\geq \|s^{-\frac{1}{\rho}}\tilde{b}(s)\|^{\tau}_{\rho,(t,1)}\left(1 - \frac{\|s^{-\frac{1}{\rho}}\tilde{b}(s)\|^{\tau}_{\rho,(y,1)}}{\|s^{-\frac{1}{\rho}}\tilde{b}(s)\|^{\tau}_{\rho,(t,1)}}\right) \\ &\geq \frac{1}{2}\|s^{-\frac{1}{\rho}}\tilde{b}(s)\|^{\tau}_{\rho,(t,1)} \quad \text{for all} \quad t \in (0,\delta).\end{aligned}$$

Together with (6.7) and (3.3), the last two estimates yield that

$$b_r(t) \gtrsim \kappa(y)b_r(t), \quad \text{for all} \quad t \in (0,\min\{\delta,\varepsilon\}).$$

(Note that a constant implicitly involved in this estimate is independent of y.) Consequently, the function κ is bounded.

(ii) Let $\kappa \in \mathcal{M}_0^+(0,1)$ and $q = \infty$. We know from part (i) of the proof of Theorem 3.2 that (3.3) is equivalent to (5.16). Since $q = \infty$, we see that $\tilde{b}(t) = b_r(t)$ and $w(t) = t^{1/p}b_r(t)\kappa(t)$ for all $t \in (0,1)$. Hence, (5.16) reads as

$$\|s^{1/p}b_r(s)\kappa(s)\|_{\infty,(0,t)} \lesssim t^{1/p}b_r(t) \quad \text{for all} \quad t \in (0,1) \tag{7.1}$$

Since (cf. [21, Lemma 5.3])

$$\|s^{1/p}b_r(s)\kappa(s)\|_{\infty,(0,t)} \geq t^{1/p}b_r(t)\kappa(t) \quad \text{for a.e. } t \in (0,1),$$

estimate (7.1) implies that

$$\|\kappa\|_{\infty,(0,1)} < \infty.$$

□

8 Proof of Theorem 3.5

Put $A := B^{0,b}_{p,r}$ and $q = \infty$. Then $b_r(t) = \tilde{b}(t)$, $t \in (0,1)$, where the function $\tilde{b}$ is given by (3.8).

First, let $\kappa \in \mathcal{M}_0(0,1;\downarrow)$ and $\kappa(t) \approx t^{-1/p} b_r(t)^{-1}$ for all $t \in (0,1)$. Since $\lim_{t\to 0+} \kappa(t) = \infty$ (cf. part 2 of Lemma 2.2), Theorem 3.4 (i) shows that the inequality

$$\|f^*(t)\|_{\infty,(0,1)} \lesssim \|f\|_A$$

cannot hold for all $f \in A$, that is, $A \not\hookrightarrow L_\infty$.

Second, denote by h the growth envelope function of the space A. Then (2.8) implies that

$$h(t)^{-1} f^*(t) \lesssim \|f\|_A \text{ for all } t \in (0,\varepsilon] \text{ and all } f \in A.$$

Extending the function h to the whole interval $(0,1)$ by putting $h(t) = h(\varepsilon)$, $t \in (\varepsilon, 1)$, we see that $h \in \mathcal{M}_0^+(0,1;\downarrow)$, h is continuous on $(0,1)$ and

$$h(t)^{-1} f^*(t) \lesssim \|f\|_A \text{ for all } t \in (0,1) \text{ and all } f \in A. \tag{8.1}$$

On setting $\kappa(t) := h(t)^{-1} t^{-1/p} b_r(t)^{-1}$, $t \in (0,1)$, we can rewrite (8.1) as

$$t^{1/p} b_r(t) \kappa(t) f^*(t) \lesssim \|f\|_A \text{ for all } t \in (0,1) \text{ and all } f \in A.$$

Since $q = \infty$ and $\tilde{b} = b_r$ on $(0,1)$, the last estimate implies that

$$\|t^{1/p} \tilde{b}(t) \kappa(t) f^*(t)\|_{\infty,(0,1)} \lesssim \|f\|_A \text{ for all } f \in A.$$

Consequently, by Theorem 3.4 (ii), $\|\kappa\|_{\infty,(0,1)} < \infty$, which show that

$$t^{-1/p} b_r(t)^{-1} \lesssim h(t) \text{ for all } t \in (0,\varepsilon]. \tag{8.2}$$

Third, by Theorem 3.3 (with $q = \infty$), $t^{1/p} b_r(t) f^*(t) \lesssim 1$ for all $t \in (0,1)$ and all $f \in A$ with $\|f\|_A \le 1$. Thus

$$\sup_{\|f\|_A \le 1} f^*(t) \lesssim t^{-1/p} b_r(t)^{-1} \text{ for all } t \in (0,1).$$

Hence,

$$h(t) \lesssim t^{-1/p} b_r(t)^{-1} \text{ for all } t \in (0,\varepsilon].$$

Together with (8.2) and (2.8), this results in

$$\sup_{\|f\|_A \le 1} f^*(t) \approx t^{-1/p} b_r(t)^{-1} \text{ for all } t \in (0,\varepsilon].$$

Further, as the function

$$t \mapsto \int_t^2 s^{-1/p-1} b_r(s)^{-1}\, ds, \quad t \in (0,1), \tag{8.3}$$

is a positive, non-increasing and continuous function equivalent to $t^{-\frac{1}{p}} b_r(t)^{-1}$ on $(0,1)$, it follows (cf. Remark 3.6) that the function (8.3) (which we now denote by h) is also a growth envelope function of the space A.

To calculate the fine index (cf. Definition 2.6), consider the function $H(t) := -\ln h(t),\ t \in (0,1)$. Since

$$H'(t) = -\frac{h'(t)}{h(t)} = -\frac{-t^{-1/p-1} b_r(t)^{-1}}{\int_t^2 s^{-1/p-1} b_r(s)^{-1} ds} \approx \frac{1}{t} \quad \text{for } \ a.e.\, t \in (0,1), \qquad (8.4)$$

we obtain $d\mu_H(t) = H'(t)\, dt \approx \frac{1}{t}\, dt$ on $(0,1)$. Thus, applying the "if" part of Theorem 3.3 with $q \in [\max\{p,r\}, \infty]$, we get

$$\begin{aligned} &\Big(\int_0^1 \Big(\frac{f^*(t)}{h(t)} \Big)^q d\mu_H(t) \Big)^{1/q} \\ \approx\ &\Big(\int_0^1 t^{\frac{q}{p}-1}\, b_r(t)^q\, f^*(t)^q\, dt \Big)^{1/q} \\ \lesssim\ &\|f\|_A \quad \text{for all } \ f \in A \end{aligned} \qquad (8.5)$$

(with the usual modification when $q = \infty$).

It remains to show that (8.5) cannot hold for $q \in (0, \max\{p,r\})$.

First, assume that $0 < q < r$. By Theorem 3.2, it is sufficient to verify that condition (3.4) or (3.6) cannot hold with

$$\omega(t) := t^{1/p-1/q} b_r(t), \qquad t \in (0,1). \qquad (8.6)$$

Since

$$\Omega_q(s) \approx s^{1/p} b_r(s) \quad \text{for all} \quad s \in (0,1), \qquad (8.7)$$

$b_r \in SV(0,1)$ and (2.3) holds, assertion 7 of Lemma 2.2 implies that there is $\varepsilon \in (0,1)$ such that

$$\|s^{-\frac{1}{p}-\frac{1}{\rho}} \Omega_q(s)\|_{\rho,(t,1)} \approx \|s^{-\frac{1}{\rho}} b_r(s)\|_{\rho,(t,1)} \gtrsim b_r(t) \quad \text{for all} \quad t \in (0,\varepsilon). \qquad (8.8)$$

Consequently, if $r < \infty$,

$$\text{LHS(3.4)} \gtrsim \int_0^\varepsilon b_r(t)^{-r} b(t^{\frac{1}{n}})^r \,\frac{dt}{t} = \int_{b_r(\varepsilon)^r}^\infty \frac{dx}{x} = \infty.$$

Similarly, if $r = \infty$, then, on using (8.8) and (5.10), we obtain that

$$\text{LHS(3.6)} \gtrsim \int_{(0,\varepsilon)} b_\infty^{**}(s)^q \, d(b_\infty^{**}(s)^{-q}) = \infty.$$

Therefore, neither (3.4) or (3.6) holds.

Second, assume that $r \le q < p$. By Theorem 3.2, it is sufficient to verify that condition (3.3) cannot hold with ω given by (8.6). Estimate (8.7) implies that

$$\text{LHS(3.3)} \gtrsim \|s^{-\frac{1}{\rho}} b_r(s)\|_{\rho,(t,1)} \quad \text{for all} \quad t \in (0,1).$$

Since $\rho < \infty$ and $b_r^\rho \in SV(0,1)$, assertion 8 of Lemma 2.2 show that (3.3) cannot hold. □

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