The Kurzweil integral with exclusion of negligible sets

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Abstract

We propose an extended version of the Kurzweil integral which contains both the Young and the Kurzweil integral as special cases. The construction is based on a reduction of the class of δ-fine partitions by excluding small sets.

KEYWORDS: Kurzweil integral, Young integral

MSC 2000: 26A39

Introduction

This work is a continuation of [4] and has been motivated by problems arising in connection with the integral formulation of evolution variational inequalities in the space of regulated functions in [5] and [2]. In this context, the Young integral as presented in [3] and [7] has an important advantage in comparison with the Kurzweil integral, namely that an arbitrary regulated function is Young integrable with respect to any function with essentially bounded variation, and this is precisely what is needed for applications to variational inequalities. The counterexample in [4] shows that this need not be the case for the Kurzweil integral.

The aim of this note is to propose a modification of the Kurzweil integral (the so-called *KN-integral*, where "N" stands for "negligible sets") the idea of which is to exclude "small" singular sets from consideration. As the main result, we prove that the *KN*-integral generalizes both the Kurzweil and the Young integral, and that it still possesses all reasonable properties, like additivity with respect to the integrands and with respect to the integration domain.

Acknowledgement. The author wishes to thank J. Kurzweil, Š. Schwabik and M. Tvrdý for stimulating suggestions and comments.

1 The Young and Kurzweil integrals

We consider a nondegenerate closed interval $[a,b] \subset \mathbb{R}$, and denote by $\mathcal{D}_{a,b}$ the set of all divisions of the form

$$d \;=\; \{t_0,\dots,t_m\}\,, \quad a = t_0 < t_1 < \ldots < t_m = b\,. \tag{1.1}$$

We say that a division $\hat{d}$ is a *refinement* of $d \in \mathcal{D}_{a,b}$ and write $\hat{d} \succ d$ if $\hat{d} \in \mathcal{D}_{a,b}$ and $d \subset \hat{d}$.

With a division $d = \{t_0,\dots,t_m\} \in \mathcal{D}_{a,b}$ we associate *partitions* D defined as

$$D = \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1,\dots,m\}\,; \qquad \tau_j \in [t_{j-1}, t_j] \quad \forall j = 1,\dots,m\,. \tag{1.2}$$

By $\mathcal{P}(d)$ we denote the set of all partitions of the form (1.2) such that

$$\tau_j \in \,]t_{j-1}, t_j[\qquad \forall j = 1,\dots,m\,. \tag{1.3}$$

The following definition is taken from [1].

Definition 1.1 *We say that a function* $g : [a,b] \to \mathbb{R}$ *is* regulated *if for every* $t \in [a,b]$ *there exist both one-sided limits* $g(t+), g(t-) \in \mathbb{R}$ *with the convention* $g(a-) = g(a)$, $g(b+) = g(b)$.

The set of all regulated functions $g : [a,b] \to \mathbb{R}$ is denoted by $G(a,b)$ according to [10].

For given functions $f : [a,b] \to \mathbb{R}$, $g \in G(a,b)$, a division $d \in \mathcal{D}_{a,b}$ and a partition $D \in \mathcal{P}(d)$ of the form (1.2), (1.3), we define the Young integral sum $Y_D(f,g)$ by the formula

$$Y_D(f,g) \;=\; \sum_{j=1}^{m} f(\tau_j)(g(t_j-) - g(t_{j-1}+)) \;+\; \sum_{j=0}^{m} f(t_j)(g(t_j+) - g(t_j-))\,. \tag{1.4}$$

Definition 1.2 *Let* $f : [a,b] \to \mathbb{R}$, $g \in G(a,b)$ *be given. We say that* $J \in \mathbb{R}$ *is the* Young integral *over* $[a,b]$ *of* f *with respect to* g *and denote*

$$J \;=\; (Y)\int_a^b f(t)\,dg(t)\,, \tag{1.5}$$

if for every $\varepsilon > 0$ *there exists* $d_\varepsilon \in \mathcal{D}_{a,b}$ *such that for every* $d \succ d_\varepsilon$ *and* $D \in \mathcal{P}(d)$ *we have*

$$|J \;-\; Y_D(f,g)| \;\le\; \varepsilon\,. \tag{1.6}$$

It is an easy exercise to check that if the value J in Definition 1.2 exists, then it is uniquely determined.

The basic concept in the Kurzweil integration theory, namely in its original version introduced in [6] which we call below the K*-integral*, as well as in its generalization (the K^**-integral* proposed in [8]), is that of a δ*-fine partition*. We define the set

$$\Gamma(a,b) \;:=\; \{\delta : [a,b] \to \mathbb{R}\,;\, \delta(t) > 0 \quad \text{for every} \quad t \in [a,b]\}\,. \tag{1.7}$$

An element $\delta \in \Gamma(a,b)$ is called a *gauge*. For $t \in [a,b]$ and $\delta \in \Gamma(a,b)$ we denote

$$I_\delta(t) \;:=\;]t - \delta(t), t + \delta(t)[\;. \tag{1.8}$$

Definition 1.3 *Let* $\delta \in \Gamma(a,b)$ *be a given gauge. A partition* D *of the form (1.2) is said to be* δ-fine *if for every* $j = 1, \dots, m$ *we have*

$$\tau_j \;\in\; [t_{j-1}, t_j] \;\subset\; I_\delta(\tau_j)\,. \tag{1.9}$$

If, moreover, a δ*-fine partition* D *satisfies the implications*

$$\tau_j = t_{j-1} \;\Rightarrow\; j = 1\,, \quad \tau_j = t_j \;\Rightarrow\; j = m\,, \tag{1.9}^*$$

then it is called a δ-fine* *partition.*

The set of all δ*-fine (*δ*-fine*) partitions is denoted by* $\mathcal{F}_\delta(a,b)$ *(*$\mathcal{F}_\delta^*(a,b)$*, respectively).*

We have indeed $\mathcal{F}_\delta^*(a,b) \subset \mathcal{F}_\delta(a,b)$. The next lemma (which belongs to the family of the so-called Cousin's Lemmas) implies in particular that these sets are nonempty for every $\delta \in \Gamma(a,b)$. In this form, it was proved in detail e. g. in [4, Lemma 1.2].

Lemma 1.4 *Let $\delta \in \Gamma(a,b)$ and a dense subset $\Omega \subset]a,b[$ be given. Then there exists $D = \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1, \ldots, m\} \in \mathcal{F}_\delta^*(a,b)$ such that $t_j \in \Omega$ for every $j = 1, \ldots, m-1$.*

For given functions $f, g : [a,b] \to \mathbb{R}$ and a partition D of the form (1.2) we define the Kurzweil integral sum $K_D(f,g)$ by the formula

$$K_D(f,g) \;=\; \sum_{j=1}^{m} f(\tau_j)\,(g(t_j) - g(t_{j-1}))\,. \tag{1.10}$$

Definition 1.5 *Let $f, g : [a,b] \to \mathbb{R}$ be given. We say that $J \in \mathbb{R}$ ($J^* \in \mathbb{R}$) is the K-*integral *(K^*-*integral*) over $[a,b]$ of f with respect to g and denote*

$$J \;=\; (K)\int_a^b f(t)\,dg(t)\,, \qquad \left(J^* \;=\; (K^*)\int_a^b f(t)\,dg(t)\right)\,, \tag{1.11}$$

if for every $\varepsilon > 0$ there exists $\delta \in \Gamma(a,b)$ such that for every $D \in \mathcal{F}_\delta(a,b)$ ($D^* \in \mathcal{F}_\delta^*(a,b)$, respectively) we have

$$|J \;-\; K_D(f,g)| \;\le\; \varepsilon\,, \qquad \Big(\; |J^* \;-\; K_{D^*}(f,g)| \;\le\; \varepsilon\,, \quad \textit{respectively}\Big)\,. \tag{1.12}$$

Using the fact that the implication

$$\delta \le \min\{\delta_1\,,\,\delta_2\} \quad \Rightarrow \quad \left\{\begin{array}{l} \mathcal{F}_\delta^*(a,b) \;\subset\; \mathcal{F}_{\delta_1}^*(a,b) \cap \mathcal{F}_{\delta_2}^*(a,b)\,, \\ \mathcal{F}_\delta(a,b) \;\subset\; \mathcal{F}_{\delta_1}(a,b) \cap \mathcal{F}_{\delta_2}(a,b) \end{array}\right. \tag{1.13}$$

holds for every $\delta\,,\,\delta_1\,,\,\delta_2 \;\in\; \Gamma(a,b)$, we easily check that the values J, J^* in Definition 1.5 are uniquely determined. Since $\mathcal{F}_\delta^*(a,b) \subset \mathcal{F}_\delta(a,b)$ for every gauge δ, we also see that if $(K)\int_a^b f(t)\,dg(t)$ exists, then $(K^*)\int_a^b f(t)\,dg(t)$ exists and they are equal.

If we compare Definitions 1.2 and 1.5, we see that the main conceptual difference between the Young and the Kurzweil integral of f with respect to g consists in the fact that while in the Young integral we require a "regular" behaviour of $f(\tau)$ with respect to small shifts of τ, as all partitions with close values of τ_j come into play, cf. also the proof of Lemma 1.6 below, in the K^*-integral we have to be able to control small variations of $g(t)$, since a partition remains δ-fine* independently of small shifts of the t_j's. The counterexample in [4] as well as our construction of the KN-integral in the next section are based on this observation. We first establish the following property of the Young integral as a variant of [7, Theorem 1.2].

Lemma 1.6 *Let $f : [a,b] \to \mathbb{R}$, $g \in G(a,b)$ be such that $(Y)\int_a^b f(t)\,dg(t)$ exists, and let $\varepsilon > 0$, $d_\varepsilon \in \mathcal{D}_{a,b}$, $d_\varepsilon = \{s_0, s_1, \ldots, s_\ell\}$ be such that for every $d \succ d_\varepsilon$ and $D \in \mathcal{P}(d)$ we have*

$$\left| Y_D(f,g) \;-\; (Y)\int_a^b f(t)\,dg(t)\right| \;\le\; \varepsilon\,. \tag{1.14}$$

For $p = 1, \ldots, \ell$ set

$$\Phi_p \;=\; \sup\{|f(\tau)|\,;\, \tau \in [s_{p-1}, s_p]\}\,. \tag{1.15}$$

Then the implication

$$\Phi_p = +\infty \quad \Rightarrow \quad g(t+) = g(t-) = g(s_{p-1}+) = g(s_p-) \qquad \forall t \in]s_{p-1}, s_p[\tag{1.16}$$

holds for all $p = 1, \ldots, \ell$.

Proof. Assume $\Phi_p = +\infty$ for some p. There exists $\sigma \in [s_{p-1}, s_p]$ and a sequence $\{\sigma_n\,;\, n \in \mathbb{N}\}$ such that $\sigma_n \in]s_{p-1}, s_p[$, $\sigma_n \to \sigma$, $|f(\sigma_n)| \to +\infty$ as $n \to \infty$.

If $\sigma > s_{p-1}$, then we fix $t \in]s_{p-1}, \sigma[$ and put $\hat{d} = d_\varepsilon \cup \{t\}$. For arbitrarily fixed $\tau_q \in]s_{q-1}, s_q[$, $q \in \{1, \dots, \ell\} \setminus \{p\}$, $\tau_p \in]s_{p-1}, t[$ and for $n \ge n_0$ sufficiently large we have

$$D_n := \{(\tau_q, [s_{q-1}, s_q]), (\tau_p, [s_{p-1}, t]), (\sigma_n, [t, s_p])\,;\, q \in \{1, \dots, \ell\} \setminus \{p\}\} \in \mathcal{P}(\hat{d}),$$

hence, by (1.14),

$$|Y_{D_n}(f, g)| \le \left|(Y) \int_a^b f(t)\, dg(t)\right| + \varepsilon \qquad \forall n \ge n_0 . \tag{1.17}$$

On the other hand, we have for $n \ge n_0$ that

$$Y_{D_n}(f, g) - Y_{D_{n_0}}(f, g) = (f(\sigma_n) - f(\sigma_{n_0}))(g(s_p-) - g(t+)), \tag{1.18}$$

hence

$$|f(\sigma_n)|\, |g(s_p-) - g(t+)| \le |f(\sigma_{n_0})|\, |g(s_p-) - g(t+)| + 2\left(\left|(Y) \int_a^b f(t)\, dg(t)\right| + \varepsilon\right). \tag{1.19}$$

Letting $n \to \infty$ we obtain from (1.19) the implication

$$\sigma > s_{p-1} \quad \Rightarrow \quad g(s_p-) = g(t+) \quad \forall t \in]s_{p-1}, \sigma[\ . \tag{1.20}$$

Analogously we have

$$\sigma < s_p \quad \Rightarrow \quad g(s_{p-1}+) = g(t-) \quad \forall t \in]\sigma, s_p[\ . \tag{1.21}$$

This yields in particular that $g(s_p-) = g(s_{p-1}+)$, and the assertion follows. ∎

2 The KN-integral

We fix a system $\mathcal{N}$ of subsets of $[a, b]$ with the following properties:

$$\overline{[a, b] \setminus A} = [a, b] \qquad \forall A \in \mathcal{N}, \tag{2.1}$$

$$A, B \in \mathcal{N} \quad \Rightarrow \quad A \cup B \in \mathcal{N}. \tag{2.2}$$

Elements of $\mathcal{N}$ will be called *negligible sets*. Typically, $\mathcal{N}$ can be for instance the system of all subsets of Lebesgue measure zero in $[a, b]$, or the system of all countable subsets of $[a, b]$.

Definition 2.1 *Let $\mathcal{N}$ be a system of negligible sets in $[a, b]$, let $\delta \in \Gamma(a, b)$ be a given gauge, and let $A \in \mathcal{N}$ be a given set. A partition D of the form (1.2) is said to be* (δ, A)-fine *if it is δ-fine* and*

$$t_j \in [a, b] \setminus A \qquad \forall j = 1, \dots, m-1 . \tag{2.3}$$

The set of all (δ, A)-fine partitions is denoted by $\mathcal{F}_{\delta, A}(a, b)$.

Note that by Lemma 1.4, the set $\mathcal{F}_{\delta, A}(a, b)$ is nonempty for every $\delta \in \Gamma(a, b)$ and $A \in \mathcal{N}$.

Definition 2.2 *Let* $f, g : [a,b] \to \mathbb{R}$ *be given. We say that* $J \in \mathbb{R}$ *is the* KN-integral *over* $[a,b]$ *of* f *with respect to* g *and denote*

$$J \;=\; (KN)\int_a^b f(t)\,dg(t)\,, \tag{2.4}$$

if for every $\varepsilon > 0$ *there exist* $\delta \in \Gamma(a,b)$ *and* $A \in \mathcal{N}$ *such that for every* $D \in \mathcal{F}_{\delta,A}(a,b)$ *we have*

$$|J \;-\; K_D(f,g)| \;\le\; \varepsilon\,. \tag{2.5}$$

Similarly as in the situation of Definition 1.5, if J satisfying (2.5) exists, then it is unique. Indeed, assume that there exist $J_1 \ne J_2$ such that for every $\varepsilon > 0$ there exist $\delta_1, \delta_2 \in \Gamma(a,b)$ and $A_1, A_2 \in \mathcal{N}$ such that for each $D_i \in \mathcal{F}_{\delta_i,A_i}(a,b)$, $i = 1,2$, we have

$$|J_i \;-\; K_{D_i}(f,g)| \;\le\; \varepsilon\,. \tag{2.6}$$

Choosing $\varepsilon < |J_1 - J_2|/2$ and putting $\delta = \min\{\delta_1, \delta_2\}$, $A = A_1 \cup A_2$ we may choose any $D \in \mathcal{F}_{\delta,A}(a,b)$. Then $D \in \mathcal{F}_{\delta_1,A_1}(a,b) \cap \mathcal{F}_{\delta_2,A_2}(a,b)$, hence $|J_i \;-\; K_D(f,g)| \;\le\; \varepsilon$ for $i = 1,2$, which is a contradiction.

Obviously, if $(K^*)\int_a^b f(t)\,dg(t)$ exists, then $(KN)\int_a^b f(t)\,dg(t)$ exists and the two integrals are equal. In the trivial case $\mathcal{N} = \{\emptyset\}$, the KN-integral and the K^*-integral coincide. Moreover, all aforementioned Kurzweil-type integrals coincide if the function g is continuous. The exact statement reads as follows.

Proposition 2.3 *Let* $f, g : [a,b] \to \mathbb{R}$ *be such that* $(KN)\int_a^b f(t)\,dg(t) = J$ *exists for some choice of* $\mathcal{N}$, *and let* g *be continuous in* $[a,b]$. *Then* $(K)\int_a^b f(t)\,dg(t)$ *exists and equals* J.

The proof of Proposition 2.3 is based on the following two auxiliary results. For a finite set S, we denote by $\#S$ the number of its elements.

Lemma 2.4 *Let* $\delta \in \Gamma(a,b)$ *be a gauge, and let* $D = \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1, \dots, m\} \in \mathcal{F}_\delta(a,b)$ *be an arbitrary partition. Let* $\mathcal{R}(D)$ *denote the set of all partitions*

$$D' \;=\; \{(\tau'_i, [t'_{i-1}, t'_i])\,;\, i = 1, \dots, m'\} \in \mathcal{F}_\delta(a,b) \tag{2.7}$$

such that

$$\bigcup_{i=1}^{m'} \{\tau'_i\} \;=\; \bigcup_{j=1}^{m} \{\tau_j\}\,, \tag{2.8}$$

$$\bigcup_{i=0}^{m'} \{t'_i\} \;\subset\; \bigcup_{j=1}^{m} \{t_j\}\,. \tag{2.9}$$

For $D' \in \mathcal{R}(D)$ *of the form (2.7) set*

$$\mu(D') \;=\; \#\{i = 1, \dots, m'-1\,;\, \tau'_i = \tau'_{i+1}\}\,, \tag{2.10}$$

and assume that $\mu(D') > 0$. *Then there exists* $D'' \in \mathcal{R}(D)$ *such that* $\mu(D'') = \mu(D') - 1$, *and for every* $f, g : [a,b] \to \mathbb{R}$ *we have* $K_{D''}(f,g) = K_{D'}(f,g)$.

Proof of Lemma 2.4. Assume that $\tau'_i = \tau'_{i+1}$ for some $i = 1, \dots, m' - 1$. It suffices to put

$$\tau''_k \;=\; \begin{cases} \tau'_k & \text{for} \quad k = 1, \dots, i\,, \\ \tau'_{k+1} & \text{for} \quad k = i+1, \dots, m'-1\,, \end{cases} \tag{2.11}$$

$$t''_k \;=\; \begin{cases} t'_k & \text{for} \quad k = 1, \dots, i-1\,, \\ t'_{k+1} & \text{for} \quad k = i, \dots, m'-1\,. \end{cases} \tag{2.12}$$

We have by hypothesis $\tau'_i = \tau'_{i+1} = t'_i$ and $[t'_{i-1}, t'_{i+1}] = [t'_{i-1}, t'_i] \cup [t'_i, t'_{i-1}] \subset I_\delta(\tau'_i)$, hence $D'' = \{(\tau''_k, [t''_{k-1}, t''_k])\,;\, k = 1, \dots, m'-1\}$ belongs to $\mathcal{F}_\delta(a,b)$, and therefore also to $\mathcal{R}(D)$. For every $f, g : [a,b] \to \mathbb{R}$ we have

$$\begin{aligned} \sum_{k=1}^{m'} f(\tau'_k)(g(t'_k) - g(t'_{k-1}) \;&=\; \sum_{k=1}^{i-1} f(\tau'_k)(g(t'_k) - g(t'_{k-1}) + f(\tau'_i)(g(t'_{i+1}) - g(t'_{i-1}) \\ &\quad + \sum_{k=i+1}^{m'-1} f(\tau'_{k+1})(g(t'_{k+1}) - g(t'_k) \\ &=\; \sum_{k=1}^{m'-1} f(\tau''_k)(g(t''_k) - g(t''_{k-1})\,, \end{aligned}$$

and Lemma 2.4 is proved. ∎

Lemma 2.5 *Let $\mathcal{N}$ be any system of negligible sets, and let $\delta \in \Gamma(a,b)$, $A \in \mathcal{N}$, and $D = \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1, \dots, m\} \in \mathcal{F}_\delta(a,b)$ be given. Assume that*

$$\tau_j < \tau_{j+1} \qquad \forall\, j = 1, \dots, m-1\,. \tag{2.13}$$

Then for every $\eta > 0$ there exists $D_\eta = \{(\tau_j, [t^*_{j-1}, t^*_j])\,;\, j = 1, \dots, m\} \in \mathcal{F}_{\delta,A}(a,b)$ such that

$$|t_j - t^*_j| \;<\; \eta \qquad \forall\, j = 0, \dots, m\,. \tag{2.14}$$

Proof of Lemma 2.5. Put $t^*_0 = t_0 = a$, $t^*_m = t_m = b$. For every $j = 1, \dots, m-1$ we have

$$t_j \in [\tau_j, \tau_{j+1}] \cap\,]\tau_{j+1} - \delta(\tau_{j+1}), \tau_j + \delta(\tau_j)[$$

by virtue of (1.9), hence for every $\eta > 0$ and every $j = 1, \dots, m-1$, the set

$$K^\eta_j \;=\;]\tau_j, \tau_{j+1}[\, \cap\,]\tau_{j+1} - \delta(\tau_{j+1}), \tau_j + \delta(\tau_j)[\, \cap\,]t_j - \eta, t_j + \eta[\tag{2.15}$$

is a nondegenerate open interval. We obtain the assertion by choosing arbitrarily $t^*_j \in K^\eta_j \setminus A$ for $j = 1, \dots, m-1$. ∎

We are now ready to prove Proposition 2.3.

Proof of Proposition 2.3. Let $\varepsilon > 0$ be given. We find $\delta \in \Gamma(a,b)$ and $A \in \mathcal{N}$ such that for every $\tilde{D} \in \mathcal{F}_{\delta,A}(a,b)$ we have

$$|K_{\tilde{D}}(f,g) \;-\; J| \;\leq\; \frac{\varepsilon}{2}\,. \tag{2.16}$$

Let $D = \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1, \dots, m\} \in \mathcal{F}_\delta(a,b)$ be arbitrary. We claim that

$$|K_D(f,g) \,-\, J| \;\le\; \varepsilon\,. \tag{2.17}$$

To check that (2.17) holds, we use Lemma 2.4 and find $D' \in \mathcal{R}(D)$ of the form (2.7) such that $\mu(D') = 0$ and

$$K_{D'}(f,g) \;=\; K_D(f,g)\,. \tag{2.18}$$

Let now $\eta > 0$ be such that the implication

$$|t-s| < \eta \quad \Rightarrow \quad |g(t) - g(s)| \sum_{i=1}^{m'} |f(\tau_i')| \;\le\; \frac{\varepsilon}{4} \tag{2.19}$$

holds for every $t, s \in [a,b]$. By Lemma 2.5 we find $D_\eta = \{(\tau_i', [t_{i-1}^*, t_i^*])\,;\, i = 1, \dots, m'\} \in \mathcal{F}_{\delta,A}(a,b)$ such that $|t_i' - t_i^*| < \eta$ for all $i = 1, \dots, m'$. Then (2.19) yields

$$|K_{D_\eta}(f,g) - K_{D'}(f,g)| \;=\; \left|\sum_{i=1}^{m'} f(\tau_i') \left(g(t_i') - g(t_{i-1}') - g(t_i^*) + g(t_{i-1}^*)\right)\right| \;\le\; \frac{\varepsilon}{2}\,. \tag{2.20}$$

On the other hand, by (2.16) we have that

$$|K_{D_\eta}(f,g) \,-\, J| \;\le\; \frac{\varepsilon}{2}\,. \tag{2.21}$$

Combining (2.20) with (2.21) and (2.18) we obtain (2.17), and Proposition 2.3 follows. ∎

We now prove as the main result of this paper that it suffices to exclude all countable subsets of $[a,b]$ as negligible sets, and the Young integral becomes a special case of the KN-integral.

Theorem 2.6 *Let $\mathcal{N}$ be the system of all countable subsets of $[a,b]$, and let $f : [a,b] \to \mathbb{R}$, $g \in G(a,b)$ be such that $J = (Y) \int_a^b f(t)\, dg(t)$ exists. Then $(KN) \int_a^b f(t)\, dg(t)$ exists and equals J.*

Proof. Let $\varepsilon > 0$ be given, and let $d_\varepsilon \in \mathcal{D}_{a,b}$, $d_\varepsilon = \{s_0, s_1, \dots, s_\ell\}$ be such that for every $d \succ d_\varepsilon$ and $D \in \mathcal{P}(d)$ we have

$$\left|Y_D(f,g) \;-\; (Y) \int_a^b f(t)\, dg(t)\right| \;\le\; \varepsilon\,. \tag{2.22}$$

Let $\Phi_1, \dots, \Phi_\ell$ be as in Lemma 1.6. Set

$$E \;=\; \{p \in \{1, \dots, \ell\}\,;\, \Phi_p < \infty\}\,, \tag{2.23}$$

$$\Phi^* \;=\; \max\{\Phi_p\,;\, p \in E\}\,, \tag{2.24}$$

$$\Delta \;=\; \min\{s_p - s_{p-1}\,;\, p = 1, \dots, \ell\}\,. \tag{2.25}$$

We fix $\delta_0 \in]0, \Delta/2[$ such that for all $p = 1, \dots, \ell$ we have

$$|g(t) - g(s_{p-1}+)| \;<\; \frac{\varepsilon}{4\ell\Phi^*} \qquad \forall t \in]s_{p-1}, s_{p-1} + \delta_0[\;, \tag{2.26}$$

$$|g(t) - g(s_p-)| \;<\; \frac{\varepsilon}{4\ell\Phi^*} \qquad \forall t \in]s_p - \delta_0, s_p[\;, \tag{2.27}$$

and put

$$\delta(t) \;=\; \begin{cases} \min\left\{\frac{\Delta}{8}, \delta_0\right\} & \text{for} \quad t \in d_\varepsilon\,, \\ \min\left\{\frac{\Delta}{8}, \operatorname{dist}(t, d_\varepsilon)\right\} & \text{for} \quad t \notin d_\varepsilon\,. \end{cases} \tag{2.28}$$

Let $A \in \mathcal{N}$ be the set of all discontinuity points of g. We now complete the proof by checking that for every $D \in \mathcal{F}_{\delta,A}(a,b)$ we have

$$|J - K_D(f,g)| \;\le\; 2\varepsilon\,. \tag{2.29}$$

In order to establish (2.29), we choose an arbitrary partition $D \in \mathcal{F}_{\delta,A}(a,b)$ of the form

$$D \;=\; \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1, \dots, m\}\,. \tag{2.30}$$

For every $p = 0, \dots, \ell$ we find $j_p \in 1, \dots, m$ such that $s_p \in [t_{j_p-1}, t_{j_p}]$. We claim that

$$s_p \;=\; \tau_{j_p} \qquad \forall p = 0, \dots, \ell\,. \tag{2.31}$$

Indeed, we have for all p that $|s_p - \tau_{j_p}| < \delta(\tau_{j_p})$ by the definition of $\mathcal{F}_{\delta,A}(a,b)$. Assuming $\tau_{j_p} \notin d_\varepsilon$ would imply according to (2.28) that $\delta(\tau_{j_p}) \le \operatorname{dist}(\tau_{j_p}, d_\varepsilon) \le |s_p - \tau_{j_p}|$, which is a contradiction, hence $\tau_{j_p} \in d_\varepsilon$, $|s_p - \tau_{j_p}| < \Delta/8$, and (2.31) follows. This implies in particular that

$$\tau_1 = s_0 = t_0 = a\,, \quad \tau_m = s_\ell = t_m = b\,. \tag{2.32}$$

We moreover have $t_{j_p} - t_{j_{p-1}-1} \ge s_p - s_{p-1} \ge \Delta$ for $p = 1, \dots\ \ell$, and $t_j - t_{j-1} < 2\delta(\tau_j) \le \Delta/4$ for $j = 1, \dots, m$, hence

$$j_{p-1} \;\le\; j_p - 4 \qquad \text{for} \;\; p = 1, \dots, \ell\,. \tag{2.33}$$

Let now M be the set

$$M \;=\; \{j \in \{1, \dots, m\}\,;\, \{j-1, j, j+1\} \cap \{j_p\,;\, p = 0, \dots, \ell\} = \emptyset\}\,, \tag{2.34}$$

and put

$$d \;=\; \{t_j\,;\, j \in M\} \cup \{t_{j_p+1}\,;\, p = 0, \dots, \ell-1\} \cup \{s_p\,;\, p = 0, \dots, \ell\}\,, \tag{2.35}$$

$$\begin{aligned} \hat{D} \;=\; & \{(\tau_j, [t_{j-1}, t_j])\,;\, j \in M\} \cup \{(\tau_{j_p+1}, [s_p, t_{j_p+1}])\,;\, p = 0, \dots, \ell-1\} \\ & \cup \{(\tau_{j_p-1}, [t_{j_p-2}, s_p])\,;\, p = 1, \dots, \ell\}\,. \end{aligned} \tag{2.36}$$

Then $d \succ d_\varepsilon$ and $\hat{D} \in \mathcal{P}(d)$ (note that $j_p - 2 \in M$ for all $p = 1, \dots, \ell$ by (2.33)!), hence by hypothesis we have

$$|J - Y_{\hat{D}}(f,g)| \;\le\; \varepsilon\,. \tag{2.37}$$

On the other hand, we have $g(t_j-) = g(t_j+) = g(t_j)$ for all $j = 1, \dots, m-1$ due to the choice of the set A, hence

$$\begin{aligned} Y_{\hat{D}}(f,g) \;=\; & \sum_{j \in M} f(\tau_j)(g(t_j) - g(t_{j-1})) + \sum_{p=0}^{\ell-1} f(\tau_{j_p+1})(g(t_{j_p+1}) - g(s_p+)) \\ & + \sum_{p=1}^{\ell} f(\tau_{j_p-1})(g(s_p-) - g(t_{j_p-2})) + \sum_{p=0}^{\ell} f(s_p)(g(s_p+) - g(s_p-))\,, \end{aligned} \tag{2.38}$$

while

$$K_D(f,g) = \sum_{j\in M} f(\tau_j)(g(t_j)-g(t_{j-1})) + \sum_{p=0}^{\ell-1} f(\tau_{j_p+1})(g(t_{j_p+1})-g(t_{j_p})) \tag{2.39}$$
$$+ \sum_{p=1}^{\ell} f(\tau_{j_p-1})(g(t_{j_p-1})-g(t_{j_p-2})) + \sum_{p=0}^{\ell} f(s_p)(g(t_{j_p})-g(t_{j_p-1}))\,.$$

Subtracting the above identities we obtain

$$K_D(f,g) - Y_{\hat D}(f,g) = \sum_{p=1}^{\ell} \Big((f(s_{p-1}) - f(\tau_{j_{p-1}+1}))(g(t_{j_{p-1}}) - g(s_{p-1}+)) \tag{2.40}$$
$$+ (f(s_p) - f(\tau_{j_p-1}))(g(s_p-) - g(t_{j_p-1}))\Big)\,.$$

We have $0 < t_{j_{p-1}} - s_{p-1} < \delta(s_{p-1}) \le \min\{\Delta/8, \delta_0\}$ and $0 < s_p - t_{j_p-1} < \delta(s_p) \le \min\{\Delta/8, \delta_0\}$, and (2.26), (2.27) yield

$$|g(t_{j_{p-1}}) - g(s_{p-1}+)| \le \frac{\varepsilon}{4\ell\Phi^*}\,, \qquad |g(s_p-) - g(t_{j_p-1})| \le \frac{\varepsilon}{4\ell\Phi^*} \tag{2.41}$$

for all $p = 1,\dots,\ell$. Moreover, it follows from Lemma 1.6 that

$$g(t_{j_{p-1}}) = g(s_{p-1}+) = g(s_p-) = g(t_{j_p-1}) \qquad \forall p \notin E\,, \tag{2.42}$$

hence

$$|K_D(f,g) - Y_{\hat D}(f,g)| \le \sum_{p\in E} \Big(|f(s_{p-1}) - f(\tau_{j_{p-1}+1})||g(t_{j_{p-1}}) - g(s_{p-1}+)| \tag{2.43}$$
$$+ |f(s_p) - f(\tau_{j_p-1})||g(s_p-) - g(t_{j_p-1})|\Big)$$
$$\le \varepsilon\,.$$

Combining (2.43) with (2.37) we obtain (2.29), and the proof is complete. ■

The KN-integral is linear with respect to both functions f and g. For the sake of completeness, we state this easy result explicitly.

Proposition 2.7 *Let $\mathcal{N}$ be any system of negligible sets.*

(i) *Let $(KN)\int_a^b f_1(t)\,dg(t)$, $(KN)\int_a^b f_2(t)\,dg(t)$ exist. Then $(KN)\int_a^b (f_1+f_2)(t)\,dg(t)$ exists and*

$$(KN)\int_a^b (f_1+f_2)(t)\,dg(t) = (KN)\int_a^b f_1(t)\,dg(t) + (KN)\int_a^b f_2(t)\,dg(t)\,. \tag{2.44}$$

(ii) *Let $(KN)\int_a^b f(t)\,dg_1(t)$, $(KN)\int_a^b f(t)\,dg_2(t)$ exist. Then $(KN)\int_a^b f(t)\,d(g_1+g_2)(t)$ exists and*

$$(KN)\int_a^b f(t)\,d(g_1+g_2)(t) = (KN)\int_a^b f(t)\,dg_1(t) + (KN)\int_a^b f(t)\,dg_2(t)\,. \tag{2.45}$$

(iii) *Let* $(KN)\int_a^b f(t)\,dg(t)$ *exist. Then* $(KN)\int_a^b \lambda f(t)\,dg(t)$, $(KN)\int_a^b f(t)\,d(\lambda g)(t)$ *exist for every constant* $\lambda \in \mathbb{R}$, *and*

$$(KN)\int_a^b \lambda f(t)\,dg(t) \;=\; (KN)\int_a^b f(t)\,d(\lambda g)(t) \;=\; \lambda (KN)\int_a^b f(t)\,dg(t)\,. \tag{2.46}$$

Proof. Let $\varepsilon > 0$ be given. We find δ_1, $\delta_2 \in \Gamma(a,b)$ and $A_1, A_2 \in \mathcal{N}$ such that for all $D_i \in \mathcal{F}_{\delta_i, A_i}(a,b)$, $i = 1,2$ we have

$$\left|(KN)\int_a^b f_i(t)\,dg(t) \;-\; K_{D_i}(f_i, g)\right| \;<\; \frac{\varepsilon}{2}\,.$$

Put $\delta := \min\{\delta_1\,,\, \delta_2\}$, $A = A_1 \cup A_2$. Then $\mathcal{F}_{\delta,A}(a,b) \subset \mathcal{F}_{\delta_1,A_1}(a,b) \cap \mathcal{F}_{\delta_2,A_2}(a,b)$, hence for every $D \in \mathcal{F}_{\delta,A}(a,b)$ we have

$$\left|(KN)\int_a^b f_1(t)\,dg(t) \;+\; (KN)\int_a^b f_2(t)\,dg(t) \;-\; K_D((f_1+f_2), g)\right| \;<\; \varepsilon\,,$$

and (2.44) follows. The same argument applies to the case (ii), while (iii) is trivial. ■

In order to analyze the behaviour of the KN-integral with respect to the variation of the integration domain, we derive the following Bolzano-Cauchy-type characterization analogous to [9, Proposition 7].

Lemma 2.8 *Let* $\mathcal{N}$ *be a system of negligible sets in* $[a,b]$, *and let* $f, g : [a,b] \to \mathbb{R}$ *be given functions. Then* $(KN)\int_a^b f(t)\,dg(t)$ *exists if and only if*

$$\forall \varepsilon > 0 \;\; \exists \delta \in \Gamma(a,b) \;\; \exists A \in \mathcal{N} \;\; \forall D, D' \in \mathcal{F}_{\delta,A}(a,b) \;:\; |K_D(f,g) - K_{D'}(f,g)| \le \varepsilon\,. \tag{2.47}$$

Proof. If $(KN)\int_a^b f(t)\,dg(t)$ exists, then (2.47) trivially holds. Conversely, assume that (2.47) is satisfied. We find $\delta_0 \in \Gamma(a,b)$ and $A_0 \in \mathcal{N}$ such that (2.47) holds with $\varepsilon = 1$. For each $n \in \mathbb{N}$ we construct by induction $\delta_n \in \Gamma(a,b)$, $\delta_n \le \delta_{n-1}$, and $A_n \in \mathcal{N}$, $A_n \supset A_{n-1}$ such that for all $D, D' \in \mathcal{F}_{\delta_n, A_n}(a,b)$ we have

$$|K_D(f,g) - K_{D'}(f,g)| \le 2^{-n}\,. \tag{2.48}$$

We fix some $D_n \in \mathcal{F}_{\delta_n,A_n}(a,b)$ for each $n \in \mathbb{N}$, and set $J_n = K_{D_n}(f,g)$. For all $m \ge n$ we have by (2.48) that $|J_n - J_m| \le 2^{-n}$, hence $\{J_n\}$ is a Cauchy sequence, $J_n \to J$ as $n \to \infty$.

Let now $\varepsilon > 0$ be given. We fix $n \in \mathbb{N}$ such that $2^{-n} \le \varepsilon$ and put $\delta = \delta_n$, $A = A_n$. It follows from (2.48) that $|K_D(f,g) - J_m| \le \varepsilon$ for all $D \in \mathcal{F}_{\delta,A}(a,b)$ and all $m \ge n$, and letting $m \to \infty$ we obtain $J = (KN)\int_a^b f(t)\,dg(t)$, which we wanted to prove. ■

We conclude the paper by proving the following result.

Proposition 2.9 *Let* $\mathcal{N}$ *be a system of negligible sets in* $[a,b]$, *and let* $f, g : [a,b] \to \mathbb{R}$ *be given functions. Let* $s \in \,]a,b[$ *be given.*

(i) *Assume that* $(KN)\int_a^b f(t)\,dg(t)$ *exists. Then* $(KN)\int_a^s f(t)\,dg(t)$, $(KN)\int_s^b f(t)\,dg(t)$ *exist.*

(ii) *Assume that* $(KN)\int_a^s f(t)\,dg(t)$, $(KN)\int_s^b f(t)\,dg(t)$ *exist. Then* $(KN)\int_a^b f(t)\,dg(t)$ *exists and*

$$(KN)\int_a^b f(t)\,dg(t) \;=\; (KN)\int_a^s f(t)\,dg(t) \,+\, (KN)\int_s^b f(t)\,dg(t)\,. \tag{2.49}$$

Proof.

(i) Assuming that $(KN)\int_a^b f(t)\,dg(t)$ exists, we prove that

$$\forall \varepsilon > 0 \;\; \exists \delta \in \Gamma(a,s) \;\; \exists A \in \mathcal{N} \;\; \forall D, D' \in \mathcal{F}_{\delta,A}(a,s) \;:\; |K_D(f,g) - K_{D'}(f,g)| \le \varepsilon\,, \tag{2.50}$$

and then use Lemma 2.8 to conclude that $(KN)\int_a^s f(t)\,dg(t)$ exists.

Let $\varepsilon > 0$ be given. We find $\delta_0 \in \Gamma(a,b)$ and $A \in \mathcal{N}$ such that for every $D_0, D_0' \in \mathcal{F}_{\delta_0,A}(a,b)$ we have

$$|K_{D_0}(f,g) - K_{D_0'}(f,g)| \le \varepsilon\,, \tag{2.51}$$

and for $t \in [a,b]$ set

$$\delta(t) \;=\; \begin{cases} \min\{\delta_0(t), |t-s|\} & \text{for} \quad t \in [a,b] \setminus \{s\}\,, \\ \delta_0(s) & \text{for} \quad t = s\,. \end{cases} \tag{2.52}$$

Let $D, D' \in \mathcal{F}_{\delta,A}(a,s)$ be arbitrary, and let $D^* \in \mathcal{F}_{\delta,A}(s,b)$ be fixed. Then

$$\begin{aligned} D &= \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1, \dots, m\}\,, \\ D' &= \{(\tau_k', [t_{k-1}', t_k'])\,;\, k = 1, \dots, m'\}\,, \\ D^* &= \{(\tau_i^*, [t_{i-1}^*, t_i^*])\,;\, i = 1, \dots, m^*\}\,, \end{aligned}$$

and we have $\tau_m = t_m = \tau_{m'}' = t_{m'}' = \tau_1^* = t_0^* = s$ by virtue of (2.52). Set

$$\begin{aligned} D_0 &= \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1, \dots, m-1\} \cup \{(s, [t_{m-1}, t_1^*])\} \qquad (2.53) \\ &\quad \cup \{(\tau_i^*, [t_{i-1}^*, t_i^*])\,;\, i = 2, \dots, m^*\}\,, \\ D_0' &= \{(\tau_k', [t_{k-1}', t_k'])\,;\, k = 1, \dots, m'-1\} \cup \{(s, [t_{m'-1}', t_1^*])\} \qquad (2.54) \\ &\quad \cup \{(\tau_i^*, [t_{i-1}^*, t_i^*])\,;\, i = 2, \dots, m^*\}\,. \end{aligned}$$

Then $D_0, D_0' \in \mathcal{F}_{\delta,A}(a,b) \subset \mathcal{F}_{\delta_0,A}(a,b)$, hence (2.51) holds. Together with the identity

$$\begin{aligned} K_{D_0}(f,g) - K_{D_0'}(f,g) &= \sum_{j=1}^{m-1} f(\tau_j)(g(t_j) - g(t_{j-1}) + f(s)(g(t_1^*) - g(t_{m-1}) \\ &\quad - \sum_{k=1}^{m'-1} f(\tau_k')(g(t_k') - g(t_{k-1}') - f(s)(g(t_1^*) - g(t_{m'-1}') \\ &= \sum_{j=1}^{m} f(\tau_j)(g(t_j) - g(t_{j-1}) - \sum_{k=1}^{m'} f(\tau_k')(g(t_k') - g(t_{k-1}') \\ &= K_D(f,g) - K_{D'}(f,g) \end{aligned}$$

this implies (2.50). We analogously check that $(KN)\int_s^b f(t)\,dg(t)$ exists, and (i) is proved.

(ii) Put $J_1 = (KN)\int_a^s f(t)\,dg(t)$, $J_2 = (KN)\int_s^b f(t)\,dg(t)$. For $\varepsilon > 0$ we find $\delta_1 \in \Gamma(a,s)$, $\delta_2 \in \Gamma(s,b)$ and $A_1, A_2 \in \mathcal{N}$ such that for every $D_1 \in \mathcal{F}_{\delta_1,A_1}(a,s)$, $D_2 \in \mathcal{F}_{\delta_2,A_2}(s,b)$ we have

$$|J_1 - K_{D_1}(f,g)| \le \varepsilon/2\,, \quad |J_2 - K_{D_2}(f,g)| \le \varepsilon/2\,. \tag{2.55}$$

Set $A = A_1 \cup A_2$, and

$$\delta(t) \;=\; \begin{cases} \min\{\delta_1(t), s-t\} & \text{for} \quad t \in [a,s[\;, \\ \min\{\delta_2(t), t-s\} & \text{for} \quad t \in]s,b]\,, \\ \min\{\delta_1(s), \delta_2(s)\} & \text{for} \quad t = s\,. \end{cases} \tag{2.56}$$

Let $D \in \mathcal{F}_{\delta,A}(a,b)$ be arbitrary, $D = \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1, \dots, m\}$. We find $k \in \{1, \dots, m\}$ such that $s \in [t_{k-1}, t_k]$. Then $s = \tau_k$ by (2.56), hence $t_{k-1} < s < t_k$, and we may put

$$\begin{aligned} D_1 &= \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1, \dots, k-1\} \cup \{(s, [t_{k-1}, s])\}\,, \\ D_2 &= \{(s, [s, t_k])\} \cup \{(\tau_j, [t_{j-1}, t_j])\,;\, j = k+1, \dots, m\}\,. \end{aligned}$$

We have $D_1 \in \mathcal{F}_{\delta_1,A_1}(a,s)$, $D_2 \in \mathcal{F}_{\delta_2,A_2}(s,b)$ and $K_D(f,g) = K_{D_1}(f,g) + K_{D_2}(f,g)$, hence

$$|J_1 + J_2 - K_D(f,g)| \le \varepsilon$$

as a consequence of (2.55), and the proof is complete. ■

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