Rainflow Counting and Energy Dissipation for Hysteresis Models in Elastoplasticity

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Abstract

The rainflow counting method is widely used in the context of fatigue analysis and damage estimation. We analyze some of its mathematical properties and provide several connections to hysteresis operators. As a consequence, we prove that the total damage obtained through the Palmgren-Miner-Rule is a continuous functional of the loading history. We then consider several constitutive laws of elastoplasticity, also including fatigue, and show that, for arbitrary loading histories, the accumulated damage and the dissipated energy can both be expressed as the total variation of the output of a hysteresis operator. Except for some remarks, we exclusively deal with the uniaxial case.

Running title. Rainflow Counting and Energy Dissipation.

1 Introduction

Elastoplastic deformations cause damage. A deformation large enough destroys a given piece of material immediately. Smaller deformations, too, lead to eventual destruction, if they are applied repeatedly. Experimental results for a given workpiece are usually condensed into an *S-N-diagram* depicting the *Wöhler line*, a plot of the (scalar) stress

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amplitude S versus the number N of cycles (oscillations between two amplitudes, e.g. 0 and S) until destruction occurs. For a sequence of cycles of varying amplitude, the famous *Palmgren-Miner-Rule of linear damage accumulation* then evaluates the total accumulated damage as the sum of the contributions $1/N$ from the individual oscillations.

The *rainflow counting method* due to Endo is widely used to decompose an arbitrary sequence of (scalar) loads or deformations into cycles and to count those cycles. In combination with the Palmgren-Miner-Rule, we thus obtain for every sequence of loads a real number which estimates the damage inflicted upon the workpiece by this loading sequence. Mechanical engineers have developed several refinements and modifications of this procedure; anyway, the single cycle respectively the corresponding *hysteresis loop* in the stress-strain plane certainly constitutes a basic event for damage assessment. It is also known (see e.g. [Clormann & Seeger, 1986]) that the memory structure of the elastoplastic constitutive law due to [Prandtl, 1928] and [Ishlinskii, 1944] directly corresponds to the decomposition performed by the rainflow method. Mathematical results on the rainflow method are included in [Rychlik, 1987], [Rychlik, 1992] and [Rychlik, 1993], its relation to fatigue and its implementation is discussed in [Dreßler & Krüger], see also the recent proceedings volume [Murakami (ed.), 1992].

Besides damage, one may also associate with every cycle or hysteresis loop the total energy dissipated during a full cycle. Obviously, the sum over the individual cycles yields the total energy dissipated during a sequence of cycles, and there is the question whether we obtain the correct value for the energy dissipation during an arbitrary loading sequence if we sum over the cycles of its rainflow decomposition.

If one associates values to individual cycles or hysteresis loops, one implicitly assumes that the underlying process is *rate independent*, meaning that only the loops themselves are important, but not the speed, with which they are traversed. Rate independent processes are described mathematically by *hysteresis operators*. Their mathematical properties have been studied a lot recently, see the monographs [Krasnosel'skii & Pokrovskii, 1983], [Mayergoyz, 1991], [Visintin, 1994], [Krejčí, 1996] and [Brokate & Sprekels]; in particular, the connection to the classical and widely used theory of *rheological models* has been well established. On the other hand, interest in a formal theory of hysteresis models is growing among scientists, since the same model often makes sense for rather different applications. For example, in [Lubarda, Sumarac & Krajcinovic, 1993] the Preisach model from ferromagnetism is used in the context of elastoplasticity, and the energy dissipation is computed as in [M, 1991] through integration over hysteresis elements with rectangular loops, which serve as a mathematical tool without intuitive counterpart in the material body.

In the present paper, we want to show that the mathematics of hysteresis operators constitutes a useful and unifying tool in the context sketched above. The organization is as follows. In Section 2, we give a formal description of the rainflow method, and we show that the accumulated damage equals the variation of the output of a certain Preisach hysteresis operator. Some of the formulas obtained are related to those of [R, 1992] and [R, 1993]. In Section 3 we use (and extend slightly) a result of Visintin [V, 1994] to show that the accumulated damage depends continuously on the input function (i.e., the loading). This result formally justifies common techniques like hysteresis filtering and range discretization in actual implementations of the rainflow method. The proofs are relegated to the appendix. In Section 4, we briefly recall the connection to the memory structure of the Preisach model. Section 5 is devoted to the derivation of hysteretic constitutive laws, dissipation formulas and rainflow count formulas for nonlinear rate independent rheological models. In particular, we express the accumulated damage, too, as the output of a certain Preisach operator, and exploit the formal similarity to the characterization of the accumulated damage in Section 2. We also discuss rheological models which take fatigue into account as a dependence upon the maximal loading amplitude. It turns out that the corresponding hysteresis operator no longer is a Preisach operator, but it belongs to the more general class of memory preserving [Krejčí, 1991/b] or Preisach type [B & S] operators.

2 The Rainflow Method

The rainflow method processes a finite sequence of numbers, which represent the discrete time history of some measured quantity as e.g. the uniaxial load or strain at a certain point. Let us denote by S the set of all finite strings of real numbers, i.e.

$$S = \{ (v_0, v_1, \ldots, v_N) : N \in \mathbf{N}_0,\ v_i \in \mathbf{R},\ 0 \le i \le N \}, \quad \mathbf{N}_0 = \mathbf{N} \cup 0, \tag{2.1}$$

and let us denote the *concatenation* of two strings $s = (v_0, \ldots, v_N)$ and $s' = (v'_0, \ldots, v'_{N'})$ by

$$(s, s') = (v_0, \ldots, v_N, v'_0, \ldots, v'_{N'}). \tag{2.2}$$

In order to describe formally the reduction and counting process of the rainflow method, we formulate two deletion rules, which operate on the set S. To remove values within a monotone part of a string, we define the *monotone deletion* by

$$(v_0, \ldots, v_N) \mapsto (v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_N), \quad \text{if } v_i \in [v_{i-1}, v_{i+1}]. \tag{2.3}$$

Here and in the following, we write $[b, c]$ for the closed interval bounded by b and c, even if $b \ge c$, so we always have $[b, c] = [c, b]$.

The second rule formalizes the deletion of an inner cycle. We say that (v_i, v_{i+1}) is a *Madelung pair* for $s = (v_0, \ldots, v_N)$, if $[v_i, v_{i+1}] \subset [v_{i-1}, v_{i+2}]$ and if neither v_i nor v_{i+1} can be removed by a monotone deletion. If (v_i, v_{i+1}) is a Madelung pair for s, then the reduction

$$(v_0, \ldots, v_N) \mapsto (v_0, \ldots, v_{i-1}, v_{i+2}, \ldots, v_N) \tag{2.4}$$

is called the *Madelung deletion* – to acknowledge the contribution [Madelung, 1905], where it may have been formulated explicitly for the first time. A string $s \in S$ is called *irreducible*, if neither deletion rule applies to it. We introduce a partial ordering on S by saying that $s' \le s$ for $s, s' \in S$, if s' can be obtained from s by a finite sequence of (arbitrarily mixed) monotone and Madelung deletions. For any such finite sequence of deletions, we define its *unsymmetric rainflow count* as a function $a_u : \mathbf{R} \times \mathbf{R} \to \mathbf{N}_0$, where $a_u(x, y)$ specifies how often the pair (x, y) is removed by a Madelung deletion. If we ignore the order within each pair, we obtain the *symmetric rainflow count* $a(x, y) = a_u(x, y) + a_u(y, x)$. Note that this definition implies that $a_u(x, x) = a(x, x) = 0$ for all $x \in \mathbf{R}$ and that $a_u(x, y) \ne 0$ for a finite number of pairs only.

In actual applications, the input values are classified in a preprocessing step, so that afterwards only finitely many, say K, different values $x_1 < \ldots < x_K$ can occur. In this case, the rainflow count reduces to the $K \times K$ *rainflow matrix* A_u with $a_{u,kl} = a_u(x_k, x_l)$. Accordingly, $A = A_u + A_u^T$ defines the *symmetric rainflow matrix*.

Lemma 2.1 *Let a string $s = (v_0, \ldots, v_N) \in S$ be given to which deletions can be applied at two different places. Then at least one of the following two possibilities occurs:*

- *The two deletions commute, i.e. they can be performed in any order, resulting in the same reduced string and the same deleted Madelung pairs.*
- *If we apply either the first or the second deletion to s, we obtain the same reduced string. In addition, the deletions are either both monotone or both Madelung; in the latter case, the deleted Madelung pairs are opposite (i.e. (y, x) instead of (x, y)).*

If $v_i \neq v_j$ for all $i \neq j$, then the two deletions always commute.

Proof: One has to check several cases. If both deletions are monotone, they commute, except possibly for the case where two adjacent input values are equal and represent a local extremum, in which case the second alternative holds. If one deletion is monotone and the other Madelung, then they always commute. Let us now assume that both deletions are of Madelung type. We denote by i_1 and i_2 the index of the left element of the respective Madelung pair (it corresponds to the index i in equation (2.4)). If $|i_2 - i_1| \geq 2$, then the deletions commute. We finally consider the case where $|i_2 - i_1| = 1$ and assume that $i_2 = i_1 + 1$. Then we must necessarily have $v_{i_1+2} = v_{i_1}$, and the second alternative holds. □

Theorem 2.2 *For any $s \in S$, there is a unique irreducible string $s_R \in S$ with $s_R \leq s$, which we term the* rainflow residual *of s. Moreover, the symmetric rainflow count is the same for all deletion sequences leading from s to s_R. If $v_i \neq v_j$ for all $i \neq j$, this is also true for the unsymmetric rainflow count.*

Proof: We use induction on N. For $N = 1$, all strings are irreducible, so the assertion holds trivially. We now assume that the theorem holds for strings of length up to $N - 1$. Let an $s = (v_0, \ldots, v_N) \in S$ be given, to which two deletions can be applied in different places. Let us denote by s_1 and s_2 the strings produced from s by either deletion. We apply the preceding lemma. If the deletions commute, the reduction of s_1 and s_2 with deletion 2 and 1 respectively, yields the same string. The induction hypothesis now implies that s_1 and s_2 uniquely reduce to the same irreducible string. If the second alternative of Lemma 2.1 holds, then $s_1 = s_2$. In both cases, the symmetric rainflow counts from s via s_1 or s_2 are identical. If all input values are different, the second alternative of Lemma 2.1 is not needed. The induction step is complete. □

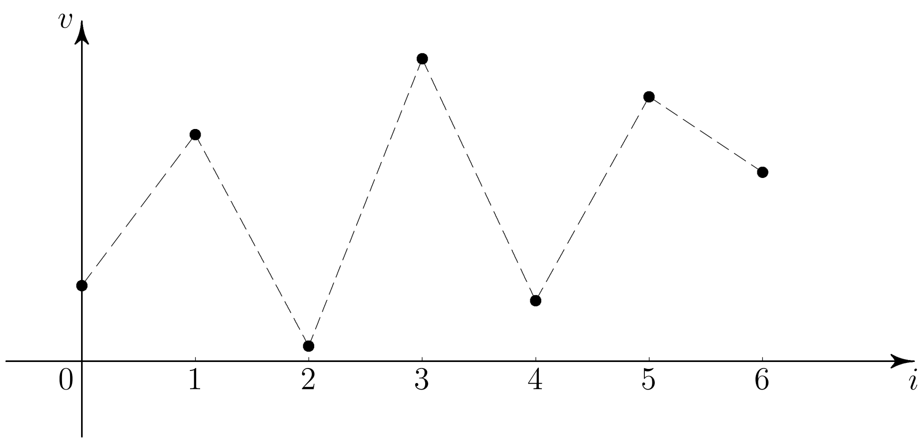

Figure 1: *Form of an irreducible string*

Irreducible strings $s = (v_0, \dots, v_N)$ have the form shown in Figure 1. They have to be *alternating*, i.e.

$$d_{i-1}d_i < 0\,, \quad 1 \le i < N\,, \tag{2.5}$$

where

$$d_i = v_{i+1} - v_i\,, \quad 0 \le i < N\,, \tag{2.6}$$

and they must satisfy

$$0 < |d_0| < \dots < |d_{J-1}| \le |d_J| > \dots > |d_{N-1}| > 0 \tag{2.7}$$

for some index J. (For $N = 1$, also $d_0 = 0$ is possible.) The pair (v_J, v_{J+1}) is obviously *maximal*, i.e.

$$v_J = \max_{0 \le i \le N} v_i\,, \quad v_{J+1} = \min_{0 \le i \le N} v_i\,, \tag{2.8}$$

or vice versa.

Proposition 2.3 *A string* $s = (v_0, \dots, v_N)$ *is irreducible if and only if it is alternating and satisfies (2.7) for some index* J, $0 \le J < N$.

Proof: Obviously, s is alternating if and only if it admits no monotone deletion. Moreover, (v_i, v_{i+1}) is a Madelung pair for s if and only if

$$0 < |d_i| \le \min\{|d_{i-1}|, |d_{i+1}|\}\,. \tag{2.9}$$

A string satisfying (2.7) therefore does not admit a Madelung deletion. Conversely, if s admits no Madelung deletion, we define J to be the smallest index such that $|d_J| > \dots > |d_{N-1}|$. Using (2.9) successively for $i = J-1, \dots, 1$ we see that (2.7) holds. □

The *rainflow algorithm* due to Endo computes the unsymmetric rainflow count and the rainflow residual from a given input string. Note however that, by Theorem 2.2, only the symmetric rainflow count of a string $s = (v_0, \dots, v_N)$ is always well defined, while the unsymmetric rainflow count may depend on the particular deletion sequence chosen. One may of course define the unsymmetric rainflow count of a string as the one produced by the deletion sequence of any variant of the rainflow algorithm. We present the version of [Krüger, Scheutzow, Beste & Petersen, 1985].

Algorithm 2.4 (The Rainflow Algorithm) The algorithm works on an input string $s_0 = (v_0, \dots, v_N)$, $N \ge 3$, which consists of alternating local maxima and minima only, i.e. it assumes that all possible monotone deletions have been performed already. It uses as intermediate variable a string s of variable length storing the current residual. By $m(s)$ we denote the substring which consists of the last four elements of s.

1. Set $s := (v_0, v_1, v_2, v_3)$, $i = 3$ and $a(x, y) = 0$ for all x, y.
2. **While** length$(s) \ge 4$ **and** the middle pair (x, y) of $m(s)$ forms a Madelung pair for $m(s)$

   begin increment $a(x, y)$ by 1; delete the middle pair (x, y) of $m(s)$ from s **end**

3. **If** $i = N$ **then** stop **else**

   begin increment i by 1; join v_i to s as the last element; **go to** 2 **end**. □

It is easy to verify that the algorithm produces a sequence of Madelung deletions for the input string s_0, and that at the beginning of step 3, the string s is irreducible. Therefore, when the algorithm stops, s equals the rainflow residual of s_0, and a equals the rainflow count of the deletion sequence.

In practice, one keeps both the rainflow count and the residual, since the latter includes – in admittedly rudimentary form – information on the order of the original sequence which has been found relevant for purposes of reconstruction and extrapolation (of loading sequences from rainflow matrices). If one is interested in a complete cycle count, however, one wants to count the residual, too. In view of the next lemma, the symmetric rainflow count of (s_R, s_R) appears as the natural way to count the residual s_R. Since this means that we have to compare counts of different strings, we will write $a(s)$ and $a(s)(x,y)$ instead of a and $a(x,y)$ to specify the string whose count is obtained.

Lemma 2.5 *For any string $s \in S$ there holds*

$$(s,s)_R = (s_R, s_R)_R = s_R\,. \tag{2.10}$$

Proof: Since irreducible strings satisfy (2.7), the maximal pair $p = (v_J, v_{J+1})$ of s_R appears in (s_R, s_R) exactly twice and in $(s_R, s_R)_R$ exactly once, except for the trivial case $N = 1, d_0 = 0$. The Madelung deletion $(p,p) \mapsto p$ being the only way to achieve this reduction, there has to exist a deletion sequence

$$(s_R, s_R) = (\ldots, p, \ldots, p, \ldots) \geq (\ldots, p, \ldots) = s_R\,, \tag{2.11}$$

removing the middle part. The assertion now follows from Theorem 2.2. □

Definition 2.6 (Periodic Rainflow Count) *For any $s \in S$, we define its* periodic rainflow count $a_{per}(s) : \mathbf{R} \times \mathbf{R} \to \mathbf{N}_0$ *as*

$$a_{per}(s) = a(s) + a(s_R, s_R)\,, \tag{2.12}$$

The basic formula for formal computations with the rainflow count is

$$a(s_1, s_2, s_3) = a(s_2) + a(s_1, s_{2R}, s_3)\,, \tag{2.13}$$

where s_1, s_2, s_3 are any (possibly empty) strings. Applying (2.13) twice to the string (s,s), for example, we obtain

$$a_{per}(s) = a(s,s) - a(s)\,, \quad s \in S\,. \tag{2.14}$$

The periodic rainflow count is a periodic invariant.

Proposition 2.7 *Let $s = (v_0, \ldots, v_N)$. Then*

$$a_{per}(\mathrm{wrap}_i(s)) = a_{per}(s)\,, \quad 0 \leq i \leq N\,, \tag{2.15}$$

where

$$\mathrm{wrap}_i(s) = (v_i, \ldots, v_N, v_0, \ldots, v_{i-1}))\,. \tag{2.16}$$

Proof: It suffices to consider the case where s is nonconstant and $v_0 = \max_{0\le i\le N}$. We fix i and set $s_1 = (v_0, \ldots, v_{i-1})$, $s_2 = (v_i, \ldots, v_N)$, so

$$s = (s_1, s_2)\,, \quad \tilde{s} := \mathrm{wrap}_i(s) = (s_2, s_1)\,. \tag{2.17}$$

We denote by a_0 the count of a single cycle of maximum amplitude, i.e.

$$a_0 = a(v_0, v_{\min}, v_0, v_{\min})\,, \quad v_{\min} = \min_{0\le i\le N} v_i\,. \tag{2.18}$$

Since obviously we have $s_R = (v_0, v_{\min}, \ldots)$ and $(s, v_0)_R = (v_0, v_{\min}, v_0)$, we get from (2.14) that

$$a_{per}(s) = a(s, v_0, s) - a(s) = a(s, v_0) + a(v_0, v_{\min}, v_0, s_R) = a(s, v_0) + a_0\,, \tag{2.19}$$

$$a(\tilde{s}, \tilde{s}) = a(s_2, s, v_0, s_1) = a(s, v_0) + a(s_2, v_0, v_{\min}, v_0, s_1)\,. \tag{2.20}$$

The proof is complete if we can show that

$$a(s_2, v_0, v_{\min}, v_0, s_1) = a(\tilde{s}) + a_0\,. \tag{2.21}$$

Either s_1 or s_2 has to contain the value $v_{\min}$. If s_1 does,

$$\begin{aligned} a(s_2, v_0, v_{\min}, v_0, s_1) &= a(s_1) + a(s_2, v_0, v_{\min}, v_0, s_{1R}) = a(s_1) + a(s_2, s_{1R}) + a_0 \\ &= a(\tilde{s}) + a_0\,. \end{aligned} \tag{2.22}$$

The other case is treated analogously. □

In [K, S, B & P, 1985], the rainflow count is related to the counting of oscillations with respect to fixed thresholds. The *relay hysteresis operator* provides a way to express such an oscillation count.

Definition 2.8 (Relay Hysteresis Operator) *Let $x, y \in \mathbf{R}$ with $x < y$, let $w_{-1} \in \{0, 1\}$ be given. We define the relay operator $\mathcal{R}_{x,y} : S \to S$ by*

$$\mathcal{R}_{x,y}(v_0, \ldots, v_N) = (w_0, \ldots, w_N)\,, \tag{2.23}$$

with

$$w_i = \begin{cases} 1\,, & v_i \ge y \\ 0\,, & v_i \le x \\ w_{i-1}\,, & x < v_i < y\,. \end{cases} \tag{2.24}$$

We write $\mathcal{R}_{x,y}(s; w_{-1})$ if we want to specify the initial value explicitly. □

For any string $s = (v_0, \ldots, v_N) \in S$, we denote its variation by

$$\mathrm{Var}\,(s) = \sum_{i=0}^{N-1} |v_{i+1} - v_i|\,. \tag{2.25}$$

The removal of a Madelung pair (x, y) from a string decreases its variation by the amount $2|y - x|$, whereas monotone deletions do not change the variation, so

$$\mathrm{Var}\,(s) = 2\sum_{x<y} (y - x) a(s)(x, y) + \mathrm{Var}\,(s_R) \tag{2.26}$$

holds for any string $s \in S$. The number $\mathrm{Var}(\mathcal{R}_{x,y}(s))$ represents the number of oscillations of s between the thresholds x and y.

Proposition 2.9 *Let $s \in S$ be given. Then there holds*

$$\mathrm{Var}\,(\mathcal{R}_{x,y}(s)) = 2 \sum_{\xi \le x < y \le \eta} a(s)(\xi,\eta) \;+\; \mathrm{Var}\,(\mathcal{R}_{x,y}(s_R))\,. \tag{2.27}$$

If moreover $w_{-1} = w_0 = w_N$, then

$$\mathrm{Var}\,(\mathcal{R}_{x,y}(s)) = 2 \sum_{\xi \le x < y \le \eta} a_{per}(s)(\xi,\eta)\,. \tag{2.28}$$

Proof: If we delete some Madelung pair (ξ,η) with $[x,y] \subset [\xi,\eta]$ from s, then for the resulting string s' there holds

$$\mathrm{Var}\,(\mathcal{R}_{x,y}(s')) = \mathrm{Var}\,(\mathcal{R}_{x,y}(s)) \;-\; 2\,, \tag{2.29}$$

while for any other single deletion we have $\mathrm{Var}\,(\mathcal{R}_{x,y}(s')) = \mathrm{Var}\,(\mathcal{R}_{x,y}(s))$. This proves (2.27). Using (2.10) and (2.14), we see that

$$\mathrm{Var}\,(\mathcal{R}_{x,y}(s,s)) = 2 \sum_{\xi \le x < y \le \eta} [a_{per}(s) + a(s)](\xi,\eta) \;+\; \mathrm{Var}\,(\mathcal{R}_{x,y}(s_R))\,. \tag{2.30}$$

Subtracting (2.27) from (2.30) we obtain (2.28), since the additional assumption on w implies that

$$\mathrm{Var}\,(\mathcal{R}_{x,y}(s,s)) = 2\,\mathrm{Var}\,(\mathcal{R}_{x,y}(s))\,. \tag{2.31}$$

□

The assumption $w_{-1} = w_0 = w_N$ for the second part of Lemma 2.9 is unpleasant, since for no choice of w_{-1} it will be satisfied for all load sequences s. We can remove it if we pass to the periodic version of the relay defined by

$$\mathcal{R}^{per}_{x,y}(s; w_{-1}) = \mathcal{R}_{x,y}(s; w^{per}_{-1})\,, \quad s \in S\,,\; w_{-1} \in \{0,1\}\,, \tag{2.32}$$

where w^{per}_{-1} equals the last value of the string $\mathcal{R}_{x,y}(s; w_{-1})$. In other words,

$$\mathcal{R}_{x,y}(s,s; w_{-1}) = (\mathcal{R}_{x,y}(s; w_{-1}), \mathcal{R}^{per}_{x,y}(s; w_{-1}))\,. \tag{2.33}$$

We then get

Corollary 2.10 *Let $s = (v_0,\ldots,v_N)$ with $v_0 = v_N$. Then there holds*

$$\mathrm{Var}\,(\mathcal{R}^{per}_{x,y}(s)) = 2 \sum_{\xi \le x < y \le \eta} a_{per}(s)(\xi,\eta)\,. \tag{2.34}$$

Proof: One checks that $(w_0,\ldots,w_N) = \mathcal{R}^{per}_{x,y}(v_0,\ldots,v_N; w_{-1})$ satisfies $w_0 = w_N = w^{per}_{-1}$ regardless of the value of w_{-1}. □

In combination with the *Palmgren-Miner-Rule* for linear damage accumulation, the rainflow method is commonly used in fatigue analysis in the following manner. Let $N(x,y)$ denote the number of times a repetition of the input cycle (x,y) leads to failure. On a unit scale,

$$\Delta(x,y) = \frac{1}{N(x,y)} \tag{2.35}$$

then represents the contribution of the single cycle (x,y) to the damage of the structure. The total damage $D(s)$ due to some input string $s \in S$ is then estimated as

$$D(s) = \sum_{x<y} a_{per}(x,y)\Delta(x,y)\,. \tag{2.36}$$

Note that here, as above and below, the values of x and y range over all real numbers, so through a suitable choice of the function Δ one can model the different behaviour in tension and compression as well as mean stress dependence (if the stress is chosen as the input variable) and amplitude. For this purpose, we may always change the coordinates from x and y to $(x+y)/2$ and $y-x$.

We want to perform linear superposition in formula (2.28) to relate the total damage to the output variation of another hysteresis operator, namely the *Preisach operator*, which has been developed in the context of ferromagnetic hysteresis [Preisach, 1935], [K & P, 1983], [Brokate & Visintin, 1989].

Within the context of fatigue analysis, there is no reason to consider arbitrarily large input values. Since it facilitates the exposition, from now on we fix an a priori bound M for admissible input values. The relevant threshold values for the relays $\mathcal{R}_{x,y}$ then lie within the triangle

$$P = \{(x,y) \in \mathbf{R}^2,\ -M \le x < y \le M\}\,. \tag{2.37}$$

Due to Corollary 2.10, the choice of the initial values $w_{-1}(x,y)$ for the relays $\mathcal{R}_{x,y}$ will be irrelevant for the results presented below; for the sake of simplicity, we will henceforth assume that

$$w_{-1}(x,y) = \begin{cases} 1\,, & x+y<0\,, \\ 0\,, & x+y \ge 0\,. \end{cases} \tag{2.38}$$

Definition 2.11 *Let $\rho \in L^1(P)$ be given. We define the Preisach operator $\mathcal{W} : S \to S$ as*

$$\mathcal{W}(s) = \int_{x<y} \rho(x,y)\mathcal{R}_{x,y}(s)\,dx\,dy\,, \tag{2.39}$$

and its periodic version $\mathcal{W}^{per} : S \to S$ by

$$\mathcal{W}^{per}(s) = \int_{x<y} \rho(x,y)\mathcal{R}^{per}_{x,y}(s)\,dx\,dy\,, \tag{2.40}$$

Here, the integral is understood to be componentwise with respect to the elements of the string $\mathcal{R}_{x,y}(s)$, and the function ρ is set to zero outside the triangle P.

An operator $\mathcal{W} : S \to S$ of the form

$$s = (v_0, \ldots, v_N) \mapsto \mathcal{W}(s) = (w_0, \ldots, w_N) \tag{2.41}$$

is called *piecewise monotone*, if

$$(w_k - w_{k-1})(v_k - v_{k-1}) \ge 0\,, \quad 1 \le k \le N\,, \tag{2.42}$$

holds for every $s = (v_0, \ldots, v_N) \in S$.

Proposition 2.12 *Let $\mathcal{W} : S \to S$ be a Preisach operator for some given density function ρ as in Definition 2.11. If $\mathcal{W}$ is piecewise monotone, then for each $s = (v_0, \ldots, v_N) \in S$ with $v_0 = v_N$ there holds*

$$\mathrm{Var}\,(\mathcal{W}^{per}(s)) = \sum_{x<y} a_{per}(x,y)\varphi(x,y)\,, \tag{2.43}$$

where

$$\varphi(x,y) = 2\int_x^y \int_x^\eta \rho(\xi,\eta)\,d\xi\,d\eta\,. \tag{2.44}$$

Proof: We have

$$\begin{aligned}&[(\mathcal{W}^{per}(s))_{i+1} - (\mathcal{W}^{per}(s))_i](\mathrm{sign}(v_{i+1}) - \mathrm{sign}(v_i)) = \\ &\quad \int_{x<y} \rho(x,y)[(\mathcal{R}^{per}_{x,y}(s))_{i+1} - (\mathcal{R}^{per}_{x,y}(s))_i](\mathrm{sign}(v_{i+1}) - \mathrm{sign}(v_i))\,dx\,dy\,.\end{aligned} \tag{2.45}$$

By (2.33) we have $\mathcal{W}(s,s) = (\mathcal{W}(s), \mathcal{W}^{per}(s))$, hence

$$(\mathcal{W}^{per}(s))_{i+1} - (\mathcal{W}^{per}(s))_i = (\mathcal{W}(s,s))_{N+i+2} - (\mathcal{W}(s,s))_{N+i+1}\,, \tag{2.46}$$

and the same formula holds true if we replace $\mathcal{W}$ by $\mathcal{R}_{x,y}$. From the piecewise monotonicity of $\mathcal{W}$ and of $\mathcal{R}_{x,y}$ it follows that, if we sum over i in (2.45),

$$\mathrm{Var}\,(\mathcal{W}^{per}(s)) = \int_{x<y} \rho(x,y)\mathrm{Var}\,(\mathcal{R}^{per}_{x,y}(s))\,dx\,dy\,. \tag{2.47}$$

We integrate (2.34) with the density function ρ over the domain where $x < y$ and obtain (2.43), since

$$\int_{-\infty}^{\infty}\int_{-\infty}^{y} \rho(x,y)\cdot 2\sum_{y\le\eta}\sum_{\xi\le x} a_{per}(\xi,\eta)\,dx\,dy = \sum_{\eta\in\mathbf{R}}\sum_{\xi<\eta} a_{per}(\xi,\eta)\cdot 2\int_\xi^\eta\int_\xi^y \rho(x,y)\,dx\,dy\,. \tag{2.48}$$

□

The function φ in equation (2.43) corresponds to the function Δ in (2.35). Since Δ is the reciprocal of N, for a real material one will always have

$$\partial_x\Delta(x,y) \le 0\,, \quad \partial_y\Delta(x,y) \ge 0\,, \quad (x,y) \in P\,. \tag{2.49}$$

Corollary 2.13 *Let $\Delta \in W^{2,1}(P)$ satisfy (2.49) as well as $\Delta(x,x) = \partial_x\Delta(x,x) = \partial_y\Delta(x,x) = 0$ for each $x \in \mathbf{R}$. Let $\mathcal{W}$ be the Preisach operator with density function*

$$\rho(x,y) = -\frac{1}{2}\partial_{xy}\Delta(x,y)\,. \tag{2.50}$$

Then for each $s = (v_0, \ldots, v_N) \in S$ with $\|s\|_\infty \le M$ and $v_0 = v_N$ the total damage $D(s)$ associated to s satisfies

$$D(s) = \sum_{x<y} a_{per}(x,y)\Delta(x,y) = \mathrm{Var}\,(\mathcal{W}^{per}(s))\,. \tag{2.51}$$

Proof: The assertion follows from Proposition 2.12 if we can prove that $\mathcal{W}$ is piecewise monotone. Let $s = (v_0, \ldots, v_N)$ with $v_N > v_{N-1}$ be given, assume that $v_N \neq v_j$ for all $j < N$. For the Preisach operator (2.39) there holds (see (4.10); note that the choice of initial values in (4.2) is consistent with (2.38))

$$\partial_N \mathcal{W}(v_0, \ldots, v_N) = 2 \int_0^{r_*} \rho(v_N - 2r, v_N)\, dr \,, \tag{2.52}$$

where $r_* > 0$ is a certain number depending on $(v_0, \ldots, v_N)$, and $\partial_N \mathcal{W}(v_0, \ldots, v_N)$ denotes the partial derivative of the last component of the output string with respect to the last input value. From (2.50) and the assumptions on Δ we get

$$\partial_N \mathcal{W}(v_0, \ldots, v_N) = \frac{1}{2}\partial_y \Delta(v_N - 2r_*, v_N) - \frac{1}{2}\partial_y \Delta(v_N, v_N) \geq 0 \,. \tag{2.53}$$

An analogous argument in the case $v_N < v_{N-1}$ yields

$$\partial_N \mathcal{W}(v_0, \ldots, v_N) = -(\frac{1}{2}\partial_x \Delta(v_N, v_N + 2r_*) - \frac{1}{2}\partial_x \Delta(v_N, v_N)) \geq 0 \,. \tag{2.54}$$

Therefore, $\mathcal{W}$ is piecewise monotone. □

Corollary 2.13 is a representation result. As a consequence, the mathematical theory developed for the Preisach operator can be used for the analysis of the damage functional D arising from the combination of rainflow counting and the Palmgren-Miner-Rule.

Finally, let us remark that if the function Δ is purely amplitude dependent, i.e. $\Delta = \Delta(y - x)$, then the associated Preisach operator becomes formally identical to the hysteresis model proposed by Prandtl [P, 1928] as a material law for certain materials. (Prandtl's model consists of a linear superposition of elastic-perfectly plastic elements, written in the terminology of rheological elements as $\mathcal{E}_0 - \sum_{r>0} \mathcal{E}_r | \mathcal{P}_r$, see Section 5 and in particular Example 5.4 below.)

3 Continuity of the Total Damage

Many materials have the property that small oscillations do not contribute to the accumulation of damage. For such a material, $\Delta(x, y) = 0$ in some neighbourhood of the main diagonal. Accordingly, let us assume that $\Delta(x, y) = 0$ if $|x - y| \leq 2r_0$ for some $r_0 > 0$. Then $\rho = -\frac{1}{2}\partial_{xy}\Delta$ will also be 0 in that region, and $\mathcal{W}(s)$ will remain constant along portions of the string s where the oscillations are small enough. Consequently, one may remove such oscillations from the input string, either in a separate preprocessing step, or within the rainflow algorithm itself. This procedure is called *hysteresis filtering*. Within the framework of mathematical analysis, a corresponding regularization procedure can be expressed via a hysteresis operator $\mathcal{F}_r$, which describes the mechanical play and therefore is called the *play operator*. In terms of input strings, $\mathcal{F}_r$ is defined by

$$\mathcal{F}_r(v_0, \ldots, v_N) = (w_0, \ldots, w_N) \,, \quad w_i = f_r(v_i, w_{i-1}) \,, \quad 0 \leq i \leq N \,, \tag{3.1}$$

where

$$f_r(v, w) = \min\{v + r, \max\{v - r, w\}\} \,, \tag{3.2}$$

and $w_{-1} \in \mathbf{R}$ represents some initial memory. The choice $w_{-1} = 0$ is consistent with (2.38), see Lemma A.1, and will henceforth be made. We also consider the periodic version $\mathcal{F}_r^{per}$ defined, in analogy to (2.33), implicitly by

$$\mathcal{F}_r(s,s) = (\mathcal{F}_r(s), \mathcal{F}_r^{per}(s)) . \tag{3.3}$$

To investigate the continuous dependence of the total damage upon the input, we view the input as a time dependent function rather than a string. It is well known that $\mathcal{F}_r$ (and, consequently, $\mathcal{F}_r^{per}$) can also be defined on the space $C[0,T]$ of continuous functions. Their close relation to the Preisach operator $\mathcal{W}$ and its periodic version $\mathcal{W}^{per}$ (see the appendix for details) yields the following result, which (for $\mathcal{W}$ instead of $\mathcal{W}^{per}$) is originally due to Visintin [V, 1994].

Theorem 3.1 *Let $\Delta \in W^{2,1}(P)$ satisfy (2.49) as well as $\Delta(x,y) = 0$ for $|x-y| \leq 2r_0$ with some given $r_0 > 0$. As in Corollary 2.13, let $\mathcal{W}$ be the Preisach operator with the density function $\rho = -\frac{1}{2}\partial_{xy}\Delta$. Then we have*

$$\lim_{n\to\infty} \mathrm{Var}\,(\mathcal{W}^{per}[v_n]) = \mathrm{Var}\,(\mathcal{W}^{per}[v]) , \tag{3.4}$$

if the functions $v_n \in C[0,T]$ converge uniformly to $v \in C[0,T]$.

Proof: See the appendix. □

This result has an important practical meaning. It shows that the total damage is stable with respect to small variations of the input, and in particular with respect to range discretization in data reduction algorithms.

4 Rainflow and Hysteresis Memory

The rainflow method is linked intimately to the memory structure of scalar rate independent hysteresis; this is well known for some widely used elastic-plastic material laws. Actually, we may establish the connection for every hysteresis model whose memory is based upon a continuous family of play (or, equivalently, relay) operators. Their collective memory evolution generated by a particular input string $s = (v_0, \ldots, v_N)$ is completely described by a string of functions $\psi_k : \mathbf{R}_+ \to \mathbf{R}$,

$$(\psi_0(r), \ldots, \psi_N(r)) = \mathcal{F}_r(s) , \quad r \geq 0 . \tag{4.1}$$

The function ψ_k represents the memory after v_k has been processed. From the definition of the play operator $\mathcal{F}_r$ in (3.1), (3.2) we see that

$$\psi_k(r) = \min\{v_k + r, \max\{v_k - r, \psi_{k-1}(r)\}\} , \quad \psi_{-1}(r) = 0 , \quad r \geq 0 . \tag{4.2}$$

We call the function ψ_N or, more precisely, its graph $\{(r, \psi_N(r)) : r \geq 0\}$ the *hysteresis memory curve* belonging to the string $s = (v_0, \ldots, v_N)$ and denote it as

$$\Phi_r(s) = \psi_N(r) . \tag{4.3}$$

If s is alternating and satisfies

$$|v_0| > |v_1| , \quad |v_{i+1} - v_i| < |v_i - v_{i-1}| , \quad 0 < i < N , \tag{4.4}$$

then its hysteresis memory curve ψ_N consists of $N+1$ pieces of straight lines of slope ± 1 starting at $(0, v_N)$ and ending at $(r_0, 0) = (|v_0|, 0)$ with corners $(r_i, \psi_N(r_i))$,

$$r_i = \frac{|v_i - v_{i-1}|}{2}, \quad \psi_N(r_i) = \frac{v_i + v_{i-1}}{2}, \quad 1 \le i \le N, \tag{4.5}$$

see Figure 2.

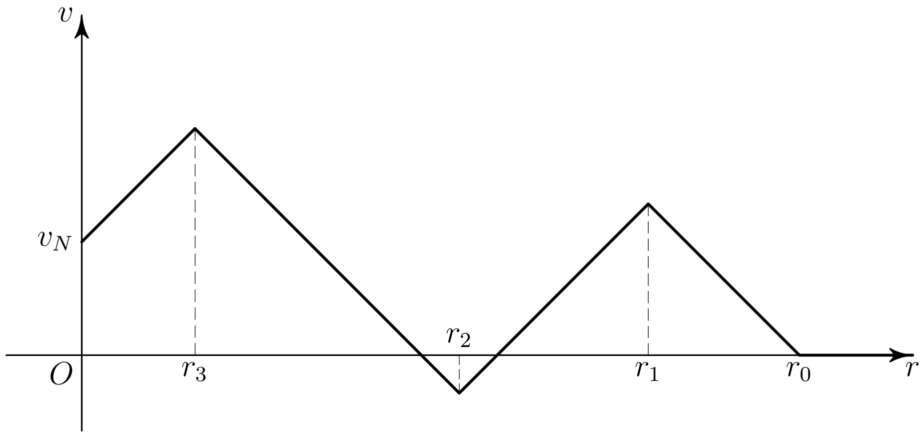

Figure 2: *The hysteresis memory curve*

The connection to the rainflow method is as follows. Let v_N tend to v_{N-2}, then r_N tends to r_{N-1}. When v_N becomes equal to v_{N-2}, the corner $(r_N, \psi_N(r_N))$ merges with $(r_{N-1}, \psi_N(r_{N-1}))$, and both vanish. At that moment, the rainflow method counts and deletes the Madelung pair (v_{N-1}, v_N) and, in the input-output-plane, a hysteresis loop is closed.

For an arbitrary string s, one may check that

$$\Phi_r(s) = \Phi_r(s_R) = \Phi_r(s_{BR}), \quad r \ge 0, \tag{4.6}$$

where we obtain the so-called *backward residual* s_{BR} from the residual $s_R = (v_0', \dots, v_{N'}')$ by deleting successively $v_0', v_1', \dots$ until the remaining string satisfies (4.4). Consequently, the hysteresis memory curve for s is constructed from the backward residual s_{BR} as described above.

We will not analyze here in detail the various properties of the hysteresis memory operator which maps input strings (or input functions) to memory curves, see e.g. [B & V, 1989], [B & S], [K & P, 1983], [V, 1994]. We only indicate how it is used to calculate the partial derivative $\partial_{v_N} \mathcal{W}(v_0, \dots, v_N)$ of the Preisach operator, which we need in the proof of Corollary 2.13. To this end, assume first that $s = (v_0, \dots, v_N)$ is alternating and satisfies (4.4). Consider the case where $v_N > v_{N-1}$, or where $N = 0$ and $v_0 > 0$. From Figure 2 we see that

$$\partial_N \mathcal{F}_r(v_0, \dots, v_N) := \partial_{v_N} \Phi_r(v_0, \dots, v_N) = \begin{cases} 1, & r < r_N, \\ 0, & r > r_N, \end{cases} \tag{4.7}$$

as well as

$$\Phi_r(v_0, \dots, v_N) = v_N - r, \quad r < r_N. \tag{4.8}$$

If we denote as in (4.7) by $\partial_N \mathcal{W}$ the partial derivative of the last component of the output string with respect to the last input value, we get from the representation (A.6) and (A.7)

$$\begin{aligned} \partial_N \mathcal{W}(v_0, \ldots, v_N) &= \int_0^\infty \partial_z p(r, \Phi_r(v_0, \ldots, v_N)) \partial_N \mathcal{F}_r(v_0, \ldots, v_N) \, dr \\ &= \int_0^{r_N} \partial_z p(r, v_N - r) \, dr = 2 \int_0^{r_N} \rho(v_N - 2r, v_N) \, dr \, . \end{aligned} \tag{4.9}$$

If the string $s = (v_0, \ldots, v_N)$ is arbitrary, the computation above applies to its backward residual s_{BR}, so we have

$$\partial_N \mathcal{W}(v_0, \ldots, v_N) = 2 \int_0^{r_*} \rho(v_N - 2r, v_N) \, dr \tag{4.10}$$

with a certain number $r_* > 0$. Formula (4.10) holds at every point where the partial derivative exists. The latter may fail to exist only at exceptional points where a corner merge occurs; at such points we must have $v_N = v_j$ for at least one $j < N$.

In an analogous manner, we deduce that in the case $v_N < v_{N-1}$

$$\partial_N \mathcal{W}(v_0, \ldots, v_N) = 2 \int_0^{r_*} \rho(v_N, v_N + 2r) \, dr \tag{4.11}$$

for some $r_* > 0$.

5 Energy Dissipation in Rate-Independent Rheological Models

In this section, we use the calculus of hysteresis operators to establish a formal analogy between energy dissipation and damage accumulation for scalar rate independent constitutive laws. Again, the construction of complex hysteresis operators from simple ones appears as the crucial tool; in fact, this technique has been developed and widely used in material science and engineering under the heading *rheological models*. As above, we are mainly interested in models with *distributed elements*, where a new element results from a given one-parameter family of elements through integration. To our knowledge, an integral formulation of an elastoplastic constitutive law of that type first appeared on p.91 of Prandtl's paper [P, 1928]. Some aspects of Prandtl's model are discussed in [Timoshenko, 1930] - in the 1940 edition it appears on p. 427f. - , [Duwez, 1935], [I, 1944] to various extent, including the connection to rheological models; the latter has been elaborated upon in particular by [Iwan, 1966 & 1967] for the distributed case. Concerning the multiaxial situation (which we do not investigate here), several variants of distributed element models for rate independent elastoplasticity have been developed. Let us mention here the multiaxial version of Prandtl's model, treated in [Visintin, 1987 & 1994], [Krejčí, 1991/a] and [Chiang & Beck, 1994], a kinematic hardening model of Prager type [I, 1967] as well as a kinematic hardening model of Mróz type [Chu 1984 & 1987], [Brokate, Dreßler & Krejčí]; for the non-distributed case, a vast literature exists.

To begin, let us first recall some basic facts. The concept of a *rheological element* consists of

- a *constitutive relation* between the stress σ and the strain ϵ, and

- an *internal energy* U. (One may also consider the free energy or the potential energy; in isothermal situations, these concepts coincide.)

Definition 5.1 (Basic Rheological Elements)

(i) $\mathcal{E}$ *denotes the* linear elastic element *with the constitutive equation and internal energy*

$$\epsilon = \frac{1}{E}\sigma\,, \quad U = \frac{1}{2}\epsilon\sigma\,, \tag{5.1}$$

where E *denotes the modulus of elasticity.*

(ii) $\mathcal{N}$ *denotes the* nonlinear elastic element *with the constitutive equation and internal energy*

$$\epsilon = g(\sigma)\,, \quad U = G(\sigma) = \sigma g(\sigma) \;-\; \int_0^\sigma g(\xi)\, d\xi\,, \tag{5.2}$$

where $g : \mathbf{R} \to \mathbf{R}$ *is a nondecreasing function with* $g(0) = 0$.

(iii) $\mathcal{P}$ *denotes the* rigid plastic element *whose constitutive relation is described by the variational inequality*

$$\dot\epsilon(t)(\sigma(t) - \tilde\sigma) \geq 0 \quad \forall\, \tilde\sigma \in [-r, r]\,, \tag{5.3}$$

$$\sigma(t) \in [-r, r]\,, \tag{5.4}$$

with the yield stress $r > 0$.

(iv) $\mathcal{B}$ *denotes the* brittle element *described by*

$$\epsilon(t) = 0 \quad \text{if } \|\sigma\|_{[0,t]} < h\,, \qquad \sigma(t) = 0 \quad \text{if } \|\sigma\|_{[0,t]} \geq h\,, \tag{5.5}$$

with the fracture stress $h > 0$. *(Here,* $\|\cdot\|_{[0,t]}$ *denotes the supremum norm over the time interval* $[0, t]$*.)*

For the elements $\mathcal{P}$ *and* $\mathcal{B}$*, the internal energy* U *is set to 0 to express the fact that no reversible power can be stored by these elements.*

The definition of $\mathcal{B}$ via (5.5) actually makes sense only if $|\sigma(t)| < h$ for all $t \geq 0$. The material remains rigid as long as $|\sigma(t)|$ stays bounded away from the value h; as soon as $|\sigma(t-)| = h$, the material breaks, σ jumps to zero and we lose any control on ϵ. Condition (5.5) can be equivalently rewritten in terms of the Heaviside function H, defined as $H(x) = 1$ if $x > 0$, and $H(x) = 0$ otherwise, as

$$\epsilon(t)H(h - \|\sigma\|_{[0,t]}) = 0\,, \quad \sigma(t)(1 - H(h - \|\sigma\|_{[0,t]}) = 0\,. \tag{5.6}$$

From two given rheological elements $\mathcal{R}_1$ and $\mathcal{R}_2$ we may form a new element $\mathcal{R}$ as the

- *series combination* $\mathcal{R} = \mathcal{R}_1 - \mathcal{R}_2$ or $\mathcal{R} = \sum_{i\in\{1,2\}} \mathcal{R}_i$, setting

$$\epsilon = \epsilon_1 + \epsilon_2\,, \quad \sigma = \sigma_1 = \sigma_2\,, \quad U = U_1 + U_2\,, \tag{5.7}$$

- *parallel combination* $\mathcal{R} = \mathcal{R}_1|\mathcal{R}_2$ or $\mathcal{R} = \prod_{i\in\{1,2\}} \mathcal{R}_i$, setting

$$\epsilon = \epsilon_1 = \epsilon_2\,, \quad \sigma = \sigma_1 + \sigma_2\,, \quad U = U_1 + U_2\,. \tag{5.8}$$

In the sequel, we will assume that ϵ, σ and U are functions of time, defined within an interval $[0,T]$. We denote by $q(t)$ the *energy dissipated during the time interval* $[0,t]$, $t \in [0,T]$. The second law of thermodynamics states that the *dissipation rate* $\dot{q}(t)$ given by

$$\dot{q} = \dot{\epsilon}\sigma - \dot{U} \tag{5.9}$$

has to satisfy

$$\dot{q}(t) \geq 0\,. \tag{5.10}$$

We consider a weak formulation of (5.10), namely

$$q(t_2) - q(t_1) = [\epsilon(t)\sigma(t) - U(t)]_{t_1}^{t_2} - \int_{t_1}^{t_2} \epsilon(t)\dot{\sigma}(t)\,dt \geq 0\,, \quad 0 \leq t_1 \leq t_2 \leq T\,. \tag{5.11}$$

A rheological element is called *thermodynamically consistent*, if (5.10) respectively (5.11) holds for all functions satisfying the constitutive relation and belonging to a suitable function space. It is easy to see that the elements $\mathcal{E}$, $\mathcal{N}$ and $\mathcal{P}$ are thermodynamically consistent. In fact, $\mathcal{E}$ and $\mathcal{N}$ are conservative, i.e. $\dot{q} = 0$. It is also obvious that any parallel or series combination of thermodynamically consistent elements is again thermodynamically consistent.

Lemma 5.2 *Let $g : \mathbf{R} \to \mathbf{R}$ be continuous and satisfy $xg(x) > 0$ if $x \neq 0$. Then the parallel combination $\mathcal{N}|\mathcal{B}$ is thermodynamically consistent for $\sigma \in W^{1,1}(0,T)$.*

Proof: Let $\sigma \in W^{1,1}(0,T)$ with $\sigma(0) < h$ be given. We have

$$\sigma = \sigma^e + \sigma^b\,, \quad \epsilon = g(\sigma^e)\,, \quad U = G(\sigma^e)\,. \tag{5.12}$$

From (5.6), we get $\sigma^b(t) = \sigma(t)H(h - \|\sigma^b\|_{[0,t]})$, hence

$$\sigma^b(t) = \sigma(t)H(h - \|\sigma\|_{[0,t]})\,. \tag{5.13}$$

Setting $t_0 = \inf\{t \in [0,T] : |\sigma(t)| = h\}$, we have

$$\epsilon = \sigma^e = U = q = 0\,, \quad \sigma^b = \sigma \qquad \text{in } [0,t_0)\,, \tag{5.14}$$

$$\sigma^b = 0\,, \quad \sigma = \sigma^e\,, \quad \epsilon = g(\sigma)\,, \quad U = G(\sigma) \qquad \text{in } [t_0,T]\,. \tag{5.15}$$

We therefore obtain

$$q(t_2) - q(t_1) = \int_0^h g(\xi)\,d\xi\,, \qquad \text{if } t_0 \in (t_1,t_2]\,, \tag{5.16}$$

and $q(t_2) = q(t_1)$ otherwise. The case $\sigma(0) \geq h$ is trivial. □

Example 5.3 (Basic Elastoplastic Elements)

(i) *The series combination $\mathcal{E} - \mathcal{P}$ is described by*

$$\epsilon = \epsilon^e + \epsilon^p\,, \quad \sigma = E\epsilon^e\,, \quad U = \frac{1}{2E}\sigma^2\,, \tag{5.17}$$

$$\sigma \in [-r,r]\,, \quad \dot{\epsilon}^p(\sigma - \tilde{\sigma}) \geq 0 \quad \forall \tilde{\sigma} \in [-r,r]\,. \tag{5.18}$$

(ii) *The parallel combination* $\mathcal{E}|\mathcal{P}$ *is described by*

$$\sigma = \sigma^e + \sigma^p \,, \quad \sigma^e = E\epsilon \,, \quad U = \frac{E}{2}\epsilon^2 \,, \tag{5.19}$$

$$\sigma^p \in [-r, r] \,, \quad \dot\epsilon(\sigma^p - \tilde\sigma) \geq 0 \quad \forall \tilde\sigma \in [-r, r] \,. \tag{5.20}$$

We observe that both $\mathcal{E} - \mathcal{P}$ and $\mathcal{E}|\mathcal{P}$ are governed by an evolution variational inequality, namely

$$(E\dot\epsilon - \dot\sigma)(\sigma - \tilde\sigma) \geq 0 \quad \forall \tilde\sigma \in [-r, r] \tag{5.21}$$

for $\mathcal{E} - \mathcal{P}$, and

$$(\dot\sigma - \dot\sigma^p)(\sigma^p - \tilde\sigma) \geq 0 \quad \forall \tilde\sigma \in [-r, r] \tag{5.22}$$

for $\mathcal{E}|\mathcal{P}$. In (5.21), the stress σ is determined from the strain ϵ, whereas in (5.22) the plastic stress σ^p, and hence $\epsilon = \frac{1}{E}(\sigma - \sigma^p)$ are determined from the stress σ. Both variational inequalities have the same form: For a given function $v : [0, T] \to \mathbf{R}$, we look for a $w : [0, T] \to \mathbf{R}$ such that

$$(\dot v(t) - \dot w(t))(w(t) - x) \geq 0 \ \text{ a.e. in } \ [0, T] \quad \forall x \in [-r, r] \,, \tag{5.23}$$

$$w(t) \in [-r, r] \ \text{ a.e. in } \ [0, T] \,, \quad w(0) = w_0 \,. \tag{5.24}$$

An elementary result in the theory of evolution variational inequalities states that (5.23), (5.24) has a unique solution $w \in W^{1,1}[0, T]$ for any $v \in W^{1,1}[0, T]$. The correspondence $v \mapsto w$ defines a hysteresis operator $\mathcal{S}_r$, called the *stop* in [K & P, 1983], which for the initial value $w_0 = \min\{r, \max\{-r, v(0)\}\}$ is related to the play operator $\mathcal{F}_r$ by the identity

$$\mathcal{S}_r + \mathcal{F}_r = id \,, \quad \text{i.e.} \quad \mathcal{S}_r[v] + \mathcal{F}_r[v] = v \,. \tag{5.25}$$

We can rewrite the constitutive relations for the two elastoplastic elements in terms of the operators $\mathcal{F}_r$ and $\mathcal{S}_r$ as

$$\mathcal{E} - \mathcal{P} : \quad \sigma = \mathcal{S}_r[E\epsilon] \,, \quad U = \frac{1}{2E}(\mathcal{S}_r[E\epsilon])^2 \,, \tag{5.26}$$

$$\mathcal{E}|\mathcal{P} : \quad E\epsilon = \mathcal{F}_r[\sigma] \,, \quad U = \frac{1}{2E}(\mathcal{F}_r[\sigma])^2 \,. \tag{5.27}$$

We now discuss rheological models with a more complex memory.

Example 5.4 (The Prandtl-Preisach-Ishlinskii Model) *We consider the infinite series combination (series-parallel model in the terminology of [I, 1967])*

$$\mathcal{N}_0 - \sum_{r>0} \mathcal{N}_r|\mathcal{P}_r \,. \tag{5.28}$$

We denote by $g(r, \cdot)$ *the constitutive function of the nonlinear elastic element* $\mathcal{N}_r$, *i.e.*

$$\epsilon_r = g(r, \sigma_r^e) \,, \quad r \geq 0 \,. \tag{5.29}$$

In the composition formula (5.7) we replace the sum with an integral, so that (5.28) is described by

$$\epsilon = \epsilon_0 + \int_0^\infty \epsilon_r \, dr \,, \quad \sigma = \sigma_r^p + \sigma_r^e \quad \forall r > 0 \,, \tag{5.30}$$

$$\sigma_r^p \in [-r, r]\,, \quad \dot{\epsilon}_r(\sigma_r^p - \tilde{\sigma}) \geq 0 \quad \forall \tilde{\sigma} \in [-r, r] \quad \forall r > 0\,, \tag{5.31}$$

$$U = G(0, \sigma_0^e) + \int_0^\infty G(r, \sigma_r^e)\, dr\,, \tag{5.32}$$

where we define G in analogy to (5.2) as

$$G(r, \sigma) = \sigma g(r, \sigma) - \int_0^\sigma g(r, \xi)\, d\xi\,. \tag{5.33}$$

The variational inequality (5.31) implies that

$$\dot{\sigma}_r^e(\sigma_r^p - \tilde{\sigma}) \geq 0 \quad \forall \tilde{\sigma} \in [-r, r]\,. \tag{5.34}$$

We therefore get

$$\sigma_r^p = \mathcal{S}_r[\sigma]\,, \quad \sigma_r^e = \mathcal{F}_r[\sigma]\,. \tag{5.35}$$

Since $\sigma_0^e = \sigma$, the constitutive relation for (5.28) can be described in Preisach operator form as

$$\epsilon = g(0, \sigma) + \int_0^\infty g(r, \mathcal{F}_r[\sigma])\, dr\,, \tag{5.36}$$

$$U = G(0, \sigma) + \int_0^\infty G(r, \mathcal{F}_r[\sigma])\, dr\,. \tag{5.37}$$

Let us assume that $g(r, 0) = 0$ for all r and that the second partial derivative $\partial_2 g(r, \xi)$ is a nonnegative, measurable and sufficiently regular (for example, the mapping $\xi \mapsto \partial_2 g(r, \xi)$ is continuous a.e. in r and $|\partial_2 g(r, \xi)| \leq c(\xi)\beta(r)$ for some functions $c \in L^\infty_{loc}(\mathbf{R})$ and $\beta \in L^1_{loc}(\mathbf{R}_+)$). We then can compute the dissipation rate as

$$\begin{aligned} \dot{q} &= \dot{\epsilon}\sigma - \dot{U} = \int_0^\infty \partial_2 g(r, \mathcal{F}_r[\sigma]) \mathcal{S}_r[\sigma] \frac{d}{dt}(\mathcal{F}_r[\sigma])\, dr \\ &= \int_0^\infty r \partial_2 g(r, \mathcal{F}_r[\sigma]) |\frac{d}{dt}(\mathcal{F}_r[\sigma])|\, dr = |\int_0^\infty r \partial_2 g(r, \mathcal{F}_r[\sigma]) \frac{d}{dt}(\mathcal{F}_r[\sigma])\, dr\,| \\ &= |\frac{d}{dt} \int_0^\infty r g(r, \mathcal{F}_r[\sigma])\, dr\,|\,. \end{aligned} \tag{5.38}$$

Here we have used the formula

$$r|\frac{d}{dt}\mathcal{F}_r[\sigma]| = \mathcal{S}_r[\sigma] \cdot \frac{d}{dt}\mathcal{F}_r[\sigma] \tag{5.39}$$

as well as the fact that the sign of $\frac{d}{dt}\mathcal{F}_r[\sigma]$ does not depend on r. (To prove (5.39), one separately considers the cases where $(t, \sigma(t))$ lies on the boundary or in the interior of the hysteresis regions of $\mathcal{F}_r$ and $\mathcal{S}_r$. For a formal proof, see [B & S].)

Formula (5.38) suggests to us to introduce the *dissipation operator*

$$\mathcal{W}_d[\sigma] = \int_0^\infty r g(r, \mathcal{F}_r[\sigma])\, dr\,. \tag{5.40}$$

The operator $\mathcal{W}_d$ is again a Preisach operator. Integrating (5.38) we obtain the total dissipation of energy as

$$q(t) = \mathrm{Var}_{[0,t]} \mathcal{W}_d[\sigma]\,. \tag{5.41}$$

Within material science, there is and has been a lot of discussion whether and how energy dissipation and damage accumulation are related to each other. In the framework of this paper, we see that the total amount of dissipated energy as well as the accumulated damage are both equal to the total variation of the output of some Preisach operator. Any relation between the accumulated damage and the total amount of dissipated energy then translates into a relation between those two Preisach operators, respectively their density functions. For example, one may propose that the accumulated damage is proportional to the amount of dissipated energy. Let us explore the consequences of this hypothesis for a representative constitutive law of power type, namely

$$g(r,\xi) = \begin{cases} \gamma r^{\kappa-2}\xi \,, & r > r_0 \,, \\ 0 \,, & 0 < r < r_0 \,, \\ \gamma\xi \,, & r = 0 \,, \end{cases} \tag{5.42}$$

where $r_0 > 0$, $\gamma > 0$ and $\kappa > 1$ are constants. As g is linear with respect to ξ, the elastic elements appearing here are linear ones. The initial loading curve $\varepsilon = F(\sigma)$ is given by the formula

$$F(\sigma) = \gamma\left(\sigma + \int_{r_0}^{\infty} r^{\kappa-2}\max\{0,\sigma - r\}\,dr\right) \,, \tag{5.43}$$

hence

$$F(\sigma) = \begin{cases} \gamma\sigma \,, & 0 \le \sigma \le r_0 \,, \\ \gamma\left[(1 - \frac{1}{\kappa-1}r_0^{\kappa-1})\sigma + \frac{1}{\kappa(\kappa-1)}\sigma^\kappa + \frac{1}{\kappa}r_0^\kappa\right] \,, & \sigma > r_0 \,. \end{cases} \tag{5.44}$$

In particular, $F'' \ge 0$ since $\kappa > 1$; actually, the constitutive law (5.29) - (5.31) with (5.42) is a special case of the one in Prandtl's paper, written in series-parallel form. If the accumulated damage is proportional to the amount of dissipated energy, we must have

$$p(r,z) = \alpha r g(r,z) \,, \quad \alpha > 0 \,, \tag{5.45}$$

where p is related to ρ and Δ by (A.6), (A.7) and (2.50), namely

$$p(r,z) = 2\int_{-\infty}^{z} \rho(\zeta - r, \zeta + r)\,d\zeta = -\int_{-\infty}^{z} \partial_{xy}\Delta(\zeta - r, \zeta + r)\,d\zeta \,. \tag{5.46}$$

From (5.42), (5.45) and (5.46) we get

$$\rho(x,y) = \begin{cases} 0 \,, & y - x \in (0, r_0] \,, \\ \frac{\alpha\gamma}{2}(y-x)^{\kappa-1} \,, & y - x > r_0 \end{cases} \tag{5.47}$$

and $\Delta(x,y) = 0$ if $y - x \in (0, r_0]$,

$$\begin{aligned} \Delta(x,y) &= \alpha\gamma \int_{r_0}^{y-x}\int_{r_0}^{s} r^{\kappa-1}\,dr\,ds \\ &= \alpha\gamma\left[\frac{1}{\kappa(\kappa+1)}(y-x)^{\kappa+1} + \frac{1}{\kappa+1}r_0^{\kappa+1} - \frac{1}{\kappa}(y-x)r_0^\kappa\right] \,. \end{aligned} \tag{5.48}$$

We see that the initial loading curve $F(\sigma)$ grows as σ^κ and the damage function $\Delta(x,y)$ grows as $|y-x|^{\kappa+1}$. This relation of the exponents, here derived from the proportionality of accumulated damage and dissipation, can be compared with experimental results on damage (in the form of $S-N-$*diagrams*) and cyclic loading (in the form of a stabilized stress-strain curve).

Note that these remarks make sense for the plastic part of the damage only, since damage occuring from cycles in the elastic region obviously cannot be related to the constitutive law (which is characterized by the single quantity E).

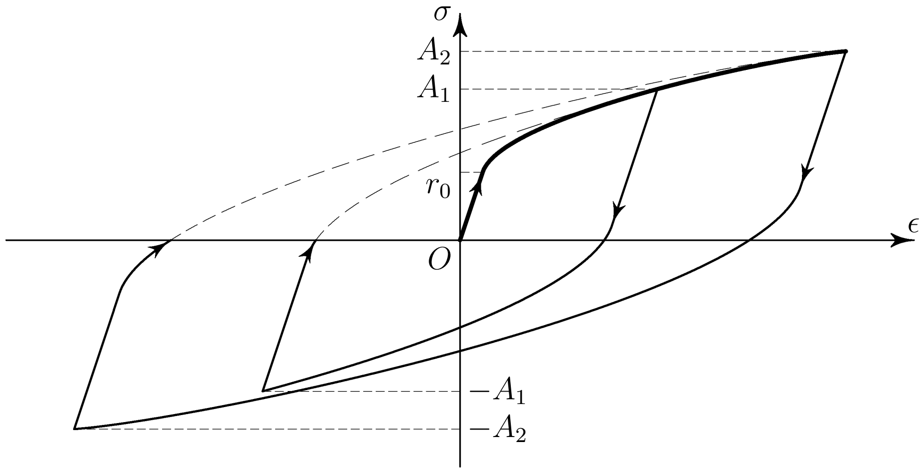

Figure 3: *The Prandtl-Preisach-Ishlinskii model 5.4.*

Figure 3 exhibits a typical strain-stress diagram corresponding to the Prandtl-Preisach-Ishlinskii model (5.36), where the function g has the form (5.42) with $\kappa = 2$. The hysteresis loops shown there arise from the input functions $\sigma_i(t) = A_i \sin t$, $i = 1, 2$, considered on the time interval $[0, 2\pi]$, where $A_1 < A_2$.

We extend the computations of the previous example to elastoplastic constitutive laws including fatigue terms of brittle type. We are motivated by experiments (see e.g. [Maugin, 1992] and [Lemaitre & Chaboche, 1990]) showing that during the accumulation of fatigue the elasticity modulus of a material may decrease. Some elastoplastic models incorporate this effect into the constitutive law through a dependence on the maximum loading amplitude. To accomplish that purpose, one may use the brittle element. We discuss the following specific model.

Example 5.5 (A Model for Fatigue) *We consider the infinite series combination*

$$\mathcal{M} = \mathcal{N}_0 - \sum_{h>0} \mathcal{N}_h | \mathcal{B}_h \,. \tag{5.49}$$

As in the previous example, the nonlinear elastic law is given by

$$\epsilon_h = g(h, \sigma_h^e) \,, \quad h \geq 0 \,. \tag{5.50}$$

According to (5.13), the elastic stress satisfies

$$\sigma_h^e(t) = \sigma(t)(1 - H(h - \|\sigma\|_{[0,t]})) \,. \tag{5.51}$$

We therefore can describe (5.49) with

$$\epsilon = \epsilon_0 + \int_0^\infty \epsilon_h \, dh \,, \quad \epsilon_0 = g(0, \sigma) \,, \quad \epsilon_h(t) = g(h, \sigma_h^e(t)) \quad \forall h > 0 \,, \tag{5.52}$$

$$U = U_0 + \int_0^\infty U_h \, dh \,, \quad U_0 = G(0, \sigma) \,, \quad U_h(t) = G(h, \sigma_h^e(t)) \quad \forall h > 0 \,. \tag{5.53}$$

Consequently, the constitutive relations for (5.49) have the form

$$\epsilon(t) = g(0, \sigma(t)) + \int_0^{\|\sigma\|_{[0,t]}} g(h, \sigma(t))\, dh\,. \tag{5.54}$$

$$U(t) = G(0, \sigma(t)) + \int_0^{\|\sigma\|_{[0,t]}} G(h, \sigma(t))\, dh\,. \tag{5.55}$$

The dissipation formula becomes

$$\dot{q}(t) = \dot{\epsilon}(t)\sigma(t) - \dot{U}(t) = \frac{d}{dt}\left(\|\sigma\|_{[0,t]}\right) \int_0^{\sigma(t)} g(\|\sigma\|_{[0,t]}, \xi)\, d\xi\,. \tag{5.56}$$

We can see in Figure 4 a strain-stress diagram for the model (5.49). The input functions are the same as for Figure 3. The function g is defined as $g(h, \xi) = \alpha(h)\xi$, where the function α has the form

$$\alpha(h) = \begin{cases} \gamma\,, & h = 0\,, \\ 0\,, & 0 < h \le r_0\,, \\ \kappa^{-1}(h^{\kappa-2} - r_0^{\kappa} h^{-2})\,, & r_0 \le h\,. \end{cases} \tag{5.57}$$

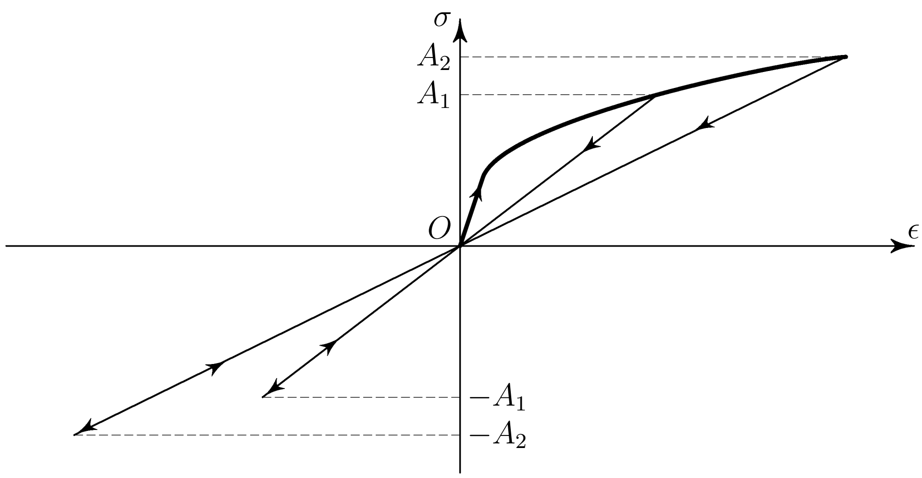

Figure 4: *The fatigue model 5.5.*

Example 5.6 (Prandtl-Preisach-Ishlinskii-Model with Fatigue) *We consider the infinite series combination*

$$\mathcal{M}_0 - \sum_{r>0} \mathcal{M}_r | \mathcal{P}_r\,. \tag{5.58}$$

Each element $\mathcal{M}_r$ has the form of Example 5.5.

Accordingly, the constitutive relations for $\mathcal{M}_r$, $r \ge 0$, are given by

$$\epsilon_r(t) = g(0, r, \sigma_r^f(t)) + \int_0^{\|\sigma_r^f\|_{[0,t]}} g(h, r, \sigma_r^f(t))\, dh\,. \tag{5.59}$$

$$U_r(t) = G(0, r, \sigma_r^f(t)) + \int_0^{\|\sigma_r^f\|_{[0,t]}} G(h, r, \sigma_r^f(t))\, dh\,. \tag{5.60}$$

Here, G is obtained from g as in formula (5.33) with the additional argument h as a parameter. The general rheological rules yield, as in Example 5.4,

$$\epsilon = \epsilon_0 + \int_0^\infty \epsilon_r \, dr \,, \quad \sigma = \sigma_0^f = \sigma_r^p + \sigma_r^f \quad \forall r > 0 \,, \tag{5.61}$$

$$\sigma_r^p \in [-r, r] \,, \quad \dot\epsilon_r (\sigma_r^p - \tilde\sigma) \ge 0 \quad \forall \tilde\sigma \in [-r, r] \quad \forall r > 0 \,, \tag{5.62}$$

$$U = U_0 + \int_0^\infty U_r \, dr \,. \tag{5.63}$$

We have $\dot\epsilon_r \dot\sigma_r^f > 0$ whenever $\dot\sigma_r^f \neq 0$, hence

$$\dot\sigma_r^f (\sigma_r^p - \tilde\sigma) \ge 0 \quad \forall \tilde\sigma \in [-r, r] \quad \forall r > 0 \,. \tag{5.64}$$

This entails

$$\sigma_r^f = \mathcal{F}_r[\sigma] \,. \tag{5.65}$$

For the play operator with zero initial condition there holds

$$\|\mathcal{F}_r[\sigma]\|_{[0,t]} = \max\{0, \|\sigma\|_{[0,t]} - r\} \,. \tag{5.66}$$

Substituting (5.59) and (5.65) into (5.61) we get

$$\begin{aligned} \epsilon(t) \;&=\; g(0,0,\sigma(t)) + \int_0^{\|\sigma\|_{[0,t]}} g(h,0,\sigma(t)) \, dh \\ &\quad + \int_0^\infty g(0,r,\mathcal{F}_r[\sigma](t)) \, dr \;+\; \int_{\Omega(\|\sigma\|_{[0,t]})} g(h,r,\mathcal{F}_r[\sigma](t)) \, dr \, dh \,. \end{aligned} \tag{5.67}$$

We can derive the formula for $U(t)$ in the same manner as that for the strain and obtain

$$\begin{aligned} U(t) \;&=\; G(0,0,\sigma(t)) + \int_0^{\|\sigma\|_{[0,t]}} G(h,0,\sigma(t)) \, dh \\ &\quad + \int_0^\infty G(0,r,\mathcal{F}_r[\sigma](t)) \, dr \;+\; \int_{\Omega(\|\sigma\|_{[0,t]})} G(h,r,\mathcal{F}_r[\sigma](t)) \, dr \, dh \,, \end{aligned} \tag{5.68}$$

where

$$\Omega(K) := \{(h,r) \in (0,\infty) \times (0,\infty) : r + h < K\} \,. \tag{5.69}$$

The "local elasticity modulus" $E(\bar\sigma)$ for a given value $\bar\sigma$ of the stress is now obtained from the formula

$$\frac{1}{E(\bar\sigma)} = \frac{\partial}{\partial s} \left(g(0,0,s) + \int_0^{\|\sigma\|_{[0,t]}} g(h,0,s) \, dh \right) \Big|_{s=\bar\sigma} \,, \tag{5.70}$$

so $E(\bar\sigma)$ is a nonincreasing function of time according to the experimental evidence.

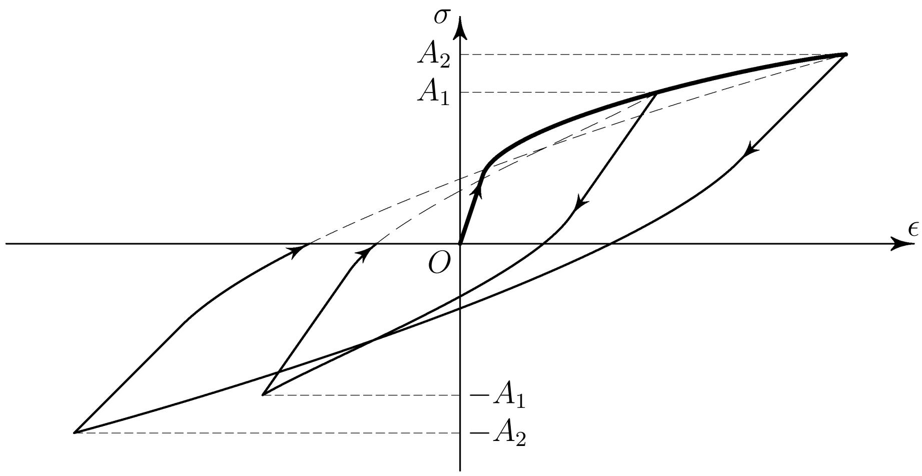

Figure 5: *The Prandtl-Preisach-Ishlinskii model with fatigue 5.6.*

In Figure 5, we present a typical diagram for the constitutive law (5.67). Again, the input functions are the same as in the two preceding diagrams. The function g has the form $g(h, r, \xi) = \alpha(h)\beta(r)\xi$, where α and β are nonnegative functions, to be chosen with respect to concrete experimental data.

The dissipation formula has the form

$$\begin{aligned} \dot{q}(t) &= \dot{\epsilon}(t)\sigma(t) - \dot{U}(t) \\ &= \frac{d}{dt}\left(\|\sigma\|_{[0,t]}\right)\left[\int_0^{\sigma(t)} g(\|\sigma\|_{[0,t]}, 0, \xi)\, d\xi + \int_0^{\|\sigma\|_{[0,t]}} \int_0^{\mathcal{F}_r[\sigma](t)} g(\|\sigma\|_{[0,t]} - r, r, \xi)\, d\xi\, dr\right] \\ &\quad + \left|\frac{d}{dt}\int_0^\infty r g(0, r, \mathcal{F}_r[\sigma](t))\, dr\right| + \left|\frac{d}{dt}\int_{\Omega(\|\sigma\|_{[0,t]})} r g(h, r, \mathcal{F}_r[\sigma](t))\, dh\, dr\right|. \end{aligned} \tag{5.71}$$

Let us assume now that the symmetry condition

$$g(h, r, -\xi) = -g(h, r, \xi) \tag{5.72}$$

holds for all arguments, as well as a regularity condition as outlined below (5.37). We then can rewrite (5.71) in the form

$$\dot{q}(t) = \frac{d}{dt} V(\|\sigma\|_{[0,t]}) + \left|\frac{d}{dt}\mathcal{W}_d[\sigma](t)\right|, \tag{5.73}$$

where we call the function V defined by

$$V(x) = \int_{|\sigma(0)|}^{x} \left[\int_0^y g(y, 0, \xi)\, d\xi + \int_0^y \int_0^{y-r} g(y - r, r, \xi)\, d\xi\, dr\right] dy \tag{5.74}$$

the *fatigue function*, which is increasing in $[|\sigma(0)|, \infty)$, and

$$\mathcal{W}_d[\sigma](t) = \int_0^\infty r g(0, r, \mathcal{F}_r[\sigma](t))\, dr + \int_{\Omega(\|\sigma\|_{[0,t]})} r g(h, r, \mathcal{F}_r[\sigma](t))\, dh\, dr \tag{5.75}$$

is the dissipation operator. We thus obtain a formula analogous to (5.41), namely

$$q(t) = V(\|\sigma\|_{[0,t]}) + \mathrm{Var}_{[0,t]}\mathcal{W}_d[\sigma]\,. \tag{5.76}$$

We can use the rainflow count to approximate the variation of $\mathcal{W}_d[\sigma]$, if we modify it in order to incorporate the dependence on the maximum input amplitude encountered so far. This makes sense, since in terms of strings, monotone deletions do not affect the final value $q(T)$, and Madelung deletions do not affect the current value of $\|\sigma\|_{[0,t]}$. Let s be a given input string, let $Q(s)$ be the total dissipated energy. The formula analogous to (2.51) reads

$$Q(s) = V(\|s\|_\infty) + \sum_{x<y,B>0} a(B,x,y)\Delta(B,x,y) + Q_R \,, \tag{5.77}$$

Here, the extended symmetric rainflow count $a(B,x,y)$ denotes the number of Madelung deletions of the pair (x,y) for the value B of the sup norm of the part of the string to the left of (x,y). The number Q_R represents the residual variation of the dissipation operator. The function Δ is related to the Preisach density ρ by a formula analogous to (2.50) or (2.44), namely

$$\rho(B,x,y) = -\frac{1}{2}\partial_{xy}\Delta(B,x,y)\,, \tag{5.78}$$

or

$$\Delta(B,x,y) = 2\int_x^y\int_x^\eta \rho(B,\xi,\eta)\,d\xi\,d\eta = 4\int_0^{\frac{y-x}{2}}\int_{x+r}^{y-r}\rho(B,\xi-r,\xi+r)\,d\xi\,dr\,. \tag{5.79}$$

We can write the operator $\mathcal{F}_d$ in the form (A.6), if we change the function p to

$$p(B,r,z) = rg(0,r,z) + \int_0^{B-r} rg(h,r,z)\,dh\,. \tag{5.80}$$

A formula analogous to (A.7), namely

$$\rho(B,z-r,z+r) = \frac{1}{2}\partial_z p(B,r,z) \tag{5.81}$$

then holds for all values of B, hence Δ can be expressed in terms of g as

$$\Delta(B,x,y) = 2\int_0^{\frac{y-x}{2}} p(B,r,y-r) - p(B,r,x+r)\,dr\,, \tag{5.82}$$

where p is given by (5.80).

Notice that the Preisach fatigue operator (5.75) has the same memory structure as the Preisach operator itself (in particular, it is compatible with the Madelung deletion rule) and belongs to the class of Preisach type operators (see [B & S]) or memory preserving operators (see [K, 1991/b]).

A continuity result analogous to Proposition 3.1 holds as well for the model with fatigue. The method of proof is analogous and we leave the details to the reader.

Proposition 5.7 *Assume that a function* $g : \mathbf{R}_+\times\mathbf{R}_+\times\mathbf{R}\to\mathbf{R}$ *satisfying (5.72) and an* $r_0 > 0$ *are given such that* $g(h,r,z) = 0$ *for all* $r\in[0,r_0]$, $h\geq 0$ *and* $z\in\mathbf{R}$. *Assume further that the partial derivative* $\partial_z g$ *is nonnegative, measurable and satisfies a regularity condition analogous to the one outlined below (5.37). Let* $\sigma_n\in C[0,T]$ *converge uniformly to* $\sigma\in C[0,T]$. *Then for the corresponding functions defining the total dissipated energy in (5.76) we have*

$$\lim_{n\to\infty} q_n(t) = q(t) \tag{5.83}$$

uniformly in $[0,T]$.

This result suggests that one might use the extended rainflow count to estimate the damage due to the combined effect of maximal loading amplitude and cyclic loading.

6 Concluding Remarks

We have presented a mathematical analysis of the scalar rainflow count and its relation to the output variation of a relay with hysteresis. We then have shown that the accumulated damage D obtained from the rainflow count and the Palmgren-Miner-Rule equals the variation of the output of a certain Preisach hysteresis operator. As a consequence, the accumulated damage defines a functional which is continuous with respect to the maximum norm. For uniaxial constitutive laws, we use the calculus of hysteresis operators to compute the energy dissipation for certain rate independent rheological models with distributed elements and show that in some examples the dissipation again equals the variation of the output of a certain Preisach operator; this in turn implies that the dissipation functional also is continuous with respect to the maximum norm, and enables us to obtain a linear accumulation formula in terms of some extended rainflow count for the dissipated energy. Thus the rainflow count, usually thought of as a purely algorithmic tool in the context of fatigue life testing and design, is brought more in contact with the framework of theoretical solid mechanics.

Numerous variants of multiaxial constitutive laws have been proposed and investigated. Aiming specifically at the distributed element case, a theory of hysteresis operators, partially built on the standard theory of variational inequalities, emerges within the mathematical sciences. The close correspondence between the rainflow count and the memory structure of the scalar constitutive law as presentend in Section 4 suggests to construct multiaxial counting methods based on multiaxial constitutive laws. However, this problem still seems to be largely open.

Acknowledgements.

We thank one of the referees as well as Renate Tobies for pointing out some references and background historical information.

A Appendix: Proof of Theorem 3.1

For the operator $\mathcal{F}_r$ defined by (3.1) and (3.2), an induction argument shows that, if

$$(w_0', \dots, w_N') = \mathcal{F}_r(v_0', \dots, v_N') = \mathcal{F}_r(s'), \tag{A.1}$$

then

$$|w_i - w_i'| \le \max_{j \le i} |v_j - v_j'|, \tag{A.2}$$

so we obtain the well known formula

$$\|\mathcal{F}_r(s) - \mathcal{F}_r(s')\|_\infty \le \|s - s'\|_\infty, \tag{A.3}$$

which implies of course that

$$\|\mathcal{F}_r^{per}(s) - \mathcal{F}_r^{per}(s')\|_\infty \le \|s - s'\|_\infty. \tag{A.4}$$

Lemma A.1 *Let the string* $s = (v_0, \dots, v_N) \in S$ *be given. For each* $r > 0$ *and* $z \in \mathbf{R}$ *there holds*

$$(\mathcal{R}_{z-r,z+r}(s))_i = \begin{cases} 0\,, & z > (\mathcal{F}_r(s))_i\,, \\ 1\,, & z < (\mathcal{F}_r(s))_i\,, \end{cases} \qquad 0 \le i \le N\,, \tag{A.5}$$

where for the relay we assume the initial value 1, if $z < 0$, *and 0 otherwise. Let* $\mathcal{W}$ *be the Preisach operator (2.39) with* $\rho \in L^1(P)$. *Then we have*

$$\mathcal{W}(s) = \int_0^\infty p(r, \mathcal{F}_r(s))\, dr\,, \quad \mathcal{W}^{per}(s) = \int_0^\infty p(r, \mathcal{F}_r^{per}(s))\, dr\,, \tag{A.6}$$

where

$$p(r,z) = 2\int_{-\infty}^{z} \rho(\zeta - r, \zeta + r)\, d\zeta\,. \tag{A.7}$$

Proof: For input functions instead of input strings, this result can be found in [Brokate, 1989] and [Krejčí, 1989]. For the convenience of the reader, we carry out explicitly the proof for the present situation. Fix z and r, set $w_i = (\mathcal{F}_r(s))_i$ and $q_i = (\mathcal{R}_{z-r,z+r}(s))_i$. For $i = -1$, i.e. for the initial values, (A.5) holds by assumption. We provide the induction step $i-1 \to i$. Assume that $v_i > v_{i-1}$. By (3.1) and (3.2) we have that

$$w_i = \max\{w_{i-1}, v_i - r\}\,. \tag{A.8}$$

According to the right hand side of (A.8), we distinguish two cases:

- Assume that $z > w_i$. Then we have $z > w_{i-1}$ and hence $q_{i-1} = 0$; on the other hand, $z + r > v_i$. Together, this implies that $q_i = 0$.
- If $z < w_i$, then we have either $w_i = w_{i-1}$ and $q_{i-1} = 1$, or $w_i = v_i - r$ and $z + r < v_i$. In both cases, $q_i = 1$ follows from (2.24).

If we insert (A.5) into (2.39) and take into account (2.33), we obtain (A.6). □

In order to estimate the difference $|\mathrm{Var}\,(\mathcal{W}^{per}(s)) - \mathrm{Var}\,(\mathcal{W}^{per}(s'))|$ of the damage attributed to the strings s and s', we introduce the notation $N_{mon}(s)$ for the number of monotonicity intervals of s, obtained by partitioning s into monotone substrings $(v_{i_j}, v_{i_j+1}, \dots, v_{i_{j+1}})$ of maximal length and counting their number.

Lemma A.2 *Let* $\mathcal{W}$ *be the Preisach operator with density function* $\rho \in L^1(P)$. *Then for any two strings* s, s' *of equal length there holds*

$$|\mathrm{Var}\,(\mathcal{W}^{per}(s)) - \mathrm{Var}\,(\mathcal{W}^{per}(s'))| \le C(s,s')\mu(\|s - s'\|_\infty)\,, \tag{A.9}$$

where

$$\mu(\delta) = \sup_{a \in \mathbf{R}} \int_0^\infty \int_a^{a+\delta} |\rho(z - r, z + r)|\, dz\, dr\,, \quad \delta > 0\,, \tag{A.10}$$

and

$$C(s,s') = 4(\max\{N_{mon}(\mathcal{W}^{per}(s)), N_{mon}(\mathcal{W}^{per}(s'))\} + 1)\,. \tag{A.11}$$

Proof: We partition $\mathcal{W}^{per}(s) = (w_0, \dots, w_N)$ into monotone substrings of maximal length $(w_{i_j}, w_{i_j+1}, \dots, w_{i_{j+1}})$, where $0 \le j < N_{mon}(\mathcal{W}^{per}(s))$. We now estimate

$$\begin{aligned}
\mathrm{Var}\,(\mathcal{W}^{per}(s)) &= \sum_{i=0}^{N-1} |w_{i+1} - w_i| = \sum_{j=0}^{N_{mon}(s)-1} |w_{i_{j+1}} - w_{i_j}| \\
&= \sum_{j=0}^{N_{mon}(s)-1} |\int_0^\infty p(r, \mathcal{F}_r^{per}(s)_{i_{j+1}}) - p(r, \mathcal{F}_r^{per}(s)_{i_j})\, dr| \\
&\le \sum_{j=0}^{N_{mon}(s)-1} |\int_0^\infty p(r, \mathcal{F}_r^{per}(s')_{i_{j+1}}) - p(r, \mathcal{F}_r^{per}(s')_{i_j})\, dr| \ + \\
&\quad + \sum_{j=0}^{N_{mon}(s)} 2|\int_0^\infty p(r, \mathcal{F}_r^{per}(s)_{i_j}) - p(r, \mathcal{F}_r^{per}(s')_{i_j})\, dr| \\
&\le \mathrm{Var}\,(\mathcal{W}^{per}(s')) + 4(N_{mon}(s)+1)\mu(\|s - s'\|_\infty)\,.
\end{aligned} \tag{A.12}$$

Reversing the role of s and s' we obtain the assertion. □

We now pass to continuous time. To define $\mathcal{F}_r$ on the space $M_{pm}[0,T]$ of all piecewise monotone functions, let $v \in M_{pm}[0,T]$ and $w_{-1} \in \mathbf{R}$ be given, and assume that $\{t_i\}_i$ is a partition of $[0,T]$ such that v is monotone on every partition interval $[t_i, t_{i+1}]$. We define $w : [0,T] \to \mathbf{R}$ by

$$w(0) = f_r(v(0), w_{-1})\,, \tag{A.13}$$

$$w(t) = f_r(v(t), w(t_{i-1}))\,, \quad \text{if } t \in (t_{i-1}, t_i]\,. \tag{A.14}$$

Obviously, $w \in M_{pm}[0,T]$ with the same monotonicity partition as v, and we have

$$(w(t_0), \dots, w(t_{i-1}), w(t)) = \mathcal{F}_r(v(t_0), \dots, v(t_{i-1}), v(t))\,, \quad \text{if } t \in (t_{i-1}, t_i]\,. \tag{A.15}$$

The resulting correspondence $v \mapsto w$ defines an operator, again denoted by $\mathcal{F}_r$, on $M_{pm}[0,T]$. Its periodic version $\mathcal{F}_r^{per}$ on $M_{pm}[0,T]$ is given by

$$\mathcal{F}_r^{per}[v](t) = \mathcal{F}_r[v](t+T)\,, \quad t \in [0,T]\,, \tag{A.16}$$

where on the right hand side $\mathcal{F}_r$ acts on $M_{pm}[0,2T]$ and v is periodically continued as $v(t+T) = v(t)$ on $[T, 2T]$. The estimates (A.3) and (A.4) become

$$\|\mathcal{F}_r[v] - \mathcal{F}_r[u]\|_\infty \le \|v - u\|_\infty\,, \quad \|\mathcal{F}_r^{per}[v] - \mathcal{F}_r^{per}[u]\|_\infty \le \|v - u\|_\infty\,, \tag{A.17}$$

for all $v, u \in M_{pm}[0,T]$, so $\mathcal{F}_r$ and $\mathcal{F}_r^{per}$ can be extended by continuity to the space $C[0,T]$ of continuous functions. We may therefore use formula (A.6) to define the Preisach operator $\mathcal{W}$ and its periodic version $\mathcal{W}^{per}$ on $C[0,T]$ by

$$\mathcal{W}[v](t) = \int_0^\infty p(r, \mathcal{F}_r[v](t))\, dr\,, \quad \mathcal{W}^{per}[v](t) = \int_0^\infty p(r, \mathcal{F}_r^{per}[v](t))\, dr\,. \tag{A.18}$$

Proof of Theorem 3.1. Let $v_n \to v$ uniformly in $C[0,T]$. Choose $\delta > 0$ such that $|v_n(t) - v_n(\tau)| \le 2r_0$ for all $|t - \tau| \le \delta$ and all n. Since $\mathcal{W}$ is piecewise monotone (as it was shown in the proof of Corollary 2.13), so is $\mathcal{W}^{per}$. Since moreover only relays $\mathcal{R}_{x,y}$

with $|x-y| > 2r_0$ are present, the input has to cross a distance of at least $2r_0$ before the output of $\mathcal{F}_{r_0}$ can change direction. Therefore

$$N_{mon}(\mathcal{W}^{per}[v_n]) \le \frac{T}{\delta} + 2\,, \quad n \in \mathbf{N}\,, \tag{A.19}$$

and the same estimate holds for the limit v in place of v_n. We now replace the functions v and v_n by strings of equal length. Formula (A.9) then yields

$$|\mathrm{Var}\,(\mathcal{W}^{per}[v]) - \mathrm{Var}\,(\mathcal{W}^{per}[v_n])| \le 4\left(\frac{T}{\delta} + 3\right)\mu(\|v - v_n\|_\infty)\,, \tag{A.20}$$

where T and δ do not depend upon n. Since by standard properties of the integral,

$$\lim_{s\downarrow 0} \mu(s) = 0\,, \tag{A.21}$$

the proof is complete. □

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