On the Mróz model

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Abstract

We treat the mathematical properties of the one parameter version of the Mróz model for plastic flow. We present continuity results and an energy inequality for the hardening rule and discuss different versions of the flow rule regarding their relation to the basic laws of thermodynamics.

1 Introduction

In a material body, which undergoes plastic deformation, the current values of the stress and the strain tensor at a given point usually do not uniquely determine each other, and one must take into account some aspects of the time history of either the stress or the strain. This is commonly achieved through the classical and well established concept of a yield surface. But since a single fixed yield surface does not describe correctly many experimentally observed phenomena, a lot of modifications and generalizations have been developed. There are various rules which specify translations and changes in size or shape of the yield surface (kinematic hardening and isotropic hardening), and there are multi-surface theories to allow for a more complex memory of the past history.

In this paper, we study a particular model for kinematic hardening due to to Mróz [15], who modified and extended the previous models of Prager and Ziegler. In fact, [15] employs a yield surface whose movement is determined from the position of a nested sequence of loading surfaces; moreover, the latter also move if the stress becomes large

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enough. We refer to Lemaitre and Chaboche [14] for an exposition of that model. More recently, Chu observed in [5], [6] that, with the standard (v. Mises) choice of spherical surfaces, the Mróz hardening rule leads to a surprisingly simple structure of memory, if one employs a continuous one parameter family of surfaces instead of a finite number of surfaces. Moreover, within the continuous setting, the characteristic feature of the Mróz model, namely the inclusion (or nonintersection) property, already uniquely determines the hardening rule. There is a marked difference here to the original (discrete) formulation, where one additionally has to fix the direction of the movement of the innermost surface, usually by some sort of geometric construction, and where special care is needed in order not to lose the inclusion property when one discretizes the loading path.

Our motivation to study the continuous one parameter version of the Mróz model is twofold. First, in the uniaxial case it reduces to a classical model for scalar rate independent hysteresis due to Prandtl [17], Preisach [18], and Ishlinskii [9]. The mathematical properties of the latter being investigated rather thoroughly (see e.g. [10], [4], [19], [2], [11] and [12], rate independent models for multiaxial behaviour begin to catch more and more the attention of applied mathematicians. Second – and this was our actual incentive to start this research – there is an intimate connection (see [7],[3]) between the memory structure of the uniaxial model and the so-called rainflow method; the latter constitutes an established counting method [16] for the evaluation of damage caused by scalar, but otherwise arbitrary, load sequences in high cycle fatigue analysis. We consequently hope that a detailed analysis of the Mróz model also helps in the development and analysis of a multiaxial counting method based on the constitutive law.

The organization of our paper is as follows. In Section 2, we present a rigorous definition of the Mróz hardening rule for the continuous one parameter model, based upon the inclusion property ratther than upon a construction involving normal vectors. This enables us to obtain some continuity and regularity theorems which indicate the well-posedness of the model. In order not to interrupt the flow of the exposition, we delegate the proofs of the two main theorems to the sections 4 and 5. In Section 3, we discuss the flow rule of the Mróz model. It turns out that the standard von Mises normality rule, if applied indifferently, may lead to a violation of the second law of thermodynamics. In analogy to the vector Prandtl-Ishlinskii model discussed e.g. in [13], we present a different flow rule and study its energy dissipation properties.

2 The hardening rule

A mathematical formulation of the constitutive stress-strain relation of plastic flow in terms of a yield surface usually has three ingredients:

- A *yield condition* to specify the form of the yield surface.
- A *hardening rule* to describe its time evolution.
- A *flow rule* to characterize the plastic strain.

Usually, hardening rules determine the yield surface evolution from the time history of the stress. Let us therefore consider a stress function $\sigma : [0,T] \to \mathbf{T}$, where we denote by $\mathbf{T}$ the space of symmetric 3×3 tensors endowed with the scalar product

$$\langle \xi, \eta \rangle \quad = \quad \xi_{ij} \, \eta_{ij}$$

and with the norm

$$| \, \xi \, | \quad = \quad \langle \xi \, , \xi \rangle^{\frac{1}{2}} \, . \tag{1}$$

For elastic-plastic constitutive laws, only the deviatoric part of the stress tensor

$$\sigma_{ij}^{(d)} \quad = \quad \sigma_{ij} \, - \, \bar{\sigma} \, \delta_{ij} \, ,$$

with the pressure $\bar{\sigma} = \frac{1}{3} \sigma_{ii}$, plays any role. Consequently, $\mathbf{T}$ is decomposed into the orthogonal direct sum

$$\mathbf{T} \quad = \quad \mathbf{T_{dia}} \oplus \mathbf{T_{dev}}$$

of spaces of diagonal tensors

$$\mathbf{T_{dia}} \quad = \quad \{ \, \xi \in \mathbf{T} : \, \xi_{ij} = \lambda \delta_{ij} \text{ for some } \lambda \in \mathbf{R} \}$$

and of deviatoric tensors

$$\mathbf{T_{dev}} \quad = \quad \{ \, \xi \in \mathbf{T} : \, \xi_{ii} = 0 \, \} \, .$$

In conformity with the von Mises yield criterion, we exclusively deal with spherical surfaces. Specifically, we consider the one parameter time dependent family $S_r(t)$ of spheres in $\mathbf{T_{dev}}$ with radius r and center $\phi(t,r)$, namely

$$S_r(t) \quad = \quad \{ \, \xi \in \mathbf{T_{dev}} : \, | \, \xi \, - \, \phi(t,r) \, | = r \, \}.$$

We further denote by

$$E_r(t) \quad = \quad \{ \, \xi \in \mathbf{T_{dev}} : \, | \, \xi \, - \, \phi(t,r) \, | < r \, \}$$

the interior of the ball bounded by the sphere $S_r(t)$. To describe their time evolution, we characterize the center function $\phi = \phi(t,r)$ for any given stress deviator $\sigma^{(d)} = \sigma^{(d)}(t)$. First, we require that the stress deviator always remains within or on every surface. This means that

$$| \, \sigma^{(d)}(t) \, - \, \phi(t,r) \, | \, \leq \, r \qquad \text{for any } \, t \in [0,T] \, , \; r \, > \, 0 \, . \tag{2}$$

Next, a surface should not move when the stress deviator lies in its interior, so

$$\frac{\partial}{\partial t} \, \phi(t,r) \; = \; 0 \qquad \text{if} \quad | \, \sigma^{(d)}(t) \, - \, \phi(t,r) \, | \; < \; r \, . \tag{3}$$

While the conditions (2) and (3) are common to most models, the *inclusion* or *nonintersection condition*

$$E_{r_1}(t) \quad \subset \quad E_{r_2}(t) \qquad \text{for any } \, 0 \, < \, r_1 \, < \, r_2, \; t \in [0,T] \, , \tag{4}$$

constitutes the distinctive feature of the model of Mróz. It can be equivalently rewritten as

$$| \, \phi(t,r_1) \, - \, \phi(t,r_2) \, | \quad \leq \quad r_2 \, - \, r_1 \qquad \text{for any } \, 0 \, < \, r_1 \, < \, r_2, \; t \in [0,T] \, . \tag{5}$$

Finally, let us assume that initially the surfaces are situated concentric around 0, so

$$\phi(0-,r) \quad = \quad 0 \qquad \text{for any } \, r \, > \, 0 \, . \tag{6}$$

We will see below that the conditions (2) – (6) uniquely define a function ϕ and therefore completely specify the movement of the surfaces for a given stress deviator $\sigma^{(d)} = \sigma^{(d)}(t)$. We call the corresponding mapping $\sigma^{(d)} \to \phi$ the *(continuous) Mróz hardening rule* . The Mróz hardening rule is a purely kinematic hardening rule, since the surfaces move but do not change shape; nevertheless it is able to model anisotropic material behaviour through its memory stored in the function $\phi(t, \cdot)$ at time t.

At this point, a remark concerning our terminology seems appropriate. Usually, the notion of a yield surface is reserved for the surface describing the onset of plastic deformation; other surfaces which appear in the description of the memory are often called loading surfaces. While this distinction appears natural for certain multi-surface models including the discrete (original) version of the Mróz model, the continuous hardening rule induces a slightly different point of view. In fact, the subsequent developments will make it quite obvious that there is no single distinguished value of the parameter r from the standpoint of the hardening rule; instead, the radius r of the surface characterizing the onset of plastic flow is implicitly fixed by the flow rule as the largest number r such that the uniaxial stress-strain curve $|\epsilon^p| = f(|\sigma^d|)$ satisfies $f' = 0$ on $[0, r]$. For this reason, we continue to speak of surfaces without further qualification.

For the mathematical treatment of the Mróz hardening rule, it is completely immaterial that we are working with deviatoric stresses as inputs, except for the scalar product structure of $\mathbf{T_{dev}}$. To emphasize this fact, we replace in the following the space $\mathbf{T_{dev}}$ with an arbitrary separable Hilbert space U and denote the input function by u instead of $\sigma^{(d)}$. But actually, nothing is lost if the reader always interprets U as the deviatoric plane and u as the stress deviator.

To start with the formal theory, we define an appropriate function space Ψ for the memory in order to have $\phi(t, \cdot) \in \Psi$. Due to (3), the surface with radius r will move away from zero only if the norm of the stress deviator exceeds the value r.

Therefore, we adopt the following definition.

Definition 2.1 *Let U be a Hilbert space with the scalar product $\langle \cdot, \cdot >\rangle$ and the norm* (1) *. We call U the input space and its elements input values. We define the space Ψ of admissible memory states by*

$$\begin{aligned}\Psi \quad = \quad \{\, \psi \,|\, \psi : [0, \infty) \to U \,,\; |\psi(r) - \psi(s)| \;\le\; |r - s| \text{ for any } r,s \ge 0, \\ \text{and there exists } R > 0 \text{ with } \psi(r) = 0 \text{ for any } r \ge R \,\}\,. \end{aligned} \tag{7}$$

For $\psi \in \Psi$ and $r > 0$, we interpret $\psi(r)$ as the center of the surface with radius r, and $\psi(0)$ as the current input value (compare (2)*). Moreover, we call $\psi(r)$ a corner of the memory state ψ, if ψ is not differentiable at r.*

Actually, Ψ is a metric space if we consider it as a subset of the space of bounded continuous functions on the nonnegative real numbers with values in U, endowed with the norm

$$\|\, \psi \,\|_\infty \quad = \quad \sup_{r \ge 0} \,|\, \psi(r) \,|\,. \tag{8}$$

We will now describe the memory update, i.e. the movement of the surfaces, for a given memory state ψ, if the input changes from its current value $\psi(0)$ to a new value v along a straight line in the input space U. We first note formally that there is a smallest radius $\alpha(v, \psi)$ such that the new input value v does not lie outside any surface with radius $r \ge \alpha(v, \psi)$.

Lemma 2.2 *Let $\psi \in \Psi$ and $v \in U$ be given. Then*

$$\alpha(v,\psi) \quad = \quad \min\{\, r \geq 0 : |\,\psi(r) \,-\, v\,| = r\,\} \tag{9}$$

is well defined, and

$$r \;<\; |\,\psi(r) \,-\, v\,| \qquad \text{if and only if } \; 0 \,\leq\, r \,<\, \alpha(v,\psi)\,. \tag{10}$$

Proof: Since

$$\Big|\,|\,\psi(r) - v\,| \;-\; |\,\psi(s) - v\,|\,\Big| \;\;\leq\;\; |\,r \,-\, s\,|$$

for any $r, s \geq 0$, the function

$$f(r) \quad = \quad r \;-\; |\,\psi(r) \,-\, v\,|$$

is nondecreasing, continuous, and satisfies $f(0) \leq 0$ as well as $\lim_{r\to\infty} f(r) = \infty$, so all assertions follow. □

Due to (3), no surface with radius $r \geq \alpha(v,\psi)$ should move if the input value changes from $\psi(0)$ to v along a straight line. On the other hand, the surfaces with smaller radius should move so as to form a new memory state in Ψ as well as to include the value v. Because of (9), their centers have to arrange themselves along the straight line connecting v and the center of the surface with radius $\alpha(v,\psi)$ with a common normal at the common boundary point v, see Figure 1.

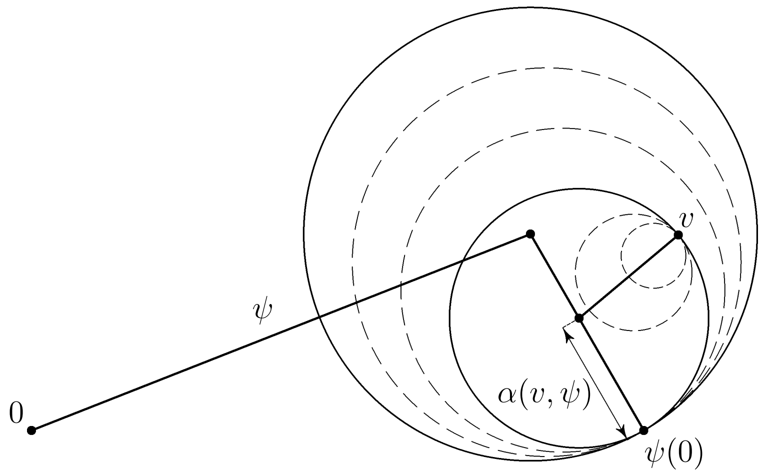

Figure 1: *Arrangement of the surfaces.*

Therefore, the following definition specifies the unique hardening rule compatible with (2) – (6).

Definition 2.3 *We define an operator $G : U \times \Psi \to \Psi$ by*

$$G(v,\psi)(r) \quad = \quad \begin{cases} \psi(r)\,, & \text{if} \quad r \,\geq\, \alpha(v,\psi) \\[2ex] v \;+\; \dfrac{r}{\alpha(v,\psi)} \Big(\, \psi(\alpha(v,\psi)) \,-\, v\,\Big)\,, & \text{otherwise} \end{cases} \tag{11}$$

for any $r \geq 0$ and any $v \in U$, $\psi \in \Psi$, where $\alpha(v,\psi)$ is defined in Lemma 2.2. We call G the incremental Mróz hardening rule.

To update a piecewise linear memory state ψ with the rule (11), we insert a (possibly degenerate) corner P at $\psi(\alpha(v,\psi))$ and connect it to the point v, thereby discarding the piecewise linear segment from $\psi(0)$ to P. We present in Figure 2 the resulting possible corner structures for various input values.

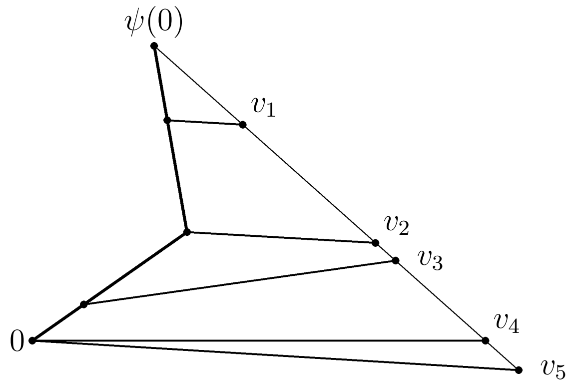

Figure 2: *Update of the memory state.*

There we imagine the input value v traveling along a straight line, and we may think of P moving along the old state ψ, deleting corners while passing through them. We also see, and easily check formally, that no surface can stay fixed while a larger one is moving. In terms of the operator G, this means that

$$G(v,\psi)(r) \neq \psi(r) \quad \Rightarrow \quad G(v,\psi)(s) \neq \psi(s) \quad \text{for any } s \leq r\,. \tag{12}$$

Next, let us consider a piecewise linear input function $u : [0,T] \to U$ represented by a sequence $\{u_k\}$ of input values. We successively apply Definition 2.3 to obtain the corresponding movement of the surfaces.

Definition 2.4 *(i) For any sequence $\{u_k\}$, $k = 0,1,2,\dots$ of input values in U we define the corresponding sequence of memory states $\{\phi_k\}$ in Ψ by*

$$\phi_k \;=\; G(u_k, \phi_{k-1})\,, \qquad \phi_{-1} \;=\; 0\,. \tag{13}$$

(ii) Let $0 = t_0 < t_1 < \ldots < t_n = T$ be a partition of $[0,T]$ and $u : [0,T] \to U$ be the piecewise linear interpolate for the values $u(t_k) = u_k$ with $u_k \in U$. Then we define the memory state function $\phi : [0,T] \times [0,\infty) \to U$ by

$$\begin{aligned} \phi(t_k, r) &= \phi_k(r)\,, && 0 \leq k \leq n\,,\; r \geq 0\,, \\ \phi(t,r) &= G(u(t), \phi_k))(r)\,, && t \in (t_k, t_{k+1})\,,\; r \geq 0\,. \end{aligned} \tag{14}$$

We write (14) in operator notation as

$$\phi \;=\; F(u)\,, \tag{15}$$

and call F the Mróz hardening rule. This is justified since for any piecewise linear $u : [0,T] \to U$, the function ϕ in (14) does not depend on the choice of the partition as long as u is linear within each interval.

From the definitions (2.3) and (2.4) it is obvious that the memory states $\{\phi_k\}$ generated from (13) are piecewise linear curves with finitely many corners in the space U. They have the length

$$L(\phi_k) \quad := \quad \int_0^\infty |\,\phi_k'(r)\,|\,dr \quad = \quad \max_{0\le j\le k} |\,u_j\,| \tag{16}$$

and satisfy

$$\phi_k(r) \;=\; 0 \qquad \text{if } r \,>\, L(\phi_k)\,, \tag{17}$$

as well as

$$|\,\phi_k'(r)\,| \;=\; 1 \qquad \text{if } r \,<\, L(\phi_k)\,, \tag{18}$$

except in corners, of course. We also note two other obvious consequences of the definitions above related to the storage and deletion of corners.

Lemma 2.5 *Let the memory states $\{\phi_k\}$ be generated by the input values $\{u_k\}$. (i) If $\phi_k(r)$ is a corner, then $\phi_{k-1}(s) = \phi_k(s)$ for any $s \ge r$. (ii) If $\alpha(\phi_k, u_{k+1}) \;\ge\; \alpha(\phi_{k-1}, u_k)$, then*

$$G(u_{k+1}, \phi_k) \;=\; \phi_{k+1} \;=\; G(u_{k+1}, \phi_{k-1})\,, \tag{19}$$

so the memory due to the input value u_k is deleted. □

Our first main result shows that the Mróz hardening rule is well posed. More precisely, the operator F is $\frac{1}{2}$-Hölder continuous with respect to the sup norm. (As a consequence, the Mróz hardening rule is stable with respect to different discretizations of some given continuous loading function.)

Theorem 2.6 *The Mróz hardening rule F defined by (15) can be extended to an operator*

$$F : C(0,T;U) \;\to\; C(0,T;\Psi)\,, \tag{20}$$

and we have

$$\max_{\substack{0\le t\le T\\ r\ge 0}} |\,\phi(t,r) \,-\, \psi(t,r)\,| \quad\le\quad \Big(\,2\,R \max_{0\le t\le T} |\,u(t) \,-\, v(t)\,|\,\Big)^{\frac{1}{2}} \tag{21}$$

for any $u, v \in C(0,T;U)$, where $\phi = F(u)$, $\psi = F(v)$, and

$$R \quad=\quad \max\{\,|u(t)|, |v(t)| \,:\, 0 \,\le\, t \,\le\, T\,\}\,.$$

Proof: This will be given in Section 4. □

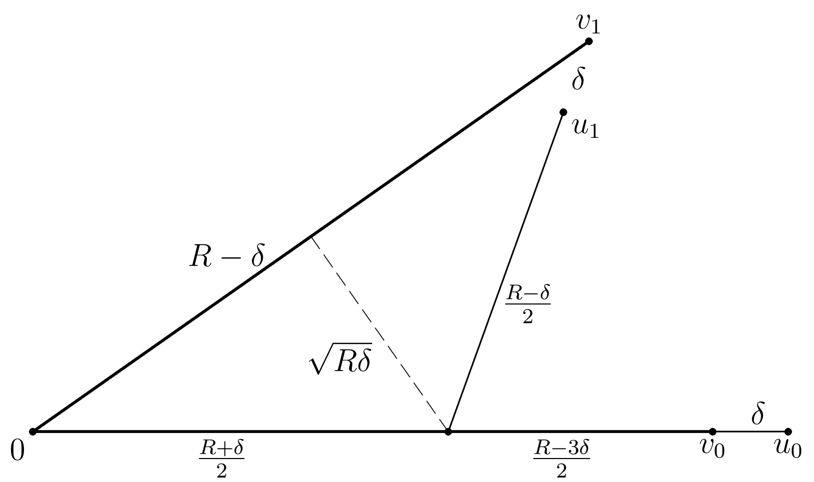

Figure 3: *Optimality of the Hölder exponent.*

The exponent $\frac{1}{2}$ in equation (21) cannot be improved in the vector case, i.e. if dim $U \geq 2$. In Figure 3 we see an example where the linear interpolates u, v of (u_0, u_1) and (v_0, v_1) satisfy (we assume $R \geq 3\delta$)

$$\| u - v \|_\infty = \delta , \quad \| F(u) - F(v) \|_\infty \geq (R\,\delta)^{\frac{1}{2}} .$$

In the scalar case dim $U = 1$, the operator F is Lipschitz continuous, see [4],[10]. Moreover, if the input function u is Lipschitz continuous (with respect to time), then so is the memory state function $\phi = F(u)$. Again, this is no longer true in the vector case. The counterexample in 2.9 below shows that $\partial_t \phi(t,u)$, i.e. the partial time derivative of the motion of the centers of the surfaces, in general does not lie in L^p if $p > 1$. For $p = 1$, the question is open. Actually, we do not even know whether $\partial_t \phi(t,u)$ exists almost everywhere if the input function u is not piecewise linear. We do however have some positive results. Let us denote by $C^\alpha(0,T;U)$ the space of α–Hölder continuous functions, where $0 < \alpha \leq 1$. Then the following theorem holds.

Theorem 2.7 *Let $u \in C(0,T;U)$ be given, set $\phi = F(u)$. Then we have*

$$|\phi(t,r) - \phi(s,r)| \;\leq\; (2\| u \|_\infty \max_{\tau \in [s,t]} |u(\tau) - u(s)|)^{\frac{1}{2}} \tag{22}$$

for any $s,t \in [0,T]$ and any $r > 0$. In particular, if $u \in C^\alpha(0,T;U)$, then $\phi(\cdot,r) \in C^{\alpha/2}(0,T;U)$.

Proof: Fix $t,s \in [0,T]$, $s \leq t$, and define $v \in C(0,T;U)$ by $v = u$ on $[0,s]$ and $v = u(s)$ on $[s,t]$. Then Theorem 2.6 implies that, setting $R = \| u \|_\infty$,

$$\begin{aligned} |\phi(t,r) - \phi(s,r)| &= |(Fu)(t,r) - (Fv)(t,r)| \;\leq\; \sqrt{2R \max_{0 \leq \tau \leq t} |u(\tau) - v(\tau)|} \\ &\leq \sqrt{2R \max_{s \leq \tau \leq t} |u(\tau) - u(s)|} \;\leq\; \sqrt{2R}\, C|t-s|^{\alpha/2} , \end{aligned}$$

so (22) follows. If now $u \in C^\alpha(0,T;U)$, then $|u(t) - u(s)| \leq C|t-s|^\alpha$ for some C independent from t and s, and (22) implies that

$$|\phi(t,r) - \phi(s,r)| \;\leq\; \sqrt{2R}\, C|t-s|^{\alpha/2} . \Box$$

We also have an estimate for $\partial_t \, \phi(t,r)$, if the input function u is piecewise linear. We will use this result later to derive continuity properties of the stress–strain law. As usual, we denote by $W^{1,1}(0,T;U)$ the Sobolev space of functions with values in U whose first derivative is Bochner integrable.

Theorem 2.8 *For any piecewise linear $u \in W^{1,1}(0,T;U)$, the function $t \mapsto \phi(t,r) = (Fu)(t,r)$ is an element of $W^{1,1}(0,T;U)$ and satisfies*

$$\int_0^T |\, \partial_t \, \phi(t,r) \,|\, dt \quad \leq \quad 3 \int_0^T |\, u'(t) \,|\, dt . \tag{23}$$

Proof: This will be given in Section 5. $\Box$

We now present the example of a Lipschitz continuous input function u whose corresponding state function ϕ does not have a time derivative in any L^p, $p > 1$.

Example 2.9 For $r = 1$ and a specific $T > 0$ to be defined below, we construct a function $u : [0,T] \to \mathbf{R}^2$ such that u is Lipschitz continuous, $u'(t) = 1$ a.e., but $\partial_t \phi(t,r) \notin L^p(0,T;U)$ for all $p > 1$. (By rescaling and embedding, the example is easily extended to arbitrary values of r, T and arbitrary input spaces U with $\dim U \geq 2$.) The idea is to let the input value $u(t)$ run through a sequence of loops ($A \to B_n \to C_n \to A$) of decreasing size but bounded total length T. A single loop is shown in Figure 4.

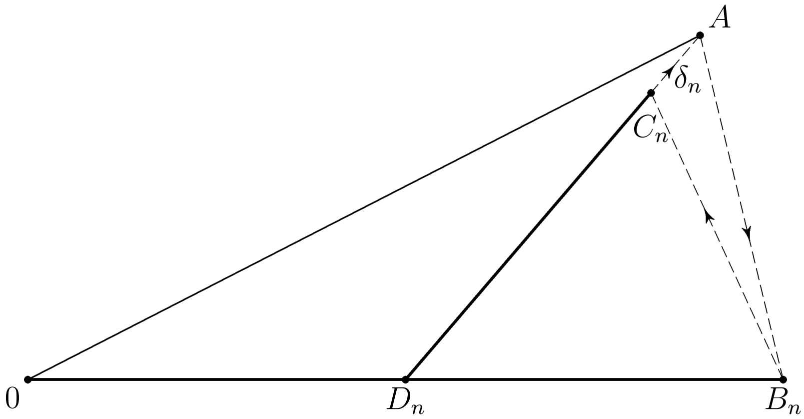

Figure 4: *Construction of the counterexample.*

We fix a point A in the plane with $|A| = 2$ and choose points B_n, C_n, D_n such that for some given $\delta_n > 0$

$$| D_n | = 1 , \quad | A - D_n | = 1 + \delta_n , \quad B_n = 2 D_n ,$$
$$C_n = \frac{\delta_n}{1+\delta_n} D_n + \frac{1}{1+\delta_n} A .$$

We easily compute that $2\langle A, D_n \rangle = 4 - 2\delta_n - \delta_n^2$ and therefore

$$| B_n - A |^2 = 2\,\delta_n (2+\delta_n) , \quad | C_n - B_n |^2 \leq 4\,\delta_n , \quad | A - C_n |^2 = \delta_n^2 . \tag{24}$$

Let us define δ_n for $n \geq 3$ by

$$\delta_n = n^{-\frac{2}{2-p_n}} , \quad p_n = 1 + \frac{\epsilon_n}{1+\epsilon_n} , \quad \epsilon_n = \frac{2\ln(\ln n)}{\ln n} . \tag{25}$$

Then the sum T of the loop lengths for $n = 3, 4, \ldots$ can be estimated as

$$\begin{aligned} T &= \sum_{n=3}^{\infty} (|B_n - A| + |C_n - B_n| + |A - C_n|) \leq 4 \sum_{n=3}^{\infty} \sqrt{\delta_n} + (1+\sqrt{2}) \sum_{n=3}^{\infty} \delta_n \\ &= 4 \sum_{n=3}^{\infty} \frac{1}{n \ln^2 n} + (1+\sqrt{2}) \sum_{n=3}^{\infty} \frac{1}{n^2 \ln^4 n} < +\infty . \end{aligned}$$

We define $u : [0,T] \to \mathbf{R}^2$ as the linear interpolate of the sequence $A, B_3, C_3, A, B_4, \ldots$ satisfying $|u'| = 1$ a.e.. If I_n denotes the time interval during which $u(t)$ moves from C_n to A, we obtain, using (24) and (25),

$$\delta_n^{p-1} \int_{I_n} | \partial_t \phi(t,1) |^p \, dt \geq \left(\int_{I_n} | \partial_t \phi(t,1) | \, dt \right)^p \geq \left(\frac{1}{2} | B_n - A | \right)^p \geq \delta_n^{p/2} ,$$

so

$$\int_0^T | \partial_t \phi(t,1) |^p \, dt \geq \sum_{n=3}^{\infty} \delta_n^{1-p+p/2} = \sum_{n=3}^{\infty} n^{-\frac{2-p}{2-p_n}} = +\infty .$$

The example is complete. $\square$

3 The flow rule

In plastic flow theory, the flow rule serves to determine the plastic strain ϵ^p from the current value of stress and the current position of the surfaces constructed from the hardening rule. The total strain ϵ is given by

$$\epsilon \;=\; \epsilon^p \;+ \epsilon^e \,, \quad \epsilon^e \;=\; A\,\sigma \,, \tag{26}$$

where the elastic strain ϵ^e is obtained from the stress via Hooke's law expressed with the symmetric positive definite matrix A. Chu [5], [6] presents a flow rule which has the following two properties:

- The plastic strain rate tensor points in the direction of the outward normal $n(t)$ common to the active surface.
- For uniaxial stress, the standard stabilized stress–strain behaviour characterized by Masing's law and the memory properties of the uniaxial version of the Mróz hardening rule (which is in fact identical with Prandtl's model in [17]) is obtained.

These properties result in the formula

$$\dot\epsilon^{(p)}(t) \;=\; f'(a(t))\langle n(t), \dot\sigma^{(d)}(t)\rangle n(t)\,, \tag{27}$$

where $a(t)$ is the radius of the largest active surface

$$a(t) \;=\; \max\{r : r \geq 0,\; |\phi(t,r) - \sigma^{(d)}(t)| = r\}\,, \quad n(t) \;=\; -\frac{\partial}{\partial r}\phi(t,0)\,, \tag{28}$$

and the function $|\epsilon^p| = f(|\sigma^{(d)}|)$ with $f(0) = f'(0) = 0$, $f'' \geq 0$, denotes the stabilized uniaxial initial stress–strain curve.

It is useful to rewrite the flow rule (27) in a derivative–free form. To this end, let us introduce an auxiliary function $\psi = \psi(t,r)$ via the Stieltjes integral

$$\psi(t,r) \;=\; \phi(0,r) \;+\; \int_0^t \frac{\sigma^{(d)}(\tau) \;-\; \phi(\tau,r)}{r}\langle\frac{\sigma^{(d)}(\tau) \;-\; \phi(\tau,r)}{r}, d_\tau\phi(\tau,r)\rangle\,. \tag{29}$$

A straightforward computation shows that equation (27) together with the initial condition

$$\epsilon^p(0) \;=\; \frac{f(a(0))}{a(0)}\sigma^{(d)}(0)\,, \quad a(0) \;=\; |\sigma^{(d)}(0)|\,, \tag{30}$$

is equivalent to the formula

$$\epsilon^p(t) \;=\; \int_0^\infty \psi(t,r)\eta(r)\,dr\,, \quad \eta(r) \;:=\; f''(r)\,, \tag{31}$$

if the functions occuring are smooth enough. This follows since we have $\partial_t\phi(t,r) = 0$ for $r > a(t)$ and

$$\langle\partial_t\phi(t,r), n(t)\rangle \quad=\quad \langle\dot\sigma^{(d)}(t) - r\dot n(t), n(t)\rangle \quad=\quad \langle\dot\sigma^{(d)}(t), n(t)\rangle \tag{32}$$

for $r < a(t)$. We obtain a continuity result for this version of the flow rule.

Proposition 3.1 *The Mróz hardening rule $\phi = F(\sigma^{(d)})$ from Definition 2.4 together with (29) and (31) defines an operator $\epsilon^p = M(\sigma^{(d)})$,*

$$M : C(0,T;\mathbf{T_{dev}}) \,\cap\, BV(0,T;\mathbf{T_{dev}}) \;\rightarrow\; C(0,T;\mathbf{T_{dev}}) \,\cap\, BV(0,T;\mathbf{T_{dev}})\,.$$

Moreover, for any sequence $(u_n)_{n\in N}$ in $C(0,T;\mathbf{T_{dev}}) \cap BV(0,T;\mathbf{T_{dev}})$ with uniformly bounded variation and $\| u_n - \sigma^{(d)} \|_\infty \to 0$ we obtain that $M(u_n)$ has uniformly bounded variation and that $\| M(u_n) - M(\sigma^{(d)}) \|_\infty \to 0$.

Proof: This follows from Theorems 2.6 and 2.8 together with the convergence result of [8], Theorem II.15.3 and its consequences. □

As many other extensions and refinements of the basic yield surface model for plastic flow, the flow rule (27) is obtained from a mixture of various guiding principles and as such is not a priori consistent with the framework of thermodynamics. In particular, one has to impose additional restrictions in order to exclude a violation of the second law. Let

$$W(t) \;=\; \int_0^t \langle \dot\epsilon(\tau), \sigma(\tau)\rangle \, d\tau \tag{33}$$

denote the total mechanical work. According to (26) and (27), we may decompose this expression as $W(t) = W^e(t) + W^p(t)$, where

$$W^e(t) \;=\; \int_0^t \langle \dot\epsilon^{(e)}(\tau), \sigma(\tau)\rangle \, d\tau \;=\; \left[\frac{1}{2}\langle A\sigma(\tau), \sigma(\tau)\rangle\right]_0^t, \tag{34}$$

$$W^p(t) \;=\; \int_0^t \langle \dot\epsilon^{(p)}(\tau), \sigma(\tau)\rangle \, d\tau \;=\; \int_0^t f'(a(\tau))\langle\dot\sigma^{(d)}(\tau), n(\tau)\rangle\langle\sigma^{(d)}(\tau), n(\tau)\rangle \, d \tag{35}$$

We now construct a cyclic process whose energy dissipation has the wrong sign.

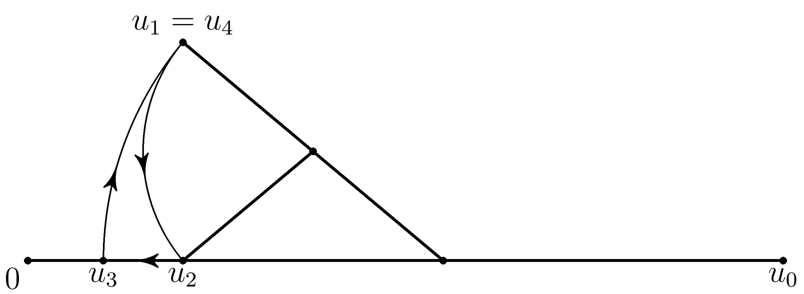

Figure 5: *A cyclic process which produces energy.*

Example 3.2 Let the situation be as in Figure 5. We define $\sigma^{(d)}(t_i) = u_i$, $0 \le i \le 4$, for some $0 = t_0 < t_1 < \ldots < t_4$. We interpolate $\sigma^{(d)}$ linearly in $[0, t_1]$ and $[t_2, t_3]$, and by a circular path with radius equal to $a(t_1)$ and $a(t_3)$ in the intervals $[t_1, t_2]$ respectively $[t_3, t_4]$. Consequently, $a(t)$ remains constant and $\langle\dot\sigma^{(d)}(t), n(t)\rangle = 0$ during the circular motion, whereas

$$\langle\dot\sigma^{(d)}(t), n(t)\rangle \;>\; 0\,, \quad \langle\sigma^{(d)}(t), n(t)\rangle \;<\; 0\,, \quad t \in (t_2, t_3)\,. \tag{36}$$

We compute the total work along the cycle $u_1 \to u_2 \to u_3 \to u_4 = u_1$ from (34), (35) and (36) as

$$W(t_4) \;-\; W(t_1) \;=\; \int_{t_2}^{t_3} f'(a(t))\langle\dot\sigma^{(d)}(t), n(t)\rangle\langle\sigma^{(d)}(t), n(t)\rangle \, dt\,. \tag{37}$$

Now if the interval $[a(t_2), a(t_3)]$ belongs to the range of plastic deformation where f' is positive, we obtain from (36) that $W(t_4) - W(t_1) < 0$. This is incompatible with the basic laws of thermodynamics as the following argument shows. (We thank Ingo Müller from Berlin for supplying it to us.) Let $\Omega \subset \mathbf{R}^3$ be a test volume and assume that the temperature on its boundary $\partial\Omega$ takes on some value T_B constant in time and uniformly over $\partial\Omega$. By the first law, the total internal energy U and the mechanical power are related by

$$\dot{U}(t) = \dot{Q}(t) + \int_\Omega \langle \partial_t \epsilon(t,x), \sigma(t,x) \rangle \, dx \,, \tag{38}$$

where

$$\dot{Q}(t) = -\int_{\partial\Omega} \langle q, n \rangle \, da \tag{39}$$

denotes the total heat flux. The second law states that for the total entropy S there holds

$$\dot{S}(t) \geq -\int_{\partial\Omega} \langle \frac{q}{T_B}, n \rangle \, da = -\frac{\dot{Q}(t)}{T_B} \,, \tag{40}$$

so that the total available free energy $F = U - T_B S$ satisfies

$$\dot{F}(t) \leq \int_\Omega \langle \partial_t \epsilon(t,x), \sigma(t,x) \rangle \, dx \,. \tag{41}$$

Now if there is a cyclic process like the one constructed above such that

$$\int_{t_1}^{t_4} \int_\Omega \langle \partial_t \epsilon(t,x), \sigma(t,x) \rangle \, dx \, dt < 0 \,, \tag{42}$$

an infinite repetition of the cycle would lead to

$$\lim_{t\to\infty} F(t) = -\infty \,. \tag{43}$$

But this is impossible, since F has to be bounded from below no matter what its specific form looks like. $\square$

A standard way to overcome this problem (see e.g. [1]) is to restrict the model to situations where

$$\dot{W}^p(t) \;=\; \langle \dot{\epsilon}^{(p)}(t), \sigma(t) \rangle \quad \geq \; 0 \tag{44}$$

almost everywhere in t. Let us characterize the situation in which (44) holds. Looking at the figures accompanying the definition of the Mróz hardening rule one notices immediately that

$$\langle \dot{\sigma}^{(d)}(t), n(t) \rangle \quad \geq \; 0 \tag{45}$$

holds for piecewise linear stress functions. We present a precise formulation and proof of (45) for general stress functions. This is unfortunately a bit tedious since the limit process underlying Theorem 2.6 does not have a clear relation to the weak limit of the corresponding normals $n(t)$.

Lemma 3.3 *Let $\sigma^{(d)} \in W^{1,1}(0,T;\mathbf{T_{dev}})$ and $t \in [0,T]$ with $a(t) > 0$ be given and assume that $\dot{\sigma}^{(d)}$ is continuous in $[s,t]$ for some $s < t$. Then we have*

$$\langle \dot{\sigma}^{(d)}(t), n(t) \rangle \;\geq\; 0 \,. \tag{46}$$

Proof: Assume that $d := -\langle \dot\sigma^{(d)}(t), n(t)\rangle \;\; > 0$. We revert to the notation $u(t) = \sigma^{(d)}(t)$. We abbreviate $a = a(t)$ and choose $s < t$ such that, with $\phi = F(u)$ and $M = \max_{s \le \tau \le t} |\dot u(\tau)|$, we have

$$|t-s| \;<\; \frac{ad}{4M^2}\,, \quad |\phi(t,a) - \phi(s,a)| \;<\; \frac{ad}{4M}\,, \quad |\dot u(t) - \dot u(\rho)| \;<\; \frac{d}{4} \quad \forall \rho \in [s,t]\,. \tag{47}$$

We then have for every $\rho, \tau \in [s,t]$

$$\begin{aligned}\langle \dot u(\rho), u(\tau) - \phi(s,a)\rangle \;&=\; \langle \dot u(t), u(t) - \phi(t,a)\rangle \\ &+\; \langle \dot u(\rho) - \dot u(t), u(t) - \phi(t,a)\rangle \;+ \langle \dot u(\rho), u(\tau) - u(t)\rangle \\ &+\; \langle \dot u(\rho), \phi(t,a) - \phi(s,a)\rangle \quad < \; -\frac{ad}{4}\,. \end{aligned} \tag{48}$$

We now want to prove that $\phi(t,a) = \phi(s,a)$. To this end, we choose an equidistant partition $s = s_0 < s_1 < \ldots < s_n = t$ and set $u_k = u(s_k)$, $\phi_0 = \phi(s,\cdot)$ and $\phi_k = G(u_k, \phi_{k-1})$. We claim that

$$\phi_k(a) \;=\; \phi_0(a) \quad \Longrightarrow \quad |u_{k+1} - \phi_k(a)| \;<\; a \quad \Longrightarrow \quad \phi_{k+1}(a) \;=\; \phi_k(a)\,. \tag{49}$$

The right implication is trivial. The left one follows from the estimate (we use (48))

$$\begin{aligned}|u_{k+1} - \phi_k(a)|^2 \;&\le\; |u_{k+1} - u_k|^2 \;+\; a^2 \;+\; 2\langle u_{k+1} - u_k, u_k - \phi_k(a)\rangle \\ &=\; |u_{k+1} - u_k|^2 \;+\; a^2 \;+\; 2\int_{s_k}^{s_{k+1}} \langle \dot u(\rho), u(s_k) - \phi_0(a)\rangle\, d\rho \\ &<\; M^2(s_{k+1} - s_k)^2 \;+\; a^2 \;-\; 2(s_{k+1} - s_k)\frac{ad}{4} \;<\; a^2\,. \end{aligned}$$

From (49) we conclude that $\phi_n(a) = \phi_0(a)$, and passing to the limit as $n \to \infty$ we obtain $\phi(t,a) = \phi(s,a)$, so we have $|u(t) - \phi(s,a)| = a$. Since (48) also implies that

$$\langle u(t) - u(s), u(t) - \phi(s,a)\rangle \;\le\; \int_s^t \langle \dot u(\rho), u(t) - \phi(s,a)\rangle\, d\rho \;\le\; -(t-s)\frac{ad}{4}\,,$$

we arrive at the contradiction

$$\begin{aligned} a^2 \;&\ge\; |u(s) - \phi(s,a)|^2 \;=\; |u(s) - u(t)|^2 \;+\; a^2 \;-\; 2\langle u(t) - u(s), u(t) - \phi(s,a)\rangle \\ &\ge\; |u(s) - u(t)|^2 \;+\; a^2 \;+\; (t-s)\frac{ad}{2} \;>\; a^2\,. \end{aligned}$$

The lemma is proved. □

We now show that condition (44) is satisfied for the flow rule (27) if and only if the maximal stress does not exceed twice the value of the yield stress.

Proposition 3.4 *Let* $\sigma^{(d)} \in W^{1,1}(0,T;\mathbf{T_{dev}})$ *be given, and assume that* $\dot\sigma^{(d)}$ *is piecewise continuous. If* $f'(r) = 0$ *in the range* $0 \le r \le \|\sigma^{(d)}\|_\infty/2$, *then condition (44) is satisfied. Conversely, if* $f'(r) > 0$ *for some* $r > 0$, *then for any* $\delta > 0$ *we may construct along the lines of Example 3.2 a function* $\sigma^{(d)}$ *with* $\|\sigma^{(d)}\|_\infty < 2r + \delta$ *such that the second law is violated.*

Proof: Because of Lemma 3.3, it suffices to prove that $\langle\sigma^{(d)}(t), n(t)\rangle \ > 0$ for any t with $a(t) \geq r_0 \ := \ \| \sigma^{(d)} \|_\infty/2$. Setting $\phi = F(\sigma^{(d)})$, we have $\phi(t, 2r_0) = \phi(t, \| \sigma^{(d)} \|_\infty) = 0$, hence $|\phi(t, r_0)| \leq r_0$ and therefore, if $a(t) \geq r_0$,

$$\langle\sigma^{(d)}(t), n(t)\rangle \ = \ \langle\phi(t, r_0) + r_0 n(t), n(t)\rangle \ \geq \ r_0 \ - \ |\phi(t, r_0)| \ \geq \ 0\,.$$

For the converse, Example 3.2 works whenever we choose $|u_0| > 2r$ and $|u_0 - u_3| = 2r$. □

Although one might choose to accept the flow rule (27) on the grounds of Proposition 3.4 or for other reasons, in our opinion Example 3.2 definitely points out a weakness in that rule, and we therefore seek a remedy. Recalling that (27) is based on uniaxial data only (namely, on the function f), we focus our attention on the energy dissipation mechanism of the uniaxial hysteresis model as it was investigated from a mathematical standpoint (see e.g. [12]), and present a multiaxial generalization of that approach (compare also [13]) in order to stimulate the discussion. We write the total mechanical power (= rate of work) in the form

$$\langle\dot\epsilon(t), \sigma(t)\rangle \ = \ \frac{d}{dt}\,(P_E(t) \ + \ P_M(t)) \ + \ D(t)\,. \tag{50}$$

Here, $D(t)$ is the dissipation rate, which should turn out to be nonnegative, $P_E(t)$ denotes the standard elastic potential

$$P_E(t) \ = \ \frac{1}{2}\langle A\sigma(t), \sigma(t)\rangle\,, \tag{51}$$

and we propose a *memory potential* $P_M(t)$ (or *hysteresis potential* in the terminology of [13]) of the form

$$P_M(t) \ = \ \frac{1}{2}\int_0^\infty |\phi(t,r)|^2 \eta(r)\, dr\,, \tag{52}$$

where, as in (31), $\eta = f'' \geq 0$. We also propose the flow rule

$$\epsilon^p(t) \ = \ \int_0^\infty \phi(t,r)\eta(r)\, dr\,. \tag{53}$$

Actually, (31) and (53) coincide in the uniaxial case. From (26), (33), (34) and (50) – (53) we compute, assuming that $\partial_t\phi(t,r)$ exists almost everywhere and is bounded for $r > 0$, which is the case if e.g. $\sigma^{(d)}$ consists of linear and circular arcs only,

$$D(t) \ = \ \int_0^\infty \langle\partial_t\phi(t,r), \sigma^{(d)}(t) - \phi(t,r)\rangle\eta(r)\, dr\,. \tag{54}$$

We will prove below that the Mróz hardening rule satisfies a certain *energy inequality* which in turn implies

$$\langle\partial_t\phi(t,r), \sigma^{(d)}(t) - \phi(t,r)\rangle \quad \geq \quad 0 \tag{55}$$

almost everywhere. This leads to a satisfactory state of affairs, since then $D(t) \geq 0$ almost everywhere, and consequently no cyclic process will violate the second law of thermodynamics regardless of the form or amplitude of the loading history. This follows from the observation that the function $t \mapsto \phi(t,r)$ (and therefore, also ϵ^p, P_E and P_M as functions of t), are T-periodic in t for T-periodic piecewise linear (and hence, by continuity, for T-periodic continous) input functions $\sigma^{(d)}$ with the possible exception of the first cycle. Before we prove (55), however, we briefly note that the flow rule (53), too, yields a continuous (i.e. well-posed) stress-strain relation.

Proposition 3.5 *Let us define the modified Mróz operator* $\epsilon^p = M^*(\sigma^{(d)})$ *as the composition of the flow rule (53) and the Mróz hardening rule* $\phi = F(\sigma^{(d)})$. *Then for any* $\sigma_1, \sigma_2 \in C(0,T;\mathbf{T_{dev}})$ *we have*

$$\| M^*(\sigma_1^{(d)}) - M^*(\sigma_2^{(d)}) \|_\infty \leq \sqrt{2R} f'(R)(\| \sigma_1^{(d)} - \sigma_2^{(d)} \|_\infty)^{\frac{1}{2}},$$

where $R = \max\{\| \sigma_1^{(d)} \|_\infty, \| \sigma_2^{(d)} \|_\infty\}$.

Proof: This is an immediate consequence of Theorem 2.6. $\square$

We now present an energy inequality for the Mróz hardening rule.

Proposition 3.6 *Let* $\sigma^{(d)} \in W^{1,1}(0,T;\mathbf{T_{dev}})$ *be given, set* $\phi = F(\sigma^{(d)})$. *Then we have*

$$\frac{1}{2}\left(|\phi(t,r)|^2 - |\phi(s,r)|^2\right) - \langle \phi(t,r), \sigma^{(d)}(t)\rangle + \langle \phi(s,r), \sigma^{(d)}(s)\rangle + \int_s^t \langle \phi(\tau,r), \dot{\sigma}^{(d)}(\tau)\rangle \, d\tau \leq 0 \tag{56}$$

for every $0 \leq s < t \leq T$ *and every* $r \geq 0$.

Dividing both sides of (56) by $t-s$ and letting s tend to t, we immediately see that (55) holds for every $t \in [0,T]$ for which $\partial_t \phi(t,r)$ exists and at which $\dot{\sigma}^{(d)}$ is continuous (actually it suffices that t is a Lebesgue point of $\dot{\sigma}^{(d)}$), so the dissipation rate $D(t)$ as defined above is always nonnegative.

We moreover remark that (56) may be of independent interest, i.e. outside the context of the flow rule.

The remainder of this section is devoted to the proof of Proposition 3.6. We first prove a discrete version of the energy inequality (56).

Lemma 3.7 *Let* $\{u_k\}_{k\geq 0}$ *be a sequence of input values in* U, *set* $\phi_k = G(u_k, \phi_{k-1})$, $\phi_{-1} = 0$. *Then we have*

$$\langle \phi_k(r) - \phi_{k-1}(r), \phi_k(r) - u_k\rangle \leq 0 \tag{57}$$

for every $k \geq 0$ *and every* $r \geq 0$, *and*

$$\frac{1}{2}\left(|\phi_m(r)|^2 - |\phi_l(r)|^2\right) - \langle \phi_m(r), u_m\rangle + \langle \phi_l(s), u_l\rangle + \sum_{k=l}^{m-1} \langle \phi_k(r), u_{k+1} - u_k\rangle \leq 0 \tag{58}$$

for every $m > l \geq 0$ *and every* $r \geq 0$.

Proof: We first prove (57). Set

$$a_k = \alpha(u_k, \phi_{k-1}) = \min\{r \geq 0 : |u_k - \phi_{k-1}(r)| = r\}.$$

Assume that $0 < r < a_k$, otherwise (57) holds trivially. Then

$$\langle \phi_k(r) - \phi_{k-1}(r), \phi_k(r) - u_k \rangle \; = \; \langle \phi_k(r) - \phi_k(a_k) + \phi_{k-1}(a_k) - \phi_{k-1}(r), \phi_k(r) - u_k \rangle$$

$$= \; -r(a_k - r) \; + \; \langle \phi_{k-1}(a_k) - \phi_{k-1}(r), \phi_k(r) - u_k \rangle \quad \leq \; 0$$

so (57) is proved. We now substitute the right hand side of

$$\phi_k(r) \; = \; \frac{1}{2}(\phi_k(r) + \phi_{k-1}(r)) \; + \; \frac{1}{2}(\phi_k(r) - \phi_{k-1}(r))$$

for the second occurrence of $\phi_k(r)$ in (57) and obtain

$$\begin{aligned} 0 \; &\geq \; \sum_{k=l+1}^{m} \langle \phi_k(r) - \phi_{k-1}(r), \frac{1}{2}(\phi_k(r) + \phi_{k-1}(r)) - u_k \rangle \\ &= \; \frac{1}{2}\left(|\phi_m(r)|^2 - |\phi_l(r)|^2 \right) \; - \; \sum_{k=l+1}^{m} \langle \phi_k(r) - \phi_{k-1}(r), u_k \rangle \, . \end{aligned}$$

Rearranging the last sum, inequality (58) follows. □

Proof of Proposition 3.6 Let $0 \leq s < t \leq T$ be given. We approximate the function $\sigma^{(d)}$ by piecewise linear interpolants $u^n : [0, T] \to U$ on the partition $\pi^n = (t_k^n)$ with $0 = t_0^n < t_1^n < \ldots < t_{k_n}^n = T$ such that t and s belong to each partition and that $\max_k(t_{k+1}^n - t_k^n) \to 0$ as n tends to infinity. Fix $n \in N$ and set $\phi^n = F(u^n)$. We apply (58) to the sequence $u_k = u^n(t_k^n)$ with m and l chosen such that $t = t_m^n$, $s = t_l^n$. Since $\phi_k = \phi^n(t_k^n)$, we get

$$\begin{aligned} E^n \; &:= \; \frac{1}{2}\left(|\phi^n(t,r)|^2 - |\phi^n(s,r)|^2 \right) - \langle \phi^n(t,r), u^n(t) \rangle \; + \; \langle \phi^n(s,r), u^n(s) \rangle \\ &\qquad + \int_s^t \langle \phi^n(\tau, r), \dot{u}^n(\tau) \rangle \, d\tau \\ &\leq \; \int_s^t \langle \phi^n(\tau, r), \dot{u}^n(\tau) \rangle \, d\tau \; - \; \sum_{k=l}^{m-1} \langle \phi^n(t_k^n, r), u_{k+1} - u_k \rangle \\ &= \; \sum_{k=l}^{m-1} \frac{1}{t_{k+1}^n - t_k^n} \int_{t_k^n}^{t_{k+1}^n} \langle \phi^n(\tau, r) - \phi^n(t_k^n, r), u_{k+1} - u_k \rangle \, d\tau \, . \end{aligned} \tag{59}$$

From Theorem 2.6 we obtain, if $t_k^n \leq \tau \leq t_{k+1}^n$,

$$\begin{aligned} |\phi^n(\tau, r) - \phi^n(t_k^n, r)| \; &\leq \; \sqrt{2\| \, u^n \, \|_\infty |u^n(\tau) - u^n(t_k^n)|} \\ &\leq \; \sqrt{2\| \, \sigma^{(d)} \, \|_\infty \max_k |\sigma^{(d)}(t_{k+1}^n) - \sigma^{(d)}(t_k^n)|} \quad =: \; \delta^n \, . \end{aligned} \tag{60}$$

From (59) and (60) we conclude that

$$E^n \; \leq \; \sum_{k=l}^{m-1} \delta^n |u_{k+1} - u_k| \; \leq \; \delta^n \int_0^T |\dot{\sigma}^{(d)}(\tau)| \, d\tau \, . \tag{61}$$

Since u^n converges to $\sigma^{(d)}$ in $W^{1,1}$ and ϕ^n converges to ϕ uniformly, E^n converges to the left hand side of (56). We also see from (60) that δ^n converges to 0. The proof is complete.

4 Proof of the continuity result

In this section we prove Theorem 2.6.

Lemma 4.1 *Let $\phi, \psi \in \Psi$ be given, set $u = \phi(0)$, $v = \psi(0)$,*

$$L_\phi = \inf \{ r : \phi|[r, \infty) = 0 \} , \quad L_\psi = \inf \{ s : \psi|[s, \infty) = 0 \} . \tag{62}$$

Then there holds, for all $s \geq 0$,

$$|\phi(0) - \psi(s)|^2 - s^2 \ \leq \ (L_\phi + L_\psi)|u - v| . \tag{63}$$

Proof: It suffices to prove (63) for $s \leq L_\psi$. For any such s we have

$$\begin{aligned} |\phi(0) - \psi(s)|^2 - s^2 \ &\leq \ (|\phi(0) - \psi(0)| + |\psi(0) - \psi(s)|)^2 - s^2 \\ &\leq \ (|u - v| + s)^2 - s^2 = |u - v|^2 + 2s|u - v| , \end{aligned} \tag{64}$$

as well as, since $\psi(L_\psi) = 0$,

$$|\phi(0) - \psi(s)| \ \leq \ L_\phi + L_\psi - s , \tag{65}$$

so

$$|\phi(0) - \psi(s)|^2 - s^2 \ \leq \ (L_\phi + L_\psi)^2 - 2s(L_\phi + L_\psi) . \tag{66}$$

Put

$$2s_* = L_\phi + L_\psi - |u - v| . \tag{67}$$

For $s \leq s_*$, (63) follows from (64), for $s \geq s_*$ we use (66). □

Lemma 4.2 *Let $\{u_k\}_{k=0}^n$, $\{v_k\}_{k=0}^n$ be a pair of input sequences, set*

$$R = \max_{0 \leq k \leq n} \{|u_k|, |v_k|\} , \quad \delta = \max_{0 \leq k \leq n} |u_k - v_k| . \tag{68}$$

Let $\{\phi_k\}_{k=0}^n$, $\{\psi_k\}_{k=0}^n$ be the corresponding memory states. Then there holds

$$|\phi_k(r) - \psi_k(s)|^2 \ \leq \ 2R\delta + (r - s)^2 , \tag{69}$$

for any $r, s \geq 0$.

Proof: We use induction on k . For $k = 0$, the assertion is trivial, if we assume that $u_0 = v_0 = 0$ (which we may do without loss of generality). Assume that (69) holds for $k - 1$. We set

$$a = \alpha(u_k, \phi_{k-1}) , \quad b = \alpha(v_k, \psi_{k-1}) . \tag{70}$$

Then $\phi_{k-1} = \phi_k$ on $[a, R]$, $\psi_{k-1} = \psi_k$ on $[b, R]$, so inequality (69) holds for $(r, s) \in [a, R] \times [b, R]$. We now prove the implication

$$\text{(69) holds for } (a, s) \quad \Rightarrow \quad \text{(69) holds for any } (r, s) , \quad r \leq a . \tag{71}$$

To prove (71), we fix $s \in [0, R]$ and consider the function $f : [0, a] \to \mathbf{R}$ defined by

$$f(r) = |\phi_k(r) - \psi_k(s)|^2 - (r - s)^2 . \tag{72}$$

Since $\phi_k(r) = u_k + re_\phi$ on $[0,a]$ for some unit vector e_ϕ, the function f is affine linear. From Lemma 4.1 and (16) - (18) we conclude that

$$f(0) \ \leq \ 2R\delta\,, \tag{73}$$

therefore

$$f(a) \ \leq \ 2R\delta \quad \Rightarrow \quad f(r) \ \leq \ 2R\delta\,,\ 0 \leq r \leq a\,. \tag{74}$$

The implication (71) now follows from (74). By symmetry,

$$\text{(69) holds for } (r,b) \quad \Rightarrow \quad \text{(69) holds for any } (r,s)\,, \quad s \leq b\,. \tag{75}$$

Applying (71) and (75) repeatedly, (69) follows for all values of r and s. □

Proof of Theorem 2.6. For a fixed $R > 0$, we define

$$U_R \ = \ \{\, u \in U : |u| \ \leq \ R\,\}\,,\ \Psi_R \ = \ \{\, \psi \in \Psi : \psi(r) = 0 \,\text{for any}\, r \ \geq \ R\,\}\,.$$

Now let $u \in C(0,T;U_R)$ be piecewise linear. We first claim that $\phi = F(u) \in C(0,T;\Psi_R)$. To this end, we observe that Lemma 4.2 implies for any $\psi = \phi(t,\cdot)$, $t \in [0,T]$, and any $v \in U$ the estimate

$$\|\, G(v,\psi) \ - \ \psi \,\|_\infty \ \leq \ \sqrt{2\,|v - \psi(0)| \,\max\{|v|, L(\psi)\}}\,, \tag{76}$$

and this in turn implies that $\phi(t,\cdot)$ depends continuously upon t. Next, we consider piecewise linear inputs $u, v \in C(0,T;U_R)$ with $\delta \ := \ \|\, u - v\,\|_\infty$. We fix $t \in [0,T]$ and let $\{t_k\}_{k=0}^n$ denote a partition of $[0,T]$ which includes the point t and is such that both u and v are linear within each interval $[t_k, t_{k+1}]$. We apply Lemma 4.2 to the admissible pair $\{u_k\}_{k=0}^n$, $\{v_k\}_{k=0}^n$, where $u_k = u(t_k)$, $v_k = v(t_k)$. From (69) we obtain the estimate

$$\max_{r \geq 0} |\,(Fu)(t,r) \ - \ (Fv)(t,r)\,| \ \leq \ \sqrt{2R\,\|\, u - v\,\|_\infty}\,. \tag{77}$$

Therefore, the operator F is uniformly continuous on the set of piecewise linear input functions, which is dense in the space $C(0,T;U_R)$, and F has values in the space $C(0,T;\Psi_R)$. Because U_R and Ψ_R are complete metric spaces, F can be extended uniquely to an operator from $C(0,T;U_R)$ to $C(0,T;\Psi_R)$, such that (71) holds for any $u, v \in C(0,T;U_R)$. This concludes the proof of the theorem. □

5 Proof of the bounded variation result

This section is devoted to the proof of Theorem 2.8. Due to the memory buildup, it may happen that $\int |\partial_t \phi(t,r)|dt$ becomes large for some time period although $|u'(t)|$ is small during that time. (It does not happen if dim(U) = 1, i.e. in the memory structure of the scalar Preisach operator.) This constitutes the essential difficulty, and we have to introduce several intermediate quantities attached to the corners of the evolving memory state to overcome it.

For the whole of this section, let us fix a piecewise linear input $u : [0,T] \to U$ together with the corresponding piecewise linear memory state $\phi(t,r) = (Fu)(t,r)$, and let us fix a number $r > 0$. To estimate $\int |\partial_t \phi(t,r)|dt$, we have to study the actual movement $t \mapsto \phi(t,r)$. Let $P_i(t)$, $0 \leq i \leq N(t)$ denote the corners of $\phi(t,\cdot)$ counted from the end

$P_0(t) = 0$, so $\phi(t, \cdot)$ has $N(t) + 1$ corners. The other end $\phi(t, 0) = u(t)$ is not counted as a corner, but we use the convention $P_{N(t)+1}(t) = u(t)$. For each corner $P_i(t)$, we define its r-coordinate $r_i(t)$ and the unit vector $e_i(t)$ pointing towards it from $P_{i+1}(t)$ by the formulas

$$e_i(t) \;=\; \frac{P_i(t) - P_{i+1}(t)}{|P_i(t) - P_{i+1}(t)|}\,, \quad P_i(t) \;=\; \phi(t, r_i(t))\,, \quad 0 \le i \le N(t)\,, \tag{78}$$

Actually, $r_0(t)$ is not uniquely specified by (78), so we set

$$r_0(t) \;=\; \max_{0 \le s \le t} |u(s)|\,. \tag{79}$$

The last corner $P_{N(t)}(t)$, which represents the midpoint of the largest currently active surface, plays a central role in our analysis. We therefore introduce the abbreviations

$$P(t) \;= P_{N(t)}(t)\,, \quad a(t) \;=\; r_{N(t)}(t)\,, \quad e(t) \;= e_{N(t)}(t)\,, \tag{80}$$

so $a(t)$ denotes the radius of the largest currently active surface and $e(t)$ the inward normal common to all active surfaces. We obviously have

$$P(t) \;=\; u(t) \;+\; a(t)e(t)\,. \tag{81}$$

We choose a partition $0 = t_0 < t_1 < \ldots < t_K = T$ of the interval $[0, T]$ such that $u'(t)$ is constant in each subinterval (t_j, t_{j+1}). We may obviously assume that $u' \neq 0$ in each subinterval (otherwise we just drop such an interval). Passing to a suitable refinement, if necessary, we may also assume that, in each subinterval $[t_j, t_{j+1}]$, one of the following five cases occurs.

Case (E) (Enlarge) For $t \in [t_j, t_{j+1}]$ we have

$$a(t) \;=\; r_0(t) \;=\; |u(t)|\,, \quad N(t) \;=\; 0\,, \tag{82}$$

and $a'(t)$ is a positive constant. (See Figure 6.)

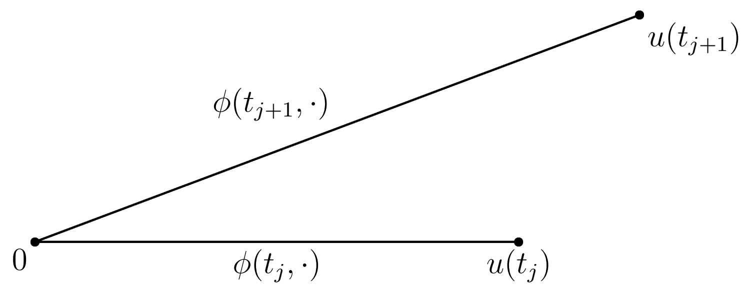

Figure 6: *Enlarge.*

Case (CM) (Create and Move) Here, $r_i(t)$ and $P_i(t)$ are constant for $i \le N(t_j)$, and

$$N(t) \;= N(t_j) \;+\; 1\,, \quad a(t) \;=\; \frac{1}{2}(t - t_j)\frac{|u'|^2}{\langle e(t_j), u' \rangle}\,, \quad t \in (t_j, t_{j+1}]\,. \tag{83}$$

Since $a(t_j) > 0$, the function a has a downward jump at $t = t_j$. (See Figure 7.)

Case (M) (Move) For $t \in [t_j, t_{j+1}]$ we have $N(t) = N(t_j) \geq 1$, and $r_i(t)$ and $P_i(t)$ are constant for $i \leq N(t_j) - 1$. The values $a(t)$ and $e(t)$ are implicitly determined by $|e(t)| = 1$ and by

$$P(t) \;=\; u(t) \,+\, a(t)e(t) \;=\; (a(t) - a(t_j))e_* \,+\, a(t_j)e(t_j) \,+\, u(t_j)\,, \qquad (84)$$

where we have abbreviated $e_* := e_{N(t_j)-1}(t_j)$. An elementary computation involving the implicit function theorem shows that the function $a = a(t)$ is continous and strictly increasing in $[t_j, t_{j+1}]$. (See Figure 8.)

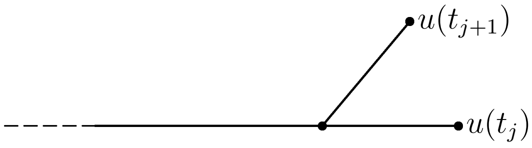

Figure 7: *Create and move.*

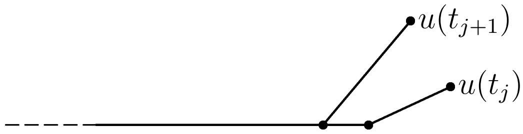

Figure 8: *Move.*

Case (MM) (Move and Merge) This is the same as case (M) except for the modification $N(t_{j+1}) = N(t_j) - 1$ which takes into account the merge at $t = t_{j+1}$. (See Figure 9.)

Case (MDM) (Move and Double Merge) Here, both corners vanish in the merge. The description of case (M) remains valid for $t < t_{j+1}$, but we have $N(t_{j+1}) = N(t_j) - 2 \geq 0$ and $a(t_{j+1}) > \lim_{t\uparrow t_{j+1}} a(t)$, so a has an upward jump at $t = t_{j+1}$. (See Figure 10.)

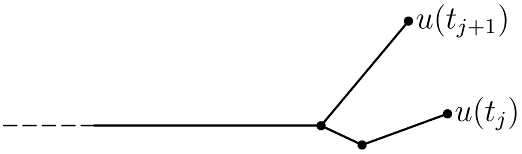

Figure 9: *Move and merge.*

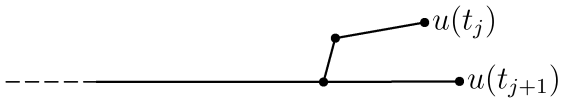

Figure 10: *Move and double merge.*

Finally, we may assume that on every partition interval (t_j, t_{j+1}), either $a(t) < r$ for all t, or $a(t) > r$ for all t.

A partition with all the properties above will be called *regular*. It is easy to see that any partition can be refined to a regular partition, and that any refinement of a regular partition is again a regular partition.

Next, for any regular partition $\{t_j\}$ we want to define the *activity period* $I_i(t_j)$ of the corner $P_i(t_j)$ as the time period prior to t_j during which P_i or a corner merged into P_i has moved. This is achieved as follows. Set $I_i(t_0) = I_i(0) = \emptyset$ for any $i \geq 0$ and define recursively for $j = 0, \ldots, K-1$

$$\begin{aligned} I_i(t_{j+1}) &= I_i(t_j)\,, & i \;<\; N(t_{j+1}) \\ I_{N(t_{j+1})}(t_{j+1}) &= (t_j, t_{j+1}) \,\cup \bigcup_{k\geq N(t_{j+1})} I_k(t_j)\,, \\ I_i(t_{j+1}) &= \emptyset\,, & i \;>\; N(t_{j+1})\,. \end{aligned} \qquad (85)$$

We see in particular that in the cases (MM) and (MDM), the activity period of the last corner swallows up the activity periods of the corners which merged into it. It is easy to see that

$$\bigcup_{i\geq 0} \overline{I_i(t_j)} \;=\; [0, t_j]\,, \qquad 1 \leq j \leq K\,, \tag{86}$$

and that, for any $i \neq k$ and any j,

$$I_i(t_j) \;\cap\; I_k(t_j) \;=\; \emptyset\,. \tag{87}$$

Next, we denote by $V_i(t_j)$ the *input variation during the activity period* $I_i(t_j)$, namely

$$V_i(t_j) \;=\; \int_{I_i(t_j)} |u'(t)|\, dt \quad . \tag{88}$$

Because of (86) and (87), we have

$$\sum_{i\geq 0} V_i(t_j) \;=\; \int_0^{t_j} |u'(t)|\, dt\,, \qquad 0 \;\leq\; j \;\leq\; K\;. \tag{89}$$

It turns out useful to extract from the memory state $\phi(t,\cdot)$ the numbers

$$\begin{aligned} M_i(t) \;&=\; r_i(t)\,|\, e_i(t) \;-\; e_{i-1}(t)\,|\,, \qquad 1 \;\leq\; i \;\leq\; N(t)\,,\\ M_i(t) \;&=\; 0\,, \qquad \text{otherwise.} \end{aligned} \tag{90}$$

which represent the smallest input variation capable of producing the corner $P_i(t)$.

Finally, we want to define the contribution to the output variation related to the movement of the corner $P_i(t)$. Since $\partial_t\phi(t,r) = 0$ if $r > a(t)$ and

$$\phi(t,r) \;=\; u(t) \;+\; r\, e(t)\,, \qquad \text{if}\;\; 0 \;\leq\; r \;\leq\; a(t)\;, \tag{91}$$

we are interested in the variation of $e(t)$ in the time period where $r \leq a(t)$. We therefore define the index set $J_i^j(r)$ by

$$J_i^j(r) \;=\; \{\, k : \,(t_k, t_{k+1}) \;\subset\; I_i(t_j)\,,\; a(t) \;\geq\; r \text{ on } (t_k, t_{k+1})\,\}\,, \tag{92}$$

and the contribution to the output variation $d_i(t_j)$ by

$$d_i(t_j) \;=\; \sum_{k\in J_i^j(r)} |e(t_{k+1}) - e(t_k)|\,. \tag{93}$$

The following lemma, which relates the various quantities just defined, constitutes the key to the proof of Theorem 2.8.

Lemma 5.1 *For every regular partition we have*

$$M_i(t_j) \;\leq\; V_i(t_j)\,, \tag{94}$$

$$J_i^j(r) \;=\; \emptyset\,, \qquad \text{if}\, r_i(t_j) \;\leq\; r \tag{95}$$

$$d_i(t_j) \;\leq\; \left(\frac{2}{r} \;-\; \frac{1}{r_i(t_j)}\right) V_i(t_j)\,, \qquad \text{if}\, r_i(t_j) \;>\; r \tag{96}$$

for any $j = 1, \ldots, K$ *and any* $i \geq 0$.

Proof: We use induction over j. It is easy to see that (94) – (96) hold for $j = 0$ (or $j = 1$ if $u(0) = 0$). Let us suppose now that (94) – (96) hold for some $j \geq 0$ and all $i \geq 0$. For $i > N(t_{j+1})$ we have from (84) that $M_i(t_{j+1}) = V_i(t_{j+1}) = 0$ and $J_i^{j+1}(r) = \emptyset$, hence (94) – (96) hold for $j+1$ in place of j. For $i < N(t_{j+1})$ we have $I_i(t_{j+1}) = I_i(t_j)$, $r_i(t_{j+1}) = r_i(t_j)$ and $e_i(t_{j+1}) = e_i(t_j)$, hence (94) – (96) with j replaced by $j+1$ follow from the induction hypothesis. It remains to perform the induction step for $i = N(t_{j+1})$. We consider the five cases (E), (CM), (M), (MM) and (MDM) separately.

(E) We have $i = 0$, so $M_i(t_{j+1}) = 0$ and there is nothing to prove for (94). If $r_0(t_{j+1}) \leq r$, then $j \notin J_0^{j+1}(r)$ and $r_0(t_j) < r$, so $J_0^{j+1}(r) = J_0^j(r) = \emptyset$. If $r_0(t_{j+1}) > r$, then $J_0^{j+1}(r) = J_0^j(r) \cup \{j\}$. In this case we have

$$e(t_{j+1}) \;=\; \frac{u(t_{j+1})}{r_0(t_{j+1})}\,, \quad e(t_j) \;=\; \frac{u(t_j)}{r_0(t_j)}\,,$$

hence

$$|e(t_{j+1}) - e(t_j)| \;\leq\; \frac{1}{r_0(t_j)}|u(t_{j+1}) - u(t_j)|\,.$$

Formula (93) then gives

$$d_0(t_{j+1}) \;\leq\; d_0(t_j) \;+\; \frac{1}{r_0(t_j)}|u(t_{j+1}) - u(t_j)|\,.$$

Together with the identity $V_0(t_{j+1}) = V_0(t_j) + |u(t_{j+1}) - u(t_j)|$ and the induction hypothesis, we obtain the induction step for (96) in both cases $r_0(t_j) = r$ and $r_0(t_j) > r$.

(CM) One immediately checks from Figure 7 and the definitions that $I_i(t_{j+1}) = (t_j, t_{j+1})$, $M_i(t_j) = V_i(t_j) = |u(t_{j+1}) - u(t_j)|$ and $J_i^{j+1}(r) = \emptyset$, so (94) – (96) hold for $j+1$.

(M) Since $N(t_{j+1}) = N(t_j)$ and $a(t) = r_i(t)$ in $[t_j, t_{j+1}]$, the basic identity (84) can be rewritten for $t = t_{j+1}$ as

$$r_i(t_{j+1})(e_i(t_{j+1}) - e_{i-1}(t_{j+1})) \;=\; r_i(t_j)(e_i(t_j) - e_{i-1}(t_j)) \;-\; (u(t_{j+1}) - u(t_j))\,,$$

and (90) yields

$$M_i(t_{j+1}) \;\leq\; M_i(t_j) \;+\; |u(t_{j+1}) - u(t_j)|\,. \tag{97}$$

We have $I_i(t_{j+1}) = I_i(t_j) \cup ((t_j), (t_{j+1}))$, so

$$V_i(t_{j+1}) \;=\; V_i(t_j) \;+\; |u(t_{j+1}) - u(t_j)|\,, \tag{98}$$

and the induction step for (94) follows easily. In the case $r_i(t_{j+1}) \leq r$, the induction step for (95) and (96) is trivial, since $J_i^{j+1}(r) = J_i^j(r) = \emptyset$. If, on the other hand, $r_i(t_{j+1}) > r$, then $J_i^{j+1}(r) = J_i^j(r) \cup \{j\}$, hence

$$d_i(t_{j+1}) \;=\; d_i(t_j) \;+\; |e(t_{j+1}) - e(t_j)|\,. \tag{99}$$

Another reformulation of (84) at $t = t_{j+1}$ gives

$$r_i(t_{j+1})(e(t_{j+1}) - e(t_j)) \;=\; (r_i(t_{j+1}) - r_i(t_j))(e_{i-1}(t_j) - e_i(t_j)) \;-\; (u(t_{j+1}) - u(t_j))\,, \tag{100}$$

which implies

$$|e(t_{j+1}) - e(t_j)| \ \le\ \left(\frac{1}{r_i(t_j)} - \frac{1}{r_i(t_{j+1})}\right) M_i(t_j) \ +\ \frac{1}{r_i(t_{j+1})}|u(t_{j+1}) - u(t_j)|\,.$$

The induction hypothesis for $r_i(t_j) > r$ (the other case is analogous) and (98) and (99) yield

$$\begin{aligned} d_i(t_{j+1}) \ &\le\ d_i(t_j) \ +\ \left(\frac{1}{r_i(t_j)} - \frac{1}{r_i(t_{j+1})}\right) V_i(t_j) \ +\ \frac{1}{r_i(t_{j+1})}|u(t_{j+1}) - u(t_j)| \\ &\le\ \left(\frac{2}{r} - \frac{1}{r_i(t_{j+1})}\right) V_i(t_{j+1}) \ -\ \left(\frac{2}{r} - \frac{2}{r_i(t_{j+1})}\right) |u(t_{j+1}) - u(t_j)|\,, \end{aligned}$$

which completes the induction step in the case (M).

(MM) Now we have

$$a(t_{j+1}) \ =\ r_i(t_{j+1}) \ =\ r_i(t_j)\,, \quad e(t_j) \ =\ e_{i+1}(t_j)\,, \quad e(t_{j+1}) \ =\ e_i(t_{j+1})\,.$$

The basic identity (84) becomes at $t = t_{j+1}$

$$u(t_{j+1}) - u(t_j) + r_i(t_{j+1})e_i(t_{j+1}) \ =\ (r_i(t_j) - r_{i+1}(t_j))e_i(t_j) + r_{i+1}(t_j)e_{i+1}(t_j)\,. \quad (101)$$

Using $r_i(t_{j+1}) = r_i(t_j)$, $e_{i-1}(t_{j+1}) = e_{i-1}(t_j)$, we easily obtain from (101) that

$$M_i(t_{j+1}) \ \le\ M_i(t_j) \ +\ M_{i+1}(t_j) \ +\ |u(t_{j+1}) - u(t_j)|\,. \quad (102)$$

We have by definition that $I_i(t_{j+1}) = I_i(t_j) \cup I_{i+1}(t_j) \cup (t_j, t_{j+1})$, hence

$$V_i(t_{j+1}) \ =\ V_i(t_j) \ +\ V_{i+1}(t_j) \ +\ |u(t_{j+1}) - u(t_j)|\,, \quad (103)$$

and the induction step for (94) follows easily from (103) and the induction hypothesis. Concerning (95) and (96), the case $r_i(t_j) \le r$ is again trivial, assume now that $r_i(t_j) > r$. We have

$$\begin{aligned} J_i^{j+1}(r) \ &=\ J_i^j(r) \cup J_{i+1}^j(r) \cup \{j\} \\ d_i(t_{j+1}) \ &=\ d_i(t_j) \ +\ d_{i+1}(t_j) \ +\ |e(t_{j+1}) - e(t_j)|\,. \end{aligned} \quad (104)$$

We rewrite (101) as

$$r_i(t_{j+1})(e(t_{j+1}) - e(t_j)) \ =\ (r_i(t_j) - r_{i+1}(t_j))(e_i(t_j) - e_{i+1}(t_j)) \ -\ (u(t_{j+1}) - u(t_j))\,,$$

therefore

$$|e(t_{j+1}) - e(t_j)| \ \le\ \left(\frac{1}{r_{i+1}(t_j)} - \frac{1}{r_i(t_j)}\right) M_{i+1}(t_j) \ +\ \frac{1}{r_i(t_{j+1})}|u(t_{j+1}) - u(t_j)|\,.$$

the induction hypothesis together with (103) and (104) now yields the induction step similarly as in the case (M).

(MDM) We have obviously $M_i(t_{j+1}) = M_i(t_j)$ and

$$I_i(t_{j+1}) \;=\; (t_j, t_{j+1}) \cup \bigcup_{k=0}^{2} I_{i+k}(t_j) \tag{105}$$

hence

$$V_i(t_{j+1}) \;=\; |u(t_{j+1}) - u(t_j)| \;+\; \sum_{k=0}^{2} V_{i+k}(t_j)\,, \tag{106}$$

and the induction step for (94) follows easily. For (95) and (96), the case $r_i(t_{j+1}) \le r$ is as simple as in the previous situations (M) and (MM). If $r_{i+1}(t_j) \le r \le r_i(t_{j+1})$, then $J_i^{j+1}(r) = J_i^j(r)$, hence $d_i(t_{j+1}) = d_i(t_j)$, and the assertion follows immediately from the induction hypothesis and (106). The last case to be considered arises when $r \le r_{i+2}(t_j) < r_{i+1}(t_j)$. From (105) we get

$$\begin{aligned} J_i^{j+1}(r) \;&=\; \{j\} \cup \bigcup_{k=0}^{2} J_{i+k}^{j}(r)\,, \\ d_i(t_{j+1}) \;&=\; |e(t_{j+1}) - e(t_j)| \;+\; \sum_{k=0}^{2} d_{i+k}(t_j)\,. \end{aligned} \tag{107}$$

The basic vector identity (84) at $t = t_{j+1}$ becomes

$$\begin{aligned} r_{i+1}(t_j)(e(t_{j+1}) - e(t_j)) \;=\; & (r_{i+1}(t_j) - r_{i+2}(t_j))(e_{i+1}(t_j) - e_{i+2}(t_j)) \\ & - (u(t_{j+1}) - u(t_j))\,, \end{aligned}$$

therefore

$$|e(t_{j+1}) - e(t_j)| \;\le\; \left(\frac{1}{r_{i+2}(t_j)} - \frac{1}{r_{i+1}(t_j)}\right) M_{i+2}(t_j) \;+\; \frac{1}{r_{i+1}(t_j)}|u(t_{j+1}) - u(t_j)|\,.$$

The induction hypothesis together with (106) and (107) now completes the induction step similarly as in the cases (M) and (MM).

Lemma 5.1 is proved. □

Proof of Theorem 2.8. First, from the description of the five cases above it is easy to see that $t \mapsto \phi(t, r)$ is absolutely continous in each partition interval of a regular partition. Next, we note that Lemma 5.1 implies that

$$d_i(t_j) \;\le\; \frac{2}{r} V_i(t_j) \tag{108}$$

for any $i \ge 0$ and any $j \ge 1$. From (91) – (93) and (108) we obtain for any regular partition

$$\begin{aligned} \sum_{j=0}^{K-1} |\phi(t_{j+1}, r) - \phi(t_j, r)| \;&=\; \sum_{i\ge 0} \sum_{k \in J_i^{K-1}(r)} |\phi(t_{k+1}, r) - \phi(t_k, r)| \\ &\le\; \int_0^T |u'(t)|\,dt \;+\; r\sum_{i\ge 0} d_i(T) \;\le\; 3\int_0^T |u'(t)|\,dt\,. \end{aligned}$$

Since we may arbitrarily refine the partition, the assertion of Theorem 2.8 is proved.

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