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# Global behaviour of solutions to the wave equation with hysteresis

Pavel Krejčí
*Matematický ústav ČSAV, Žitná 25*
*115 67 Praha 1, Czechoslovakia*

*Abstract.* The wave equation with a Preisach hysteresis operator can be considered as a one-dimensional projection of Maxwell's equations in a ferromagnetic medium. An initial-boundary value problem for this equation is solved here with emphasizing the fact that under a bounded forcing term the solutions remain bounded. This is due to the strong dissipation of hysteresis energies. New proofs of hysteresis energy inequalities are given without referring to the structure of hysteresis memory.

**Introduction.**

Hyperbolic equations with hysteresis operators appear in various problems of mathematical physics (Maxwell's equations, elastoplastic oscillations etc.). We present here a qualitative study of an initial-boundary value problem for the equation

$$W(u_t)_t - u_{xx} = g(x,t),$$

where $g$ is a given function and $W$ is a Preisach hysteresis operator. It has been proved in [5] that this equation is hyperbolic in the sense of finite speed of propagation of waves.

The present paper is divided into 9 sections. Sections 1-3 are devoted to the investigation of properties of the Preisach operator (representation, continuity, superposition and inversion, energy inequalities). We introduce here a new approach which does not make use of the structure of memory. This enables us to replace the assumptions of oddness and "virgin initial state" (cf. [5]) by weaker ones. The Preisach operator $W$ is locally represented by a superposition (Nemytskii) operator $\Phi$ (Lemma (1.18)). The two hysteresis energy potentials still play a crucial role here. The assumption of convexity of loops is interpreted in terms of $\Phi$ as the requirement that $u \to \Phi(u)$ is convex if $u$ increases and concave if $u$ decreases. Indeed, for a general Preisach operator this is true only if $u$ remains small during the whole history of the process (Lemma (3.1)). In §4 we investigate parameter-dependent Preisach operators.

The main results of the paper are formulated in §5, namely the existence of global solutions, sufficient conditions for uniqueness, regularity and asymptotic behaviour. Their proofs are given in §§6-9.

## 1. Preisach operator

Let $u \in W^{1,1}(0,T)$ be a given function and $h > 0$, $x_h^0$ given numbers, $|x_h^0| \le h$. The problem of finding a function $x_h \in W^{1,1}(0,T)$ such that

$$
\begin{aligned}
(1.1) \qquad & (i) \quad x_h(t) \in [-h,h], t \in [0,T], \\
& (ii) \quad (x_h'(t) - u'(t))(x_h(t) - \varphi) \le 0 \qquad \text{a.e.} \qquad \forall \varphi \in [-h,h], \\
& (iii) \quad x_h(0) = x_h^0
\end{aligned}
$$

has a unique solution (cf. e.g. [6]).

Let $\bar{h} > 0$ be a given number. We introduce the sets

$$\Lambda := \{\lambda \in W^{1,\infty}(0,\infty); \quad |\lambda'(h)| \le 1 \text{ a.e.}\},$$

$$\Lambda(\bar{h}) := \{\lambda \in \Lambda; \quad \lambda(h) = 0 \text{ for } h \ge \bar{h}\}$$

of admissible initial states and we put

$$(1.2) \qquad x_h^0 := sign\ (u(0) - \lambda(h))\ min\ \{h, |u(0) - \lambda(h)|\}$$

for some $\lambda \in \Lambda$.

The initial condition characterized by the function $\lambda \equiv 0$ is called *reference* (or *virgin*) *state*.

The existence and uniqueness result for (1.1), (1.2) enables us to define an operator $f_h(., \lambda(h)) : W^{1,1}(0,T) \to W^{1,1}(0,T)$ for every $h > 0$ and $\lambda \in \Lambda$ by the formula

$$(1.3) \qquad f_h(u, \lambda(h))(t) := x_h(t), \qquad t \in [0,T],$$

where $x_h$ is the solution of (1.1), (1.2).

The operator $f_h$ is called *stop*. We further introduce the operator ($I$ denote the identity)

$$(1.4) \qquad l_h(., \lambda(h)) := I - f_h(., \lambda(h))$$

which is called *play* (cf. [3], [9]).

It can be shown easily (cf. e.g. [6]) that $l_h(., \lambda(h)), f_h(., \lambda(h))$ are Lipschitz continuous in $W^{1,1}(0,T)$ and that for every $\lambda, \mu \in \Lambda$, $u, v \in W^{1,1}(0,T)$ we have

$$(1.5) \qquad |l_h(u, \lambda(h))(t) - l_h(v, \mu(h))(t)| \le max\{|\lambda(h) - \mu(h)|, ||u - v||_{[0,t]}\},$$

where we denote $||w||_{[0,t]} := max\{|w(s)|, 0 \le s \le t\}$.

This implies immediately that $l_h(., \lambda(h)), f_h(., \lambda(h))$ can be considered as Lipschitz continuous operators in $C([0,T])$.

**(1.6) Lemma.** *Let $\lambda \in \Lambda(\bar{h}), u \in C([0,T]), t \in [0,T]$ be given, $||u||_{[0,t]} \le \bar{h}$. Put $\mu(h) := l_h(u, \lambda(h))(t)$ for every $h > 0$. Then $\mu \in \Lambda(\bar{h}), \mu(0) = u(t)$.*

**Proof.** The Lipschitz continuity of $l_h(.,\lambda(h))$ and the closedness of $\Lambda(\bar{h})$ with respect to the uniform convergence imply that it suffices to assume $u$ to be smooth and piecewise monotone.

More precisely, we assume that $0 = t_0 < t_1 < ... < t_N = T$ is a partition of $[0,T]$ such that $u'(t) \neq 0$ in $(t_{i-1}, t_i), i = 1, ..., N$.

For $t \in (t_{i-1}, t_i)$ (1.1) yields

$$x_h(t) = \begin{cases} \min\{x_h(t_{i-1}) + u(t) - u(t_{i-1}), h\} & \text{if } u'(t) > 0, \\ \max\{x_h(t_{i-1}) + u(t) - u(t_{i-1}), -h\} & \text{if } u'(t) < 0, \end{cases} \tag{1.7}$$

hence

$$l_h(u,\lambda(h))(t) = \begin{cases} \max\{l_h(u,\lambda(h))(t_{i-1}), u(t) - h\} & \text{if } u'(t) > 0, \\ \min\{l_h(u,\lambda(h))(t_{i-1}), u(t) + h\} & \text{if } u',(t) < 0. \end{cases} \tag{1.8}$$

An easy induction argument completes the proof. ■

(1.9) **Remark.** Relations (1.7), (1.2) represent the standard definition of the stop for piecewise monotone inputs (cf. [3]). The extension to arbitrary continuous inputs is then possible by (1.5).

(1.10) **Definition.** *Let* $p : R^1 \to R^1, \nu : R^1 \times [0,\infty) \to R^1, \eta \in L^1_{loc}(0,\infty), \eta_0 \in L^1(0,\infty), \alpha > 0$ *be given such that*

$$\begin{aligned} (1.11) \qquad & (i) && p' \in L^\infty_{loc}(R^1), \frac{\partial \nu}{\partial \rho} \in L^1_{loc}(R^1 \times (0,\infty)), \\ & (ii) && \nu(0,h) \equiv 0, \\ & (iii) && \eta(h) \geq \frac{\partial \nu}{\partial \rho}(\rho,h) \geq -\eta_0(h), \quad \eta(h) \geq \eta_0(h) \geq 0 \ a.e., \\ & (iv) && p'(\rho) \geq \alpha \quad a.e. \end{aligned}$$

*Let* $\lambda \in \Lambda(\bar{h})$ *be a given initial state. The operators* $W_\lambda$ *defined by the formula*

$$W_\lambda(u)(t) := p(u(t)) + \int_0^\infty \nu(l_h(u,\lambda(h))(t), h)dh \tag{1.12}$$

*is called a Preisach operator.*

*If* $p, \nu$ *are linear with respect to* $\rho$*, i.e.*

$$p(\rho) = \alpha\rho, \qquad \nu(\rho,h) = \rho.\eta(h) \tag{1.13}$$

*then the Preisach operator* $W_\lambda$ *is called an Ishlinskii operator.*

In the sequel we assume

$$\begin{aligned} (1.14) \qquad & (i) && \int_0^r \eta_0(h)dh < \alpha \qquad \text{for every } r > 0, \\ & (ii) && \lim_{h\to\infty} \left[ h(\alpha - \int_0^\infty \eta_0(a)\ da) + \int_0^h \int_b^\infty \eta_0(a)\ da\ db \right] = +\infty \end{aligned}$$

(1.15) **Remarks.**

(i) It is easy to see that the Preisach operator $W_\lambda$ is continuous in $C([0,T])$.

(ii) It can be shown ([5]) that formula (1.12) is equivalent to the standard definition of the Preisach operator (cf. e.g. [2],[8]).

(iii) In general we need not require so much regularity for $\rho, \nu$ (cf. [2]). Here, in application to hyperbolic PDE's, this regularity plays an important role.

We present here an alternative approach to the Preisach operator *without referring to the structure of memory.* The philosophy is close to [2] in spite of important differences.

We first represent the Preisach operator locally by means of Nemytskii (superposition) operators. According to [7] we introduce the *identification function* $S(\varrho, h)$ of the operator $W_\lambda$ as the solution of the Cauchy problem

$$\text{(1.16)} \qquad \begin{aligned} &(i) && S_{hh} - S_{\varrho\varrho} = \nu(\varrho, h) \\ &(ii) && S_h(\varrho, 0) = p(\varrho) \\ &(iii) && S(\varrho, 0) = 0 \end{aligned}$$

We have

$$\text{(1.17)} \qquad S(\varrho, h) = \frac{1}{2}\int_{\varrho-h}^{\varrho+h} \rho(\xi)d\xi + \frac{1}{2}\int_0^h \int_{\varrho-h+b}^{\varrho+h-b} \nu(a,b)da\ db.$$

(1.18) **Lemma.** *Let $u \in C([0,T])$ and $[t_1, t_2] \subset [0,T]$ be given such that $u$ is monotone in $[t_1, t_2]$. Let $W_\lambda$ be a Preisach operator* (1.12)*. Then there exists an absolutely continuous increasing function $\Phi$ depending only on $\{l_h\big(u, \lambda(h)\big)(t_1); h > 0\}$ such that for every $t \in [t_1, t_2]$ we have $W_\lambda(u)(t) = \Phi(u(t))$.*

**Proof.** (i) Let $u$ be non-decreasing in $[t_1, t_2]$. Put $\lambda_1(h) := l_h(u, \lambda(h))(t_1)$, $h^* := max\{\bar{h}, ||u||_{[0,T]}\}$. By Lemma (1.6) we have $\lambda_1 \in \Lambda(h^*)$, hence for every $v \in [u(t_1), u(t_2)]$ there exists $q \in (0, \infty)$,such that $\lambda_1(q) + q = v$. Put

$$\text{(1.19)} \qquad R^+(v) := max\{q > 0; \quad v = q + \lambda_1(q)\}$$

Indeed, formula (1.8) holds for $t \in [t_1, t_2]$, hence

$$l_h(u, \lambda(h))(t) = \begin{cases} u(t) - h & \text{for } h < R^+(u(t)), \\ \lambda_1(h) & for h \geq R^+(u(t)), \end{cases} \quad t \in [t_1, t_2].$$

This yields

$$W_\lambda(u)(t) = p\big(u(t)\big) + \int_0^{R^+\left(u(t)\right)} \nu\big(u(t) - h, h\big) + \int_{R^+\left(u(t)\right)}^{h^*} \nu\big(\lambda_1(h), h\big)dh.$$

It is easy to see that the function $v \to R^+(v)$ is increasing and $R^+(v) \leq h^*$ for $v \in [u(t_1), u(t_2)]$.

Putting

$$\Phi(v) = p(v) + \int_0^{R^+(v)} \nu(v-h,h)dh + \int_{R^+(v)}^{h^*} \nu\big(\lambda_1(h),h\big)dh \tag{1.20}$$

we obtain using (1.16)

$$\Phi(v) = (S_h + S_\varrho)(0,h^*) - \int_{R^+(v)}^{h^*} \big(1+\lambda_1'(h)\big)(S_{h\varrho}+S_{\varrho\varrho})\big(\lambda_1(h),h\big)dh. \tag{1.21}$$

It remains to prove that $\Phi$ is increasing and absolutely continuous in $[u(t_1),u(t_2)]$. Let $v_1, v_2 \in [u(t_1),u(t_2)]$ be arbitrarily chosen, $v_1 < v_2$.

Then

$$\begin{cases} v_2 - v_1 = \displaystyle\int_{R^+(v_1)}^{R^+(v_2)} \big(1+\lambda_1'(h)\big)dh, \\ \Phi(v_2) - \Phi(v_1) = \displaystyle\int_{R^+(v_1)}^{R^+(v_2)} \big(1+\lambda_1'(h)\big)(S_{h\varrho}+S_{\varrho\varrho})\big(\lambda_1(h),h\big)dh. \end{cases} \tag{1.22}$$

We have

$$(S_{\varrho h} + S_{\varrho\varrho})(\varrho,h) = p'(\varrho+h) + \int_0^h \frac{\partial\nu}{\partial\varrho}(\varrho+h-a,a)da,$$

hence (1.11),(1.14) yield

$$c_1(v_2-v_1) \le \Phi(v_2) - \Phi(v_1) \le c_2(v_2-v_1)$$

for some positive constants $c_1, c_2$.

(ii) Let $u$ be non-increasing. We proceed as before putting

$$R^-(v) = max\{q>0; v = -q+\lambda_1(q)\} \tag{1.23}$$

for $v \in [u(t_2),u(t_1)]$. The function $R^-$ is decreasing in $[u(t_2),u(t_1)]$ and by (1.8)

$$l_h\big(u,\lambda(h)\big)(t) = \begin{cases} u(t)+h & \text{for } h < R^-\big(u(t)\big) \\ \lambda_1(h) & \text{for } h \ge R^-\big((t)\big) \end{cases}$$

$t \in [t_1,t_2]$. Analogously to (1.21) the function $\Phi$ is defined for $v < u(t_1)$ by the formula

$$\Phi(v) = (S_h - S_\varrho)(0,h^*) - \int_{R^-(v)}^{h^*} \big(1-\lambda_1'(h)\big)(S_{h\varrho}-S_{\varrho\varrho})\big(\lambda_1(h),h\big)dh \tag{1.24}$$

with the same conclusion.

Lemma (1.18) is proved. ■

(1.25) **Remark.** The same argument can be used for deriving the "primary curve" of the operator $W_\lambda$. Indeed, the value of $W_\lambda(u)(0)$ depends only on $\lambda$ and $u(0)$. Replacing $\lambda_1$ by $\lambda$ in the computation above we obtain $W_\lambda(u)(0) = \Phi_0\big(u(0)\big)$, where the function $\Phi_0$ is given by (1.21) for $u(0) \ge \lambda(0)$ and (1.24) for $u(0) < \lambda(0)$ with $\lambda_1$ replaced by $\lambda$.

## 2. Properties of the Preisach operator

In this section we still assume that (1.11),(1.14) hold and that $W_\lambda$ is a given Preisach operator (1.12).

The two following lemmas establish a superposition formula for hysteresis operators.

(2.1) **Lemma.** *Let $S$ be the function (1.17) and let $u \in W^{1,1}(0,T), \lambda \in \Lambda(\bar{h}), r > 0$ be given, $\bar{h} \geq ||u||_{[0,T]}$. Then there exists a function $h_r \in W^{1,1}(0,T)$ such that for every $t \in [0,T)$ we have*

$$S_\varrho\big(l_{h_r(t)}(u, \lambda(h_r(t)))(t), h_r(t)\big) = r.$$

**Proof.** Let $t \in [0,T]$ be fixed. The function $\mu(h) := l_h\big(u, \lambda(h)\big)(t)$ belongs to $\Lambda(\bar{h})$ by Lemma (1.6) and $\frac{d}{dh}\big(S_\varrho(\mu(h),h)\big) \geq \alpha - \int_0^h \eta_0(a)da > 0$, $\lim_{h\to\infty} S_\varrho\big(\mu(h),h\big) = +\infty$, $S_\varrho\big(\mu(0),0\big) = 0$ by (1.14), (1.16)(iii) (let us note that for $h \geq \bar{h}$ we have $\mu(h) = 0$).

Denoting by $h_r(t)$ the unique solution $h$ of the equation $S_\varrho\big(l_h(u,\lambda(h))(t),h\big) = r$ we obtain for every $t_1 < t_2, h_i := h_r(t_i), i = 1,2$

$$\begin{aligned} &S_\varrho(l_{h_2}(u,\lambda(h_2))(t_2),h_2) - S_\varrho(l_{h_1}(u,\lambda(h_1))(t_2),h_1) = \\ &= S_\varrho(l_{h_1}(u,\lambda(h_1))(t_1),h_1) - S_\varrho(l_{h_1}(u,\lambda(h_1))(t_2),h_1), \end{aligned}$$

hence

$$|h_2 - h_1| \leq \frac{1}{\alpha(r)} \left| \int_{t_1}^{t_2} \frac{\partial}{\partial t} S_\varrho\big(l_{h_1}(u,\lambda(h_1))\big((t),h_1\big)\, dt \right|,$$

where $\alpha(r) := \inf\{\frac{\partial}{\partial h} S_\varrho\big(l_h(u,\lambda(h))\big((t),h\big); 0 < h \leq \bar{h}_r, t \in [0,T]\}, \bar{h}_r := \max\{h_r(t); t \in [0,T]\}$. Therefore, $h_r$ is absolutely continuous in $[0,T]$ and Lemma (2.1) is proved. ■

(2.2) **Lemma.** *Let $u \in W^{1,1}(0,T), \lambda \in \Lambda(\bar{h}), \bar{h} \geq ||u||_{[0,T]}$, $r > 0$ be given and let $h_r \in W^{1,1}(0,T)$ be the function introduced in Lemma (2.1). Put*

$$\lambda_0(h) := l_h\big(u,\lambda(h)\big)(0),$$

$$\mu(r) := S_h(\lambda_0(h_r^0), h_r^0) + \int_{h_r^0}^\infty \nu(\lambda_0(h),h)dh,$$

*where $h_r^0$ is the solution of the equation $S_\varrho(\lambda_0(h_r^0),h_r^0) = r$. For $t \in [0,T]$ put*

$$U_r(t) := S_h\big(l_{h_r(t)}(u,\lambda(h_r(t)))(t), h_r(t)\big) + \int_{h_r(t)}^\infty \nu\big(l_h(u,\lambda(h))(t),h\big)dh.$$

*Then $\mu \in \Lambda$, $\mu(0) = W_\lambda(u)(0)$, $U_0(t) = W_\lambda(u)(t)$ and $U_r(t) = l_r\big(U_0,\mu(r)\big)(t)$ for $t \in [0,T]$.*

**Proof.** The identities $\mu(0) = W_\lambda(u)(0), U_0(t) = W_\lambda(u)(t)$ follow from the fact that $h_r(t) = 0$ for $r = 0$. We further have

$$\frac{d}{dr}\mu(r) = \big(S_{\varrho\varrho} + \lambda'(h_r^0)S_{h\varrho}\big)\big(S_{h\varrho} + \lambda'(h_r^0)S_{\varrho\varrho}\big)^{-1},$$

hence $\left|\frac{d}{dr}\mu(r)\right| \le 1$ a.e.

By definition we have $U_r(0) = \mu(r)$. Let $[t_1, t_2]$ be a subinterval of $[0, T]$ such that $u$ is non-decreasing in $[t_1, t_2]$. Put $\lambda_1(h) := l_h(u, \lambda(h))(t_1)$ for each $h > 0$. By Lemma (1.8) $U_0$ is non-decreasing in $[t_1, t_2]$. Let $R^+$ be the function (1.19). We have $\frac{\partial}{\partial r} h_r(t) > 0$ for every fixed $t \in [t_1, t_2]$, hence there exists a unique $\hat{r}(t)$ such that $h_r(t) \le R^+\big(u(t)\big)$ for $r \le \hat{r}(t), h_r(t) > R^+\big(u(t)\big)$ for $r > \hat{r}(t)$.

Thus for $r \le \hat{r}(t)$ we obtain

$$\begin{aligned} U_r(t) =& S_h\big(l_{h_r(t)}(u, \lambda(h_r(t)))(t), h_r(t)\big) \\ &+ \int_{h_r(t)}^{R^+(u(t))} \nu(u(t) - h, h)dh + \int_{R^+(u(t))}^{\bar{h}} \nu(\lambda_1(h), h)dh. \end{aligned}$$

Using (1.16) we obtain analogously to (1.21)

$$U_r(t) = U_0(t) - r \quad \text{ for } r \le \hat{r}(t) \tag{2.3}$$

For $r > \hat{r}(t)$ we have

$$U_r(t) = S_h\big(\lambda_1(h_r(t)), h_r(t)\big) + \int_{h_r(t)}^{\infty} \nu(\lambda_1(h), h)dh,$$

$$S_\varrho\big(\lambda_1(h_r(t)), h_r(t)\big) = r.$$

This yields $\frac{d}{dt} h_r(t) = 0$ for $r > \hat{r}(t)$ for a.e. $t \in (t_1, t_2)$, in particular

$$U_r(t) = U_r(t_1) \text{ for } r > \hat{r}(t). \tag{2.4}$$

We have indeed $U_{\hat{r}(t)}(t) = U_0(t) - \hat{r}(t)$.

Therefore, (2.3),(2.4) yield $U_r(t) = \max\{U_r(t_1), U_0(t) - r\}$.

We verify in a similar way that if $u$ is non-increasing in $[t_1, t_2]$ (and, consequently, $U_0$ is non-increasing in $[t_1, t_2]$), then

$$U_r(t) = \min\{U_r(t_1), U_0(t) + r\} \text{ for } t \in [t_1, t_2].$$

By induction we conclude that $U_r(t) = l_r(U_0, \mu(r))(t)$ for every piecewise monotone function $u \in W^{1,1}(0, T)$. The assertion now follows from (1.8),(1.9).

■

(2.5) **Proposition.** *Let us assume (1.11), (1.14) and let $\lambda, \mu \in \Lambda(\bar{h})$ be given. Let $W_\lambda, W_\mu$ be the Preisach operators (1.12). Then*

(i) *for every $u, v \in C([0, T])$ and $t \in [0, T]$ we have $|W_\lambda(u)(t) - W_\mu(v)(t)| \le \varphi(r)||u - v||_{[0,t]} + \int_0^{\bar{h}} |\lambda(h) - \mu(h)|\eta(h)dh$, where $r := \max\{\bar{h}, ||u||_{[0,t]}, ||v||_{[0,t]}\}$ and $\varphi(r) :=$ supess $\{p'(s), s \in [0, r]\} + \int_0^r \eta(h)dh$,*

(ii) *if $p'$ is bounded in $R^1$ and $\eta \in L^1(0, \infty)$, then $W_\lambda$ is Lipschitz in $C([0, T])$,*

(iii) *the operator $W_\lambda$ is invertible in $C([0, T])$ and $W_\lambda^{-1}$ is locally Lipschitz.*

*If moreover $\int_0^\infty \eta_0(h)dh < \alpha$, then $W_\lambda^{-1}$ is Lipschitz with the constant* $2(\alpha - \int_0^\infty \eta_0(h)dh)^{-1}$,

(iv) *if (1.13) holds and $W_\lambda$ is an Ishlinskii operator, then $W_\lambda^{-1}$ is also an Ishlinskii operator.*

**Remark.** A more complete information about the inversion and superposition of Preisach operators in the case $\lambda \equiv 0$ can be found in [7].

**Proof of (2.5)**

(i) For $\lambda \in \Lambda(\bar{h})$ we have by (1.5),(1.6)

$$|l_h(u, \lambda(h))(t) - l_h(v, \lambda(h))(t)| \leq ||u - v||_{[0,t]}, \quad |l_h(u, \lambda(h))(t)| \leq \max\{\bar{h}, ||u||_{[0,t]}\},$$

hence (i),(ii) follow easily from (1.14).

(iii) Let $u, v \in W^{1,1}(0,T)$ be given piecewise monotone functions and put $U = W_\lambda(u), V = W_\lambda(v)$. Our aim is to prove that there exists a function $\psi$ such that for every $t \in [0,T]$ we have

$$|u(t) - v(t)| \leq \psi(max\{||u||_{[0,t]}, ||v||_{[0,t]}, \bar{h}\}||U - V||_{[0,t]}. \tag{2.6}$$

This implies already the local Lipschitz continuity of $W_\lambda^{-1}$ in $C([0,T])$. Indeed, for $||u||_{[0,t]} > \bar{h}$ we have either $||u||_{[0,t]} = u(\tau_+)$ for some $\tau_+ \in [0,t]$ or $||u||_{[0,t]} = -u(\tau_-)$ for some $\tau_- \in [0,t]$. Putting $\mu(h) = l_h(u, \lambda(h))(\tau_\pm)$ we obtain using Lemma (1.6) $\mu(|u(\tau_\pm)|) = 0, \mu(0) = u(\tau_\pm)$, hence $\mu(h) = u(\tau_\pm) \mp h$ for $h \in (0, |u(\tau_\pm)|)$. Consequently, $||U||_{[0,t]} \geq |U(\tau_\pm)| \geq \xi_0(||u||_{[0,t]})$, where $\xi_0(h)$ is the function in (1.14) (ii).

Moreover, by Lemma (1.18) for every $U, V$ continuous and piecewise monotone we can find continuous and piecewise monotone functions $u, v$ such that $U = W_\lambda(u), V = W_\lambda(v)$. Assuming (2.6) we obtain the local Lipschitz continuity of $W_\lambda^{-1}$ by a standard density argument.

It remains to prove (2.6). We can assume $u(t) - v(t) = ||u - v||_{[0,t]} > 0$. Put $h^* := \min\{h > 0; l_h(u, \lambda(h))(t) \leq l_h(v, \lambda(h))(t)\}$ and

$$r^* := S_\varrho(l_{h^*}(u, \lambda(h^*))(t), h^*) = S_\varrho(l_{h^*}(v, \lambda(h^*))(t), h^*),$$

where $S$ is the identification function (1.17).

We have

$$U(t) - V(t) = p\big(u(t)\big) - p\big(v(t)\big) + \int_0^\infty \big(\nu(l_h(u, \lambda(h))(t), h) - \nu(l_h(v, \lambda(h))(t), h)\big)dh$$

and by Lemma (2.2)

$$l_{r^*}\big(U, \mu(r^*)\big)(t) - l_{r^*}\big(V, \mu(r^*)\big)(t) = \int_{h^*}^\infty \big(\nu(l_h(u, \lambda(h))(t), h\big) - \nu\big(l_h(v, \lambda(h))(t), h)\big)dh,$$

hence using (1.5),(1.11) we obtain

$$\alpha\big(u(t)-v(t)\big)\le 2||U-V||_{[0,t]}+\int_0^{h^*}\big(l_h(u,\lambda(h))(t)-l_h(v,\lambda(h))(t)\big)\eta_0(h)dh,$$

hence

$$\big(\alpha-\int_0^{h^*}\eta_0(h)dh\big)\big(u(t)-v(t)\big)\le 2||U-V||_{[0,t]}.$$

We have $h^*\le\max\{||u||_{[0,t]},||v||_{[0,t]},\bar h\}$ and (2.6) follows easily from (1.14)(i).

(iv) Formula (1.17) yields $S(\varrho,h)=\varrho\varphi(h)$, where $\varphi(h):=\alpha h+\int_0^h\int_0^a\eta(b)\,db\,da$.
Let $u\in C([0,T])$ and $\lambda\in\Lambda(\bar h)$ be given. Put $U=W_\lambda(u),\lambda_0(h)=l_h\big(u,\lambda(h)\big)(0)$.
We have $\varphi'(h)>0,\lim\limits_{h\to\infty}\varphi(h)=+\infty$ and Lemma (2.1) implies $h_r(t)=\varphi^{-1}(r)$, hence $h_r(t)$ is independent of $t$.
Let $\beta>0,\xi\in L^1_{loc}(0,\infty)$ and $\mu\in\Lambda(\bar r)$ be arbitrarily chosen and let $Z_\mu:C([0,T])\to C([0,T])$ be the Ishlinskii operator

$$Z_\mu(v)(t)=\beta v(t)+\int_0^\infty l_r\big(v,\mu(r)\big)(t)\xi(r)dr.$$

According to Lemma (2.2) it is convenient to put

$$\mu(r):=\lambda_0(h)\varphi'(h)+\int_h^\infty\lambda_0(h)\eta(a)da$$

where $h=\varphi^{-1}(r)$.
Then $\mu\in\Lambda\big(\varphi(\bar h)\big)$ and

$$l_r\big(U,\mu(r)\big)=l_h\big(u,\lambda(h)\big)(t)\cdot\varphi'(h)+\int_h^\infty l_a\big(u,\lambda(a)\big)(t)\eta(a)da.$$

Put $\sigma(r):=\beta r+\int_0^r\int_0^a\xi(b)\,db\,da$. We have $Z_\mu(U)(t)=\alpha\beta u(t)+\int_0^\infty l_h\big(u,\lambda(h)\big)(t)\delta(h)dh$ where $\delta(h)=\frac{d^2}{dh^2}\big(\sigma(\varphi(h))\big)$.
For $\sigma=\varphi^{-1}$ we obtain $\beta=\frac{1}{\alpha}$ and $Z_\mu\circ W_\lambda=I$, hence $Z_\mu=W_\lambda^{-1}$ is the Ishlinskii operator generated by the function $\sigma=\varphi^{-1}$. ■

(2.7) **Proposition** (Monotonicity). *Let $u,v\in W^{1,1}(0,T),\lambda\in\Lambda(\bar h)$ be given functions and let $W_\lambda$ be the Preisach operator (1.12) satisfying (1.11) with $\eta_0\equiv 0$. Put $N_h(u)(t):=\nu\big(l_h(u,\lambda(h))(t),h\big),N_h(v)(t):=\nu\big(l_h(v,\lambda(h))(t),h\big)$ for $h>0$. Then*

(i) $[\big(N_h(u)\big)'(t)-\big(N_h(v)\big)'(t)][l_h\big(u,\lambda(h)\big)(t)-l_h\big(v,\lambda(h)\big)(t)]\le$
$\le[\big(N_h(u)\big)'(t)-\big(N_h(v)\big)'(t)]\big(u(t)-v(t)\big)$ *a.e.*

(ii) *If $W_\lambda$ is an Ishlinskii operator and (1.13) holds, then*

$$\Big[\big(W_\lambda(u)\big)'(t)-\big(W_\lambda(v)\big)'(t)]\big(u(t)-v(t)\big)\ge$$
$$\ge\frac{1}{2}\frac{d}{dt}[\alpha\big(u(t)-v(t)\big)^2+\int_0^\infty\big(l_h(u,\lambda(h))(t)-l_h(v,\lambda(h))(t)\big)^2\eta(h)dh\Big]\qquad a.e.$$

Before proving (2.7) we state an easy lemma.

(2.8) **Lemma.** *Let $W_\lambda$ be the Preisach operator (1.12), $\frac{\partial\nu}{\partial\varrho} > 0$ a.e. Let $u \in W^{1,1}(0,T)$ be a given function such that $u'(t) \neq 0$ exists and $\big(W_\lambda(u)\big)'(t)$ exists for some $t \in (0,T)$. Then there exists $\hat{h}(t) \geq 0$ such that for $h > \hat{h}(t)$ we have $\frac{d}{dt} l_h\big(u, \lambda(h)\big)(t) = 0$, for $h < \hat{h}(t)$ we have $\frac{d}{dt} l_h\big(u, \lambda(h)\big)(t) = u'(t), l_h\big(u, \lambda(h)\big)(t) = u(t) \pm h$.*

**Remark.** The implication $u'(t) = 0 \Rightarrow \big(W_\lambda(u)\big)'(t) = 0$ is trivial.

**Proof of (2.8)** Let $x_h$ be solution of (1.1),(1.2). Let us suppose that for some $h_1 < h_2$ we have $x_{h_1}(t) \in (-h_1, h_1), |x_{h_2}(t)| = h_2$. Lemma (1.6) and (1.3),(1.4) give $\big|\frac{d}{dh} x_h(t)\big| \leq 1$, which is a contradiction. Put $\hat{h}_2 = \inf\{h > 0; |x_h(t)| < h\}$.

Let us suppose that for some $h_3 < \hat{h}_2$ the derivative $\frac{d}{dt} x_{h_3}(t)$ does not exist. This means $x'_{h_3+}(t) = 0, x'_{h_3-}(t) = u'(t)$ for the right and left derivatives, respectively. The same argument as above shows that the same is true for all $h \in [h_3, \hat{h}_2]$. Put $\hat{h}_1 = \sup\{h < \hat{h}_2; \frac{d}{dt} x_h(t)$ does not exist $\}$. For $h < \hat{h}_1$ we have indeed $\frac{d}{dt} x_h(t) = 0$.

A standard use of the Lebesgue dominated convergence theorem yields

$$\big(W_\lambda(u)\big)'_+(t) - \big(W_\lambda(u)\big)'_-(t) = u'(t) \int_{\hat{h}_1}^{\hat{h}_2} \frac{\partial\nu}{\partial\varrho}\big(l_h(u, \lambda(h)(t), h\big) dh,$$

hence $\hat{h}_1 = \hat{h}_2$ and (2.8) follows easily. ■

**Proof of (2.7)** We obtain from (1.1)(ii),(1.3),(1.4) putting $\varphi := h_h\big(v, \lambda(h)\big)(t)$

$$\big(N_h(u)\big)'(t) \cdot [f_h\big(u, \lambda(h)\big)(t) - f_h\big(v, \lambda(h)\big)(t)] \geq 0$$

and similarly

$$\big(N_h(v)\big)'(t) \cdot [f_h\big(v, \lambda(h)\big)(t) - f_h\big(u, \lambda(h)\big)(t)] \geq 0,$$

which gives (i). Part (ii) follows immediately from (i) and Lemma (2.8). ■

## 3. Energy Potentials

The role of the convexity of hysteresis loops in the theory of hyperbolic equations with hysteresis has been pointed out several times (cf. e.g. [4],[5]). We present here a different approach which consists in determining sufficient conditions for the function $\Phi$ from Lemma (1.18) to be convex when $u$ increases and concave when $u$ decreases.

(3.1) **Lemma.** *Let $\lambda \in \Lambda(\bar{h})$ be given and let $W_\lambda$ be the Preisach operator (1.12) with $p(u) = \alpha u$ and $\frac{\partial\nu}{\partial\varrho}, \frac{\partial^2\nu}{\partial\varrho^2}$ continuous in $R^1 \times [0,\infty)$, $\frac{\partial\nu}{\partial\varrho}(0,0) > 0$.*

*Let $\Phi$ be the function (1.21) and let us denote by $\Phi'_+, \Phi'_-$ its right and left derivatives, respectively. Then there exists $U_0 \in (0, +\infty]$ and a continuous non-increasing function $\gamma : [0, U_0) \to R^1_+$ such that for every $u \in C([0,T])$ the following implications hold:*

(i) *If $u$ is non-decreasing in $[t_1, t_2]$ and $\max\{\bar{h}, ||u||_{[0,t_2]}\} < U < U_0$, then $\Phi'_-(v_2) - \Phi'_+(v_1) \geq 2\gamma(U)(v_2 - v_1)$ for all $v_2 > v_1, \quad v_1, v_2 \in [u(t_1), u(t_2)]$;*

(ii) *If $u$ is non-increasing in $[t_1, t_2]$ and $\max\{\bar{h}, ||u||_{[0,t_2]}\} < U < U_0$ then $\Phi'_-(v_2) - \Phi'_+(v_1) \leq -2\gamma(U)(v_2 - v_1)$ for all $v_2 > v_1, \quad v_2, v_1 \in [u(t_2), u(t_1)]$.*

**Proof.**

(i) Let $R^+$ be the function (1.19) and put $\lambda_1(h) := l_h\big(u, \lambda(h)\big)(t_1)$ for $h > 0$ as in the proof of Lemma (1.18). Let $u(t_1) \leq v_1 < v_2 \leq u(t_2)$ be given and let us choose an arbitrary sequence $v_n \downarrow v_1$. Then $R^+(v_n) \searrow R^+(v_1)$ and (1.22) yields

$$\Phi'_+(v_1) = (S_{h\varrho} + S_{\varrho\varrho})\big(\lambda(R^+(v_1)), R^+(v_1)\big) = \alpha + \int_0^{R^+(v_1)} \frac{\partial\nu}{\partial\varrho}(v_1 - h, h)dh.$$

Similarly, putting $R_0^+(v) = \min\{q > 0; v = q + \lambda_1(q)\}$ we obtain

$$\Phi'_-(v_2) = \alpha + \int_0^{R_0^+(v_2)} \frac{\partial\nu}{\partial\varrho}(v_2 - h, h)dh.$$

Therefore,

$$\Phi'_-(v_2) - \Phi'_+(v_1) = \int_0^{R^+(v_1)} \int_{v_1}^{v_2} \frac{\partial^2\nu}{\partial\varrho^2}(a - h, h)dadh + \int_{R_0^+(v_2)}^{R^+(v_1)} \frac{\partial\nu}{\partial\varrho}(v_2 - h, h)dh.$$

Put

$$\delta_1(U) := \min\{\left|\frac{\partial\nu}{\partial\varrho}(\varrho, h)\right|; |\varrho| + h \leq U\},$$
$$\delta_2(U) := \max\{\left|\frac{\partial^2\nu}{\partial\varrho^2}(\varrho, h)\right|; |\varrho| + h \leq U\}.$$

Then $\delta_1$ is non-increasing, $\delta_2$ is non-decreasing and for $\bar{h}, |v_1|, |v_2| \in (0, U]$ we have $R^+(v_1), R_0^+(v_2) \in (0, U]$ and $\Phi'_-(v_2) - \Phi'_+(v_1) \geq \big(R_0^+(v_2) - R^+(v_1)\big)\delta_1(U) - U\delta_2(U) \cdot (v_2 - v_1)$. We have $v_2 - v_1 \leq 2\big(R_0^+(v_2) - R^+(v_1)\big)$, hence putting $U_0 = \inf\{U > 0; \frac{1}{2}\delta_1(U) - U\delta_2(U) > 0\}, \gamma(U) = \frac{1}{2}\big(\frac{1}{2}\delta_1(U) - U\delta_2(U)\big)$ we obtain (i).

(ii) The argument is similar. We use the functions $R^-$ defined by (1.23) and $R_0^-(v) := \min\{q > 0; -q + \lambda_1(q) = v\}$ and we obtain the formula

$$\Phi'_-(v_2) - \Phi'_+(v_1) = \int_0^{R^-(v_2)} \int_{v_1}^{v_2} \frac{\partial^2\nu}{\partial\varrho^2}(a + h, h)da\ dh - \int_{R^-(v_2)}^{R_0^-(v_1)} \frac{\partial\nu}{\partial\varrho}(v_1 + h, h)dh,$$
$$2\big(R_0^-(v_1) - R^-(v_2)\big) \geq v_2 - v_1,$$

hence

$$\Phi'_-(v_2) - \Phi'_+(v_1) \leq -\big(\frac{1}{2}\delta_1(U) - U\delta_2(U)\big)(v_2 - v_1).$$

(3.2) **Lemma.** *Let $\Psi$ be an absolutely continuous increasing function and let $\Psi'_+, \Psi'_-$ exist at every point of its domain of definition. Let $K > 0$ be a given constant and $u \in W^{1,\infty}(0,T)$ a given function such that $\Psi(u) \in W^{2,1}(0,T)$. Then the following implications hold:*

(i) If $u$ is non-decreasing in $[t_1, t_2]$ and $\Psi'_-(v_2) - \Psi'_+(v_1) \geq K(v_2 - v_1)$ for all $v_1 < v_2$, $v_1, v_2 \in [u(t_1), u(t_2)]$, then $(\frac{1}{2}\Psi(u)' \cdot u') \in BV(t_1, t_2)$ and

$$\int_{t_1}^{t_2} \big(\Psi(u)\big)''(t) u'(t) dt \geq \Big[\frac{1}{2}\big(\Psi(u(t))\big)' u'(t)\Big]_{t_1}^{t_2} + \frac{1}{2} K \int_{t_1}^{t_2} |u'(t)|^3 dt. \tag{3.3}$$

(ii) If $u$ is non-increasing in $[t_1, t_2]$ and $\Psi'_-(v_2) - \Psi'_+(v_1) \leq -K(v_2 - v_1)$ for all $v_1 < v_2$, $v_1, v_2 \in [u(t_2), u(t_1)]$, then (3.3) holds.

**Proof.** The problem consists in justifying the integration by parts at the left-hand side of (3.3). Put $w(t) = \Psi\big(u(t)\big)$ and $\Psi_n(v) = \int_{\infty}^{\infty} n\varrho\big(n(v-\sigma)\big)\Psi(\sigma)d\sigma$, where $u \in N$ is an arbitrary integer and $\varphi \in \mathcal{D}(-1,1)$ is a nonnegative mollifier, $\int_{-1}^{1} \varphi(\sigma)d\sigma = 1$. Put $u_n(t) = \Psi_n^{-1}\big(w(t)\big)$. The functions $\Psi_n$ converge to $\Psi$ locally uniformly, $\Psi'_n$ are locally bounded away from 0 and $\Psi_n(u_n) - \Psi_n(u) = \Psi(u) - \Psi_n(u)$, hence $u_n \to u$ uniformly. The identity $\Psi'_n(u_n)u'_n = \Psi'(u)u'$ a.e. yields $|u'_n(t)| \leq const.|u'(t)|$ a.e., hence $u'_n \to u'$ in $L^\infty$-weak *. The function $\Psi'$ is monotone in $[u(t_1), u(t_2)]$ (or $[u(t_2), u(t_1)]$), hence it has at most countably many points of discontinuity.

If $u(t)$ is a point of continuity of $\Psi'$, then $\Psi'_n\big(u_n(t)\big) \to \Psi'\big(u(t)\big)$, hence $u'_n(t) \to u'(t)$. Put $M := \{t \in (t_1, t_2); u'_n(t) \to u'(t)\}$. We have meas $u(M) = \big|\int_M u'(t)dt\big| = 0$ since $\Psi'(v)$ is discontinuous for every $v \in u(M)$. Consequently, $u'(t) = 0$ for a.e. $t \in M$ and meas $M = 0$. This implies $u_n \to u$ in $W^{1,p}(0,T)$-strong for every $p \in [1, \infty)$.

In the case (i) we have $\liminf_{n\to\infty} \Psi''_n\big(u_n(t)\big) \geq K$ for every $t \in [t_1.t_2]$ hence the identity

$$\int_{\tau_1}^{\tau_2} \big(\Psi_n(u_n(t))\big)'' u'_n(t) dt = \Big[\frac{1}{2}\big(\Psi_n(u_n(t))\big)' u'_n(t)\Big]_{\tau_1}^{\tau_2} + \frac{1}{2}\int_{\tau_1}^{\tau_2} \Psi''_n(u_n(t))|u'_n(t)|^3 dt$$

for $t_1 \leq \tau_1 < \tau_2 \leq t_2$ yields

$$\int_{\tau_1}^{\tau_2} \big(\Psi(u(t))\big)'' u'(t) dt \geq \Big[\frac{1}{2}\big(\Psi(u(t))\big)' u'(t)\Big]_{\tau_1}^{\tau_2} + \frac{K}{2}\int_{\tau_1}^{\tau_2} |u'(t)|^3 dt$$

for a.e. $\tau_1, \tau_2 \in [t_1, t_2], \tau_1 < \tau_2$.

The function $\tau \longmapsto \frac{1}{2}\big[\big(\Psi(u(t))\big)' u'(t)\big]_{t_1}^{\tau} + \frac{K}{2}\int_{t_1}^{\tau} |u'(t)|^3 dt - \int_{t_1}^{\tau} (\Psi(u(t))'' n'(t) dt$ is non-increasing, hence $\frac{1}{2}\big(\Psi(u)'u'\big) \in BV(t_1, t_2)$ and (3.3) holds. The case (ii) is analogous.
■

In what follows we reduce the class of Preisach operators (1.12). We assume

(3.4) $(i)$ $\dfrac{\partial \nu}{\partial \varrho}, \dfrac{\partial^2 \nu}{\partial \varrho^2}$ are continuous in $R^1 \times [0, \infty)$,

$(ii)$ $\eta$ is continuous in $[0, \infty)$,

$(iii)$ $p(\varrho) = \alpha\varrho$ for $\varrho \in R^1$,

$(iv)$ $\dfrac{\partial \nu}{\partial \varrho} > 0, \quad \eta(h) \geq \dfrac{\partial \nu}{\partial \varrho}(\varrho, h) \geq 0 \quad \forall (\varrho, h) \in R^+ \times [0, \infty)$.

We further define the function

$$B(\varrho, h) = \int_0^\varrho \sigma \frac{\partial \nu}{\partial \sigma}(\sigma, h) d\sigma. \tag{3.5}$$

For $u \in W^{1,1}(0,T)$ we introduce the energy potential

$$(3.6) \qquad \begin{aligned} &(i) \quad P_1(u)(t) = \frac{\alpha}{2} u^2(t) + \int_0^\infty B\big(l_h(u, \lambda(h))(t), h\big) dh \\ &(ii) \quad P_2(u)(t) = \frac{1}{2} W_\lambda(u)'(t) u'(t), \end{aligned}$$

where $B$ is given by (3.5) and $W_\lambda$ is the operator (1.12).

The potential $P_2$ does not correspond to the usual physical notion of energy. Its physical meaning does not seem obvious.

(3.7) **Theorem.** *Put* $\xi(r) := \alpha + \int_0^{\max\{\bar{h}, r\}} \eta(h) dh$. *For every* $u \in W^{1,1}(0,T)$ *we have* $P_1(u) \in W^{1,1}(0,T)$ *and the inequalities*

$$\begin{aligned} &(i) \qquad \big(W_\lambda(u)(t)\big)^2 \le 2\xi(||u||_{[0,T]}) \cdot P_1(u)(t), \\ &(ii) \qquad \big(P_1(u)\big)'(t) \le \big(W_\lambda(u)\big)'(t) u(t) \end{aligned}$$

*hold (almost) everywhere in* $(0,T)$.

**Proof.** For all $(\varrho, h) \in R^1 \times [0, \infty)$ we have

$$\nu(\varrho, h) \frac{\partial \nu}{\partial \varrho}(\varrho, h) \text{ sign } \varrho \le |\varrho| \eta(h) \frac{\partial \nu}{\partial \varrho}(\varrho, h),$$

hence $\nu^2(\varrho, h) \le 2\eta(h) B(\varrho, h)$. Put $\beta = \int_0^{\max\{\bar{h}, ||u||_{0,T]}\}} \eta(h) dh$. For $\varepsilon = \frac{\alpha}{\beta}$ Hölder's inequality yields

$$\big(W_\lambda(u)(t)\big)^2 \le (1 + \frac{1}{\varepsilon}) \big(\alpha u(t)\big)^2 + (1 + \varepsilon) \Big( \int_0^\infty \nu(l_h(u, \lambda(h))(t), h) dh \Big)^2$$

and

$$\Big( \int_0^\infty \nu(l_h(u, \lambda(h))(t), h) dh \Big)^2 \le \beta \int_0^\infty \frac{1}{\eta(h)} \nu^2 \big(l_h\big(u, \lambda(h)\big)(t), h\big) dh$$

which implies (i).

Part (ii) is an easy consequence of (2.8) and (2.7)(i) for $v \equiv 0$. ■

(3.8) **Theorem.**

(i) *Let* $\xi(r)$ *be as in (3.7). Then for every* $u \in W^{1,1}(0,T)$ *the inequalities* $[(W_\lambda(u))'(t)]^2 \le 2\xi(||u||_{[0,T]}) P_2(u)(t)$, $\frac{1}{2}\alpha(u'(t))^2 \le P_2(u)(t)$ *hold almost everywhere in* $(0,T)$.

(ii) *Let* $U_0, \gamma$ *be as in Lemma (3.1). Let* $u \in W^{1,\infty}(0,T)$ *be such that* $W_\lambda(u) \in W^{2,1}(0,T)$. *If* $\max\{\bar{h}, ||u||_{[0,T]}\} \leq U < U_0$, *then we have* $P_2(u) \in BV(0,T)$ *and* $[P_2(u)(t)]_{t_1}^{t_2} \leq \int_{t_1}^{t_2}(W_\lambda(u))''(t)u'(t)dt - \gamma(U)\int_{t_1}^{t_2}|u'(t)|^3 dt$ *for all* $0 \leq t_1 < t_2 \leq T$.

**Remark.** We see an important formal similarity between (3.7) and (3.8). This justifies the "energy" terminology.

**Proof of (3.8)**

(i) It is easy to see that $W_\lambda(u)$ is absolutely continuous, hence (i) follows from (2.8).

(ii) The function $\big(W_\lambda(u)\big)'$ is absolutely continuous, hence the set $Z := \{t \in [t_1, t_2];\ \big(W_\lambda(u)\big)'(t) \neq 0\}$ is open, $Z = \bigcup_{k=1}^{\infty}(a_k, b_k)$. By Lemmas (3.1), (3.2) (ii) holds if $t_1$ is replaced by $a_k$ and $t_2$ by $b_k$. Moreover, for $t_1 < a_i < t_2, t_1 < b_j < t_2$ we have by (i) $P_2(u)(a_i-) = P_2(u)(a_i+) = 0, P_2(u)(b_j-) = P_2(u)(b_j+) = 0$. The same argument as at the end of the proof of Lemma (3.2) shows that $P_2(u) \in BV(0,T)$. The assertion now follows from the additivity of the Lebesgue integral.

(3.9) **Corollary.** *Let* $W_\lambda$ *be the Ishlinskii operator (1.13) and let* $\eta : [0,\infty) \to [0,\infty)$ *be continuous and positive in* $[0,\infty)$. *Then the conclusion of Theorem (3.8) holds for* $U_0 = +\infty$ *and* $\gamma(U) = \frac{1}{4}\min\{\eta(h), 0 \leq h \leq U\}$.

**Proof.** The formulas for $U_0$ and $\gamma(U)$ are given in the proof of Lemma (3.1). ■

## 4. Dependence on parameters.

We have to consider hysteresis operators acting on functions of one "time" variable and several "spatial" variables. For our purposes it suffices to consider functions $u : [0,1] \times [0,T] \to R^1$ such that for every $x \in [0,1]$ the function $u(x,\cdot)$ belongs to $C([0,T])$. The initial state $\lambda$ may also depend on x.

We assume

(4.1) $\qquad (i) \quad \lambda : [0,1] \times [0,\infty) \to R^1$ is continuous,

$\qquad\quad (ii) \quad \lambda(x,\cdot) \in \Lambda(\bar{h})$ for every $x \in [0,1]$,

and we define for every $u \in C([0,1] \times [0,T])$ and $(x,t) \in [0,1] \times [0,T]$

$$W(u)(x,t) := W_{\lambda(x,\cdot)}(u(x,\cdot))(t), \tag{4.2}$$

where $W_{\lambda(x,\cdot)}$ is the operator (1.12).

(4.3) **Proposition.** *Let us assume (3.4) and let* $\lambda$ *satisfying (4.1) be given. Then* $W$ *given by (4.2) is a locally Lipschitz operator in* $C([0,1] \times [0,T])$ *which is invertible and* $W^{-1}$ *is Lipschitz.*

**Proof.** For $u \in C([0,1] \times [0,T])$ put $U(x,t) := W(u)(x,t), ||u|| := \max\{u(x,t)|, x \in [0,1], t \in [0,T]\}, r = \max\{\bar{h}, ||u||\}$. We have to prove first that $U \in C([0,1] \times [0,T])$. Let $0 \le s < t \le T, x, y \in [0,1]$ be given. We have by (2.5)(i)

$$|U(x,t) - U(y,s)| \le |U(x,t) - U(x,s)| + \varphi(r)||u(x,\cdot) - u(y,\cdot)||_{[0,s]} + \\ + \int_0^{\bar{h}} |\lambda(x,h) - \lambda(y,h)|\, \eta(h)dh,$$

hence $U$ is continuous. The local Lipschitz continuity of $W$ follows easily from (2.5)(i).

Let further $U \in C([0,1] \times [0,T])$ be given and put $u(x,t) := W^{-1}_{\lambda(x,\cdot)}(U(x,\cdot))(t)$ (the invertibility of $W_{\lambda(x,\cdot)}$ is ensured by (2.5)(iii)). We have

$$||W_{\lambda(x,\cdot)}(u(x,\cdot)) - W_{\lambda(x,\cdot)}(u(y,\cdot))||_{[0,s]} \le \frac{\alpha}{2}||u(x,\cdot) - u(y,\cdot)||_{[0,s]},$$

$$||W_{\lambda(x,\cdot)}(u(y,\cdot)) - W_{\lambda(y,\cdot)}(u(y,\cdot))||_{[0,s]} \le \int_0^{\bar{h}} |\lambda(x,h) - \lambda(y,h)|\eta(h)dh,$$

hence $u \in C([0,1] \times [0,T])$. The Lipschitz continuity of $W^{-1}$ is an immediate consequence of (2.5)(iii). ■

(4.4) **Remarks.**

(i) The operator $W$ given by (4.2) depends continuously on $\lambda$. If $\lambda_1, \lambda_2$ are two functions satisfying (4.1) and $W_i$ is the operator corresponding to $\lambda_i$, $i = 1, 2$, then for every $u, v \in C([0,1] \times [0,T])$ we have

$$||W_1(u) - W_2(v)|| \le \varphi(r)||u - v|| + \max\{\int_0^{\bar{h}} |\lambda_1(x,h) - \lambda_2(x,h)|\eta(h)dh; x \in [0,1]\}.$$

(ii) There is a slight ambiguity in the formula (4.2), where the dot in $u(x,\cdot)$ replaces the "time" variable $t$ and in $\lambda(x,\cdot)$ the "memory" variable $h$. Here, the "memory" character of $h$ is not as important as in [7].

## 5. Statement of the problem.

Our aim here is to solve the problem

$$\text{(5.1)} \qquad \begin{aligned} &(i) && W(u_t)_t - u_{xx} = g(x,t), \\ &(ii) && u(0,t) = u_x(1,t) = 0, \\ &(iii) && u(x,0) = u^o(x), u_t(x,0) = u^1(x), \end{aligned}$$

where $u^o, u^1, g$ are given functions and $W$ is the Preisach operator (4.2) satisfying the assumptions of Proposition (4.3).

We first give a list of assumptions.

(5.2)

$g \in L^\infty_{loc}(0,\infty; L^2(0,1)), G : [0,\infty) \to [0,\infty)$ are given functions such that

$(i)$ $G$ is nonincreasing in $[0,\infty]$,

$(ii)$ $g_t \in L^\infty(0,\infty; L^2(0,1))$,

$(iii)$ $\int_0^1 |g_t(x,t)|^2 dx \le G(t)$ a.e.,

(5.3) $u^o \in W^{2,2}(0,1),\ u^1 \in W^{1,2}(0,1)$ are given functions such that $u^o(0) = u^1(0) = {u^o}'(1) = 0$.

We put $\tilde{E}(0) := \frac{1}{2}\int_0^1 \left[\frac{1}{\alpha}({u^o}''(x) + g(x,0))^2 + |{u^1}'(x)|^2\right] dx,$

(5.4) there exist $U \in (0, U_0)$ and $\delta > 0$ such that $\bar{h} \le U$ and

(i) $G(0)\frac{\gamma(U)}{\xi(U)} < \frac{3}{8}$,

(ii) $6\tilde{E}(0) + 3G(0)\left(4\frac{\xi(U)}{\gamma(U)} + G(0)\right) < (1-\delta)U^2$, where $U_0, \gamma, \xi$ are introduced in Lemma (3.1) and Theorem (3.8).

(5.5) **Remark.** The condition (5.4) needs some comment. For an arbitrary operators $W$ and $U < U_0$ (5.4) holds if the data $\bar{h}, u^o, u^1, g_t$ are sufficiently small in appropriate norms. On the other hand, if $W$ is an Ishlinskii operator satisfying the assumptions of (3.9) and

$$\lim_{U\to\infty} \frac{\gamma(U)}{\xi(U)} = 0, \qquad \lim_{U\to\infty} \frac{U^2\gamma(U)}{\xi(U)} = +\infty,$$

then (5.4) holds for arbitrary data and U sufficiently large.

We can easily see that Ishlinskii operators satisfying (5.6) exist. Putting in (1.13) $\eta(h) = h^{\sigma-2}$ for some $\sigma \in (1,2)$, we have $\gamma(U) = \frac{1}{4}\eta(U) = \frac{1}{4}U^{\sigma-2}, \xi(U) = \alpha + \frac{1}{\sigma-1}U^{\sigma-1}$ for $U$ sufficiently large, hence (5.6) holds.

The main results of this paper are the following:

(5.7) **Theorem.** (Existence). *Let (5.2)-(5.4) hold. Then there exists a continuous function* $u : [0,1]\times[0,\infty) \to R^1$ *such that*

$$u_{tt}, u_{xt} \in L^\infty\big(0,\infty; L^2(0,1)\big), u_{xx} \in L^\infty_{loc}(0,\infty; L^2(0,1)),$$

*(5.1)(ii),(iii) hold for all* $t \ge 0$ *and* $x \in [0,1]$, *(5.1)(i) holds almost everywhere in* $(0,1)\times(0,\infty)$ *and* $|u_t(x,t)| < U$ *for all* $(x,t) \in [0,1]\times[0,\infty)$.

(5.8) **Theorem.** (Uniqueness). *Let (5.2)-(5.4) and let $u, v$ be two solutions of (5.1) satisfying Theorem (5.7).*

*(i) If $W$ is an Ishlinskii operator, then $u = v$.*

*(ii) If $W$ is a general Preisach operator and $u_{tt}, v_{tt} \in L^1_{\text{loc}}(0, \infty; L^\infty(0,1))$, then $u = v$.*

(5.9) **Theorem.** (Asymptotic behaviour). *Let (5.2)-(5.4) hold and let $u$ be a solution of (5.1) satisfying Theorem (5.7). let us assume $\lim\limits_{t\to\infty} G(t) = 0$. Then there exists a function $\kappa : [0,\infty) \to [0,\infty)$ such that $\lim\limits_{t\to\infty} \kappa(t) = 0$ and*

$$|u_t(x,t)| \leq \kappa(t) \qquad \forall (x,t) \in [0,1] \times [0,\infty).$$

*If moreover $G(t) = 0$ for $t \geq t_0$, then there exists a function $v \in W^{2,2}(0,1)$ and a constant $K > 0$ such that $v(0) = v'(1) = 0$ and*

$$|u_t(x,t)| + |u_x(x,t) - v'(x)| \leq \frac{K}{t} \qquad \forall (x,t) \in [0,1] \times [0,\infty).$$

**Remark.** The qualitative analysis of the 'ordinary' equation $u'' + W^{-1}(u) = 0$ (cf.[4]) shows that the estimate $\frac{K}{t}$ can hardly be improved.

(5.10) **Proposition.** (Regularity). *Let (5.7) hold. Then the functions $W(u_t)_t, u_{xt} : [0,\infty] \to L^2(0,1)$ are weakly continuous.*

## 6. Approximation and estimates.

We apply the space-discretization method. Let $n > 0$ be a given integer. For $t \geq 0$ put $g_j(t) := n \int_{\frac{j}{n}}^{\frac{j+1}{n}} \int_0^t g(\xi, \tau)\, d\tau\, d\xi, \quad j = 0, 1, \cdots, n-1.$

Let us consider the system of differential equations for $j = 1, \cdots, n-1$.

$$\begin{aligned} W_j(u_j')(t) =& \Delta_j v(t) + g_j(t), \\ v_j'(t) =& \Delta_{j-1} u(t), \end{aligned} \tag{6.1}$$

for unknown functions $u_1, ..., u_{n-1}, v_1, \cdots, v_{n-1}$, where we put $u_0 = v_n = 0,\ v_0 = v_1,\ u_n = u_{n-1},\ \Delta_j v := n(v_{j+1} - v_j),\ \Delta_{j-1} u = n(u_j - u_{j-1}),\ W_j := W_{\lambda(\frac{j}{n}, \cdot)}$, with initial conditions

$$u_j(0) = u^0\left(\frac{j}{n}\right),\ \Delta_j v\,(0) = \Phi_0^j\left(u^1\left(\frac{j}{n}\right)\right),\ v_n(0) = 0, \tag{6.2}$$

where $\Phi_0^j$ is the function corresponding to $W_j$ in the sense of (1.25).

By (2.5)(iii), (3.4) the operators $W_j^{-1}$ are Lipschitz in $C([0,T])$ for every $T > 0$ and $j \geq 1$ with the Lipschitz constant $\frac{2}{\alpha}$, hence the system (6.1),(6.2) has a unique global classical solution $\{u_j, v_j; j = 1, \cdots, n-1\}$ and (3.8)(i) implies $u_j, v_j \in W^{2,\infty}_{loc}(0,\infty)$.

We obtain from (6.1),(6.2) $u'_j(0) = u^1\left(\frac{j}{n}\right)$ and from (5.3),(5.4) $|u^1(\xi)| \leq (2\tilde{E}(0))^{\frac{1}{2}} \leq [(1-\delta)/3]^{\frac{1}{2}} U$ for each $\xi \in [0,1]$, hence $u'_j(t) < U$ for every $j = 1, \cdots, n-1$ and $t$ sufficiently small. Put

$$T_n = \inf\{t > 0; u'_j(t) > U \text{ for some } j = 1, \cdots, n-1\}$$

and

$$E^{(n)}(t) = \frac{1}{n}\sum_{j=1}^{n-1}\left(P_2^j(u'_j)(t) + \frac{1}{2}(v''_j(t))^2\right)$$

for $t \in (0, T_n)$, where $P_2^j$ is the potential (3.6)(ii) corresponding to $W_j$.

Equations (6.1) and Theorem (3.8)(i) yield

$$E^{(n)}(t) \leq \frac{1}{2n}\sum_{j=1}^{n-1}\left[\frac{1}{\alpha}(n(\Delta_j u(t) - \Delta_{j-1}u(t)) + g'_j(t))^2 + \right.$$
$$\left. + n^2(W_j^{-1}(\Delta_j v + g_j)(t) - W_{j-1}^{-1}(\Delta_{j-1}v + g_{j-1})(t))^2\right].$$

The right-hand side of the last inequality is continuous with respect to $t$, hence

$$E^{(n)}(0+) \leq \frac{1}{2n}\sum_{j=1}^{n-1}\left[\frac{1}{\alpha}(n(\Delta_j u(0) - \Delta_{j-1}u(0)) + g'_j(0))^2 + \right.$$
$$\left. + n^2((\Phi_0^j)^{-1}(\Delta_j v(0)) - (\Phi_0^{j-1})^{-1}(\Delta_{j-1}v(0)))^2\right].$$

This yields

$$\limsup_{n\to\infty} E^{(n)}(0+) \leq \tilde{E}(0)$$

hence (5.4) (ii) remains valid if $\tilde{E}(0)$ is replaced by $E^{(n)}(0+)$ for $n$ sufficiently large. We can differentiate (6.1) twice with respect to $t$, hence

$$(6.3) \qquad \begin{array}{ll} (i) & W_j(u'_j)'' = \Delta_j v'' + g''_j, \\ (ii) & v'''_j = \Delta_{j-1}u'' \end{array}$$

holds in the sense of distributions and almost everywhere.

Let $\sigma \in (0, T_n)$ be arbitrarily chosen. Multiplying (6.3) by $u''_j$ and (ii) by $v''_j$ we obtain from (3.8)(ii) for all $\sigma \leq s < t < T_n$

$$E^{(n)}(t-) - E^{(n)}(s+) + \gamma(U)\int_s^t \frac{1}{n}\sum_{j=1}^{n-1}|u''_j(\tau)|^3 d\tau \leq$$
$$\leq \frac{1}{n}\sum_{j=1}^{n-1}\int_s^t u''_j(\tau)g''_j(\tau)d\tau.$$

We have indeed $\frac{1}{n}\sum_{j=1}^{n-1}|g_j''(t)|^2 \le \int_0^1 |g_t(x,t)|^2 dx$ for every $\tau \le 0$, hence (3.8)(i), (5.2)(iii) and Hölder's inequality yields

$$\text{(6.4)} \qquad E^{(n)}(t-) - E^{(n)}(s+) + B\frac{1}{n}\sum_{j=1}^{n-1}\int_s^t \left|P_2^j(u_j')(\tau)\right|^{\frac{3}{2}} d\tau \le$$
$$\le K_1 G^{\frac{3}{2}}(\sigma)(t-s),$$

where $B : \sqrt{2}\gamma(U)\xi(U)^{-\frac{3}{2}}, K_1 := \left(\frac{2}{3}\right)^{\frac{3}{2}}\gamma(U)^{-\frac{1}{2}}$.

We further multiply (6.3)(i) by $u_j'$ and we obtain after integration

$$\frac{1}{n}\sum_{j=3}^{n-1}\int_s^t \left(v_j''(\tau)\right)^2 d\tau = \frac{2}{n}\sum_{j=1}^{n-1}\int_s^t P_2^j(u_j')(\tau)\, d\tau +$$
$$+\frac{1}{n}\sum_{j=1}^{n-1}\int_s^t g_j''(\tau)u_j'(\tau)\, d\tau + V^{(n)}(s) - V^{(n)}(t),$$

where $V^{(n)}(\tau) := \frac{1}{n}\sum_{j=1}^{n-1} W_j(u_j')'(\tau)u_j'(\tau)$.

The relation $\frac{1}{2}\sum_{j=1}^{n-1}\left|u_j'(\tau)\right|^2 \le \frac{1}{n}\sum_{j=1}^{n-1}|\Delta_{j-1}u'(\tau)|^2 = \frac{1}{n}\sum_{j=1}^{n-1}\left|v_j''(\tau)\right|^2$ then gives

$$\text{(6.5)} \qquad V^{(n)}(t) - V^{(n)}(s) +$$
$$+\int_s^t E^{(n)}(\tau)d\tau \le 3(t-s)^{\frac{1}{3}}\left[\frac{1}{n}\sum_{j=1}^{n-1}\int_s^t \left|P_s^j(u_j')(\tau)\right|^{\frac{3}{2}} d\tau\right]^{\frac{2}{3}} +$$
$$+\frac{1}{2}G^2(\sigma)(t-s).$$

Let us choose a number $\varepsilon(\sigma) > 0$ such that

$$\text{(6.6)} \qquad \left[\frac{1}{3}G(\sigma)\frac{\xi(U)}{\gamma(U)}\right]^{\frac{1}{2}} \le \varepsilon(\sigma) \le \left[\frac{1}{3}G(0)\frac{\xi(U)}{\gamma(U)}\right]^{\frac{1}{2}}.$$

Then (6.5) implies

$$\text{(6.7)} \qquad V^{(n)}(t) - V^{(n)}(s) + \int_s^t E^{(n)}(\tau)d\tau \le \left(\frac{1}{2}G^2(\sigma) + 4\varepsilon^2(\sigma)\right)(t-s) +$$
$$+\frac{1}{\varepsilon(\sigma)}\cdot\frac{1}{n}\sum_{j=1}^{n-1}\int_s^t \left|P_2^j(u_j')(\tau)\right|^{\frac{3}{2}} d\tau.$$

Let us denote $F_\sigma^{(n)}(\tau) := E^{(n)}(\tau) + B\varepsilon(\sigma)V^{(n)}(\tau)$ for $\tau > \sigma$.

We have by (3.8)(i) $V^{(n)}(\tau) \le \xi(U)^{\frac{1}{2}} E^{(n)}(\tau)$, hence (5.4)(i) implies

$$\frac{1}{2} E^{(n)}(\tau) \le F_\sigma^{(n)}(\tau) \le \frac{3}{2} E^{(n)}(\tau)$$

for a.e. $\tau \ge \sigma$.

Putting (6.4) and (6.7) together we obtain

$$F_\sigma^{(n)}(t-) - F_\sigma^{(n)}(s+) + B\varepsilon(\sigma) \int_s^t E^{(n)}(\tau) d\tau \le K_2(\sigma)(t-s),$$

where

$$K_2(\sigma) := K_1 G^{\frac{3}{2}}(\sigma) + B\varepsilon(\sigma) \left( \frac{1}{2} G^2(\sigma) + 4\varepsilon^2(\sigma) \right),$$

consequently

$$F_\sigma^{(n)}(t-) - F_\sigma^{(n)}(s+) + \frac{2}{3} B\varepsilon(\sigma) \int_s^t F_\sigma^{(n)}(\tau) d\tau \le K_2(\sigma)(t-s). \tag{6.8}$$

We see that the function

$$t \to F_\sigma^{(n)}(t) + \frac{2}{3} B\varepsilon(\sigma) \int_\sigma^t F_\sigma^{(n)}(\tau) d\tau - K_2(\sigma) t$$

is nonincreasing in $(\sigma, T_n)$, hence its derivative in the sense of distributions is non-positive. For every smooth positive function $\varphi$ and every $t \in (\sigma, T_n)$ this yields

$$\varphi(t) F_\sigma^{(n)}(t-) - \varphi(\sigma) F_\sigma^{(n)}(\sigma+) + \int_\sigma^t \Big[ -F_\sigma^{(n)}(\tau)\varphi'(\tau) + \\ + \frac{2}{3} B\varepsilon(\sigma) F_\sigma^{(n)}(\tau)\varphi(\tau) - K_2(\sigma)\varphi(\tau) \Big] d\tau \le 0.$$

In particular, for $\varphi(t) = e^{\frac{2}{3} B\varepsilon(\sigma)(t-\sigma)}$ this gives

$$F_\sigma^{(n)}(t-) \le e^{-\frac{2}{3} B\varepsilon(\sigma)(t-\sigma)} F_\sigma^{(n)}(\sigma+) + \frac{3K_2(\sigma)}{2B\varepsilon(\sigma)} \left( 1 - e^{-\frac{2}{3} B\varepsilon(\sigma)(t-\sigma)} \right),$$

hence

$$E^{(n)}(t-) \le 3 \left[ e^{-\frac{2}{3} B\varepsilon(\sigma)(t-\sigma)} E^{(n)}(\sigma+) + \frac{K_2(\sigma)}{B\varepsilon(\sigma)} \right] \tag{6.9}$$

holds for every $0 \le \sigma < t < T_n$.

For $\sigma = 0$ (6.6),(6.9) imply

$$E^{(n)}(t-) \le 3 \left[ E^{(n)}(0+) + G(0) \left( 2 \frac{\xi(U)}{\gamma(U)} + \frac{1}{2} G(0) \right) \right]$$

hence by (5.4)(ii)

$$E^{(n)}(t-) \le \frac{1}{2}(1-\delta)U^2. \tag{6.10}$$

Let us suppose $T_n < +\infty$. For almost every $t < T_n$ we have

$$\max_j |u_j'(t)| \le \left( \frac{1}{n} \sum_{j=1}^{n-1} |\Delta_{j-1} u'(t)|^2 \right)^{\frac{1}{2}} = \left( \frac{1}{n} \sum_{j=1}^{n-1} |v_j''(t)|^2 \right)^{\frac{1}{2}} \le$$
$$\le \left(2E^{(n)}(t)\right)^{\frac{1}{2}} \le \sqrt{1-\delta}U,$$

hence

$$\limsup_{t \to T_n} |u_j'(t)| < U \qquad \text{for all } j = 1, \cdots, n-1,$$

which is a contradiction.

Consequently, $T_n = +\infty$ for $n$ sufficiently large and (6.9),(6.10) hold for all $0 \le \sigma < t < +\infty$. ∎

**7. Proof of existence and regularity.**

Let $\{u_j, v_j;\ j = 1, \cdots, n-1\}$ be the solution of (6.1),(6.2). For $x \in \left[\frac{j}{n}, \frac{j+1}{n}\right)$ and $t \le 0$ we put

$$\begin{aligned}
u^{(n)}(x,t) :=& u_j(t) + \left(x - \frac{j}{n}\right) \Delta_j u(t),\\
v^{(n)}(x,t) :=& v_j(t) + \left(x - \frac{j}{n}\right) \Delta_j v(t),\\
\tilde{u}^{(n)}(x,t) :=& u_j(t),\\
\tilde{v}^{(n)}(x,t) :=& v_{j+1}(t),\\
\tilde{g}^{(n)}(x,t) :=& g_j'(t),\\
\lambda^{(n)}(x,h) :=& \lambda\left(\frac{j}{n}, h\right)
\end{aligned}$$

and for every function $z : [0,1] \times [0,\infty) \to R^1$ such that $z(x,\cdot)$ is continuous for every $x \in [0,1]$ we put

$$W^{(n)}(z)(x,t) := W_{\lambda^{(n)}(x,\cdot)}\big(z(x,\cdot)\big)(t).$$

The system (6.1) can be rewritten in the form

$$\begin{aligned}
W^{(n)}\left(\tilde{u}_t^{(n)}\right) &= v_x^{(n)} + \int_0^t \tilde{g}^{(n)}(x,\tau)d\tau,\\
\tilde{v}_t^{(n)} &= u_x^{(n)},
\end{aligned} \tag{7.1}$$

for $t > 0,\ x \in (0,1) \setminus \left\{ \frac{1}{n}, \frac{2}{n}, \cdots, \frac{j-1}{n} \right\}$.

The estimate (6.10) shows that $u_{xt}^{(n)}, v_{xt}^{(n)}, u_{tt}^{(n)}, v_{tt}^{(n)}$ are bounded in $L^\infty\big(0,\infty; L^2(0,1)\big)$ independently of $n$. For every $T > 0$ there exist functions $z, w \in C\big([0,1] \times [0,T]\big)$ such that $z_x, z_t, w_x, w_t \in L^\infty\big(0,T; L^2(0,1)\big)$ and subsequences $\left\{u^{(m)}, v^{(m)}\right\}$ of $\left\{u^{(n)}, v^{(n)}\right\}$ such that $u_t^{(m)} \to z, v_t^{(m)} \to w$ uniformly, $u_{xt}^{(m)} \to z_x,\ u_{tt}^{(m)} \to z_t, v_{xt}^{(m)} \to w_x, v_{tt}^{(m)} \to w_t$ in $L^\infty\big(0,T; L^2(0,1)\big)$-weak *. Moreover, there exist a constant $c > 0$ such that $\left|\tilde{u}_t^{(n)}(x,t) - u_t^{(n)}(x,t)\right|^2 \leq \frac{c}{n}, \left|\tilde{v}_t^{(n)}(x,t) - v_t^{(n)}(x,t)\right| \leq \frac{c}{n}$, hence $\tilde{u}_t^{(m)} \to z, \tilde{v}_t^{(m)} \to w$ uniformly.

Remark (4.4)(i) holds also for functions $\lambda_i, u, v$ which are piecewise continuous with respect to $x$. Therefore, $W^{(m)}\left(\tilde{u}_t^{(m)}\right) \to W(z)$ uniformly and $W^{(m)}\left(\tilde{u}_t^{(m)}\right)_t \to W(z)_t$ in $L^\infty\big(0,T; L^2(0,1)\big)$ - weak *.

We conclude from (7.1) that $W(z)_t = w_x + g,\ z_x = w_t$ almost everywhere and putting $u(x,t) := \int_o^x w(\xi,t)d\xi$ we obtain $w = u_x,\ \frac{\partial}{\partial x}(z - u_t) = 0$, hence $z = u_t$ and (5.1)(i) holds almost everywhere in $(0,1) \times (0,T)$. The initial and boundary conditions (5.1)(ii),(iii) are satisfied trivially.

We can now repeat the same procedure in the interval $[0,2T]$ choosing a convergent subsequence $\left\{u^{(r)}, v^{(r)}\right\}$ of $\left\{u^{(m)}, v^{(m)}\right\}$. By induction we construct a sequence $\left\{u^{(k)}\right\}$ of solutions of (5.1) such that $u^{(k)}$ is defined in $[0,1] \times [0,kT]$ and $u^{(k)}|_{[0,lT]} = u_l$ for $l < k$. Let us note that (6.10) is independent of $T$, hence $u$ can be extended to $[0,1] \times [0,\infty]$ in such a way that (5.7) holds.

Proposition (5.10) follows from a standard argument ([1]). Let $t_n \to t$ be an arbitrary sequence and let $\varepsilon > 0$ be given. For $\psi \in L^2(0,1)$ we find $\tilde{\psi} \in W^{1,2}(0,1)$ such that $\int_0^1 \left|\tilde{\psi} - \psi\right|^2 dx < \varepsilon$. The functions $u_{xx}, u_{tx}, W(u_t)_t : [0,\infty) \to L^2(0,1)$ are (locally) bounded, hence there exists a constant $c > 0$ such that e.g.

$$\left| \int_0^1 (u_{xx}(x,t_n) - u_{xx}(x,t))\psi(x)dx \right| \leq c\varepsilon + \\ + \left| \int_0^1 (u_x(x,t_n) - u_x(x,t))\tilde{\psi}'(x)dx \right|$$

and (5.10) follows easily.

## 8. Proof of uniqueness.

Let us assume that the hypotheses of Theorem (5.8) hold. We have

$$\int_0^1 \left[ \big(W(u_t)_t - W(v_t)_t\big)(u_t - v_t) + (u_x - v_x)(u_{xt} - v_{xt}) \right] dx = 0 \tag{8.1}$$

for a.e. $t > 0$.

In the case (5.8)(i) the assertion follows immediately from (8.1) and (2.7)(ii). In the case (5.8)(ii) we put $u^h(x,t) := l_h(u_t(x,\cdot), \lambda(x,h))(t), v^h(x,t) := l_h(v_t(x,\cdot), \lambda(x,h))(t)$. Then

(8.1) and (2.7) (i) yield

$$(8.2)\qquad \int_0^1 \int_0^\infty \left[\frac{\partial}{\partial t}\big(\nu(u^h(x,t),h) - \nu(v^h(x,t),h)\big)\right]\big(u^h(x,t) - v^h(x,t)\big)\, dh\, dx +$$

$$+\frac{1}{2}\int_0^1 \frac{\partial}{\partial t}\left[\big(u_x(x,t) - v_x(x,t)\big)^2 + \alpha\big(u_t(x,t) - v_t(x,t)\big)^2\right] dx \le 0$$

for a.e. $t \ge 0$.

The expression $\frac{\partial \nu}{\partial \varrho}(\varrho, h)$ is bounded away form 0 for $|\varrho| + h \le U$ (we have $\gamma(U) > 0$, cf the proof of (3.1)). Putting

$$M(t) := \frac{1}{2}\int_0^1 \left(\left[\int_0^\infty \frac{\partial \nu}{\partial \varrho}\big(u^h(x,t),h\big)\big(u^h(x,t) - v^h(x,t)\big) dh\right] + \right.$$
$$\left. + \big(u_x(x,t) - v_x(x,t)\big)^2 + \alpha\big(u_t(x,t) - v_t(x,t)\big)^2\right) dx$$

we see that there exists a function $k \in L^1_{loc}(0,\infty)$ such that $\frac{d}{dt}M(t) \le k(t)M(t)$ a.e. and Gronwall's lemma completes the proof of Theorem (5.8). ■

**9. Asymptotic behaviour.**

The proof of Theorem (5.9) relies on the inequality (6.9). We can choose $\varepsilon$ in (6.6) such that $\lim\limits_{t\to\infty} \varepsilon(t) = 0$. Let us construct a sequence $\{t_k\}, t_k \to +\infty$ by induction: $t_1 = 0, t_{k+1} - t_k = \frac{1}{\varepsilon(t_k)}\left|\log\left(\frac{\frac{1}{2}(1-\delta)U^2}{\varepsilon^2(t_k)}\right)\right|$. We have $\frac{K_2(t_k)}{B\varepsilon(t_k)} \le c_1\varepsilon^2(t_k)$, where $c_1$ is a positive constant.

For an arbitrary integer $n$ and for $t \le t_{k+1}$ we obtain from (6.9),(6.10)

$$E^{(n)}(t-) \le c_2\varepsilon(t_k),$$

where $c_2 > 0$ is a constant independent of $n$ and $k$. It suffices to put $\kappa(t) := [2c_2\varepsilon(t_{k-1})]^{1/2}$ for $t \in [t_k, t_{k+1})$.

Let us suppose now $G(t) = 0$ for $t \le t_0$. Putting $\varepsilon(t) = \frac{\delta}{t}$ for $t \ge t_0$ we obtain from (6.9) $E^{(n)}(\tau-) \le c\left(e^{-\frac{2}{3}B\delta}E^{(n)}(t+) + \frac{\delta^2}{t^2}\right)$ for every $\tau \ge 2t$, where $c$ is a positive constant.

Choosing $\delta$ sufficiently large (taking a larger $t_o$, if necessary) we obtain

$$E^{(n)}(\tau-) \le \frac{1}{32}E^{(n)}(t+) + \frac{c\delta^2}{t^2}$$

for all $\tau \ge 2t$.

Put $t_k := 2^k t_0$. We choose $K \ge 32c\delta^2$ such that $E^{(n)}(t_0+) \le \frac{K}{t_0^2}$. By induction we obtain

$$E^{(n)}(\tau+) \le \frac{K}{4t_k^2} \le \frac{K}{\tau^2}$$

for $\tau \in [t_k, t_{k+1})$.

We have $g(x,t) = g^o(x)$ for $t \le t_0$. It suffices to put $v(x) = \int_0^x \int_a^1 g^o(b)\, db\, da$. We have $|u_t^{(n)}(x,t)|^2 \le \int_0^1 |u_{xt}^{(n)}(x,t)|^2 dx \le 2E^{(n)}(t)$ a.e., $|u_x^{(n)}(x,t) - v'(x)|^2 \le \int_0^1 |W^{(n)}(u_t^{(n)})_t|^2 dx$ $\le 2\xi(U)E^{(n)}(t) + o(n)$ for a.e. $t \le t_0$ and Theorem (5.9) is proved.

**References.**


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