On uniqueness in evolution quasivariational inequalities

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Abstract

We consider a rate independent evolution quasivariational inequality in a Hilbert space X with closed convex constraints having nonempty interior. We prove that there exists a unique solution which is Lipschitz dependent on the data, if the dependence of the Minkowski functional on the solution is Lipschitzian with a small constant and if also the gradient of the square of the Minkowski functional is Lipschitz continuous with respect to all variables. We exhibit an example of nonuniqueness if the assumption of Lipschitz continuity is violated by an arbitrarily small degree.

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1 Introduction

In 1973, Moreau has introduced the *sweeping process,* [10, 11]. It describes the movement $\xi = \xi(t)$ of a point in a Hilbert space X induced by a time-dependent closed convex set $C = C(t)$ according to

$$-\dot{\xi}(t) \in N_{C(t)}(\xi(t))\,, \quad \xi(0) = \xi_0\,, \tag{1.1}$$

where $N_K(x)$ denotes the normal cone to a convex set K at a point x. The evolution variational inequality

$$\langle \dot{v}(t), w - v(t)\rangle \geq \langle f(t), w - v(t)\rangle \quad \forall w \in \Gamma\,, \quad v(t) \in \Gamma\,, \quad v(0) = v_0\,, \tag{1.2}$$

$\Gamma \subset H$ closed and convex, constitutes a special case of (1.1), if we set

$$\xi(t) = v(t) - \int_0^t f(s)\,ds\,, \quad \xi_0 = v_0\,, \quad C(t) = \Gamma - \int_0^t f(s)\,ds\,. \tag{1.3}$$

In the same manner, the evolution *quasivariational* inequality

$$\langle \dot{v}(t), w - v(t)\rangle \geq \langle f(t), w - v(t)\rangle \quad \forall w \in \Gamma(v(t))\,, \quad v(t) \in \Gamma(v(t))\,, \quad v(0) = v_0\,, \tag{1.4}$$

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becomes a special case of the *implicit* or *state-dependent* sweeping process

$$-\dot{\xi}(t) \in N_{C(t,\xi(t))}(\xi(t)), \quad \xi(0) = \xi_0, \tag{1.5}$$

if we set

$$\xi(t) = v(t) - \int_0^t f(s)\,ds, \quad \xi_0 = v_0, \quad C(t,\xi) = \Gamma\left(\xi + \int_0^t f(s)\,ds\right) - \int_0^t f(s)\,ds. \tag{1.6}$$

An appropriate meaning has to be given to the time derivative if discontinuous processes are taken into consideration. The Young integral formulation was investigated in [5]. Another approach to nonsmooth evolution differential inclusions based on energy considerations was recently proposed by Mielke and Theil in [9].

While the sweeping process (1.1) has been an object of extensive study, see the survey [7], much less is known about the implicit process (1.5). The paper [2] seems to be the first result in this direction, in a more general setting actually, but restricted to the case $\dim X = 1$. Kunze and Monteiro Marques [8] have proved existence for (1.5), if C satisfies a Lipschitz condition with respect to the Hausdorff distance,

$$d_H(C(t,\xi), C(s,\eta)) \leq L_1|t-s| + L_2|\xi - \eta|, \tag{1.7}$$

if $L_2 < 1$ and give examples for nonexistence if $L_2 > 1$. However, no matter how small L_2 is chosen, uniqueness may fail to hold; Ballard [1] has given an example in the context of quasi-static friction problems.

Indeed, it is a general feature of quasivariational inequalities that the loss of monotonicity, as caused by the state dependence of the constraint C, is accompanied by a loss of uniqueness of their solution. On the other hand, there is no a priori reason why it should be impossible to enforce uniqueness if the constraint behaves in a sufficiently regular manner. We will show in this paper that uniqueness holds if the *normal vectors* to the constraint $C(t,\xi)$ satisfy a Lipschitz condition with respect to (t,ξ). Under the assumptions below this means that we require the *gradient* of the Minkowski functional M of C to be Lipschitz continuous as a function of (t,ξ), whereas (1.7) is under suitable hypotheses, see Lemma 3.2 below, equivalent to the Lipschitz continuity of M itself.

Our proof of uniqueness (and, incidentally, of existence at the same time) is based on the contraction principle, thus the possible loss of monotonicity does not play any role here. In order to obtain a contraction, results concerning time regularity for the solution operator $f \mapsto \xi$ of the standard variational inequality (1.1) are required which are stronger than the basic estimate

$$|\xi_1(t) - \xi_2(t)| \leq |\xi_{01} - \xi_{02}| + \int_0^t |f_1(s) - f_2(s)|\,ds. \tag{1.8}$$

Such stronger estimates have been provided within the context of *hysteresis operators*, see [4] for a recent exposition. However, to apply those results we need the constraint to be sandwiched uniformly between balls with fixed radius around zero. In particular, constraints with empty topological interior are excluded.

In addition, we will relate the nonuniqueness in the quasivariational inequality to the generic nonuniqueness phenomenon for scalar ordinary differential equations. More precisely, to every concave increasing function $\psi : \mathbb{R} \to \mathbb{R}$ with $\psi(0) = 0$ which gives rise to a positive solution of the scalar problem

$$y' = \psi(y), \quad y(0) = 0, \tag{1.9}$$

besides the trivial one, there corresponds a quasivariational inequality with nonunique solution, where the gradient (with respect to the state ξ) of the Minkowski functional varies like $\psi(|\xi|)$.

2 Problem statement

We consider a separable Hilbert space X endowed with a scalar product $\langle\cdot,\cdot\rangle$, a norm $|x| = \sqrt{\langle x,x\rangle}$ for $x \in X$, and a family of bounded convex closed sets $Z(\varrho) \subset X$ with a smooth boundary parametrized by $\varrho \in \mathcal{R} \subset Y$, where Y is a reflexive Banach space endowed with a norm $|\cdot|_Y$ and $\mathcal{R}$ is a convex closed set with non-empty interior $\mathcal{R}^\circ$. By Y' we mean the dual of Y, $(\!(\cdot,\cdot)\!)$ is the duality pairing between Y and Y', and $|\cdot|_{Y'}$, $|\cdot|_{\mathcal{L}(X,Y)}$ denote natural norms in the corresponding spaces.

Throughout the paper we assume that there exist $0 < c \le C$ such that

$$B_c(0) \ \subset\ Z(\varrho) \ \subset\ B_C(0) \qquad \forall \varrho \in \mathcal{R}\,. \tag{2.1}$$

We consider two problems, namely a variational inequality where the constraint Z depends on an additional given function (Problem $(\boldsymbol{\mathcal{P}})$), and a quasivariational inequality (Problem $(\boldsymbol{\mathcal{I}})$).

For given functions $u \in W^{1,1}(0,T\,;X)$, $r \in W^{1,1}(0,T\,;\mathcal{R})$ and an initial condition $x_0 \in Z(r(0))$ we look for a function $\xi \in W^{1,1}(0,T\,;X)$ such that

$$\begin{aligned}
(\boldsymbol{\mathcal{P}})\ \ &\text{(i)} && u(t)-\xi(t) \in Z(r(t)) \quad \forall t \in [0,T]\,,\\
&\text{(ii)} && u(0)-\xi(0) = x_0\,,\\
&\text{(iii)} && \Big\langle \dot\xi(t), u(t)-\xi(t)-y\Big\rangle \ \ge\ 0 \quad \forall y \in Z(r(t)) \quad \text{for a.e. } t \in\,]0,T[\,.
\end{aligned}$$

Besides serving as the crucial tool for the solution of problem $(\boldsymbol{\mathcal{I}})$, problem $(\boldsymbol{\mathcal{P}})$ is of interest in itself (consider, for example, a constitutive stress-strain law where the yield function depends on the temperature).

We now formulate problem $(\boldsymbol{\mathcal{I}})$. Let $g : [0,T]\times X \times X \to \mathcal{R}$ be a mapping satisfying Hypothesis **(G)** stated below at the beginning of Section 6. For a given function $u \in W^{1,1}(0,T\,;X)$ and an initial condition $x_0 \in Z(g(0,u(0),u(0)-x_0))$ (for instance, any $x_0 \in B_c(0)$ satisfies this inclusion) we look for a solution $\xi \in W^{1,1}(0,T\,;X)$ of the implicit problem

$$\begin{aligned}
(\boldsymbol{\mathcal{I}})\ \ &\text{(i)} && u(t)-\xi(t) \in Z(g(t,u(t),\xi(t))) \quad \forall t \in [0,T]\,,\\
&\text{(ii)} && u(0)-\xi(0) = x_0\,,\\
&\text{(iii)} && \Big\langle \dot\xi(t), u(t)-\xi(t)-y\Big\rangle \ \ge\ 0 \quad \forall y \in Z(g(t,u(t),\xi(t))) \quad \text{for a.e. } t \in\,]0,T[\,.
\end{aligned}$$

The implicit sweeping process (1.5) becomes a special case of $(\boldsymbol{\mathcal{I}})$, if we set $u = 0$, $g(t,u,\xi) = (t,\xi)$, $x_0 = -\xi_0$ and $C = -Z$. On the other hand, $(\boldsymbol{\mathcal{I}})$ becomes a special case of (1.5), if we set $C(t,\xi) = u(t) - Z(g(t,u(t),\xi))$ and $\xi_0 = u(0) - x_0$. The quasivariational inequality (1.4) is subsumed under $(\boldsymbol{\mathcal{I}})$ either going through (1.6), or by setting

$$\xi(t) = v(t) + u(t)\,, \quad u(t) = \int_0^t f(s)\,ds\,, \quad x_0 = -v_0\,, \quad Z = -\Gamma\,, \quad g(t,u,\xi) = \xi - u\,. \tag{2.2}$$

The formulation (1.5) certainly looks more compact than $(\boldsymbol{\mathcal{I}})$. However, in applications a driving function like u (or its derivative f) often appears, and it is useful to study the dependence of ξ on u with respect to standard function spaces. (In (1.5) one has to deal with metric properties of the set-valued mapping $t \mapsto C(t,\xi)$.)

Our main results include the existence, uniqueness, and Lipschitz continuous input-output dependence for both Problems $(\boldsymbol{\mathcal{P}})$ and $(\boldsymbol{\mathcal{I}})$ in Section 4. Before, we recall some basic notions from convex analysis as presented in [12, 4].

3 Preliminaries: Convex sets

Consider a Hilbert space X as in the previous section and a convex closed set $Z \subset X$. The mapping $M_Z : X \to \mathbb{R}_+$ defined by the formula

$$M_Z(x) = \inf\left\{s > 0\,;\ \frac{x}{s} \in Z\right\} \tag{3.1}$$

is called the *Minkowski functional associated with* Z. The *polar set* Z^* to Z is defined by the formula

$$Z^* \ = \ \{x \in X\,;\ \langle x, y\rangle \le 1 \quad \forall y \in Z\}\,. \tag{3.2}$$

We first summarize in Lemma 3.1 below some results of [4, Sections 3 and 6].

Lemma 3.1

(i) *Assume that there exist* $C > c > 0$ *such that*

$$B_c(0) \ \subset \ Z \ \subset \ B_C(0)\,. \tag{3.3}$$

Then we have

$$B_{1/C}(0) \ \subset \quad Z^* \quad \subset \ B_{1/c}(0)\,, \tag{3.4}$$

$$\frac{|x|}{C} \ \le \ M_Z(x) \ \le \ \frac{|x|}{c}\,, \tag{3.5}$$

$$c\,|x| \ \le \ M_{Z^*}(x) \ \le \ C\,|x|\,. \tag{3.6}$$

(ii) *Let the derivative* $\partial_x M_Z(x) \in X$ *exist for each* $x \in X \setminus \{0\}$, *and set* $J_Z(0) = 0$, $J_Z(x) = M_Z(x)\partial_x M_Z(x)$ *for* $x \ne 0$. *Then the unit outward normal* $n_Z(x)$ *to* Z *is uniquely determined at each point* $x \in \partial Z$, *and we have*

$$n_Z(x) \ = \ \frac{J_Z(x)}{|J_Z(x)|} \qquad \forall x \in \partial Z\,, \tag{3.7}$$

$$M_{Z^*}(J_Z(x)) \ = \ M_Z(x) \qquad \forall x \in Z\,. \tag{3.8}$$

(iii) *Let there exist* $L > 0$ *such that*

$$|n_Z(x) - n_Z(y)| \ \le \ L\,|x-y| \qquad \forall x, y \in \partial Z\,, \tag{3.9}$$

and let (3.3) hold. Then we have

$$|J_Z(x) - J_Z(y)| \ \le \ \left(\frac{1}{c^2} + \frac{L}{c}\left(1 + \frac{C}{c}\right)^2\right)\,|x-y| \qquad \forall x, y \in X\,. \tag{3.10}$$

We can measure the distance of two sets Z_1, Z_2 in the system $\mathcal{C}$ of all closed convex subsets Z of X either as the *Hausdorff distance*

$$d_H(Z_1, Z_2) \;=\; \max\{\sup_{z_1\in Z_1} \operatorname{dist}(z_1, Z_2), \sup_{z_2\in Z_2} \operatorname{dist}(z_2, Z_1)\}\,, \tag{3.11}$$

or the *Minkowski distance*

$$d_M(Z_1, Z_2) \;=\; \sup_{|x|=1} |M_{Z_1}(x) - M_{Z_2}(x)|\;. \tag{3.12}$$

We first show that these concepts are equivalent in the class of sets satisfying (3.3).

Lemma 3.2 *Let $Z_1, Z_2 \in \mathcal{C}$ be such that (3.3) holds for $Z = Z_i$, $i = 1, 2$. Then we have*

$$c^2 d_M(Z_1, Z_2) \;\le\; d_H(Z_1, Z_2) \;\le\; C^2 d_M(Z_1, Z_2)\,. \tag{3.13}$$

Proof. Assume first that there exists $x \in Z_1 \setminus Z_2$. Using (3.5) we obtain

$$\begin{aligned}\operatorname{dist}(x, Z_2) \;&\le\; \left| x - \frac{x}{M_{Z_2}(x)} \right| \le \frac{|x|^2}{M_{Z_1}(x) M_{Z_2}(x)} \left(M_{Z_2}\left(\frac{x}{|x|}\right) - M_{Z_1}\left(\frac{x}{|x|}\right)\right)\\ &\le\; C^2 d_M(Z_1, Z_2)\,,\end{aligned}$$

and reversing the roles of Z_1 and Z_2 we obtain the right inequality in (3.13). To prove the left estimate in (3.13), we divide the unit sphere $\partial B_1(0)$ into the sets

$$\begin{aligned} A_0 \;&=\; \{x \in \partial B_1(0)\,;\, M_{Z_1}(x) = M_{Z_2}(x)\}\,,\\ A_1 \;&=\; \{x \in \partial B_1(0)\,;\, M_{Z_1}(x) > M_{Z_2}(x)\}\,,\\ A_2 \;&=\; \{x \in \partial B_1(0)\,;\, M_{Z_1}(x) < M_{Z_2}(x)\}\,. \end{aligned}$$

For $x \in A_2$ set $\bar{x} = x/M_{Z_1}(x)$, and let $Q_{Z_2}\bar{x}$ be the orthogonal projection of $\bar{x}$ onto Z_2, that is,

$$Q_{Z_2}\bar{x} \;\in Z_2\,, \qquad |P_{Z_2}\bar{x}| \;=\; \operatorname{dist}(\bar{x}, Z_2)\,, \tag{3.14}$$

where we denote $P_{Z_2}\bar{x} = \bar{x} - Q_{Z_2}\bar{x}$. We have $M_{Z_2}(\bar{x}) > M_{Z_1}(\bar{x}) = 1$, hence $\bar{x} \notin Z_2$ and $d := |P_{Z_2}\bar{x}| > 0$. Put $m = 1 + d/c$. Then the vector

$$\frac{1}{m}\,\bar{x} \;=\; \frac{c}{c+d}\, Q_{Z_2}\bar{x} \,+\, \frac{d}{c+d}\,\frac{cP_{Z_2}\bar{x}}{d}$$

is a convex combination of elements of Z_2, hence $M_{Z_2}(\bar{x}) \le m$. This yields

$$M_{Z_2}(\bar{x}) - M_{Z_1}(\bar{x}) \;\le\; m - 1 \;\le\; \frac{1}{c} \operatorname{dist}(\bar{x}, Z_2) \;\le\; \frac{1}{c} d_H(Z_1, Z_2)\,.$$

Using (3.5) we conclude that

$$M_{Z_2}(x) - M_{Z_1}(x) \;\le\; \frac{1}{c^2} d_H(Z_1, Z_2)\,,$$

and arguing similarly for $x \in A_0 \cup A_1$ we complete the proof. ■

Another distance criterion for convex sets involving the mapping J_Z introduced in Lemma 3.1 is used in Sections 5 and 7. Here, we derive the following estimate.

Lemma 3.3 *Let the hypotheses of Lemma 3.2 hold, and let the outward normal vectors* n_{Z_i} *satisfy (3.9) for* $Z = Z_i$, $i = 1, 2$. *Let* L_J *be the Lipschitz constant on the right-hand side of (3.10). Then*

$$|J_{Z_1}(x) - J_{Z_2}(x)| \;\leq\; \frac{2\sqrt{2}}{c}\Big(d_M(Z_1, Z_2)\,(cL_J + d_M(Z_1, Z_2))\Big)^{1/2} \qquad \forall x \in \partial B_1(0)\,. \tag{3.15}$$

Proof. Let $x \in X$ with $|x| = 1$ and let $J_{Z_1}(x) \neq J_{Z_2}(x)$. We define

$$x_s = x + s\frac{J_{Z_2}(x) - J_{Z_1}(x)}{|J_{Z_2}(x) - J_{Z_1}(x)|} \qquad \text{for} \quad s \geq 0\,. \tag{3.16}$$

We may assume that $\langle x_s - x, x\rangle \leq 0$, otherwise we interchange Z_1 and Z_2. The functions $\lambda_i(s) := \frac{1}{2}M^2_{Z_i}(x_s)$ are convex and satisfy

$$\lambda_i(0) + s\lambda_i'(0) \;\leq\; \lambda_i(s) \;\leq\; \lambda_i(0) + s\lambda_i'(s) \qquad \text{for} \quad s \geq 0\,. \tag{3.17}$$

Thus,

$$\begin{aligned}\lambda_2(s) - \lambda_1(s) \;&\geq\; \lambda_2(0) - \lambda_1(0) \,+\, s\,(\lambda_2'(0) - \lambda_1'(s)) \qquad (3.18)\\ &=\; \lambda_2(0) - \lambda_1(0) \,+\, s\,(\lambda_2'(0) - \lambda_1'(0)) \,+\, s\,(\lambda_1'(0) - \lambda_1'(s))\,.\end{aligned}$$

Note that

$$\lambda_i'(s) = \left\langle J_{Z_i}(x_s), \frac{J_{Z_2}(x) - J_{Z_1}(x)}{|J_{Z_2}(x) - J_{Z_1}(x)|}\right\rangle \qquad \text{for} \quad s \geq 0\,,$$

hence

$$\lambda_2'(0) - \lambda_1'(0) \;=\; |J_{Z_2}(x) - J_{Z_1}(x)|\,, \tag{3.19}$$

$$|\lambda_1'(s) - \lambda_1'(0)| \;\leq\; |J_{Z_1}(x_s) - J_{Z_1}(x)| \;\leq\; sL_J\,. \tag{3.20}$$

We further have by (3.5) for all $s \geq 0$ that

$$|\lambda_2(s) - \lambda_1(s)| \leq \frac{|x_s|}{c}\,|M_{Z_2}(x_s) - M_{Z_1}(x_s)| \leq \frac{|x_s|^2}{c}\,d_M(Z_1, Z_2) \leq \frac{1 + s^2}{c}\,d_M(Z_1, Z_2)\,. \tag{3.21}$$

Combining (3.18)–(3.21) we obtain for all $s > 0$ that

$$|J_{Z_2}(x) - J_{Z_1}(x)| \;\leq\; \frac{2 + s^2}{sc}\,d_M(Z_1, Z_2) \,+\, sL_J\,. \tag{3.22}$$

The right-hand side attains its minimum for $s = \sqrt{2\,d_M(Z_1, Z_2)/(cL_J + d_M(Z_1, Z_2))}$, and the assertion follows. ■

We conclude this section with the following result as a simplified variant of [6, Lemma 3.2].

Lemma 3.4 *Let* $Z \subset X$ *be an arbitrary convex closed set, and for* $\theta > 0$ *set* $Z_\theta = Z + B_\theta(0)$. *Then the outward unit normal* $n_\theta(x)$ *to* Z_θ *is uniquely determined at each point* $x \in \partial Z_\theta$, *and we have*

$$|n_\theta(x) - n_\theta(y)| \;\leq\; \frac{1}{\theta}\,|x - y| \qquad \forall x, y \in \partial Z_\theta\,. \tag{3.23}$$

Proof. Let $Q_Z : X \to Z$ be the orthogonal projection onto Z. Putting $P_Z x = x - Q_Z x$ we have $|P_Z x| = \mathrm{dist}\,(x, Z)$ for each $x \in X$, and

$$\langle P_Z x, Q_Z x - z \rangle \;\geq\; 0 \qquad \forall x \in X\,, \quad \forall z \in Z\,. \tag{3.24}$$

For $x \in \partial Z_\theta$ and $\hat{x} \in Z_\theta$ this yields $|P_Z x| = \theta$, $|P_Z \hat{x}| \leq \theta$, and

$$\langle P_Z x, x - \hat{x} \rangle \;=\; \langle P_Z x, Q_Z x - Q_Z \hat{x} \rangle \,+\, |P_Z x|^2 \,-\, \langle P_Z x, P_Z \hat{x} \rangle \;\geq\; 0\,.$$

Let $\tilde{n}$ be an arbitrary unit vector such that $\langle \tilde{n}, x - \tilde{x} \rangle \geq 0$ for every $\tilde{x} \in Z_\theta$. Putting $\tilde{x} = Q_Z x + \theta \tilde{n}$ we obtain $\langle \tilde{n}, P_Z x \rangle \geq \theta$, hence $\tilde{n} = (1/\theta) P_Z x = n_\theta(x)$. From (3.24) it follows for all $x, y \in X$ that $\langle P_Z x - P_Z y, Q_Z x - Q_Z y \rangle \geq 0$, hence $|P_Z x - P_Z y| \leq |x - y|$, and (3.23) follows. ■

4 Existence and uniqueness for the explicit problem

By $M(\varrho, x)$ we denote the Minkowski functional $M_{Z(\varrho)}(x)$ associated with $Z(\varrho)$ as in (3.1). In addition to (2.1), we make the following hypothesis:

(E) There exists a function $\mu : \mathcal{R} \times X \to \mathbb{R}_+$ and a constant $\kappa > 0$ such that

$$|\mu(\varrho, x) - \mu(\sigma, x)| \;\leq\; \kappa |\varrho - \sigma|_Y \quad \forall \varrho, \sigma \in \mathcal{R} \quad \forall x \in X\,, \tag{4.1}$$

$$\frac{|x|}{M(\varrho, x)} \;=\; \mu(\varrho, x) \quad \forall \varrho \in \mathcal{R} \quad \forall x \in X \setminus \{0\}\,. \tag{4.2}$$

A typical example is $Z(\varrho) = \varrho\, Z$ for $\varrho \geq 0$, where $Y = \mathbb{R}$, $\mathcal{R} = [c', C']$ for some $C' \geq c' > 0$, $Z \subset X$ is a fixed bounded convex closed set with $B_{c''}(0) \subset Z \subset B_{C''}(0)$ for some $0 < c'' \leq C''$. Then (4.1), (4.2) are trivially satisfied, as μ is linear in ϱ.

It is obvious that if Problem $(\mathcal{P})$ has a solution, it is unique. Indeed, if ξ, η are two solutions, then $\left\langle \dot{\xi}, \eta - \xi \right\rangle \geq 0$, $\langle \dot{\eta}, \xi - \eta \rangle \geq 0$, hence $\left\langle \dot{\xi} - \dot{\eta}, \xi - \eta \right\rangle \leq 0$ and the assertion follows.

Proposition 4.1 *Let Hypotheses (2.1) and* **(E)** *hold. Then Problem* $(\mathcal{P})$ *admits a unique solution* $\xi \in W^{1,1}(0, T\,;\, X)$ *for every given functions* $u \in W^{1,1}(0, T\,;\, X)$, $r \in W^{1,1}(0, T\,;\, \mathcal{R})$ *and every initial condition* $x_0 \in Z(r(0))$.

Proof. We construct the solution by time-discretization. Fix some $n \in \mathbb{N}$ and the corresponding equidistant partition $0 = t_0 < t_1 < \ldots < t_n = T$ with $t_k = kT/n$, $k = 0, 1, \ldots, n$. For $\varrho \geq 0$ we denote by $Q_\varrho : X \to Z(\varrho)$ the orthogonal projection of X onto $Z(\varrho)$, and set $P_\varrho = I - Q_\varrho$ where $I : X \to X$ is the identity. Analogously as in the proof of Lemma 3.4 we have

$$\langle P_\varrho x, Q_\varrho x - z \rangle \;\geq\; 0 \qquad \forall x \in X\,, \quad \forall z \in Z\,. \tag{4.3}$$

For $k = 0, \ldots, n$ we define the sequences $u_k = u(t_k)$, $r_k = r(t_k)$, $\xi_k = u_k - x_k$, $x_k = Q_{r_k}(x_{k-1} + u_k - u_{k-1})$ for $k \geq 1$ starting from the given initial condition x_0. Then we have

$$\xi_k - \xi_{k-1} \;=\; P_{r_k}(x_{k-1} + u_k - u_{k-1})\,, \tag{4.4}$$

hence

$$\langle \xi_k - \xi_{k-1}, x_k - y_k \rangle \ \geq \ 0 \quad \forall k = 1, \dots, n\,, \quad \forall y_k \in Z(r_k) \tag{4.5}$$

according to (4.3). We have $M(r_k, x_k) \leq 1$ for all k, and from (4.5) it follows that

$$\left\langle \xi_k - \xi_{k-1}, x_k - \frac{M(r_{k-1}, x_{k-1})}{M(r_k, x_{k-1})} x_{k-1} \right\rangle \ \geq \ 0\,.$$

This yields

$$\begin{aligned} |\xi_k - \xi_{k-1}|^2 \ &\leq \ \langle \xi_k - \xi_{k-1}, u_k - u_{k-1} \rangle \\ &\quad + M(r_{k-1}, x_{k-1}) \left\langle \xi_k - \xi_{k-1}, \left(\frac{1}{M(r_{k-1}, x_{k-1})} - \frac{1}{M(r_k, x_{k-1})} \right) x_{k-1} \right\rangle \\ &\leq \ |\xi_k - \xi_{k-1}| \left(|u_k - u_{k-1}| + \kappa\, |r_k - r_{k-1}|_Y \right), \end{aligned}$$

hence

$$|\xi_k - \xi_{k-1}| \ \leq \ |u_k - u_{k-1}| + \kappa\, |r_k - r_{k-1}|_Y \ \leq \ \int_{t_{k-1}}^{t_k} \left(|\dot u| + \kappa\, |\dot r|_Y \right) dt\,. \tag{4.6}$$

We now define the piecewise linear functions

$$\begin{aligned} u^{(n)}(t) \ &= \ u_{k-1} + \frac{n}{T}(t - t_{k-1})(u_k - u_{k-1})\,, \\ r^{(n)}(t) \ &= \ r_{k-1} + \frac{n}{T}(t - t_{k-1})(r_k - r_{k-1})\,, \\ x^{(n)}(t) \ &= \ x_{k-1} + \frac{n}{T}(t - t_{k-1})(x_k - x_{k-1})\,, \\ \xi^{(n)}(t) \ &= \ \xi_{k-1} + \frac{n}{T}(t - t_{k-1})(\xi_k - \xi_{k-1}) \end{aligned}$$

for $t \in [t_{k-1}, t_k[\,$, $k = 1, \dots, n$, continuously extended to $t = T$. Then $u^{(n)} \to u$ strongly in $W^{1,1}(0, T\,; X)$, $r^{(n)} \to r$ strongly in $W^{1,1}(0, T\,; \mathcal{R})$ as $n \to \infty$, and the first inequality in (4.6) has the form

$$|\dot\xi^{(n)}(t)| \leq |\dot u^{(n)}(t)| + \kappa |\dot r^{(n)}(t)| \qquad \text{a. e.} \tag{4.7}$$

For every $t \in [t_{k-1}, t_k[\,$, $x^{(n)}(t)$ is a convex combination of x_k and x_{k-1}, hence

$$\begin{aligned} M(r(t), x^{(n)}(t)) \ &\leq \ \alpha M(r(t), x_k) + (1-\alpha) M(r(t), x_{k-1}) \\ &\leq \ 1 \ + \ \alpha(M(r(t), x_k) - M(r_k, x_k))) + (1-\alpha)(M(r(t), x_{k-1}) - M(r_{k-1}, x_{k-1}))) \\ &=: \ 1 \ + \ \alpha_n(t)\,. \end{aligned} \tag{4.8}$$

By (2.1), (3.5), and (4.1)–(4.2) we have for every $\varrho, \sigma \in \mathcal{R}$ and $x \neq 0$ that

$$|M(\varrho, x) - M(\sigma, x)| \ \leq \ \kappa \frac{M(\varrho, x)\, M(\sigma, x)}{|x|} |\varrho - \sigma|_Y \ \leq \ \frac{\kappa}{c} M(\sigma, x)\, |\varrho - \sigma|_Y\,,$$

hence

$$|\alpha_n(t)| \ \leq \ \frac{\kappa}{c} \max\{ |r(t) - r_k|_Y, |r(t) - r_{k-1}|_Y \} \ \leq \ \frac{\kappa}{c} \int_{t-(T/n)}^{t+(T/n)} |\dot r(\tau)|_Y \, d\tau \tag{4.9}$$

for all $t \in [0,T]$, with $\dot r(t)$ extended by 0 outside of $[0,T]$. In particular, $\alpha_n(t) \to 0$ for every $t \in [0,T]$ as $n \to \infty$. Furthermore, for $y \in L^\infty(0,T\,;\,X)$ such that $y(t) \in Z(r(t))$ for a. e. $t \in [0,T]$, and for a. e. $t \in]t_{k-1}, t_k[$ we have by (4.5) and (4.6) that

$$
\begin{aligned}
\left\langle \dot\xi^{(n)}(t), x^{(n)}(t) - y(t)\right\rangle \;&=\; \frac{n}{T}\left\langle \xi_k - \xi_{k-1}, x_k - \frac{M(r(t),y(t))}{M(r_k,y(t))}y(t)\right\rangle \\
&- \frac{n}{T}\left\langle \xi_k - \xi_{k-1}, x_k - x^{(n)}(t) + M(r(t),y(t))\left(\frac{1}{M(r(t),y(t))} - \frac{1}{M(r_k,y(t))}\right)y(t)\right\rangle \\
&\geq\; -\frac{n}{T}|\xi_k - \xi_{k-1}|(|x_k - x_{k-1}| + \kappa\,|r_k - r_{k-1}|_Y) \\
&\geq\; -\frac{2n}{T}(|u_k - u_{k-1}| + \kappa\,|r_k - r_{k-1}|_Y)\int_{t_{k-1}}^{t_k}(|\dot u| + \kappa\,|\dot r|_Y)\,dt\,,
\end{aligned}
$$

hence for every $t \in [0,T]$ we have

$$
\begin{aligned}
\int_0^t \left\langle \dot\xi^{(n)}, x^{(n)} - y\right\rangle(s)\,ds \;&\geq\; -2\sum_{k=1}^n(|u_k - u_{k-1}| + \kappa\,|r_k - r_{k-1}|_Y)\int_{t_{k-1}}^{t_k}(|\dot u| + \kappa\,|\dot r|_Y)\,ds \qquad (4.10)\\
&\geq\; -2\,\max_k(|u_k - u_{k-1}| + \kappa\,|r_k - r_{k-1}|_Y)\int_0^T(|\dot u| + \kappa\,|\dot r|_Y)\,ds \\
&=:\; -\beta_n\,,
\end{aligned}
$$

with $\beta_n \geq 0$, $\lim_{n\to\infty}\beta_n = 0$. For $n \in \mathbb{N}$ and $t \in [0,T]$ put

$$\gamma_n(t) \;=\; \min\left\{1, \frac{1}{M(r(t), x^{(n)}(t))}\right\}.$$

Then

$$\gamma_n(t)\,x^{(n)}(t) \in Z(r(t))\,, \quad 0 \;\leq\; 1 - \gamma_n(t) \;\leq\; \frac{|\alpha_n(t)|}{1 + |\alpha_n(t)|} \;\leq\; 1\,, \quad \lim_{n\to\infty}\gamma_n(t) = 1\,. \qquad (4.11)$$

for all $t \in [0,T]$. This yields in particular

$$
\begin{aligned}
\int_0^t \left\langle \dot\xi^{(n)}(s), x^{(n)}(s) - \gamma_m(s)x^{(m)}(s)\right\rangle\,ds \;&\geq\; -\beta_n\,,\\
\int_0^t \left\langle \dot\xi^{(m)}(s), x^{(m)}(s) - \gamma_n(s)x^{(n)}(s)\right\rangle\,ds \;&\geq\; -\beta_m
\end{aligned}
$$

for $n, m \in \mathbb{N}$ and $t \in [0,T]$. Summing up the two above inequalities we obtain

$$
\begin{aligned}
\int_0^t \left\langle \dot\xi^{(n)} - \dot\xi^{(m)}, x^{(n)} - x^{(m)}\right\rangle(s)\,ds \;+\; \int_0^t \left(\left\langle \dot\xi^{(n)}, (1-\gamma_m)\,x^{(m)}\right\rangle + \left\langle \dot\xi^{(m)}, (1-\gamma_n)\,x^{(n)}\right\rangle\right)(s)\,ds \\
\geq\; -\beta_n - \beta_m\,,
\end{aligned}
$$

hence

$$
\begin{aligned}
\frac{1}{2}|x^{(n)}(t) - x^{(m)}(t)|^2 \;&\leq\; \int_0^T |\dot u^{(n)} - \dot u^{(m)}||x^{(n)} - x^{(m)}|(s)\,ds \\
&+ C\int_0^T \left(|\dot\xi^{(n)}|(1-\gamma_m) + |\dot\xi^{(m)}|(1-\gamma_n)\right)(s)\,ds + \beta_n + \beta_m\,.
\end{aligned}
$$

Since $t \in [0,T]$ is arbitrary, we conclude that $\{x^{(n)}\}$ is a Cauchy sequence in $C(0,T\,;\,X)$. There exist therefore $x, \xi \in C(0,T\,;\,X)$ such that $x + \xi = u$, $x^{(n)} \to x$, $\xi^{(n)} \to \xi$ uniformly in $C(0,T\,;\,X)$ as $n \to \infty$, and (4.11) implies that $x(t) \in Z(r(t))$ for every $t \in [0,T]$. From (4.7) it follows that the sequence $\{\dot\xi^{(n)}\}$ is equibounded and equiintegrable in $W^{1,1}(0,T\,;\,X)$, and by the Dunford-Pettis theorem (see [3]) we have $\dot\xi^{(n)} \to \dot\xi$ weakly in $L^1(0,T\,;\,X)$. This enables us to pass to the limit in (4.10) as $n \to \infty$ and obtain

$$\int_0^T \left\langle \dot\xi(t), x(t) - y(t) \right\rangle dt \;\geq\; 0 \tag{4.12}$$

for every $y \in L^\infty(0,T\,;\,X)$ such that $y(t) \in Z(r(t))$ for a.e. $t \in [0,T]$. To see that (4.12) is equivalent to condition (iii) of $(\boldsymbol{\mathcal{P}})$, let us consider a Lebesgue point $\bar t \in]0,T[$ of $\dot\xi$ and any $y \in Z(r(\bar t))$, and put

$$y_\delta(t) \;=\; \begin{cases} y\frac{M(r(\bar t),y)}{M(r(t),y)} & \text{for} \quad t \in]\bar t - \delta, \bar t + \delta[\ , \\ x(t) & \text{for} \quad t \in [0,T] \setminus]\bar t - \delta, \bar t + \delta[\end{cases}$$

for $\delta > 0$ sufficiently small, with obvious modifications if $y = 0$. Then $y_\delta \in L^\infty(0,T\,;\,X)$, $y_\delta(t) \in Z(r(t))$ for all $t \in [0,T]$, hence

$$\int_{\bar t - \delta}^{\bar t + \delta} \left\langle \dot\xi(t), x(t) - y_\delta(t) \right\rangle dt \;\geq\; 0\,.$$

Dividing the above inequality by 2δ and letting $\delta \to 0+$ we obtain the assertion. ■

5 Lipschitz estimates

Under the assumption (2.1), we denote by $Z^*(\varrho)$ the polar set to $Z(\varrho)$ defined in (3.2) for $\varrho \in [a,b]$, and by $M^*(\varrho,\cdot)$ its Minkowski functional. By Lemma 3.1 we have $B_{1/C}(0) \subset Z^*(\varrho) \subset B_{1/c}(0)$ for every $\varrho \in \mathcal{R}$, and the inequalities

$$\frac{|x|}{C} \;\leq\; M(\varrho,x) \;\leq\; \frac{|x|}{c}\,, \tag{5.1}$$

$$c\,|x| \;\leq\; M^*(\varrho,x) \;\leq\; C\,|x| \tag{5.2}$$

hold for every $x \in X$ and $\varrho \in \mathcal{R}$. The following conditions are assumed to hold:

(L1) The partial derivatives $\partial_\varrho M(\varrho,x) \in Y'$, $\partial_x M(\varrho,x) \in X$ exist for every $x \in X \setminus \{0\}$ and $\varrho \in \mathcal{R}^\circ$, and the mappings

$$J(\varrho,x) \;=\; M(\varrho,x)\partial_x M(\varrho,x) \;:\; \mathcal{R}^\circ \times X \setminus \{0\} \to X\,, \tag{5.3}$$

$$K(\varrho,x) \;=\; M(\varrho,x)\partial_\varrho M(\varrho,x) \;:\; \mathcal{R}^\circ \times X \setminus \{0\} \to Y' \tag{5.4}$$

admit continuous extensions to $x = 0$ and $\varrho \in \mathcal{R}$.

(L2) For every $x, x' \in B_C(0)$ and $\varrho, \varrho' \in \mathcal{R}$ we have

$$|K(\varrho,x)|_{Y'} \;\leq\; K_0\,, \tag{5.5}$$

$$|J(\varrho,x) - J(\varrho',x')| \;\leq\; C_J\,(|\varrho - \varrho'|_Y + |x - x'|)\,, \tag{5.6}$$

$$|K(\varrho,x) - K(\varrho',x')|_{Y'} \;\leq\; C_K\,(|\varrho - \varrho'|_Y + |x - x'|) \tag{5.7}$$

with some fixed constants $K_0, C_J, C_K > 0$.

Note that **(E)** follows from (5.1) and (5.5). Indeed, for all $x \neq 0$, $\varrho \in \mathcal{R}$, and $p > 0$ we have $\mu(\varrho, px) = \mu(\varrho, x)$. For $|x| = C$ we have in particular $M(\varrho, x) \geq 1$ for all $\varrho \in \mathcal{R}$ by (5.1), and for all $\varrho, \sigma \in \mathcal{R}$ we thus obtain

$$\begin{aligned} |\mu(\varrho,x) - \mu(\sigma,x)| &= \frac{2C}{M(\varrho,x)\, M(\sigma,x)\, (M(\varrho,x) + M(\sigma,x))} \left| \frac{1}{2} M^2(\varrho,x) - \frac{1}{2} M^2(\sigma,x) \right| \\ &\leq CK_0 |\varrho - \sigma| \,. \end{aligned}$$

The following two lemmas constitute the main steps towards the desired Lipschitz estimates.

Lemma 5.1 *Let* **(L1)** *hold, let* $(r,u) \in W^{1,1}(0,T\,;\mathcal{R}) \times W^{1,1}(0,T\,;X)$ *and* $x_0 \in Z(r(0))$ *be given, and let* $\xi \in W^{1,1}(0,T\,;X)$ *be the solution to Problem* $(\mathcal{P})$. *For* $t \in]0,T[$ *set*

$$\begin{aligned} A[r,u](t) &= \left\langle \dot{\xi}(t), J(r(t), x(t)) \right\rangle , \\ B[r,u](t) &= \frac{1}{2} M^2(r(t), x(t)) \,, \\ G[r,u](t) &= \langle \dot{u}(t), J(r(t), x(t)) \rangle + \langle\!\langle K(r(t), x(t)), \dot{r}(t) \rangle\!\rangle \,, \end{aligned}$$

with $x(t) = u(t) - \xi(t)$. *Then for a. e.* $t \in]0,T[$ *we have either*

(i) $\dot{\xi}(t) = 0$, $\frac{d}{dt} B[r,u](t) = G[r,u](t)$

or

(ii) $\dot{\xi}(t) \neq 0$, $x(t) \in \partial Z(r(t))$, $A[r,u](t) = G[r,u](t) > 0$, $B[r,u](t) = \max_{[0,T]} B[r,u] = 1/2$, $\frac{d}{dt} B[r,u](t) = 0$, *and*

$$\dot{\xi}(t) = \frac{A[r,u](t)}{|J(r(t), x(t))|^2}\, J(r(t), x(t)) \,. \tag{5.8}$$

Proof. Let $L \subset]0,T[$ be the set of Lebesgue points of all functions $\dot{u}$, $\dot{r}$, $\dot{\xi}$, $\frac{d}{dt} B[r,u]$. Then L has full measure in $[0,T]$, and for $t \in L$ we have

$$\frac{d}{dt} B[r,u](t) = \langle \dot{x}(t), J(r(t), x(t)) \rangle + \langle\!\langle K(r(t), x(t)), \dot{r}(t) \rangle\!\rangle \,. \tag{5.9}$$

If $\dot{\xi}(t) = 0$, then $\dot{x}(t) = \dot{u}(t)$, and (i) follows from (5.9). If $\dot{\xi}(t) \neq 0$, then $x(t) \in \partial Z(r(t))$, hence $M(r(t), x(t)) = 1 = \max_{s \in [0,T]} M(r(s), x(s))$. We therefore have $B[r,u](t) = 1/2 = \max_{[0,T]} B[r,u]$, $\frac{d}{dt} B[r,u](t) = 0$. As a consequence of $(\mathcal{P})$ (iii) we have $\dot{\xi}(t) = k\, n(r(t), x(t))$ with a constant $k > 0$, where $n(r(t), x(t))$ is the unit outward normal to $Z(r(t))$ at the point $x(t)$, hence $k = \left\langle \dot{\xi}(t), n(r(t), x(t)) \right\rangle$, and (5.8) follows from Lemma 3.1 (ii). Furthermore, (5.9) yields $\langle \dot{x}(t), J(r(t), x(t)) \rangle = - \langle\!\langle K(r(t), x(t)), \dot{r}(t) \rangle\!\rangle$, hence

$$\begin{aligned} \left\langle \dot{\xi}(t), J(r(t), x(t)) \right\rangle &= \langle \dot{u}(t), J(r(t), x(t)) \rangle - \langle \dot{x}(t), J(r(t), x(t)) \rangle \\ &= \langle \dot{u}(t), J(r(t), x(t)) \rangle + \langle\!\langle K(r(t), x(t)), \dot{r}(t) \rangle\!\rangle \,, \end{aligned}$$

and the proof is complete. ■

In the situation of Lemma 5.1, we always have

$$|G[r,u](t)| \;\leq\; |\dot u(t)||J(r(t),x(t))| + K_0|\dot r(t)|_Y\,, \tag{5.10}$$

$$|\dot\xi(t)| \;\leq\; |\dot u(t)| + CK_0|\dot r(t)|_Y\,. \tag{5.11}$$

Indeed, (5.11) is trivial if $\dot\xi(t) = 0$; otherwise we have $|\dot\xi(t)| = A[r,u](t)/|J(r(t),x(t))| = G[r,u](t)/|J(r(t),x(t))|$ with $x(t) \in \partial Z(r(t))$. Lemma 3.1 then yields $C\,|J(r(t),x(t)))| \geq M^*(r(t),J(r(t),x(t))) = M(r(t),x(t)) = 1$, and (5.11) follows from (5.10).

Lemma 5.2 *Let* **(L1)** *and (5.5) hold, let* $(r,u),(s,v) \in W^{1,1}(0,T\,;\mathcal{R}) \times W^{1,1}(0,T\,;X)$ *and* $x_0 \in Z(r(0))$, $y_0 \in Z(s(0))$ *be given, let* $\xi,\eta \in W^{1,1}(0,T\,;X)$ *be the respective solutions to Problem* $(\boldsymbol{\mathcal{P}})$, *and set* $x = u-\xi$, $y = v-\eta$. *Then for a. e.* $t \in]0,T[$ *we have*

$$|A[r,u](t) - A[s,v](t)| + \frac{d}{dt}|B[r,u](t) - B[s,v](t)| \;\leq\; |G[r,u](t) - G[s,v](t)|\,, \tag{5.12}$$

$$\begin{aligned} |\dot\xi(t) - \dot\eta(t)| \;\leq\; & C\,(|\dot u(t)| + CK_0|\dot r(t)|_Y)|J(r(t),x(t)) - J(s(t),y(t))| \\ & + C\,|A[r,u](t) - A[s,v](t)|\,. \end{aligned} \tag{5.13}$$

Proof. The assertion follows directly from Lemma 5.1 if $\dot\xi(t) = \dot\eta(t) = 0$. Assume now

- $\dot\xi(t) \neq 0$, $\dot\eta(t) \neq 0$.

Then (5.12) is again an immediate consequence of Lemma 5.1. To prove (5.13), we use (5.8) and the elementary vector identity

$$\left|\frac{z}{|z|^2} - \frac{z'}{|z'|^2}\right| \;=\; \frac{1}{|z||z'|}\,|z - z'| \qquad \text{for} \quad z,z' \in X \setminus \{0\}\,,$$

to obtain

$$\begin{aligned} |\dot\xi(t) - \dot\eta(t)| \;\leq\;& |A[r,u](t)|\left|\frac{J(r(t),x(t))}{|J(r(t),x(t))|^2} - \frac{J(s(t),y(t))}{|J(s(t),y(t))|^2}\right| \\ & + \frac{1}{|J(s(t),y(t))|}\,|A[r,u](t) - A[s,v](t)| \\ =\;& \frac{1}{|J(r(t),x(t))|\,|J(s(t),y(t))|}\,|G[r,u](t)|\;|J(r(t),x(t)) - J(s(t),y(t))| \\ & + \frac{1}{|J(s(t),y(t))|}\,|A[r,u](t) - A[s,v](t)|\,. \end{aligned}$$

Lemma 3.1 yields $|J(r(t),x(t))| \geq 1/C$, $|J(s(t),y(t))| \geq 1/C$. Combining the above inequalities with (5.10) we obtain the assertion.

Let us consider now the case

- $\dot\xi(t) \neq 0$, $\dot\eta(t) = 0$.

Then $|A[r,u](t) - A[s,v](t)| = A[r,u](t)$, $B[r,u](t) - B[s,v](t) = 1/2 - B[s,v](t) \geq 0$, hence

$$\begin{aligned} |A[r,u](t) - A[s,v](t)| + \frac{d}{dt}|B[r,u](t) - B[s,v](t)| \;&=\; A[r,u](t) - \frac{d}{dt}B[s,v](t) \\ &=\; G[r,u](t) - G[s,v](t)\,, \end{aligned}$$

hence (5.12) is fulfilled. We further have similarly as above that

$$|\dot{\xi}(t) - \dot{\eta}(t)| \;=\; |\dot{\xi}(t)| \;\le\; C\, A[r,u](t) \;=\; C\,|A[r,u](t) - A[s,v](t)|\,,$$

hence (5.13) holds. The remaining case

- $\dot{\xi}(t) = 0\,,\ \dot{\eta}(t) \neq 0$

is analogous, and Lemma 5.2 is proved. ∎

We are now ready to prove the following crucial estimate.

Proposition 5.3 *Let* **(L1)**, **(L2)** *hold, let* $(r,u),(s,v) \in W^{1,1}(0,T\,;\mathcal{R}) \times W^{1,1}(0,T\,;X)$ *and* $x_0 \in Z(r(0))$, $y_0 \in Z(s(0))$ *be given, let* $\xi, \eta \in W^{1,1}(0,T\,;X)$ *be the respective solutions to Problem* **($\mathcal{P}$)**, *and set* $x = u - \xi$, $y = v - \eta$. *Then for a. e.* $t \in \,]0,T[$ *we have*

$$|\dot{\xi}(t) - \dot{\eta}(t)| \;+\; C\,\frac{d}{dt}|B[r,u](t) - B[s,v](t)| \;\le\; \frac{C}{c}|\dot{u}(t) - \dot{v}(t)| + CK_0\,|\dot{r}(t) - \dot{s}(t)|_Y \qquad (5.14)$$
$$+\; C\Big(2C_J|\dot{u}(t)| + (C_K + CC_J K_0)\,|\dot{r}(t)|_Y\Big)(|r(t) - s(t)|_Y + |x(t) - y(t)|)\,.$$

Proof. By Lemma 3.1 we have $c\,|J(r(t),x(t))| \le M^*(r(t), J(r(t),x(t))) = M(r(t),x(t)) \le 1$, hence $|J(r(t),x(t))| \le 1/c$ for every $t \in [0,T]$. By Lemma 5.2, we can estimate the left-hand side of (5.14) by

$$C\,\Big(|G[r,u](t) - G[s,v](t)| + (|\dot{u}(t)| + CK_0|\dot{r}(t)|_Y)|J(r(t),x(t)) - J(s(t),y(t))|\Big)$$

which together with the assumptions (5.5) – (5.7) yields the assertion. ∎

6 Implicit model

We now consider Problem **($\mathcal{I}$)** under the following hypothesis on the mapping g.

(G) $g : [0,T] \times X \times X \to Y$ is continuous and $g(t,u,\xi) \in \mathcal{R}$ for each $(t,u,\xi) \in [0,T] \times X \times X$. Its partial derivatives $\partial_t g, \partial_u g, \partial_\xi g$ exist and satisfy the inequalities

$$|\partial_\xi g(t,u,\xi)|_{\mathcal{L}(X,Y)} \;\le\; \gamma\,, \qquad (6.1)$$
$$|\partial_u g(t,u,\xi)|_{\mathcal{L}(X,Y)} \;\le\; \omega\,, \qquad (6.2)$$
$$|\partial_t g(t,u,\xi)|_Y \;\le\; a(t)\,, \qquad (6.3)$$
$$|\partial_\xi g(t,u,\xi) - \partial_\xi g(t,v,\eta)|_{\mathcal{L}(X,Y)} \;\le\; C_g\,(|u-v| + |\xi - \eta|)\,, \qquad (6.4)$$
$$|\partial_u g(t,u,\xi) - \partial_u g(t,v,\eta)|_{\mathcal{L}(X,Y)} \;\le\; C_u\,(|u-v| + |\xi - \eta|)\,, \qquad (6.5)$$
$$|\partial_t g(t,u,\xi) - \partial_t g(t,v,\eta)|_Y \;\le\; b(t)\,(|u-v| + |\xi - \eta|) \qquad (6.6)$$

for every $u,v,\xi,\eta \in X$ and a. e. $t \in \,]0,T[$ with given functions $a,b \in L^1(0,T)$ and given constants $\gamma, \omega, C_g, C_u > 0$ such that

$$\delta \;=\; CK_0\,\gamma \;<\; 1\,, \qquad (6.7)$$

where C, K_0 are as in Hypotheses (2.1) and **(L2)**.

Let us start our analysis with the following necessary condition.

Lemma 6.1 *Let Hypotheses* **(L1)**, **(L2)**, *and* **(G)** *hold, and let* $\xi \in W^{1,1}(0,T\,;\,X)$ *be a solution to Problem* $(\boldsymbol{\mathcal{I}})$ *with some* $u \in W^{1,1}(0,T\,;\,X)$ *and* $x_0 \in Z(g(0,u(0),u(0)-x_0))$. *Then we have*

$$|\dot\xi(t)| \le \frac{1}{1-\delta}\,((1+CK_0\omega)|\dot u(t)| + CK_0\,a(t)) \quad a.\,e. \tag{6.8}$$

Proof. Inequality (6.8) is an easy consequence of (5.11) with $r(t) = g(t,u(t),\xi(t))$. Indeed, using (6.1), (6.2) we obtain $|\dot r(t)|_Y \le a(t) + \omega|\dot u(t)| + \gamma|\dot\xi(t)|$ and (6.8) follows. ∎

We now prove the converse as the main result of this section.

Theorem 6.2 *Let Hypotheses* **(L1)**, **(L2)**, *and* **(G)** *hold. Then for every* $u \in W^{1,1}(0,T\,;\,X)$ *and every* $x_0 \in Z(g(0,u(0),u(0)-x_0))$ *there exists a unique solution* $\xi \in W^{1,1}(0,T\,;\,X)$ *to Problem* $(\boldsymbol{\mathcal{I}})$ *in the set*

$$\Omega \;=\; \left\{\eta \in W^{1,1}(0,T\,;\,X)\,;\; \begin{array}{rcl} |\dot\eta(t)| & \le & \frac{1}{1-\delta}\,((1+CK_0\omega)|\dot u(t)| + CK_0\,a(t)) \;\; a.\,e. \\ \eta(0) & = & u(0)-x_0 \end{array}\right\}.$$

Proof. Let $S : \Omega \to W^{1,1}(0,T\,;\,X)$ be the mapping which with each $\eta \in \Omega$ associates the solution ξ to Problem $(\boldsymbol{\mathcal{P}})$ with $r(t) = g(t,u(t),\eta(t))$. By (5.11) we have

$$\begin{aligned} |\dot\xi(t)| &\le |\dot u(t)| + CK_0|\dot r(t)|_Y \;\le\; (1+CK_0\omega)|\dot u(t)| + CK_0\,a(t) + \delta|\dot\eta(t)| \\ &\le \frac{1}{1-\delta}\,((1+CK_0\omega)|\dot u(t)| + CK_0\,a(t))\,, \end{aligned} \tag{6.9}$$

hence $S(\Omega) \subset \Omega$. The set Ω is convex and closed in $W^{1,1}(0,T\,;\,X)$. We now check that $S : \Omega \to \Omega$ is a contraction with respect to a suitable norm in $W^{1,1}(0,T\,;\,X)$.

Let $\eta_1,\eta_2 \in \Omega$ be given. By Proposition 5.3, the functions $\xi_i = S(\eta_i)$ for $i = 1,2$ satisfy almost everywhere the inequality

$$\begin{aligned} |\dot\xi_1(t) - \dot\xi_2(t)| + \dot\beta(t) \;&\le\; \delta|\dot\eta_1(t) - \dot\eta_2(t)| \\ &\quad + C_\delta(|\dot u(t)| + a(t) + b(t))(|\eta_1(t)-\eta_2(t)| + |\xi_1(t)-\xi_2(t)|) \end{aligned} \tag{6.10}$$

with $\beta(t) = C|B[g(\cdot,u,\eta_1),u](t) - B[g(\cdot,u,\eta_2),u](t)| \ge 0$, $\beta(0) = 0$, and with a constant $C_\delta > 0$ independent of η_1,η_2.

Let now $\varepsilon > 0$ be chosen so small that

$$\frac{\delta + \varepsilon C_\delta}{1 - \varepsilon C_\delta} \;=\; \delta^* \;<\; 1\,, \tag{6.11}$$

and let us define an auxiliary function

$$w(t) \;=\; e^{-\frac{1}{\varepsilon}\int_0^t (|\dot u(\tau)| + a(\tau) + b(\tau))\,d\tau} \qquad \text{for} \quad t \in [0,T]\,. \tag{6.12}$$

We have $w(t) > 0$ for every $t \in [0,T]$ and $\dot w(t) \le 0$ a. e. We test the inequality (6.10) by $w(t)$ and integrate over $[0,T]$. Taking into account the relations

$$\begin{aligned}
\int_0^T \dot\beta(t)\, w(t)\, dt &= [\beta(t)\, w(t)]_0^T - \int_0^T \beta(t)\, \dot w(t)\, dt \ \ge\ 0\,, \\
\int_0^T w(t)\, (|\dot u(t)| + a(t) + b(t))(|\eta_1(t) - \eta_2(t)| &+ |\xi_1(t) - \xi_2(t)|)\, dt \\
&\le -\varepsilon \int_0^T \dot w(t) \int_0^t (|\dot\eta_1(\tau) - \dot\eta_2(\tau)| + |\dot\xi_1(\tau) - \dot\xi_2(\tau)|)\, d\tau\, dt \\
&= -\varepsilon \left[w(t) \int_0^t (|\dot\eta_1(\tau) - \dot\eta_2(\tau)| + |\dot\xi_1(\tau) - \dot\xi_2(\tau)|)\, d\tau\right]_0^T \\
&\quad + \varepsilon \int_0^T w(t)\, (|\dot\eta_1(t) - \dot\eta_2(t)| + |\dot\xi_1(t) - \dot\xi_2(t)|)\, dt \\
&\le \varepsilon \int_0^T w(t)\, (|\dot\eta_1(t) - \dot\eta_2(t)| + |\dot\xi_1(t) - \dot\xi_2(t)|)\, dt\,,
\end{aligned}$$

we obtain from (6.10) that

$$\int_0^T w(t)\, |\dot\xi_1(t) - \dot\xi_2(t)|\, dt \ \le\ (\delta + \varepsilon C_\delta) \int_0^T w(t)\, |\dot\eta_1(t) - \dot\eta_2(t)|\, dt + \varepsilon C_\delta \int_0^T w(t)\, |\dot\xi_1(t) - \dot\xi_2(t)|\, dt$$

hence

$$\int_0^T w(t)\, |\dot\xi_1(t) - \dot\xi_2(t)|\, dt \ \le\ \delta^* \int_0^T w(t)\, |\dot\eta_1(t) - \dot\eta_2(t)|\, dt\,. \tag{6.13}$$

We thus checked that S is a contraction on Ω with respect to the weighted norm

$$\|\eta\| = |\eta(0)| + \int_0^T w(t)\, |\dot\eta(t)|\, dt\,,$$

hence S admits a unique fixed point ξ which is a solution of $(\boldsymbol{\mathcal{I}})$. ■

7 Lipschitz continuity

In this section we prove local Lipschitz continuity estimates for both Problems $(\boldsymbol{\mathcal{P}})$ and $(\boldsymbol{\mathcal{I}})$ as a combination of Proposition 5.3 and a Gronwall-type argument.

Theorem 7.1 *Let the assumptions of Proposition 5.3 be fulfilled. Then there exist positive constants C_0, C_1 such that for every $R > 0$, every $(r,u), (s,v) \in W^{1,1}(0,T\,;\,\mathcal{R}) \times W^{1,1}(0,T\,;\,X)$ with $\int_0^T (|\dot u| + |\dot r|_Y) dt \le R$, $\int_0^T (|\dot v| + |\dot s|_Y) dt \le R$, and every $x_0 \in Z(r(0))$, $y_0 \in Z(s(0))$, the respective solutions $\xi, \eta \in W^{1,1}(0,T\,;\,X)$ of problem $(\boldsymbol{\mathcal{P}})$ satisfy the inequality*

$$\int_0^T |\dot\xi - \dot\eta|\, dt \ \le\ C_1 e^{C_0 R} \left(|x_0 - y_0| + |r(0) - s(0)|_Y + \int_0^T (|\dot u - \dot v| + |\dot r - \dot s|_Y) dt \right)\,. \tag{7.1}$$

Proof. By Proposition 5.3, there exists a constant $C_0 > 0$ such that

$$\begin{aligned}
|\dot x(t) - \dot y(t)| + \dot\beta(t) \ &\le\ C_0\Big(|\dot u(t) - \dot v(t)| + |\dot r(t) - \dot s(t)|_Y \\
&\quad + (|\dot u(t)| + |\dot r(t)|_Y)(|x(t) - y(t)| + |r(t) - s(t)|_Y)\Big)
\end{aligned} \tag{7.2}$$

with $\beta(t) = C\,|B[r,u](t) - B[s,v](t)|$. We argue similarly as in the proof of Theorem 6.2 and test (7.2) by the function

$$w_1(t) \;=\; e^{-C_0 \int_0^t (|\dot u| + |\dot r|_Y)\, d\tau}\,.$$

This yields

$$\begin{aligned}\frac{d}{dt}\left(w_1(t)\int_0^t |\dot x - \dot y|\, d\tau\right) + w_1(t)\,\dot\beta(t) \;&\le\; C_0\, w_1(t)\Big(|\dot u(t) - \dot v(t)| + |\dot r(t) - \dot s(t)|_Y\Big) \qquad (7.3)\\ &\quad -\dot w_1(t)\left(|x_0 - y_0| + |r(0) - s(0)|_Y + \int_0^t |\dot r - \dot s|_Y\, d\tau\right).\end{aligned}$$

Note that

$$\begin{aligned}\int_0^T w_1(t)\,\dot\beta(t)\, dt \;&=\; [w_1(t)\,\beta(t)]_0^T - \int_0^T \dot w_1(t)\,\beta(t)\, dt \;\ge\; -w_1(0)\,\beta(0) \qquad (7.4)\\ &\ge\; -\frac{C}{2}|M^2(r(0),x_0) - M^2(s(0),y_0)| \;\ge\; -\left(CK_0|r(0)-s(0)|_Y + \frac{C}{c}|x_0 - y_0|\right).\end{aligned}$$

On the other hand, integrating (7.3) from 0 to T and using the fact that for every $t \in [0,T]$ we have $1 \ge w_1(t) \ge w_1(T) \ge e^{-C_0 R}$ we obtain

$$\begin{aligned}e^{-C_0 R}\int_0^T |\dot x - \dot y|\, dt \;&\le\; -\int_0^T w_1(t)\,\dot\beta(t)\, dt + |x_0 - y_0| + |r(0) - s(0)|_Y \qquad (7.5)\\ &\quad +\, (C_0+1)\int_0^T (|\dot u - \dot v| + |\dot r - \dot s|_Y)\, dt\,,\end{aligned}$$

and the assertion follows from (7.4), (7.5). ■

Theorem 7.2 *Let the assumptions of Theorem 6.2 be fulfilled. Then there exist positive constants C_2, C_3 such that for every $R > 0$, every $u, v \in W^{1,1}(0,T\,;\,X)$ with $\int_0^T |\dot u|\, dt \le R$, $\int_0^T |\dot v|\, dt \le R$, and every $x_0 \in Z(g(0,u(0),u(0)-x_0))$, $y_0 \in Z(g(0,v(0),v(0)-y_0))$, the respective solutions $\xi, \eta \in W^{1,1}(0,T\,;\,X)$ of problem $(\boldsymbol{\mathcal{I}})$ satisfy the inequality*

$$\int_0^T |\dot\xi - \dot\eta|\, dt \;\le\; C_3 e^{C_2 R}\left(|x_0 - y_0| + |u(0) - v(0)| + \int_0^T |\dot u - \dot v|\, dt\right). \qquad (7.6)$$

Proof. We use Lemma 6.1 and Proposition 5.3 with $r(t) = g(t,u(t),\xi(t))$, $s(t) = g(t,v(t),\eta(t))$, and find a constant $C^* > 0$ such that

$$\begin{aligned}(1-\delta)|\dot\xi(t) - \dot\eta(t)| + \dot\beta(t) \;&\le\; C^*\Big(|\dot u(t) - \dot v(t)| \qquad (7.7)\\ &\quad +\, (|\dot u(t)| + a(t) + b(t))(|u(t) - v(t)| + |\xi(t) - \eta(t)|)\Big)\end{aligned}$$

with $\beta(t) = C\,|B[g(\cdot,u,\xi),u](t) - B[g(\cdot,v,\eta),v](t)|$. Repeating the procedure from the proof of Theorem 7.1 with

$$C_2 \;=\; \frac{C^*}{1-\delta}\,, \quad w_2(t) = e^{-C_2\int_0^t (|\dot u(\tau)| + a(\tau) + b(\tau))\, d\tau} \qquad (7.8)$$

we easily obtain the assertion. ■

8 Nonuniqueness

Let $\psi : [0,1] \to [0,1]$ be an increasing concave function with $\psi(0) = 0$. Then the initial value problem

$$\dot{w} = \delta\psi(w)\,, \quad w(0) = 0\,, \tag{8.1}$$

has a positive solution for any $\delta > 0$ in addition to the trivial one, if and only if

$$\int_0^1 \frac{1}{\psi(\varepsilon)}\, d\varepsilon < \infty\,. \tag{8.2}$$

Let us assume that (8.2) holds. Then

$$\lim_{\varepsilon\to 0+} \psi'(\varepsilon) = \lim_{\varepsilon\to 0+} \frac{\psi(\varepsilon)}{\varepsilon} \;=\; \infty\,. \tag{8.3}$$

Moreover assume that there exists $K > 0$ such that

$$\psi'(\varepsilon) \le \frac{K}{\sqrt{\varepsilon}} \qquad \text{for a. e. } \varepsilon \in]0,1[\ . \tag{8.4}$$

Then

$$\psi(\varepsilon) \;\le\; 2K\sqrt{\varepsilon} \qquad \forall \varepsilon \in [0,1]\,. \tag{8.5}$$

A typical example is $\psi(\varepsilon) = \varepsilon^\beta$, $1/2 \le \beta < 1$.

We denote

$$\theta \;=\; \frac{1}{1+8K^2} \tag{8.6}$$

with K from (8.4), and consider for $\alpha \ge 0$ and $0 \le \eta \le 1-\theta$ the system of convex sets

$$V(\eta,\alpha) = B_{1-\theta}(0) \cap H(\eta,\alpha)\,, \tag{8.7}$$

in $X = \mathbb{R}^2$, where $H(\eta,\alpha)$ is the half-plane

$$H(\eta,\alpha) = \{z \in \mathbb{R}^2;\ \langle z,\nu_\alpha\rangle \le 1-\theta-\eta\}\,, \quad \nu_\alpha = \begin{pmatrix} 1 \\ -\alpha \end{pmatrix}\,. \tag{8.8}$$

We further define two one-parametric families of convex sets

$$\tilde{Z}(\varepsilon) \;=\; V(\varepsilon + \theta(\sqrt{1+\psi^2(\varepsilon)} - 1), \psi(\varepsilon))\,, \tag{8.9}$$

$$Z(\varepsilon) \;=\; \tilde{Z}(\varepsilon) + B_\theta(0)\,, \tag{8.10}$$

for $\varepsilon \in [0,\varepsilon_0]$, where $\varepsilon_0 \in]0,1]$ is chosen in such a way that for every $\varepsilon \in]0,\varepsilon_0]$ we have

$$\varepsilon + \theta\sqrt{1+\psi^2(\varepsilon)} \;<\; 1\,, \tag{8.11}$$

$$\theta^2\,(1+\psi^2(\varepsilon))\,\frac{\psi^2(\varepsilon)}{\varepsilon} + 2\theta\left((1-\varepsilon)\,\frac{\psi^2(\varepsilon)}{\varepsilon} + \frac{1}{\varepsilon}\left(1 - \frac{1-\varepsilon}{\sqrt{1+\psi^2(\varepsilon)}}\right)\right) \;\le\; 2-\varepsilon\,. \tag{8.12}$$

Let $\Lambda(\theta,\varepsilon)$ denote the left-hand side of (8.12). To see that condition (8.12) is meaningful, it suffices to use (8.5) which entails that

$$\limsup_{\varepsilon\to 0+} \Lambda(\theta,\varepsilon) \ \leq\ 4K^2\theta^2 \ +\ 2\theta\,(1+6K^2) \ =\ 2\left(1-\left(\frac{4K^2}{1+8K^2}\right)^2\right) \ <\ 2\,. \tag{8.13}$$

The set $\tilde{Z}(\varepsilon)$ can also be characterized as

$$z\in\tilde{Z}(\varepsilon) \quad\Leftrightarrow\quad |z|\leq 1-\theta\,, \quad \left\langle z,\nu_{\psi(\varepsilon)}\right\rangle \ \leq\ 1-\varepsilon-\theta\sqrt{1+\psi^2(\varepsilon)}\,, \tag{8.14}$$

see Figure 1. As the next step, we check that the segment

$$S(\varepsilon) \ =\ \left\{z=\begin{pmatrix}x\\ y\end{pmatrix}\ ;\ x-\psi(\varepsilon)y \ =\ 1-\varepsilon\,,\ |y|\leq\theta\,\psi(\varepsilon)\right\} \tag{8.15}$$

satisfies

$$S(\varepsilon)\ \subset\ \partial Z(\varepsilon) \qquad \forall\varepsilon\in[0,\varepsilon_0]\,. \tag{8.16}$$

The limit case $\varepsilon=0$ is easy: we have $Z(0)=B_1(0)$, and $S(0)$ contains only one element $e_1=\begin{pmatrix}1\\ 0\end{pmatrix}$. For $\varepsilon>0$ and $z\in S(\varepsilon)$ we define

$$z_\theta \ =\ z-\frac{\theta}{\sqrt{1+\psi^2(\varepsilon)}}\,\nu_{\psi(\varepsilon)}\,.$$

Then

$$|z_\theta - z| \ =\ \theta\,, \tag{8.17}$$

$$\left\langle z_\theta,\nu_{\psi(\varepsilon)}\right\rangle \ =\ 1-\varepsilon-\theta\sqrt{1+\psi^2(\varepsilon)}\,, \tag{8.18}$$

$$|z_\theta|^2 = |z|^2\ +\ \theta^2\ -\ 2\theta\frac{1-\varepsilon}{\sqrt{1+\psi^2(\varepsilon)}}\,, \tag{8.19}$$

where $|z|^2\leq(1-\varepsilon+\theta\psi^2(\varepsilon))^2+\theta^2\psi^2(\varepsilon)$ by (8.15). Developing the computation and using (8.12) we obtain from (8.19) that

$$\begin{aligned}
|z_\theta|^2 \ &\leq\ (1-\varepsilon)^2+\theta^2\,(1+\psi^2(\varepsilon)+\psi^4(\varepsilon))+2\theta\,(1-\varepsilon)\,\psi^2(\varepsilon)-2\theta\,\frac{1-\varepsilon}{\sqrt{1+\psi^2(\varepsilon)}}\\
&=\ (1-\theta)^2+\theta^2\,(1+\psi^2(\varepsilon))\,\psi^2(\varepsilon)+2\theta\left((1-\varepsilon)\,\psi^2(\varepsilon)+\left(1-\frac{1-\varepsilon}{\sqrt{1+\psi^2(\varepsilon)}}\right)\right)+\varepsilon^2-2\varepsilon\\
&\leq\ (1-\theta)^2,
\end{aligned}$$

hence $z_\theta\in\tilde{Z}(\varepsilon)$ and $z\in Z(\varepsilon)$.

To see that $z\in\partial Z(\varepsilon)$, we notice using (8.14) that for arbitrary $\tilde{z}\in\tilde{Z}(\varepsilon)$ we have

$$|z-\tilde{z}|\ \geq\ \frac{1}{\sqrt{1+\psi^2(\varepsilon)}}\left\langle z-\tilde{z},\nu_{\psi(\varepsilon)}\right\rangle\ \geq\ \theta\,.$$

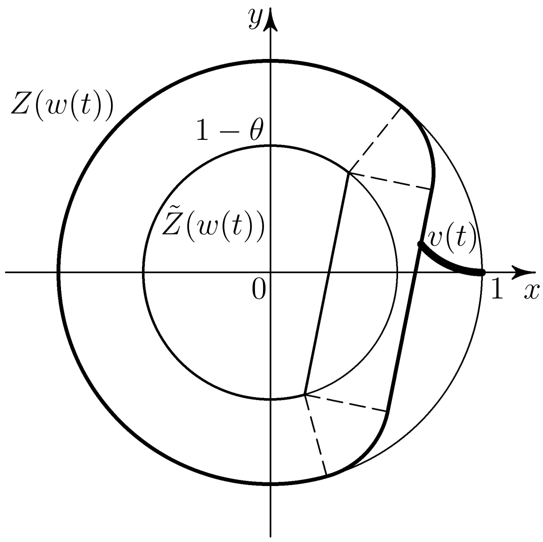

Figure 1: Trajectory of the nontrivial solution $v(t)$.

In other words, $\operatorname{dist}(z, \tilde{Z}(\varepsilon)) = \theta$, and (8.16) is verified.

We now construct the example of nonuniqueness for the evolution quasivariational inequality (1.4). Let $w : [0, t_0] \to \mathbb{R}$ be the positive solution to the initial value problem (8.1) for a fixed $\delta > 0$, where we choose t_0 sufficiently small such that

$$w(t_0) \le \varepsilon_0\,, \quad \frac{w(t_0)}{\psi(w(t_0))} \le \delta\,\theta\,, \quad \delta\psi(w(t_0)) < 1\,. \tag{8.20}$$

We define $v(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}$ with

$$v_1(t) = 1 - w(t) + \frac{1}{\delta} w(t)\psi(w(t))\,, \quad v_2(t) = \frac{1}{\delta} w(t)\,, \tag{8.21}$$

and set

$$\Gamma(v) = Z(\delta v_2)\,. \tag{8.22}$$

Then

$$\frac{v_2(t)}{\psi(w(t))} \;\le\; \frac{v_2(t_0)}{\psi(w(t_0))} \;<\; \theta\,, \tag{8.23}$$

hence $v(t) \in S(w(t)) \subset \partial Z(w(t)) = \partial\Gamma(v(t))$ for all $t \in [0, t_0]$, see Figure 1. We set

$$f(t) = (f_1(t), 0)\,, \quad f_1(t) = 1 + \dot{v}_1(t)\,, \tag{8.24}$$

then $f(t) - \dot{v}(t) = \nu_{\psi(w(t))}(v(t)) \in N_{\Gamma(v(t))}(v(t))$, and $f_1(t) > 0$ for $t \le t_0$. Therefore, both $v(t)$ as defined and $\hat{v}(t) \equiv e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ are solutions of (1.4). Thus, there is nonuniqueness, despite the fact that the outward normal vectors $n_{Z(\varepsilon)}(z)$ are Lipschitz continuous in z for every ε with a Lipschitz constant independent of ε as a consequence of Lemma 3.4. Moreover, the orthogonal projections $Q_{\tilde{Z}(\varepsilon)}$ of X onto $\tilde{Z}(\varepsilon)$ (and therefore also $n_{Z(\varepsilon)}$ by virtue of the argument in the proof of Lemma 3.4) depend Lipschitz-continuously on ε in each interval $[\varepsilon_1, \varepsilon_0]$

with $0 < \varepsilon_1 < \varepsilon_0$. For $\varepsilon = 0$ there is indeed no problem in any direction different from e_1, as the boundary of $Z(\varepsilon)$ does not move here for ε small. However, the dependence of $J_{Z(\varepsilon)}(z)$ on ε is not globally Lipschitz and the Lipschitz constant behaves like $\psi(\varepsilon)/\varepsilon$ for $z = e_1$ near $\varepsilon = 0$. Indeed, for fixed $\varepsilon > 0$ and $z \in S(\varepsilon)$ we have $M_{Z(\varepsilon)}(z) = \left\langle z, \nu_{\psi(\varepsilon)} \right\rangle /(1-\varepsilon)$. We easily find $q(\varepsilon) > 0$ which guarantees that for every $y \in B_{q(\varepsilon)}(0)$ we can find $p(\varepsilon, y) > 0$ such that $p(\varepsilon, y)(e_1 + y) \in S(\varepsilon)$. For all such y we thus have $M_{Z(\varepsilon)}(e_1 + y) = \left\langle e_1 + y, \nu_{\psi(\varepsilon)} \right\rangle /(1-\varepsilon)$, hence $J_{Z(\varepsilon)}(e_1) = \nu_{\psi(\varepsilon)}/(1-\varepsilon)^2$. Obviously, $J_{Z(0)}(x) = x$ for every $x \in X$, and we conclude that

$$|J_{Z(\varepsilon)}(e_1) - J_{Z(0)}(e_1)| \;=\; \frac{\sqrt{\psi^2(\varepsilon) + \varepsilon^2(2-\varepsilon)^2}}{(1-\varepsilon)^2}\,, \tag{8.25}$$

where the right-hand side is of the order of ε^β if for example $\psi(\varepsilon) = \varepsilon^\beta$, $1/2 \le \beta < 1$. On the other hand, we will see that $Z(\varepsilon)$ does depend Lipschitz-continuously on ε in the sense of the Hausdorff distance. This implies in turn that the map $v \to \Gamma(v)$ is Lipschitz with an arbitrarily small constant similarly as in the example of [1] provided δ is chosen sufficiently small (in fact, to make Figure 1 more transparent, we have chosen a trajectory of the solution $v(t)$ corresponding to a 'very large' δ). We conclude the paper by establishing this Lipschitz continuity result as a separate lemma.

Lemma 8.1 *There exists a constant $R > 0$ such that for every $0 \le \varepsilon_2 < \varepsilon_1 \le \varepsilon_0$ we have*

(i) $Z(\varepsilon_1) \;\subset\; Z(\varepsilon_2)$,

(ii) $d_H(Z(\varepsilon_1), Z(\varepsilon_2)) \;\le\; R\,(\varepsilon_1 - \varepsilon_2)$.

It is interesting to compare formula (8.25) with Lemmas 3.2, 3.3, and 8.1. We see that the hypothesis (8.5) is sharp in the sense that if one sets $\psi(\varepsilon) = \varepsilon^\beta$ with $\beta < 1/2$ in the example above, one cannot expect that $d_H(Z(\varepsilon), Z(0))$ remains Lipschitz continuous in ε.

Proof. To prove (i), let us consider an arbitrary $z = \begin{pmatrix} x \\ y \end{pmatrix} \in \tilde{Z}(\varepsilon_1)$ and assume that $z \notin \tilde{Z}(\varepsilon_2)$. Then

$$x - \psi(\varepsilon_2) y \;>\; 1 - \varepsilon_2 - \theta\sqrt{1 + \psi^2(\varepsilon_2)}\,, \tag{8.26}$$

$$x - \psi(\varepsilon_1) y \;\le\; 1 - \varepsilon_1 - \theta\sqrt{1 + \psi^2(\varepsilon_1)}\,, \tag{8.27}$$

$$x \;\le\; 1 - \theta\,. \tag{8.28}$$

For $\alpha \in [0, \psi(\varepsilon_0)]$ we define an auxiliary function

$$\Psi(\alpha) \;=\; \psi^{-1}(\alpha) + \theta\,(\sqrt{1 + \alpha^2} - 1)$$

and set $\alpha_i = \psi(\varepsilon_i)$, $i = 0, 1, 2$. From (8.26)–(8.28) it follows that

$$\frac{\Psi(\alpha_2)}{\alpha_2} \;>\; y \;>\; \frac{\Psi(\alpha_1) - \Psi(\alpha_2)}{\alpha_1 - \alpha_2}\,. \tag{8.29}$$

Since Ψ is convex, increasing, and $\Psi(0) = 0$, (8.29) is contradictory and we conclude that $\tilde{Z}(\varepsilon_1) \subset \tilde{Z}(\varepsilon_2)$, hence (i) holds.

It remains to prove the Lipschitz estimate in (ii). Let $z \in \tilde{Z}(\varepsilon_2) \setminus \tilde{Z}_(\varepsilon_1)$ be arbitrary. With the above notation we have

$$\langle z, \nu_{\alpha_2} \rangle \;\le\; 1 - \theta - \Psi(\alpha_2)\,, \tag{8.30}$$

$$\langle z, \nu_{\alpha_1} \rangle \;>\; 1 - \theta - \Psi(\alpha_1)\,. \tag{8.31}$$

We find $\alpha_* \in [\alpha_2, \alpha_1[$ such that

$$\langle z, \nu_{\alpha_*} \rangle \;=\; 1 - \theta - \Psi(\alpha_*)\,, \tag{8.32}$$

and put $\varepsilon_* = \psi^{-1}(\alpha_*)$, $z_1 = z - \sigma\nu_{\alpha_1}$, where $\sigma > 0$ is chosen in such a way that

$$\langle z_1, \nu_{\alpha_1} \rangle \;=\; 1 - \theta - \Psi(\alpha_1)\,, \tag{8.33}$$

that is,

$$\sigma \;=\; \frac{1}{1+\alpha_1^2}\left(\langle z, \nu_{\alpha_1}\rangle - 1 + \theta + \Psi(\alpha_1)\right).$$

We have

$$|z_1|^2 \;=\; |z|^2 - \sigma^2\,|\nu_{\alpha_1}|^2 - 2\sigma\,(1 - \theta - \Psi(\alpha_1)) \;<\; |z|^2 \;\le\; (1-\theta)^2\,,$$

hence $z_1 \in \tilde{Z}(\varepsilon_1)$. Our goal now is to find $R > 0$ independent of $\varepsilon_1, \varepsilon_2$ such that

$$|z - z_1| \;\le\; R\,(\varepsilon_1 - \varepsilon_*)\,, \tag{8.34}$$

and it will suffice to pass from $\tilde{Z}(\varepsilon_i)$ to $Z(\varepsilon_i)$, $i = 1, 2$, to complete the proof. Inequality (8.34) will follow from a series of estimates, where $C_1, C_2, \dots$ will denote arbitrary positive constants depending only on K. First of all, we have

$$\begin{aligned} |z - (1-\theta)\nu_{\alpha_*}|^2 \;&\le\; (1 + |\nu_{\alpha_*}|^2)\,(1-\theta)^2 - 2(1-\theta)\,(1 - \theta - \Psi(\alpha_*)) \\ &\le\; (1-\theta)\left(\alpha_*^2 + 2\psi^{-1}(\alpha_*)\right) \;\le\; C_1\,\varepsilon_*\,. \end{aligned} \tag{8.35}$$

On the other hand, we have

$$\begin{aligned} |z - z_1| \;&=\; \sigma\,|\nu_{\alpha_1}| \;\le\; \sigma\,|\nu_{\alpha_1}|^2 \;=\; \langle z - z_1, \nu_{\alpha_1}\rangle \\ &=\; \langle z - (1-\theta)\nu_{\alpha_*}, \nu_{\alpha_1} - \nu_{\alpha_*}\rangle \;+\; (1-\theta)\,\langle \nu_{\alpha_*}, \nu_{\alpha_1} - \nu_{\alpha_*}\rangle \;+\; \Psi(\alpha_1) - \Psi(\alpha_*) \\ &\le\; \sqrt{C_1\,\varepsilon_*}\,|\nu_{\alpha_1} - \nu_{\alpha_*}| \;+\; \frac{1-\theta}{2}\,\left(|\nu_{\alpha_1}|^2 - |\nu_{\alpha_*}|^2\right) \;+\; \Psi(\alpha_1) - \Psi(\alpha_*) \\ &\le\; C_2\,\left(\sqrt{\varepsilon_*}(\alpha_1 - \alpha_*) + (\varepsilon_1 - \varepsilon_*) + (\alpha_1^2 - \alpha_*^2)\right)\,, \end{aligned} \tag{8.36}$$

where, by (8.4)–(8.5), we have $\alpha_1 - \alpha_* \le 2K(\sqrt{\varepsilon_1} - \sqrt{\varepsilon_*})$ and

$$\alpha_1^2 - \alpha_*^2 \;=\; 2\int_{\varepsilon_*}^{\varepsilon_1} \psi(\varepsilon)\,\psi'(\varepsilon)\,d\varepsilon \;\le\; 4K^2\,(\varepsilon_1 - \varepsilon_*)\,,$$

and (8.34) immediately follows. ■

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