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Vector hysteresis models

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Abstract. Following Krasnosel'skii & Pokrovskii 1983 we express the constitutive law for the Prandtl-Reuss elastoplastic model in terms of a hysteresis operator and we introduce the vector Ishlinskii model. We investigate some properties (continuity, energy inequalities, dependence on spatial variables) of these operators.

Introduction

In this paper we will investigate the Prandtl-Reuss elastoplastic model (cf. Duvaut & Lions 1972, Nečas & Hlaváček 1981). We denote by S^n the vector space of symmetric tensors $n \times n$ (we have $\dim S^n = \frac{1}{2}n(n+1)$) and we fix a closed convex (usually unbounded) set $Z \subset S^n$. Its interior represents the elasticity domain and its boundary is the surface of plasticity. The strain tensor $\varepsilon = \{\varepsilon_{ij}\}$ is decomposed into two parts $\varepsilon_{ij} = \varepsilon^e_{ij} + \varepsilon^p_{ij}$. The elastic part ε^e satisfies the linear Hooke's law $\varepsilon^e_{ij} = A_{ijk\ell}\sigma_{k\ell}$, where $A = \{A_{ijk\ell}\}$ is a given constant symmetric positive definite matrix with respect to the scalar product $< \xi, \eta >= \xi_{ij}\eta_{ij}$ in S^n and $\sigma = \{\sigma_{ij}\}$ is the stress tensor. The values of σ cannot leave the domain Z and the plastic strain ε^p satisfies the inequality $< \dot{\varepsilon}^p, \sigma - \varphi > \geq 0 \quad \forall \varphi \in Z$, where the dot denotes the time derivative.

Assuming ε to be known we find σ as the solution of the inequality

$$< A\dot{\sigma} - \dot{\varepsilon},\ \sigma - \varphi > \geq 0 \qquad \forall \varphi \in Z$$

with a given initial condition $\sigma(0) \in Z$.

Putting $u = A^{-1/2}\varepsilon,\ x = A^{1/2}\sigma,\ \tilde{Z} = A^{1/2}(Z)$ we see that $x(t) \in \tilde{Z}$ satisfies the inequality

$$< \dot{x} - \dot{u}, x - \varphi > \leq 0 \quad \forall \varphi \in \tilde{Z}.$$

The correspondence $u \to x$ has been investigated in Krasnosel'skii & Pokrovskii 1983, where the notion of a hysteresis operator $x = f(u)$ is introduced. The aim of this paper is to derive new properties of the operator f.

Using this notation we can express the Prandtl-Reuss constitutive law in the form

$$\sigma = A^{-1/2} f(A^{-1/2}\varepsilon).$$

Let us consider an elastoplastic body occupying a domain $\Omega \subset R^n$ and satisfying the Ishlinskii constitutive law

$$\sigma = A^{-1/2} \int_0^\infty f(A^{-1/2} \frac{1}{h} \varepsilon)\, h\, d\mu(h) = A^{-1/2} F(A^{-1/2}\varepsilon),$$

(cf. (5.3) in Section 5).

The strain tensor is given by the formula

$$\varepsilon_{ij}(x,t) = \frac{1}{2}\Big(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\Big),$$

where $u = \{u^i\}$ is the displacement vector. The dynamical behavior of the body is determined by the equation of motion

$$\frac{\partial^2 u^i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} + h^i,$$

where $h = \{h^i\}$ is a given forcing term (which may depend on u, ε and σ), and by appropriate boundary and initial conditions.

Let us denote $(Du)_{ij} = \frac{1}{2}\big(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\big)$, $(D^*\sigma)^i = \frac{\partial \sigma_{ij}}{\partial x_j}$. The problem consists in solving the equation

$$\frac{\partial^2 u}{\partial t^2} - D^* A^{-1/2} F(A^{-1/2} Du) = h \tag{0.1}$$

in $\Omega \times (0,T)$. This is a natural way of transforming evolution differential inequalities into equations with hysteresis (cf. also Krejčí 1989). The equation (0.1) under suitable boundary and initial conditions can be solved by standard methods (e.g Faedo-Galerkin). Let us note that we can derive a priori estimates for the second derivatives of an approximate solution u_k (differentiating (0.1) with respect to t, testing with $\frac{\partial^2 u_k}{\partial t^2}$ and using (5.6) (ii) and then testing with $D^*D\frac{\partial u_k}{\partial t}$ and using (5.8), cf. Section 5). We can next use the Minty-Browder argument based on the inequality (2.6) in Section 2 (cf. Krejčí 1988) and the continuity of F for passing to the limit. A similar existence result has been proved (by different methods) by Visintin 1987.

1 Geometry of convex sets

We denote by X^N an N-dimensional real vector space with a scalar product $< .,. >$, $|x| =< x,x >^{1/2}$ for $x \in X^N$. We assume that $Z \subset X^N$ is a closed convex set (bounded or unbounded) containing 0 in its interior. We fix a number $m > 0$ such that

$$(1.1) \qquad |x| \leq m \Rightarrow x \in Z.$$

We denote by Q the orthogonal projection of X^N onto Z such that $|x - Qx| = \min\{|x - \xi|,\ \xi \in Z\}, \quad Px = x - Qx$.

Let us recall the following properties of P, Q.

(1.2) **Lemma** *For every $x, y \in X^N$ we have*

(i) $< Px,\, Qx - \varphi > \geq 0 \qquad \forall \varphi \in Z,$

(ii) $< Px - Py,\, Qx - Qy > \geq 0,$

(iii) $< Px, x > \geq m|Px| + |Px|^2,$ *where m is given by (1.1),*

(iv) $Q(x + \alpha Px) = Qx$ *for every* $\alpha \geq -1.$

Proof (i) Let us assume $< Px, Qx - \varphi > < 0$ for some $\varphi \in Z$ and $x \in X^N$. For $\alpha \in [0,1]$ we put $\xi_\alpha = \alpha\varphi + (1-\alpha)\, Qx \in Z$. We have $|x - \xi_\alpha|^2 = |x - Qx|^2 + \alpha(< Px, Qx - \varphi > +\alpha|Qx - \varphi|^2) < |x - Qx|^2$ for α sufficiently small, and this contradicts the definition of Q. Part (ii) follows immediately from (i) and (iii) is obvious for $x \in Z$. For $x \notin Z$ we find $\alpha > 0$ such that $|\alpha Px| = m$ and put $\varphi = \alpha Px \in Z$. We have $< \varphi, Qx - \varphi > \geq 0$, hence $< Px, x > = \frac{1}{\alpha} < \varphi, Qx + Px > \geq \frac{1}{\alpha}|\varphi|^2 + |Px|^2 = m|Px| + |Px|^2$.
(iv) For an arbitrary $\varphi \in Z$ we have $|x + \alpha Px - \varphi|^2 = |Qx - \varphi|^2 + (1+\alpha)^2|Px|^2 + (1+\alpha) < Px, Qx - \varphi >$, hence the minimum of $|x + \alpha Px - \varphi|$ is attained for $\varphi = Qx$.

Let us show some specific properties of unbounded domains.

(1.3) **Lemma** *Let Z be unbounded. Then there exists a non-empty convex closed cone $C \subset Z$ with vertex 0 such that*
(i) for every $x \in C$ and $y \in X^N$, $|y| \leq m$ we have $x + y \in Z$,
(ii) the function $K(r) = \max\{\text{dist}(z, C), z \in Z, |z| = r\}$ is nondecreasing for r sufficiently large and $\lim\limits_{r\to\infty} K(r)/r = 0$.

Proof Let us denote by M the set of all points $x \in Z$ such that $|x| = 1$ and $\alpha x \in Z$ for every $\alpha \geq 0$. Let $\{z_n\} \subset Z$ be an arbitrary unbounded sequence. We see immediately that every limit point of $\frac{z_n}{|z_n|}$ belongs to M, hence M is non-void. Put $C =$

$\{\alpha x; x \in M, \alpha \geq 0\}$. The convexity and closedness of C follow from the convexity and closedness of Z.

Let us choose $x \in C$ and $|y| \leq m$. For an arbitrary sequence $\{y_n\}$, $|y_n| < m$, $y_n \to y$ we choose positive numbers $r_n > \frac{m}{m-|y_n|} > 1$. We have $|\frac{r_n}{r_n-1} y_n| < m$, hence $x + y_n = \frac{1}{r_n}(r_n x) + (1 - \frac{1}{r_n})(\frac{r_n}{r_n-1} y_n) \in Z$ and (i) follows easily.

Let us denote by Q_c the projection of X^N onto C, $P_c x = x - Q_c x$. Repeating the proof of (i) and using (1.2) (iv) we can easily verify that the set $N_c = \{x \in Z; Q_c x = 0\}$ is bounded. Put $r_c = \max\{|x|; x \in N_c\}$.

We next prove that for every $r_2 > r_1 > r_c$ and for every $z_1 \in Z$ such that $|z_1| = r_1$ there exists $z_2 \in Z, |z_2| = r_2$ such that $\text{dist}(z_2, C) \geq \text{dist}(z_1, C)$.

Put $x_1 = Q_c z_1$. We have $\text{dist}(z_1, C) = |z_1 - x_1| = |P_c z_1|, < z_1, x_1 > \geq 0$. Put $z_2 = z_1 + \beta x_1$, where $\beta > 0$ is such that $|z_2| = r_2$. We have $(1 - \frac{1}{n}) z_1 + \frac{1}{n}(n\beta x_1) \in Z$ for every $n \geq 1$, hence $z_2 \in Z$. We further put $x_2 = Q_c z_2$. Let us assume $|z_2 - x_2| < |z_1 - x_1|$. For $\alpha = -\frac{\beta}{1+\beta} > -1$ we obtain

$$|P_c(z_1 + \alpha P_c z_1)| \leq |z_1 + \alpha P_c z_1 - (1+\alpha) x_2| = (1+\alpha)|z_2 - x_2| < (1+\alpha)|P_c z_1|,$$

which contradicts (1.2) (iv).

Thus we have proved that the function $K(r)$ is nondecreasing for $r \geq r_c$. Let us assume $\limsup_{r \to \infty} K(r)/r > 0$. Then there exists $\varepsilon > 0$ and a sequence $r_n \nearrow \infty$ such that $K(r_n) \geq \varepsilon r_n$. We find $z_n \in Z$, $|z_n| = r_n$ such that $\text{dist}(z_n, C) \geq \varepsilon |z_n|$.

The sequence $\frac{z_n}{|z_n|}$ has a limit point $x \in M$ and $|\frac{z_n}{|z_n|} - x| = \frac{1}{|z_n|}|z_n - |z_n| x| \geq \frac{1}{|z_n|}\text{dist}(z_n, C) \geq \varepsilon$ which is a contradiction. Lemma (1.3) is proved.

2 A differential inequality

Let $[0,T]$ be a given interval. We denote by $L^p_N, W^{k,p}_N, C_N$ the spaces $L^p(0,T; X^N)$, $W^{k,p}(0,T; X^N), C([0,T]; X^N)$ with usual norms $\|.\|_{0,p}, \|.\|_{k,p}, \|.\|_{0,\infty}$, respectively.

We further introduce the set $\mathcal{K} = \{u \in L^\infty_N; u(t) \in Z \text{ a.e.}\}$.

(2.1) **Problem** For a given $u \in W^{1,1}_N$ find $x \in W^{1,1}_N \cap \mathcal{K}$ such that

$$\text{(i)} \qquad < x' - u', x - \varphi > \leq 0 \quad \text{a.e.} \quad \text{for every} \quad \varphi \in Z,$$
$$\text{(ii)} \qquad x(0) = Qu(0).$$

This choice of the initial condition corresponds to the "reference state". It enables us to avoid the difficulties in more complex hysteresis models.

(2.2) **Proposition** *Problem (2.1) has a unique solution.*

Proof *Existence.* For every fixed $n \in \mathbf{N}$ put

$$u_j = u(\frac{j}{n}T), \quad j = 0, \ldots, n$$
$$x_{j+1} = Q(u_{j+1} - u_j + x_j), \quad x_0 = Q(u_0).$$

Lemma (1.2)(i) yields

$$(2.3) \qquad < x_{j+1} - x_j - u_{j+1} + u_j, x_{j+1} - \varphi > \leq 0 \quad \forall \varphi \in Z, \text{ hence}$$

$$(2.4) \qquad |x_{j+1} - x_j| \leq |u_{j+1} - u_j|, \quad |x_{j+1} - u_{j+1} - x_j + u_j| \leq |u_{j+1} - u_j|.$$

For $t \in [\frac{j}{n}T, \frac{j+1}{n}T)$ put

$$u^{(n)}(t) = u_j + n\big(\frac{t}{T} - \frac{j}{n}\big)\big(u_{j+1} - u_j\big),$$

$$(2.5) \qquad x^{(n)}(t) = x_j + n\big(\frac{t}{T} - \frac{j}{n}\big)\big(x_{j+1} - x_j\big).$$

We have $u^{(n)} \in W_N^{1,1}$, $x^{(n)} \in W_N^{1,1} \cap \mathcal{K}$, $\|u^{(n)} - u\|_{1,1} \to 0$ and $|x^{(n)'}(t)| \leq |u^{(n)'}(t)|$ a.e.

By Arzelà-Ascoli theorem and Dunford-Pettis theorem (Edwards 1965) there exists $x \in W_n^{1,1} \cap \mathcal{K}$ and a subsequence $\{x^{(m)}\}$ of $\{x^{(n)}\}$ such that $x^{(m)'} \to x'$ in L_N^1-weak and $x^{(m)} \to x$ uniformly in C_N. It remains to verify that x satisfies (2.1)(i).

Let $\varphi \in Z$ be arbitrarily chosen. Using (2.3) we obtain for $\tau \in \big(\frac{j}{n}T, \frac{j+1}{n}T\big)$:

$$< x^{(n)'}(\tau) - u^{(n)'}(\tau), \quad x^{(n)}(\tau) - \varphi > \leq \frac{n}{T}|u_{j+1} - u_j|^2.$$

This yields for every $s < t$

$$\int_s^t < x'(\tau) - u'(\tau), \ x(\tau) - \varphi > d\tau \leq 0$$

and (2.1)(i) follows easily.

Uniqueness. Let $u, v \in W_N^{1,1}$ be given and let $x, y \in W_N^{1,1} \cap \mathcal{K}$ satisfy the inequalities $< x' - u', x - \varphi > \leq 0, < y' - v', y - \psi > \leq 0 \quad \forall \varphi, \psi \in Z$. We obtain immediately

$$(2.6) \qquad < x' - y', \ x - y > \leq < u' - v', \ x - y > \qquad \text{a.e.,}$$

hence $u = v$ implies $x = y$.

(2.7) **Remarks**

(i) We shall see in the next section that in fact $x^{(n)}$ converge strongly to x in $W_N^{1,1}$ (Remark (3.3))

(ii) Putting $\varphi = x(t \pm \delta)$ in (2.1)(i) we obtain for $\delta \to 0^+$ $< x' - u', x' >= 0$ a.e.

Let us denote $\xi = u - x$. In Krasnosel'skii & Pokrovskii 1983 ξ and x are called multidimensional play and multidimensional stop, respectively. From the mechanical point of view this corresponds to the decomposition of the strain u into the sum of the plastic strain ξ and elastic strain x. Geometrically, $x'(t)$ and $\xi'(t)$ are the projections of $u'(t)$ on the tangential and normal cones to Z, respectively, at the point $x(t)$.

3 Hysteresis operators in Sobolev spaces

Proposition (2.2) enables us to define an operator $f : W_N^{1,1} \to W_N^{1,1}$ by the relation $x = f(u) \Leftrightarrow x$ is the solution of (2.1). The term "hysteresis operator" is used by analogy to the one-dimensional case.

(3.1) **Proposition** *The operator $f : W_N^{1,1} \to W_N^{1,1}$ is continuous.*

We first prove an easy lemma.

(3.2) **Lemma** *Let $g_n, g \in L^1(\Omega, \mu)$, $f_n, f \in L^1(\Omega, \mu; X^N)$ be given functions, $n = 1, 2, \ldots,$ where (Ω, μ) is a measurable space and μ is a positive measure. Let us assume*

$$\int_\Omega |g_n - g| \, d\mu \to 0, \qquad \int_\Omega < f_n - f, \varphi > d\mu \to 0 \qquad \forall \varphi \in L^\infty(\Omega, \mu; X^N),$$

$$|f_n(x)| \le g_n(x) \quad \mu - \text{a.e,} \qquad |f(x)| = g(x) \quad \mu - \text{a.e.}$$

Then $\int_\Omega |f_n - f| \, d\mu \to 0$.

Proof Put $\varphi(x) = 0$ if $f(x) = 0$, $\quad \varphi(x) = \frac{f(x)}{|f(x)|}$, if $f(x) \ne 0$. We have $\varphi \in L^\infty(\Omega, \mu; X^N)$, hence

$$\int_\Omega < f_n, \varphi > d\mu \to \int_\Omega |f| \, d\mu.$$

Put $\psi_n(x) = g_n(x) - < f_n(x), \varphi(x) >$. We have $\psi_n(x) \ge 0$ a.e., $\int_\Omega |\psi_n| \, d\mu \to 0$, therefore there exists a subsequence ψ_k such that $\psi_k(x) \to 0 \quad \mu-$a.e. This means $< f_k(x), f(x) > \to |f(x)|^2 \quad \mu-$a.e. On the other hand,

$$|f_k(x) - f(x)|^2 \le g_k^2(x) + g^2(x) - 2 < f_k(x), f(x) > \to 0 \quad \mu - \text{a.e.,}$$

hence (cf. Edwards 1965 ,§4.21) $\int_\Omega |f_k - f| \, d\mu \to 0$ and (3.2) follows immediately.

Proof of (3.1) Let $\{u_n\}$ be a sequence in $W_N^{1,1}$, $\|u_n - u\|_{1,1} \to 0$, $x_n = f(u_n), x = f(u)$. Put $y_n = 2x_n - u_n$, $y = 2x - u$. We have by (2.7)(ii) $|y_n'(t)| = |u_n'(t)|, |y'(t)| = |u'(t)|$ a.e. and (2.6) yields $< x_n' - x', x_n - x > \le < u_n' - u', x_n - x >$. Consequently, $y_n \to y$

uniformly in C_N and by Dunford-Pettis theorem $y_n' \to y'$ in L_N^1 weak. Using lemma (3.2) for $y_n' = f_n, |u_n'| = g_n$ we conclude that $y_n \to y$ strongly in $W_N^{1,1}$, hence $\|x_n - x\|_{1,1} \to 0$.

(3.3) **Remark** The same argument shows that the discrete approximations (2.5) converge to x strongly in $W_N^{1,1}$.

(3.4) **Corollary.** *The operator f maps $W_N^{1,p}$ into $W_N^{1,p}$ for $1 \le p \le \infty$ and is continuous for $1 \le p < \infty$.*

(3.5) **Remark** For $N = 1$ the operator f is Lipschitz in $W^{1,1}$, but not Lipschitz on bounded sets in $W^{1,p}$ for $p > 1$ and not continuous in $W^{1,\infty}$ (cf. e.g. Krejčí & Lovicar 1990). On the other hand, no proof of the conjecture of Krasnosel'skii & Pokrovskii 1983 saying that $f : W_N^{1,1} \to W_N^{1,1}$ is locally Lipschitz on bounded and smooth characteristics Z has been published. Indeed, the Lipschitz continuity of $f : W_N^{1,1} \to C_N$ follows trivially from (2.6).

4 Hysteresis operators in the space of continuous functions

(4.1) **Theorem** *The operator f is closable in C_N and its closure $f : C_N \to C_N$ is continuous.*

Remark This result is known for $N = 1$ (then f is Lipschitz) or if Z is bounded (cf. Krasnosel'skii & Pokrovskii 1983). We extend the method of proof to unbounded characteristics Z.

We first prove three lemmas.

(4.2) **Lemma** *Let $u \in W_N^{1,1}$ be given, $x = f(u), \xi = u - x$. Let us assume $|u(t) - u(t_1)| \le \frac{m}{2}$ for $t \in [t_1, t_2]$, where m is defined by (1.1).*

(i) *Let Z be bounded. Then $\int_{t_1}^{t_2} |\xi'(t)|\, dt \le \frac{1}{m}(\mathrm{diam} Z)^2$.*

(ii) *Let Z be unbounded. Then $\int_{t_1}^{t_2} |\xi'(t)|\, dt \le \frac{1}{m} K^2(|x(t_1)|)$, where K is defined in (1.3)(ii).*

Proof We have $< \xi'(t), u(t) - \xi(t) - \varphi > \ge 0$ a.e. for every $\varphi \in Z$. We fix $\bar{x} \in Z$ such that $\bar{x} = 0$ in the case (i) and $\bar{x} \in C$ (cf. (1.3)) such that $|x(t_1) - \bar{x}| \le K(|x(t_1)|)$ in the case (ii). Choosing $y = \frac{m}{2} \frac{\xi'(t)}{|\xi'(t)|} + u(t) - u(t_1)$ for $\xi'(t) \ne 0$ and $y = u(t) - u(t_1)$ for $\xi'(t) = 0$ we have $|y| \le m$, hence we can put $\varphi = \bar{x} + y$. This yields

$$m|\xi'(t)| \le -\frac{d}{dt}|\xi(t) - u(t_1) + \bar{x}|^2, \quad \text{hence}$$

$$\int_{t_1}^{t_2} |\xi'(t)|\,dt \le \frac{1}{m}|x(t_1)-\bar{x}|^2 \quad \text{and (4.2) is proved.}$$

(4.3) **Lemma** *Let $u_n \in W^{1,1}_N, n = 1, 2, \dots$ and $u \in C_N$ be given such that $\|u_n - u\|_{0,\infty} \to 0$. Put $x_n = f(u_n), \xi_n = u_n - x_n$. Then there exist $n_0 > 0$ and a constant $c > 0$ depending only on Z and u such that*

$$\int_0^T |\xi_n'(t)|\,dt \le c \quad \textit{for} \quad n \ge n_0.$$

Proof We construct a partition $0 = t_0 < t_1 < \cdots < t_M = T$ of $[0, T]$ such that for $t \in [t_k, t_{k+1}]$ we have $|u(t) - u(t_k)| \le \frac{m}{6}$ We fix n_0 such that for $n \ge n_0$ we have $\|u_n - u\|_{0,\infty} < \frac{m}{6}$. For arbitrary $t \in [t_k, t_{k+1}]$ and $n \ge n_0$ we obtain $|u_n(t) - u_n(t_k)| \le \frac{m}{2}$, hence by (4.2) we have

$$\text{(i)} \qquad \int_{t_k}^{t_{k+1}} |\xi_n'(t)|\,dt \le \frac{1}{m}(\mathrm{diam} Z)^2 \quad \text{if } Z \text{ is bounded,}$$

therefore $\int_0^T |\xi_n'(t)|\,dt \le \frac{M}{m}(\mathrm{diam} Z)^2$,

$$\text{(ii)} \qquad \int_{t_k}^{t_{k+1}} |\xi_n'(t)|\,dt \le \frac{1}{m}K^2(|x_n(t_k)|) \quad \text{if } Z \text{ is unbounded,}$$

hence $\int_0^T |\xi_n'(t)|\,dt \le \frac{M}{m} \max_k K^2(|x_n(t_k)|)$.

In the case (i) we put simply $c = \frac{M}{m}(\mathrm{diam} Z)^2$. In the case (ii) we integrate the inequality $< \xi_n', \xi_n > \le < \xi_n', u_n >$ and we obtain

$$|\xi_n(t)|^2 \le |\xi_n(0)|^2 + 2\|u_n\|_{0,\infty} \int_0^T |\xi_n'(t)|\,dt$$

for each $t \in [0, T]$. This yields

$$|x_n(t)|^2 \le (|u_n(t)| + |\xi_n(t)|)^2 \le c_1 + c_2 \max_k K^2(|x_n(t_k)|),$$

hence (1.3)(ii) implies $\|x_n\|_{0,\infty} \le c_3$, where c_1, c_2, c_3 are constants depending only on Z and u (we take a larger n_0, if necessary). Consequently,

$$\int_0^T |\xi_n'(t)|\,dt \le \text{const. and (4.3) is proved.}$$

(4.4) **Lemma** *Let the assumptions of Lemma (4.3) be fulfilled. Then there exist $n_0 > 0$ and a constant $c > 0$ depending only on Z and u such that for every $m, n \ge n_0$ we have*

$$\|\xi_n - \xi_m\|_{0,\infty} \le c\big(\|u_n - u_m\|_{0,\infty} + \|u_n - u_m\|_{0,\infty}^{1/2}\big).$$

Proof The inequality (2.6) for $u = u_n, v = u_m$ has the form $< \xi'_n - \xi'_m, \xi_n - \xi_m > \leq < \xi'_n - \xi'_m, u_n - u_m >$, hence

$$|\xi_n(t) - \xi_m(t)|^2 \leq |\xi_n(0) - \xi_m(0)|^2 + 2\int_0^T (|\xi'_n| + |\xi'_m|)\, dt \cdot \|u_n - u_m\|_{0,\infty}$$

and (4.4) follows immediately from (4.3).

Proof of (4.1) Let $u \in C_N$ be given. We choose a sequence $\{u_n\} \subset W_N^{1,1}$ such that $\|u_n - u\|_{0,\infty} \to 0$. We see from (4.4) that the sequence $x_n = f(u_n)$ is fundamental in C_N and its limit is independent of the choice of $\{u_n\}$. Therefore, we can define $f(u) = \lim_{n\to\infty} x_n$. The continuity of f in C_N follows from standard considerations. Let $\{u_n\} \subset C_N$ be an arbitrary sequence, $\|u_n - u\|_{0,\infty} \to 0$. The above argument shows that for each $n = 1, 2, \ldots.$ we can find $\tilde{u}_n \in W_N^{1,1}$ such that $\|\tilde{u}_n - u_n\|_{0,\infty} < \frac{1}{n}$, $\|f(\tilde{u}_n) - f(u_n)\|_{0,\infty} < \frac{1}{n}$. We have $\|\tilde{u}_n - u\|_{0,\infty} \to 0$, hence $\|f(\tilde{u}_n) - f(u)\|_{0,\infty} \to 0$ and $\|f(u_n) - f(u)\|_{0,\infty} \to 0$.

(4.5) **Remarks**

(i) It follows from Lemma (4.4) that the operator $f : C_N \to C_N$ is $\frac{1}{2}$-Hölder continuous on compact sets. We present two examples showing that this result cannot be in general considerably improved. Nevertheless, on special characteristics Z the operator f has better continuity properties (it is globally Hölder continuous on strictly convex sets and Lipschitz on polygons, cf. Krasnosel'skii & Pokrovskii 1983).

(ii) The multidimensional play operator $I - f$, where I is the identity, has a particular smoothing property: it maps continuously C_N-strong into $BV(0,T;X^N)$-weak $*$. This follows from the argument of the proof of (4.1) applied to (4.3).

(4.6) **Examples**

(i) *f does not map in general bounded sets in C_N into bounded sets.*

Put $Z = \{(a,b) \in R^2; 0 < a \leq 1, b \geq -\log a\}, T = 1$. Let us define a sequence $\{u_n\}$ in the following way: $u_n(t) = \begin{pmatrix} r_n(t) \\ 0 \end{pmatrix}$, where r_n is continuous and piecewise linear, $r_n\left(\frac{2k}{n}\right) = 1$, $r_n\left(\frac{2k+1}{n}\right) = 0$, $k = 0, 1, \ldots, [\frac{n}{2}]$, $r'_n(t) = n(-1)^j$ for $t \in \left(\frac{j-1}{n}, \frac{j}{n}\right)$, $j = 1, \ldots, n$. In fact, it would be more accurate to shift the domain Z and the functions u_n in order to fulfil the assumption $0 \in Int Z$.

We determine the vector $x_n(t) = f(u_n)(t) = \begin{pmatrix} \alpha_n(t) \\ \beta_n(t) \end{pmatrix}$ using (2.7) (ii). Denote $\alpha_n^0 = 1, \alpha_n^k = \alpha_n\left(\frac{2k-1}{n}\right)$, $g(\alpha) = -\log \alpha$. For $t \in (0, \frac{1}{n})$ we have $\beta_n(t) = g(\alpha_n(t))$ and $x'_n(t)$ is the orthogonal projection of $u'_n(t)$ on the tangent to ∂Z at the point $x_n(t)$, hence

$$x'_n(t) = \frac{r'_n(t)}{1 + g'^2(\alpha_n(t))} \begin{pmatrix} 1 \\ g'(\alpha_n(t)) \end{pmatrix}.$$

In other words, α_n is the solution of the equation

$$\alpha_n'(1 + {g'}^2(\alpha_n(t))) = r_n', \qquad \alpha_n(0) = 1. \tag{4.7}$$

We obtain $\alpha_n^1 = \frac{\sqrt{5}-1}{2}$. For $t \in \left(\frac{2k-1}{n}, \frac{2k}{n}\right)$ we have $\beta_n(t) = \text{const.} = -\log \alpha_n^k$, $\alpha_n\left(\frac{2k}{n}\right) = 1$. We find $\tau_k \in \left(\frac{2k}{n}, \frac{2k+1}{n}\right)$ such that $r_n(\tau_k) = \alpha_n^k$. We have $\alpha_n'(t) = r_n'(t), \beta_n'(t) = 0$ for $t \in \left(\frac{2k}{n}, \tau_k\right)$ and (4.7) for $t \in \left(\tau_k, \frac{2k+1}{n}\right)$, $\alpha_n(\tau_k) = \alpha_n^k$, hence α_n^{k+1} is the positive root of the equation

$$\left(\alpha_n^{k+1}\right)^2 + \frac{\alpha_n^{k+1}}{\alpha_n^k} = 1.$$

We see immediately that the sequence $\alpha_n^{[\frac{n}{2}]}$ tends to 0 as $n \to \infty$, hence the sequence $\{x_n\}$ is unbounded.

(ii) *The Hölder continuity is optimal if Z is a ball.*

Put $Z = \{z \in X^N; |z| \le h\}$, where $h > 0$ is a given number. In this case the operator f is $\frac{1}{2}$-Hölder continuous in C_N (cf. Krasnosel'skii & Pokrovskii 1983).

Let us choose $r > h$ and for $t \in [0,1]$ put

$$u_n(t) = \begin{pmatrix} r \quad \cos nt \\ r \quad \sin nt \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad v_n(t) = \begin{pmatrix} h \quad \cos nt \\ h \quad \sin nt \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

$x_n = f(u_n), y_n = f(v_n)$. We have indeed $y_n = v_n$. The same argument as above shows that x_n is the solution of the equation

$$x_n'(t) = u_n'(t) - \frac{1}{h^2} < x_n(t), u_n'(t) > x_n(t), \quad x_n(0) = v_n(0).$$

We find out that x_n has the form $x_n(t) = \begin{pmatrix} h \quad \cos(nt + \varrho(t)) \\ h \quad \sin(nt + \varrho(t)) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, where ϱ is the solution of the equation $\varrho' + n\left(\frac{r}{h}\cos\varrho - 1\right), \varrho(0) = 0$, hence

$$\varrho(t) = 2\text{arctan}\left(\sqrt{\frac{r-h}{r+h}}\text{tanh}\left(\frac{\sqrt{r^2-h^2}}{2h}nt\right)\right).$$

We have

$$|x_n(t) - y_n(t)|^2 = 4h^2 \frac{(r-h)\text{tanh}^2\left(\frac{\sqrt{r^2-h^2}}{2h}nt\right)}{(r+h) + (r-h)\text{tanh}^2\left(\frac{\sqrt{r^2-h^2}}{2h}nt\right)},$$

hence $\lim_{n\to\infty} |x_n(1) - y_n(1)| = h\sqrt{\frac{2(r-h)}{r}}$.

5 Properties of hysteresis operators

A. Superposition and inversion formulas.

(5.1) **Proposition** *For every $u \in W_N^{1,1}$ and $\alpha \geq -1$ we have $f((1+\alpha)u - \alpha f(u)) = f(u)$.*

Proof Put $v = f(u), w = f((1+\alpha)u - \alpha v)$. We have (cf. (1.2) (iv)) $w(0) = Q(u(0) + \alpha Pu(0)) = Qu(0) = v(0)$ and $< w' - (1+\alpha)u' + \alpha v', w - \varphi > \leq 0$, $(1+\alpha) < v' - u', v - \psi > \leq 0$ for every $\varphi, \psi \in Z$. Putting $\varphi = v, \psi = w$ we obtain $< w' - v', w - v > \leq 0$, hence $w = v$.

(5.2) **Corollary** *For every $\alpha > 0, \alpha + \beta > 0, h > 0$ we have $w = \alpha u + \beta h\, f\big(\frac{1}{h}u\big) \iff u = \hat{\alpha} w + \hat{\beta}\hat{h} f\big(\frac{1}{\hat{h}}w\big)$, where $\hat{\alpha} = \frac{1}{\alpha}, \hat{\alpha} + \hat{\beta} = \frac{1}{\alpha+\beta}, \quad \hat{h} = h(\alpha + \beta)$.*

Proof By (5.1) we have $f\big(\frac{1}{\hat{h}}w\big) = f\big(\frac{\alpha}{\alpha+\beta} \cdot \frac{1}{h}u + \frac{\beta}{\alpha+\beta}\, f\big(\frac{1}{h}u\big)\big) = f\big(\frac{1}{h}u\big)$ and (5.2) follows easily.

B. Vector Ishlinskii operator.

Let μ be a positive measure on $(0, \infty), \mu((0, \infty)) < \infty$ and let f be the multidimensional stop corresponding to Z. We define the vector Ishlinskii operator by the formula

$$F(u) = \int_0^\infty f\big(\frac{1}{h}u\big) \cdot h \quad d\mu(h). \tag{5.3}$$

Let us note that for $h > \frac{1}{m}\|u\|_{0,\infty}$ we have $f\big(\frac{1}{h}u\big)(t) = \frac{1}{h}u(t)$, hence the integral in (5.3) is always finite. We check easily that the vector Ishlinskii operator F is continuous in C_N and $W_N^{1,p}$, $1 \leq p < \infty$.

C. Energy inequalities

We define the hysteresis potentials for the operator (5.3) by the following formulas

(5.4) (i) $$P_1(u) = \frac{1}{2}\int_0^\infty \big(f\big(\frac{1}{h}u\big)\big)^2\ h^2\ d\mu(h) \quad \text{for} \quad u \in C_N,$$

(ii) $$P_2(u) = \frac{1}{2} < F(u)', u' > \quad \text{for} \quad u \in W_N^{1,1}.$$

The Hölder inequality yields immediately

$$|F(u)|^2 \leq 2\mu((0,\infty)) \cdot P_1(u). \tag{5.5}$$

The following energy inequalities are satisfied.

(5.6) **Proposition**

(i) *For every* $u \in W_N^{1,1}$ *and for almost all* $t \in [0,T]$ *we have*

$$P_1(u)'(t) \le < F(u)(t), u'(t) > .$$

(ii) *For every* $u \in W_N^{2,1}$ *and for almost all* $0 \le s < t \le T$ *we have*

$$P_2(u)(t) - P_2(u)(s) \le \int_s^t < F(u)'(\tau),\, u''(\tau) > d\tau.$$

Proof The function $h \to h\, f\big(\frac{1}{h}u\big)'$ is continuous and bounded as function $R^1 \to L_N^1$, hence we can differentiate in (5.3) and (5.4) (i) with respect to t. We obtain

$$P_1(u)'(t) - < F(u)(t), u'(t) > = \int_0^\infty < f\big(\frac{1}{h}u\big)'(t) - \frac{1}{h}u'(t),\; f\big(\frac{1}{h}u\big)(t) > h^2 d\mu(h) \le 0$$

by (2.1)(i) with $\varphi = 0$.

For proving (ii) we denote $x_h(\tau) = h\, f\big(\frac{1}{h}u\big)(\tau)$. We have

$$< x_h'(\tau) - u'(\tau), x_h(\tau) - h\varphi > \le 0, \quad < x_h'(\tau+\delta) - u'(\tau+\delta), x_h(\tau+\delta) - h\psi > \le 0$$

for every $\varphi, \psi \in Z$. Putting $\varphi = \frac{1}{h}x_h(\tau+\delta)$, $\psi = \frac{1}{h}x_h(\tau)$ we obtain

$$< x_h'(\tau+\delta) - x_h'(\tau), x_h(\tau+\delta) - x_h(\tau) > \le < u'(\tau+\delta) - u'(\tau), x_h(\tau+\delta) - x_h(\tau) > .$$

After integration $\int_s^t d\tau$ this yields

$$\frac{1}{2\delta^2}|x_h(t+\delta) - x_h(t)|^2 - \frac{1}{2\delta^2}|x_h(s+\delta) - x_h(s)|^2 \le \frac{1}{\delta^2}\int_s^t < u'(\tau+\delta) - u'(\tau), x_h(\tau+\delta) - x_h(\tau) > d\tau,$$

hence for $\delta \to 0$ we obtain (cf. Edwards 1965, §4.21)

$$\frac{1}{2}|x_h'(t)|^2 - \frac{1}{2}|x_h'(s)|^2 \le \int_s^t < u''(\tau), x_h'(\tau) > d\tau \quad \text{for almost all} \quad 0 \le s < t \le T.$$

(Notice that for $u \in W_N^{2,1}$ we have $x_h \in W_N^{1,\infty}$).

By (2.7) (ii) we obtain $|x_h'(t)|^2 = < x_h'(t), u'(t) >$ and (5.6) (ii) follows easily.

Remarks

(i) The potential P_1 (as well as the operator F) is rate independent and corresponds to the usual physical notion of energy. On the other hand, the potential P_2 has no natural interpretation and the term "energy" is maybe not physically justified. We use it in order to emphasize the formal similarity between (5.6) (i) and (ii).

(ii) In the case $N = 1, Z = [-1, 1]$ we can make use of the simple structure of the hysteresis memory for getting more precise information about the dissipation of energies corresponding to P_1 and P_2. The dissipation of $P_1(u)$ is characterized by the area of the hysteresis loop in the plane with coordinates $u, F(u)$. Moreover, we can obtain an estimate from below of the dissipation of energy for $P_2(u)$ which is of the order of $\int |u'(t)|^3 \, dt$. It is shown for instance in Krejčí 1988 how this estimate can be used for solving hyperbolic equations with hysteresis operators.

D. Dependence on parameters.

Let us consider input functions depending on a parameter $x \in J$, where $J \subset R^1$ is a fixed interval and let us assume $u(x, .) \in C_N$ for almost every $x \in J$. We can define the value $f(u)(x, t)$ of the operator f to be equal to $f(u(x, .))(t)$ for every $t \in [0, T]$. We see immediately that the operator $f : L^p(J, W_N^{1,1}) \to L^p(J; C_N)$ is Lipschitz for every $p \in [1, \infty]$.

Let $u \in L^1(J; W_N^{1,1})$ be given such that $\dfrac{\partial u}{\partial x} \in L^1(J; W_N^{1,1})$. We have by (2.6) for $x, y \in J$, $x > y$ and $0 \le s < t \le T$

$$\frac{1}{2}|f(u)(x,t) - f(u)(y,t)|^2 - \frac{1}{2}|f(u)(x,s) - f(u)(y,s)|^2 \le \le \int_s^t < f(u)(x,\tau) - f(u)(y,\tau), \frac{\partial u}{\partial t}(x,\tau) - \frac{\partial u}{\partial t}(y,\tau) > d\tau. \tag{5.7}$$

For $s = 0$ this yields

$$|f(u)(x,t) - f(u)(y,t)|^2 \le$$
$$\le |Qu(x,0) - Qu(y,0)|^2 + 2\int_0^t |f(u)(x,\tau) - f(u)(y,\tau)||\frac{\partial u}{\partial t}(x,\tau) - \frac{\partial u}{\partial t}(y,\tau)|d\tau.$$

The inequality (1.2) (ii) implies $|Qu(x,0) - Qu(y,0)| \le |u(x,0) - u(y,0)| \le \int_y^x |\frac{\partial u}{\partial x}(\xi, 0)| d\xi$, hence

$$|f(u)(x,t) - f(u)(y,t)| \le \int_y^x |\frac{\partial u}{\partial x}(\xi,0)|\, d\xi + 2\int_0^t \int_y^x |\frac{\partial^2 u}{\partial t\,\partial x}(\xi,\tau)| d\xi\, d\tau,$$

Thus $f(u)(., t)$ is absolutely continuous for all $t \in [0, T]$ and

$$|\frac{\partial}{\partial x} f(u)(x,t)| \le |\frac{\partial u}{\partial x}(x,0)| + 2\int_0^t |\frac{\partial^2 u}{\partial t \partial x}(x,\tau)|\, d\tau \quad \text{a.e.}$$

Dividing (5.7) by $(x - y)^2$ we obtain for $y \to x$

$$\frac{1}{2}|\frac{\partial}{\partial x} f(u)(x,t)|^2 - \frac{1}{2}|\frac{\partial}{\partial x} f(u)(x,s)|^2 \le \int_s^t < \frac{\partial}{\partial x} f(u)(x,\tau), \frac{\partial^2 u}{\partial t \partial x}(x,\tau) > d\tau \tag{5.8}$$

for almost every $x \in J$.

We derive easily analogous inequalities for the Ishlinskii operator. The generalization to a larger number of parameters is obvious.

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Conclusion

This paper presents an elementary approach to the vector hysteresis operators with closed convex characteristics Z introduced by Krasnosel'skii & Pokrovskii 1983. It contains some new results, namely the continuity in the space of continuous functions and in Sobolev spaces $W^{1,p}$ without any assumption on the boundedness and smoothness of Z. We also introduce energy potentials for the Prandtl-Reuss and Ishlinskii hysteresis operators and we prove two energy inequalities. It is shown how an evolution differential inequality can be transformed into an equation with hysteresis operator and how the energy inequalities can be used for solving such problems.

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