Hysteresis rarefaction in the Riemann problem

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Abstract. We consider the wave equation with Preisach hysteresis and Riemann initial data as a model for wave propagation in hysteretic (e. g. elastoplastic) media. The main result consists in proving that in the convex hysteresis loop domain, there exists a unique self-similar locally Lipschitz continuous solution. In other words, smooth rarefaction waves propagate in both directions from the initial jump discontinuity.

Introduction

Wave propagation in media with Preisach hysteresis (elastoplastic, ferromagnetic, piezoelectric, etc.) has some very particular features. The propagation speed is bounded above by the speed corresponding to the linearized system, see [5]. Unlike the viscosity, the hysteresis dissipation thus slows down the wave propagation. Moreover, regular scalar Preisach constitutive operators admit a nontrivial *convexity domain* $(-h, h)$, which is manifested by the fact that if the input function moves between $-h$ and h for all times, then all increasing hysteresis branches are convex and all decreasing branches are concave, see Figure 2. Some classes of operators (including the so-called Prandtl-Ishlinskii operators arising in elastoplasticity, see [7]) are globally convex with $h = \infty$. In [6, Chapter III] it is shown how this convexity property implies the validity of a higher order energy inequality, which can be used in turn to derive results on existence and long time behavior of regular solutions to the wave equation with hysteresis. Shocks thus do not appear under convex hysteresis if the data are smooth. A generalized form of the convexity-based higher order energy inequality has been exploited in [2] in the context of magnetohydrodynamic flow with hysteresis.

We prove here that in fact, even for discontinuous Riemann data, the solution in the convexity domain becomes Lipschitz continuous for all positive times, that is, rarefaction waves propagate in both directions. Only in the hysteresis output term, a stationary discontinuity at $x = 0$ may persist for $t > 0$.

This is in agreement with the observation made in [8], that maximally dissipating solutions to the Riemann problem without hysteresis have the tendency to follow a convex/concave shock path along the constitutive graph. Here, since smooth convex/concave paths are available due to the hysteretic constitutive law, shocks have no reason to occur. Indeed, things will be different if we leave the convexity domain. Then rarefaction waves will be followed by shocks and the analysis will have to take into account additional "hysteresis entropy" conditions. This, however, goes beyond the scope of this paper.

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The following text is divided into three sections. In Section 1, we state the problem and the main existence and uniqueness result. Section 2 is devoted to an overview of properties of the Preisach hysteresis model and a detailed proof of the convexity/concavity property of small amplitude loops. The proof of the main result is given in Section 3.

1. Main result

Consider the problem of wave propagation in media with hysteresis governed by the system

$$\left\{ \begin{array}{rcl} v_t & = & u_x \\ (u+w)_t & = & v_x \end{array} \right. \tag{1.1}$$

in the halfplane $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, with a hysteretic constitutive relation between u and w of the form

$$w(x,t) \;=\; F[\lambda(x,\cdot), u(x,\cdot)](t)\,, \tag{1.2}$$

where F a Preisach operator defined in (2.1) below, with initial memory configuration λ.

System (1.1) can be interpreted as a dimensionless 1D problem for longitudinal elastoplastic waves, where v is the velocity, u is the stress (identified with the linear elastic strain component), and w is the plastic strain component.

The *Riemann problem* consists in choosing the initial conditions

$$v(x,0) = \left\{ \begin{array}{ll} V_+ & \text{for} \quad x>0, \\ V_- & \text{for} \quad x<0, \end{array} \right. \qquad u(x,0) = \left\{ \begin{array}{ll} U_+ & \text{for} \quad x>0, \\ U_- & \text{for} \quad x<0, \end{array} \right. \tag{1.3}$$

where $V_\pm, U_\pm$ are given constants, and the initial memory configuration

$$\lambda(x,r) = \left\{ \begin{array}{ll} \lambda_+(r) & \text{for} \quad x>0, \\ \lambda_-(r) & \text{for} \quad x<0, \end{array} \right. \tag{1.4}$$

for some elements $\lambda_\pm$ from the set

$$\Lambda = \bigcup_{h>0} \Lambda_h\,, \quad \Lambda_h = \{\lambda \in W^{1,\infty}(\mathbb{R}_+) : \lambda(r) = 0 \quad \text{for} \quad r>h,\, |\lambda'(r)| \le 1 \text{ a.e.}\}\,. \tag{1.5}$$

The set Λ is the *state space* for the Preisach model and r is the *memory variable*.

If F is the usual superposition operator $F[\lambda, u] = g(u)$ with a monotone Lipschitz continuous function g, then Problem (1.1)–(1.3) is invariant with respect to the one-parametric semigroup of transformations

$$P_\alpha : (u,v) \mapsto (u_\alpha, v_\alpha) : u_\alpha(x,t) = u(\alpha x, \alpha t)\,,\ v_\alpha(x,t) = v(\alpha x, \alpha t)\,,$$

parameterized by $\alpha > 0$. Invariant solutions with respect to $\{P_\alpha : \alpha > 0\}$ are called *self-similar* and can be represented by functions of one variable $z = x/t$. We will show in the next sections that the invariance of (1.1)–(1.4) with respect to $\{P_\alpha : \alpha > 0\}$ is preserved in the case of Preisach hysteresis, too. However, hysteresis is an irreversible phenomenon, and the concept of self-similar solution is meaningful only if we pass to the "forward self-similar" variable $\tau = 1/|z| = t/|x|$ separately for $x>0$ and $x<0$. The common boundary condition at $x=0$ is one of the unknowns of the problem and has to be identified. The same irreversibility argument has been applied in [8] to the maximal entropy solutions of the p-system without hysteresis.

We state the main result of this paper in the following form.

Theorem 1.1 *Let F be the Preisach operator (2.1) and let Hypothesis 2.1 hold. Let $(-h,h)$ be the convexity domain of F, and let the numbers $U_+, U_-, \frac{1}{2}(U_+ + U_- + V_+ - V_-)$ belong to $(-h,h)$. Then there exists a unique pair (u,v) of self-similar solutions to Problem (1.1)–(1.4), which are Lipschitz continuous at any positive distance from the origin.*

In the next Section 2, we give more details about the Preisach operator. We specify in particular sufficient conditions for the existence of a nontrivial convexity interval.

There is no reason to expect that also $w(x,t)$ be continuous across $x=0$. In general, we have $w(0-,t) \neq w(0+,t)$, so that a stationary shock in w persists for all times. In the aforementioned applications to elastoplasticity, it is natural to admit that on the contact between the domains $x>0$ and $x<0$ with different loading histories, only the velocities and stresses coincide, while the strain may exhibit a jump.

2. The Preisach operator

We denote by $C[0,\infty)$ the set of continuous functions $[0,\infty) \to \mathbb{R}$, and fix a function g satisfying the following hypothesis.

Hypothesis 2.1 *The function g maps $\mathbb{R}_+ \times \mathbb{R}$ into $\mathbb{R}$ and*

(i) *is locally Lipschitz continuous in both variables, $g(r,0)=0$;*
(ii) *The partial derivative $\partial g(r,v)/\partial v$ is Lipschitz continuous in v;*
(iii) *There exist $m>0$ and $c>0$ such that $\partial g(r,v)/\partial v \geq m > 0$ for $r+|v| \leq c$.*

For a function $\hat{u} \in C[0,\infty)$ and initial memory configuration $\lambda \in \Lambda_h$ (cf. (1.5)), we define the output value $\hat{w}(t) = F[\lambda,\hat{u}](t)$ of the Preisach operator F at time t by the formula

$$\hat{w}(t) = F[\lambda,\hat{u}](t) := \int_0^\infty g(r, \mathfrak{p}_r[\lambda,\hat{u}](t))\,\mathrm{d}r\,, \tag{2.1}$$

where $\mathfrak{p}_r : (\lambda,\hat{u}) \mapsto \xi_r$ for a fixed $r>0$ is the solution mapping of the variational inequality

$$\begin{array}{lll} \text{(i)} & |\hat{u}(t) - \xi_r(t)| \leq r & \forall t > 0, \\ \text{(ii)} & \int_0^T (\hat{u}(t) - \xi_r(t) - y(t))\,\mathrm{d}\xi_r(t) \geq 0 & \forall T>0\ \forall y \in C([0,T];[-r,r])\,, \\ \text{(iii)} & \xi_r(0) = \lambda_0(r) := \max\{\hat{u}(0) - r, \min\{\lambda(r), \hat{u}(0)+r\}\}. & \end{array} \tag{2.2}$$

The integral in (ii) is to be interpreted as the Stieltjes integral. This definition is meaningful, since $\xi_r|_{[0,T]} \in C[0,T] \cap BV(0,T)$ for every $r>0$ and $T>0$, and $\xi_r(t)=0$ for r sufficiently large. The mapping $\mathfrak{p}_r$ is called the *play operator*, see Figure 1, and has the following properties (for a proof, see [6, Section II]).

Lemma 2.2 *Let $\lambda_i \in \Lambda$, $\hat{u}_i \in C[0,\infty)$, $i=1,2$ be given.*

(i) *For all $t>0$ we have*

$$|\mathfrak{p}_r[\lambda_1,\hat{u}_1](t) - \mathfrak{p}_r[\lambda_2,\hat{u}_2](t)| \leq \max\{|\lambda_1(r) - \lambda_2(r)|, \max\{|\hat{u}_1(s) - \hat{u}_2(s)| : s \in [0,t]\}\}.$$

(ii) *For $\hat{u} \in C[0,\infty)$ and $\alpha > 0$ set $\hat{u}_\alpha(t) = \hat{u}(\alpha t)$. Then $\mathfrak{p}_r[\lambda,\hat{u}_\alpha](t) = \mathfrak{p}_r[\lambda,\hat{u}](\alpha t)$ for all $t \geq 0$ and $\lambda \in \Lambda$.*

(iii) *If* $\hat{u}_i$ *are absolutely continuous, then so are* $\mathfrak{p}_r[\lambda_i, \hat{u}_i]$ *, and the inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathfrak{p}_r[\lambda_1, \hat{u}_1] - \mathfrak{p}_r[\lambda_2, \hat{u}_2]\right)^+(t) \le H(\hat{u}_1(t) - \hat{u}_2(t))\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathfrak{p}_r[\lambda_1, \hat{u}_1] - \mathfrak{p}_r[\lambda_2, \hat{u}_2]\right)(t)$$

holds a. e., where $y^+ = \max\{y, 0\}$ *for* $y \in \mathbb{R}$ *, and* H *is the left continuous Heaviside function*

$$H(y) = \begin{cases} 0 & \textit{for } \ y \le 0\,, \\ 1 & \textit{for } \ y > 0\,. \end{cases}$$

Property (i) is the Lipschitz continuity of the play with respect to the sup-norm, property (ii) is called the *rate independence*, and (iii) is the *Hilpert inequality* established originally in [3].

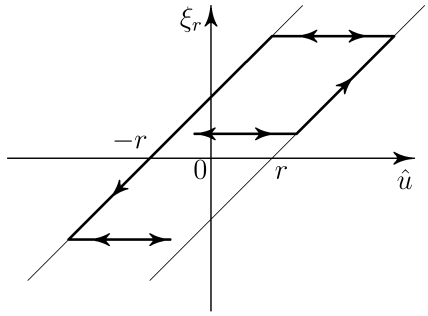

Figure 1. A hysteresis diagram of the play operator.

Formula (2.1) has been established in [5] as an equivalent variational reformulation of the original Preisach construction in [10], which is much more popular in the literature, see [4, 9]. The variational character of (2.1) turns out to be useful for the investigation of the qualitative behavior of solutions to evolution equations with hysteresis.

As an immediate consequence of the Hilpert inequality, we have the implications

$$\begin{cases} \hat{u}(t) > \hat{u}(t_0) \quad \text{for } \ t \in (t_0, t_1) & \Longrightarrow \quad \mathfrak{p}_r[\lambda, \hat{u}](t_1) \ge \mathfrak{p}_r[\lambda, \hat{u}](t_0)\,, \\ \hat{u}(t) < \hat{u}(t_0) \quad \text{for } \ t \in (t_0, t_1) & \Longrightarrow \quad \mathfrak{p}_r[\lambda, \hat{u}](t_1) \le \mathfrak{p}_r[\lambda, \hat{u}](t_0)\,. \end{cases} \tag{2.3}$$

Indeed, we use Lemma 2.2 (iii) with $\hat{u}_1(t) = \hat{u}(t)$, $\hat{u}_2(t) = \hat{u}(t)$ for $t \in [0, t_0]$, $\hat{u}_2(t) = \hat{u}(t_0)$ for $t > t_0$. Then $\mathfrak{p}_r[\lambda, \hat{u}_2](t) = \mathfrak{p}_r[\lambda, \hat{u}](t_0)$ in $[t_0, t_1]$, and it suffices to integrate over $[t_0, t_1]$.

If $\hat{u}$ in monotone (nonincreasing or nondecreasing) in an interval $[0, T]$, then ξ_r admits the representation formula

$$\xi_r(t) = \max\{\hat{u}(t) - r, \min\{\lambda_0(r), \hat{u}(t) + r\}\}, \tag{2.4}$$

cf. Figure 1. On monotone inputs, the operator F thus behaves like a superposition operator

$$\hat{w}(t) = F[\lambda, \hat{u}](t) = \varphi(\hat{u}(t)) \tag{2.5}$$

with a function φ given by the formula

$$\varphi(u) = \begin{cases} \int_0^\infty g(r, \max\{\lambda_0(r), u - r\})\,\mathrm{d}r & \text{for } \quad u \ge \hat{u}(0), \\ \int_0^\infty g(r, \min\{\lambda_0(r), u + r\})\,\mathrm{d}r & \text{for } \quad u \le \hat{u}(0). \end{cases} \tag{2.6}$$

We give here a simple proof of the following result.

Lemma 2.3 *There exist $h > 0$ and $\kappa > 0$ such that if $\lambda \in \Lambda_h$, then*

$$\frac{\varphi'(u_1) - \varphi'(u_2)}{u_1 - u_2} \geq \kappa \qquad \text{if} \quad -h < \hat{u}(0) < u_2 < u_1 < h\,, \tag{2.7}$$

$$\frac{\varphi'(u_1) - \varphi'(u_2)}{u_1 - u_2} \leq -\kappa \qquad \text{if} \quad -h < u_2 < u_1 < \hat{u}(0) < h\,. \tag{2.8}$$

The meaning of Lemma 2.3 is the following. Increasing branches of hysteresis loops in the interval $(-h, h)$ are strictly convex, and decreasing branches are strictly concave, see Figure 2.

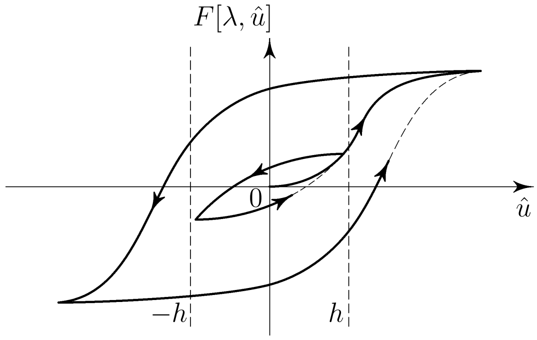

Figure 2. Preisach diagram with convexity domain $(-h, h)$.

Proof. For $u \in \mathbb{R}$ set
$$M(u) = \inf\{r \geq 0 : |u - \lambda_0(r)| \leq r\}\,.$$
The definition of M is meaningful, since $\lambda_0(r)$ vanishes for large r. We have in particular $\lambda_0(0) = \hat{u}(0)$, hence $M(\hat{u}(0)) = 0$.

Using the fact that $\lambda_0 \in \Lambda$, we obtain for $u > \hat{u}(0)$ that

$$\varphi(u) = \int_0^{M(u)} g(r, u - r)\,\mathrm{d}r + \int_{M(u)}^{\infty} g(r, \lambda_0(r))\,\mathrm{d}r\,,$$

hence, by Fubini's theorem,

$$\varphi(u) - \varphi(\hat{u}(0)) = \int_0^{M(u)} (g(r, u - r) - g(r, \lambda_0(r)))\,\mathrm{d}r = \int_{\hat{u}(0)}^{u} \int_0^{M(v)} \frac{\partial g}{\partial v}(r, v - r)\,\mathrm{d}r\,\mathrm{d}v\,. \tag{2.9}$$

For $s \geq 0$ set
$$\psi_+(s) = \int_0^s \frac{\partial g}{\partial v}(r, \lambda_0(s) + s - r)\,\mathrm{d}r\,. \tag{2.10}$$
Then $\varphi'(u) = \psi_+(M(u))$ for a.e. $u > \hat{u}(0)$. The function ψ_+ is continuously differentiable and $\psi_+'(0+) = \partial g/\partial v(0, \hat{u}(0))$. If $|\hat{u}(0)| \leq c/2$, we infer from Hypothesis 2.1 that there exists $\sigma_+ > 0$ such that
$$\frac{\psi_+(s_1) - \psi_+(s_2)}{s_1 - s_2} \geq \frac{m}{2} \qquad \text{for} \;\; 0 < s_2 < s_1 < \sigma_+\,. \tag{2.11}$$
As a counterpart of (2.9), we obtain for $u < \hat{u}(0)$ the representation formula

$$\varphi(u) - \varphi(\hat{u}(0)) = \int_0^{M(u)} (g(r, u + r) - g(r, \lambda_0(r)))\,\mathrm{d}r = -\int_u^{\hat{u}(0)} \int_0^{M(v)} \frac{\partial g}{\partial v}(r, v + r)\,\mathrm{d}r\,\mathrm{d}v\,. \tag{2.12}$$

As above, we define for $s \geq 0$ the function

$$\psi_-(s) = \int_0^s \frac{\partial g}{\partial v}(r, \lambda_0(s) - s + r)\,\mathrm{d}r\,, \tag{2.13}$$

so that $\varphi'(u) = \psi_-(M(u))$ for a. e. $u < \hat{u}(0)$. Assuming still that $|\hat{u}(0)| \leq c/2$, we conclude that there exists $\sigma_- > 0$ such that

$$\frac{\psi_-(s_2) - \psi_-(s_1)}{s_2 - s_1} \geq \frac{m}{2} \qquad \text{for } 0 < s_1 < s_2 < \sigma_-\,. \tag{2.14}$$

We now put $h = \min\{c/2, \sigma_-, \sigma_+\}$, and assume that $\lambda \in \Lambda_h$ and $|\hat{u}(0)| \leq h$. For $\hat{u}(0) < u_2 < u_1 < h$ set $s_i = M(u_i)$, $i = 1,2$. Then $u_1 - u_2 = \lambda_0(s_1) - \lambda_0(s_2) + s_1 - s_2 \leq 2(s_1 - s_2)$, hence

$$M(u_1) - M(u_2) \geq \frac{1}{2}(u_1 - u_2). \tag{2.15}$$

We have $\lambda_0(\sigma_+) = 0$, hence $h \leq \lambda_0(\sigma_+) + \sigma_+$, which yields that $0 < s_2 < s_1 < \sigma_+$. As a consequence of (2.11) and (2.15), we obtain the inequality

$$\frac{\varphi'(u_1) - \varphi'(u_2)}{u_1 - u_2} \geq \frac{m}{4}\,. \tag{2.16}$$

For $-h < u_2 < u_1 < \hat{u}(0)$ and $s_i = M(u_i)$, $i = 1,2$, we now have $s_2 > s_1$ and $u_1 - u_2 = \lambda_0(s_1) - \lambda_0(s_2) - s_1 + s_2 \leq 2(s_2 - s_1)$. Hence,

$$M(u_2) - M(u_1) \geq \frac{1}{2}(u_1 - u_2). \tag{2.17}$$

The formula $\varphi'(u_i) = \psi_-(s_i)$ yields

$$\frac{\varphi'(u_1) - \varphi'(u_2)}{u_1 - u_2} \leq -\frac{m}{4}, \tag{2.18}$$

and it suffices to put $\kappa = m/4$. ■

3. Self-similar solutions

Consider first a general identity

$$\theta_t = \chi_x \tag{3.1}$$

between two locally integrable functions $\theta, \chi : \mathbb{R} \times (0,\infty) \to \mathbb{R}$. The weak formulation of (3.1) reads

$$\int_{-\infty}^{\infty} \int_0^{\infty} (\theta\, \varrho_t - \chi\, \varrho_x)(x,t)\,\mathrm{d}t\,\mathrm{d}x \;=\; 0 \tag{3.2}$$

for all Lipschitz continuous test functions ϱ with compact support in $\mathbb{R} \times (0,\infty)$. Following [1], we assume that θ, χ are *self-similar* of the form

$$\theta(x,t) = \hat{\theta}\left(\frac{x}{t}\right)\,, \quad \chi(x,t) = \hat{\chi}\left(\frac{x}{t}\right)\,. \tag{3.3}$$

We have the following characterization for $\hat{\theta}, \hat{\chi}$.

Lemma 3.1 *Let* θ, χ *be as in (3.3). Then Eq. (3.2) holds if and only if the function* $z \mapsto z\hat{\theta}(z) + \hat{\chi}(z)$ *is absolutely continuous and the identity*

$$(z\hat{\theta}(z) + \hat{\chi}(z))' \;=\; \hat{\theta}(z) \tag{3.4}$$

holds almost everywhere, where prime denotes the derivative with respect to $z = x/t$ *.*

In particular, under the hypotheses of Lemma 3.1, we have the following "Rankine-Hugoniot condition for stationary shocks"

$$\chi(0-, t) = \chi(0+, t)\,. \tag{3.5}$$

Proof. Let (3.4) hold, and let ϱ be an arbitrary admissible test function. In the left hand side of (3.2), we substitute $z = x/t$, and put

$$\eta(z) = \int_0^\infty \varrho(tz, t)\,\mathrm{d}t \tag{3.6}$$

for $z \in \mathbb{R}$. We obtain the integral

$$\int_{-\infty}^{\infty} (\hat{\theta}(z)\,(z\eta(z))' + \hat{\chi}(z)\,\eta'(z))\,\mathrm{d}z\,, \tag{3.7}$$

which equals to 0 by virtue of (3.4). Conversely, for an arbitrary test function η with compact support in $\mathbb{R}$ we can find ϱ such that (3.6) holds. Indeed, it suffices to put

$$\varrho(x, t) = \eta\left(\frac{x}{t}\right)\frac{\mu(t)}{t}$$

with a nonnegative function μ with compact support in $(0, \infty)$ such that $\int_0^\infty \mu(t)\,\mathrm{d}t = 1$. By the same substitution, we thus conclude that (3.2) implies (3.4). ∎

As already mentioned in Section 1, hysteretic processes cannot go backward in time. We therefore look for self-similar solutions to Problem (1.1)–(1.4) in the form

$$\begin{cases} u(x,t) \;=\; u\left(1, \frac{t}{x}\right), & v(x,t) \;=\; v\left(1, \frac{t}{x}\right) \quad \text{for } x > 0\,, \\ u(x,t) \;=\; u\left(-1, -\frac{t}{x}\right), & v(x,t) \;=\; v\left(-1, -\frac{t}{x}\right) \quad \text{for } x < 0\,. \end{cases} \tag{3.8}$$

By virtue of the rate independence, the operator F is compatible with the self-similar structure. For $\alpha > 0, x \in \mathbb{R}, t \geq 0$ set $u_\alpha(x,t) = u(\alpha x, \alpha t)$, $w_\alpha = F[\lambda_\pm, u_\alpha]$, and assume that u satisfies (3.8). Then $u = u_\alpha$, and by Lemma 2.2 we have

$$w(x,t) = w_\alpha(x,t) = F[\lambda_\pm, u(\alpha x, \cdot)](\alpha t) = w(\alpha x, \alpha t)\,,$$

hence

$$\begin{cases} w(x,t) \;=\; w\left(1, \frac{t}{x}\right) & \text{for } x > 0\,, \\ w(x,t) \;=\; w\left(-1, -\frac{t}{x}\right) & \text{for } x < 0\,. \end{cases} \tag{3.9}$$

For $\tau \geq 0$ we now set

$$\begin{cases} u_+(\tau) \;=\; u(1,\tau), & u_-(\tau) \;=\; u(-1,\tau)\,, \\ v_+(\tau) \;=\; v(1,\tau), & v_-(\tau) \;=\; v(-1,\tau)\,, \\ w_+(\tau) \;=\; w(1,\tau), & w_-(\tau) \;=\; w(-1,\tau)\,. \end{cases} \tag{3.10}$$

By (3.9), we have

$$w_{\pm}(\tau) = F[\lambda_{\pm}, u_{\pm}](\tau) \qquad \text{for} \quad \tau \geq 0\,. \tag{3.11}$$

In this setting, condition (3.5) becomes a boundary condition at $\tau = +\infty$. By Lemma 3.1, we derive for the unknown functions $u_{\pm}, v_{\pm}$ in (3.10) of the forward variable $\tau = 1/|z|$ the system of boundary value problems

$$\left\{ \begin{array}{rcl} \dfrac{\mathrm{d}}{\mathrm{d}\tau}\left(\dfrac{1}{\tau}v_{+}(\tau) + u_{+}(\tau)\right) & = & -\dfrac{1}{\tau^2}v_{+}(\tau) \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}\left(\dfrac{1}{\tau}(u_{+} + w_{+})(\tau) + v_{+}(\tau)\right) & = & -\dfrac{1}{\tau^2}(u_{+} + w_{+})(\tau) \end{array} \right. \tag{3.12}$$

$$\left\{ \begin{array}{rcl} \dfrac{\mathrm{d}}{\mathrm{d}\tau}\left(-\dfrac{1}{\tau}v_{-}(\tau) + u_{-}(\tau)\right) & = & \dfrac{1}{\tau^2}v_{-}(\tau) \\ \dfrac{\mathrm{d}}{\mathrm{d}\tau}\left(-\dfrac{1}{\tau}(u_{-} + w_{-})(\tau) + v_{-}(\tau)\right) & = & \dfrac{1}{\tau^2}(u_{-} + w_{-})(\tau) \end{array} \right. \tag{3.13}$$

with boundary conditions

$$u_{\pm}(0) = U_{\pm}\,,\ v_{\pm}(0) = V_{\pm}\,,\ u_{\pm}(+\infty) = U_0\,,\ v_{\pm}(+\infty) = V_0 \tag{3.14}$$

with unknown values of U_0, V_0 that also have to be found. Eliminating $v_{\pm}$ from the above equations, we obtain the system

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\tau^2}(u_{\pm} + w_{\pm})(\tau) - u_{\pm}(\tau)\right) \;=\; -\frac{2}{\tau^3}(u_{\pm} + w_{\pm})(\tau)\,. \tag{3.15}$$

The above considerations enable us to reduce the proof of Theorem 1.1 to the proof of the following statement.

Proposition 3.2 *Let h be as in Lemma 2.3, and let the numbers $U_+, U_-, \frac{1}{2}(U_+ + U_- + V_+ - V_-)$ belong to $(-h, h)$. Then there exists a unique Lipschitz continuous solution to Problem (3.12)–(3.14).*

Proof. For simplicity, we denote by a dot the derivative with respect to τ. Absolutely continuous solutions to (3.12)–(3.14) satisfy almost everywhere the identities

$$\dot{w}_{+}(\tau) \;=\; (\tau^2 - 1)\dot{u}_{+}(\tau)\,, \tag{3.16}$$

$$\dot{w}_{-}(\tau) \;=\; (\tau^2 - 1)\dot{u}_{-}(\tau)\,. \tag{3.17}$$

We first check that both u_+, u_- are monotone in $(0, \infty)$. Assume for example that there exist $\tau_1 < \tau_2$ such that $u_+(\tau_2) = u_+(\tau_1)$ and $u_+(\tau) > u_+(\tau_1)$ for $\tau \in (\tau_1, \tau_2)$. It follows from (2.3) that $w_+(\tau_2) \geq w_+(\tau_1)$. Integrating (3.16) over $[\tau_1, \tau_2]$ yields

$$0 \leq w_{+}(\tau_2) - w_{+}(\tau_1) = 2\int_{\tau_1}^{\tau_2} \tau(u_{+}(\tau_1) - u_{+}(\tau))\,\mathrm{d}\tau < 0\,,$$

which is a contradiction. The other cases are similar.

On monotone inputs, we have the representation formula (2.5)–(2.6) for the Preisach operator. Let $\varphi_{\pm}$ be the corresponding functions associated with $w_{\pm}$. Using the chain rule, we rewrite (3.16)–(3.17) as

$$(\varphi_{\pm}'(u_{\pm}(\tau)) + 1 - \tau^2)\,\dot{u}_{\pm}(\tau) \;=\; 0\,. \tag{3.18}$$

We have by hypothesis $U_+, U_- \in (-h,h)$. For $U_0 \in (-h,h)$ put $\tau_\pm = \sqrt{1+\varphi_\pm'(U_\pm)}$, $\tau_\pm^0 = \sqrt{1+\varphi_\pm'(U_0)}$. The unique absolutely continuous solution to (3.18) with boundary conditions $u_\pm(0) = U_\pm$, $u_\pm(+\infty) = U_0$ has the form of two rarefaction waves

$$u_\pm(\tau) \;=\; \begin{cases} U_\pm & \text{for} \quad \tau \in [0,\tau_\pm) \\ (\varphi_\pm')^{-1}(\tau^2-1) & \text{for} \quad \tau \in [\tau_\pm, \tau_\pm^0) \\ U_0 & \text{for} \quad \tau \ge \tau_\pm^0 \,. \end{cases} \tag{3.19}$$

By Lemma 2.3, $(\varphi_\pm')^{-1}$ is Lipschitz continuous, and nondecreasing if $U_0 > U_\pm$, nonincreasing if $U_0 < U_\pm$, hence formula (3.19) is meaningful. The functions $v_\pm$ are obtained by integrating directly the equations (3.12)–(3.13), that is,

$$\begin{cases} v_+(\tilde\tau) = V_+ - \tilde\tau u_+(\tilde\tau) + \int_0^{\tilde\tau} u_+(\tau)\,\mathrm{d}\tau, \\ v_-(\tilde\tau) = V_- + \tilde\tau u_-(\tilde\tau) - \int_0^{\tilde\tau} u_-(\tau)\,\mathrm{d}\tau \end{cases} \tag{3.20}$$

for every $\tilde\tau \ge 0$.

We now show that there is a unique way to choose U_0 in such a way that v is continuous across $x=0$. Putting $\tilde\tau = \max\{\tau_+^0, \tau_-^0\}$ in (3.20), we obtain

$$v_+(\tilde\tau) - v_-(\tilde\tau) = V_+ - V_- + \int_0^{\tilde\tau} (u_+(\tau) + u_-(\tau) - 2U_0)\,\mathrm{d}\tau \,. \tag{3.21}$$

The continuity condition $v_+(+\infty) = v_-(+\infty)$ thus reads

$$V_+ - V_- = \int_0^{\tau_+^0} (U_0 - u_+(\tau))\,\mathrm{d}\tau + \int_0^{\tau_-^0} (U_0 - u_-(\tau))\,\mathrm{d}\tau \,. \tag{3.22}$$

We have

$$\int_0^{\tau_+^0} (U_0 - u_+(\tau))\,\mathrm{d}\tau \;=\; \tau_+^0 U_0 - \tau_+ U_+ - \int_{\tau_+}^{\tau_+^0} (\varphi_+')^{-1}(\tau^2-1)\,\mathrm{d}\tau \;=\; \int_{U_+}^{U_0} \sqrt{1+\varphi_+'(z)}\,\mathrm{d}z \,,$$

and similarly

$$\int_0^{\tau_-^0} (U_0 - u_-(\tau))\,\mathrm{d}\tau \;=\; \tau_-^0 U_0 - \tau_- U_- - \int_{\tau_-}^{\tau_-^0} (\varphi_-')^{-1}(\tau^2-1)\,\mathrm{d}\tau \;=\; \int_{U_-}^{U_0} \sqrt{1+\varphi_-'(z)}\,\mathrm{d}z \,.$$

The right hand side of the equation

$$V_+ - V_- = \int_{U_+}^{U_0} \sqrt{1+\varphi_+'(z)}\,\mathrm{d}z + \int_{U_-}^{U_0} \sqrt{1+\varphi_-'(z)}\,\mathrm{d}z \tag{3.23}$$

is a continuous increasing function of U_0, hence U_0 is in (3.23) uniquely determined. It remains to check that $U_0 \in (-h,h)$ under our hypotheses. This is obvious if U_0 is between U_+ and U_-. Assume that $U_0 > \max\{U_+, U_-\}$. Then (3.23) yields $V_+ - V_- \ge 2U_0 - U_+ - U_-$, hence $2U_0 \le V_+ - V_- + U_+ + U_- < 2h$. Similarly, if $U_0 < \min\{U_+, U_-\}$, then

$$V_- - V_+ = \int_{U_0}^{U_+} \sqrt{1+\varphi_+'(z)}\,\mathrm{d}z + \int_{U_0}^{U_-} \sqrt{1+\varphi_-'(z)}\,\mathrm{d}z \,,$$

hence $V_- - V_+ \ge U_+ + U_- - 2U_0$ and $2U_0 \ge V_+ - V_- + U_+ + U_- > -2h$. The proof is complete. ■

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