A uniqueness criterion for the Riemann problem *†

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Abstract

A new entropy criterion (maximal dissipation condition) for the quasilinear wave equation with generally non-monotone nonlinearity is introduced and tested on self-similar solutions of the corresponding Riemann problem. It is shown that the maximally dissipating solution exists and it is uniquely determined. The relation between the maximal dissipation principle and other entropy criteria is discussed.

Keywords: Nonlinear wave equation, Riemann problem, entropy condition.

AMS-classification: 35L70, 35L65, 35L67, 35G25

Introduction

Introducing hysteresis into hyperbolic equations makes the problem easier. This fact has already been recognized recently [Kre-1989, 1993]. The present paper is based on another point of view: we do not assume any a priori hysteretic structure in a quasilinear wave equation and we show that nevertheless, convex hysteresis appears as a natural consequence of the maximal dissipation principle.

It is well known that systems of hyperbolic conservation laws of the type

(i) $\quad u_t + f(u)_x = 0,\ x \in R^1,\ t > 0,$

where $u(x,t)$ is the vector $(u_1(x,t), \ldots, u_n(x,t))$, $n \in \mathbf{N}$, and $f : R^n \to R^n$ is a given function, do not admit in general global regular solutions satisfying a given initial condition $u(x,0) = u_0(x)$ even if the data are smooth, and that weak solutions may be multiple

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if f is nonlinear, unless additional criteria are fulfilled [Lax-1957]. These criteria (usually called entropy conditions) have been tested on the well-known Riemann problem for equation (i) which consists in choosing initial data

$$\text{(ii)} \qquad u(x,0) := \begin{cases} u_- & \text{for } x < 0, \\ u_+ & \text{for } x \geq 0, \end{cases}$$

where $u_-, u_+ \in R^n$ are given constant vectors.
When listing the existing entropy conditions, we must remind at least the following ones:

(iii) Lax' shock condition [Lax-1957],
(iv) condition $\eta(u)_t + q(u)_x \leq 0$ with a given entropy pair (η, q) [Kru-1970, Lax-1971],
(v) viscosity admissibility criterion [Liu-1976],
(vi) Liu's shock admissibility criterion [Liu-1981],
(vii) maximal entropy rate criterion [Daf-1973].
(viii) viscosity-capillarity criterion [She-1986].

Under various assumptions on f and/or u_-, u_+, existence and uniqueness for the problem (i), (ii) has been established. For a survey of the results see e.g. [Chang] and the references cited there. While for $n = 1$ the theory is fairly complete, starting from $n = 2$ considerable difficulties can appear. This has been many times demonstrated on the so-called p-system (e.g. [Smo])

$$\text{(ix)} \qquad \begin{cases} w_t + p(v)_x = 0, \\ v_t - w_x = 0. \end{cases}$$

Here, the hyperbolicity is implied by $p'(v) < 0$ and the complication begins when p is not convex (concave) or even monotone in the whole range of v. The Riemann problem for (ix) in the case of nonconvex p has been treated under various entropy conditions and various assumptions on p [Lei-1974, Liu-1974, Wen-1972, Zum-1993 etc.].
In this paper we concentrate our attention to the equation

$$\text{(x)} \qquad u_{tt} - g(u_x)_x = 0$$

arising from (ix) for $g = -p$ by a formal elimination of v and w. In fact, all what is assumed about g is that it is locally Lipschitz continuous.

We find a general uniqueness condition based on the maximal dissipation principle which yields a unique solution of the Riemann problem for (x) in the class of regulated self-similar solutions. Our approach is close to that of Dafermos [Daf-1973] and Leibovich [Lei-1974]. We apply the maximal dissipation principle locally in space, while Dafermos requires that the admissible solution maximizes the total entropy rate. It appears that maximally dissipating solutions are those which minimize the L^2-norm. Geometrically, shocks are allowed only along the boundary of the convex hull of the graph of g similarly as in [Lei-1974]. Maximal dissipation principle thus imposes a convex hysteretic behavior to the equation (x). This also explains, why equation (x), where a single-valued function g is replaced by a convex hysteresis operator, admits regular solutions [Kre-1989, 1993].

An important feature of the technique developed below is that it fits also to the case where g is not monotone, which corresponds to the non-strictly hyperbolic case studied e.g. in [Fan-1992], [Hat-1986], [Key-1986], [She-1982], [She-1986]. Keyfitz [Key-1986] replaces the one-valued constitutive relation by a hysteretic one, while Shearer [She-1982] applies Liu's criterion or a so-called viscosity-capillarity criterion [She-1986]. The viscosity-capillarity approach is also applied in [Fan-1992] to obtain a unique solution of the Riemann problem with g non-monotone. Dafermos maximal entropy rate criterion is applied in [Hat-1986] to a special class of solutions. In what follows we enlarge results of this kind by constructing a unique minimal solution of the Riemann problem according to our minimality condition (see Definition 4.3 below).
Our paper is organized as follows. In Sect. 1 we present general well-known facts about self-similar solutions in order to preserve the self-consistency. In Sect. 2 we show how to construct a continuum of piecewise constant self-similar solutions of the Riemann problem for equation (x). Sect. 3 is then devoted to the correspondence between intervals of monotonicity of self-similar solutions and convex (concave) trajectories along the graph of the constitutive function g. In Sect. 4 the maximal dissipation principle is applied to the set of self-similar solutions to derive the L^2-minimality condition. The aim of Sect. 5 is to show that our minimality condition ensures existence and uniqueness of solutions to the Riemann problem and provide an explicit construction of the minimal solution. In Sect. 6 it is shown that this minimal solution always satisfies the entropy condition (iv) above. In the convex (concave) case, both criteria are equivalent. In Sect. 7 we prove that the minimal solution is a pointwise limit of viscous solutions. Finally in Sect. 8, the L^2-minimality criterion is compared with entropy conditions (iii), (vi), (vii). The question of relationship between the minimality and viscosity-capillarity criteria is open.

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1. Self-similar solutions.

In this section we introduce the problem, reformulate it for self-similar solutions and derive some well-known facts related to the Riemann problem.

Assumption 1.1. *We are given a function* $g : (U_-, U_+) \to R^1$ *which is locally Lipschitz continuous and such that* $G_- := g(U_-+) < g(u) < g(U_+-) =: G_+$ *for all* $u \in (U_-, U_+)$ *where* $U_-, G_- \in R^1 \cup \{-\infty\}, U_+, G_+ \in R^1 \cup \{\infty\}$. *For an arbitrary compact set* $K \subset (U_-, U_+)$ *we define*

$$L_K := \sup\left\{\frac{g(u)-g(v)}{u-v};\ u,v\in K,\ u\neq v\right\}. \tag{1}$$

Throughout the paper, this will be the only assumption concerning g (except for some technical auxiliary lemmas and part of Sect. 6).

Definition 1.2. *A function $u : R^1 \times [0,T] \to R^1$ is called* a weak solution *to the equation*

$$u_{tt} - g(u_x)_x = 0, \tag{2}$$

if u is continuous in $R^1 \times [0,T]$, $u_t, u_x \in L^\infty(R^1\times(0,T)) \cap C([0,T]; L^2_{loc}(R^1))$, $u_x(x,t)$ belongs to a compact subset of (U_-,U_+) for a.e. $(x,t)\in(0,T)\times R^1$ and the identity

$$\int_0^T\int_{-\infty}^{\infty}(u_t\varphi_t - g(u_x)\varphi_x)dxdt = 0 \tag{3}$$

holds for every $\varphi \in \mathcal{D}(R^1\times(0,T))$, where $\mathcal{D}(\Omega)$ denotes the set of C^∞ functions with a compact support in Ω.

Definition 1.3. *A weak solution u to equation (2) is called* self-similar*, if there exists a function $f : R^1 \to R^1$ such that*

$$u(x,t) = tf\left(\frac{x}{t}\right) \tag{4}$$

for every $(x,t)\in R^1\times(0,T)$.

Proposition 1.4. *Let u be a weak self-similar solution to (2). Then f is absolutely continuous, $v(z) = f'(z)$ belongs to a compact subset of (U_-,U_+), the function $z \longmapsto z^2v(z) - g(v(z))$ is equal to an absolutely continuous function almost everywhere and the equation*

$$\left(z^2v(z) - g(v(z))\right)' = 2zv(z) \tag{5}$$

is satisfied almost everywhere.

Proof. The function f is absolutely continuous, since $u(\cdot,t)$ is absolutely continuous for almost all t. On the other hand, we have $f'(z) = u_x(zt,t)$, hence $v = f'$ belongs to $L^\infty(R^1)$, and $u_t(zt,t) = f(z) - zf'(z)$. We can rewrite identity (3) in the form

$$\int_0^T\int_{-\infty}^{\infty}\left[(f(z)-zf'(z))t\varphi_t(zt,t) - g(f'(z))t\varphi_x(zt,t)\right]dzdt = 0$$

for all $\varphi\in\mathcal{D}(R^1\times(0,T))$. The function $\psi(z,t) := \varphi(zt,t)$ belongs also to $\mathcal{D}(R^1\times(0,T))$, hence the identity

$$\int_{-\infty}^{\infty}\Big[\left(zf(z) - z^2f'(z) + g(f'(z))\right)\left(\int_0^T\psi_z(z,t)dt\right) \\ - (f(z)-zf'(z))\left(\int_0^T t\psi_t(z,t)dt\right)\Big]dz = 0$$

holds for any $\psi \in \mathcal{D}(R^1 \times (0,T))$.
Putting $\eta(z) = \int_0^T \psi(z,t)dt$ we obtain

$$\int_{-\infty}^{\infty} \Big[\Big(zf(z) - z^2 f'(z) + g(f'(z))\Big)\eta'(z) + (f(z) - zf'(z))\,\eta(z)\Big]\,dz = 0 \tag{6}$$

for any $\eta \in \mathcal{D}(R^1)$.
Identity (6) entails that the function $z \longmapsto zf(z) - z^2 f'(z) + g(f'(z))$ is equal to an absolutely continuous function almost everywhere and its derivative is equal to $f(z) - zf'(z)$ a.e. Since f is absolutely continuous, we obtain easily the assertion. □

Lemma 1.5. *(Rankine-Hugoniot condition) Let v satisfy equation (5) and let us assume that there exist $z_0 \in R^1$ and sequences $z_1^{(n)} \to z_0$, $z_2^{(n)} \to z_0$ such that $v(z_i^{(n)} \to v_i$, $i = 1, 2$, $v_1 \neq v_2$. Then we have*

$$z_0^2 = \frac{g(v_2) - g(v_1)}{v_2 - v_1}. \tag{7}$$

Proof. We just notice that (7) is a necessary and sufficient condition for the continuity of the function $z \longmapsto z^2 v(z) - g(v(z))$ at the point $z = z_2$. □

Remark 1.6. We always assume that the values of $v(z)$ are chosen in such a way that the function $z \mapsto z^2 v(z) - g(v(z))$ is absolutely continuous.

Proposition 1.7. *Let $K \subset (U_-, U_+)$ be a compact interval and let v be a solution of equation (5) such that $v(z) \in K$ a.e. Then there exist constants $V_+, V_- \in K, b > 0, G_0 \in (G_-, G_+)$ such that $g(v(0+)) = g(v(0-)) = g(v(0)) = G_0$ and*

$$v(z) = \begin{cases} V_- & \textit{for} \quad z < -\sqrt{b}, \\ V_+ & \textit{for} \quad z > \sqrt{b}. \end{cases} \tag{8}$$

Proof. Put $b := L_K$, where L_K is given by (1). Assume that $\varepsilon > 0$ is arbitrarily chosen and put $z_0 = \sqrt{b + \varepsilon}$. For every $z > z_0$ (5) entails

$$z^2 \left(v(z) - v(z_0)\right) - \left(g(v(z)) - g(v(z_0))\right) = \int_{z_0}^{z} 2\xi \left(v(\xi) - v(z_0)\right) d\xi.$$

The assumption $\mid g(v(z)) - g(v(z_0)) \mid \leq b \mid v(z) - v(z_0) \mid$ implies

$$\varepsilon \mid v(z) - v(z_0) \mid \leq \int_{z_0}^{z} 2\xi \mid v(\xi) - v(z_0) \mid d\xi$$

and Gronwall's lemma yields $v(z) = v(z_0)$, hence v is constant in $(\sqrt{b}, \infty)$. The same argument works in $(-\infty, -\sqrt{b})$. The continuity at 0 follows from Lemma 1.5 and Remark 1.6. □

Identity (5) is a necessary condition for u given by (4) to be a weak self-similar solution to (2). We will see that this condition is also sufficient.

Theorem 1.8. *Let v satisfy the hypotheses of Proposition 1.7. Put $\bar{V} := \int_{-\infty}^{\infty}(v(z) - P_0(z))dz$, where P_0 is the function*

$$P_0(z) := \begin{cases} V_+, & z \geq 0, \\ V_-, & z < 0, \end{cases} \tag{9}$$

and V_-, V_+ are as in (8). Then the function $u(x,t)$ given by (4), where f is defined by the formula

$$f(z) := \int_0^z v(\xi)d\xi + \int_{-\infty}^0 (v(\xi) - P_0(\xi))\, d\xi + K$$

with an arbitrary constant $K \in R^1$, is a weak solution to equation (2) in $R^1 \times (0,\infty)$ and satisfies the initial conditions

$$u(x,0) = \begin{cases} V_+x & \text{for } x \geq 0 \\ V_-x & \text{for } x < 0 \end{cases}, \qquad u_t(x,0) = \begin{cases} D_+ & \text{for } x \geq 0 \\ D_- & \text{for } x < 0 \end{cases}, \tag{10}$$

where $D_+ = K + \bar{V}$, $D_- = K$.

Proof. We only have to verify that the integrals

$$\int_{-\infty}^{\infty} | u_t(x,t) - u_t(x,0) |^2\, dx, \quad \int_{-\infty}^{\infty} | u_x(x,t) - u_x(x,0) |^2\, dx$$

tend to 0 as $t \to 0+$. Then $u(\cdot,t) \to u(\cdot,0)$ locally uniformly as $t \to 0+$, since

$$\begin{aligned} | u(x,t) - u(x,0) | &\leq | u(0,t) - u(0,0) | + \int_0^x | u_x(\xi,t) - u_x(\xi,0) |\, d\xi \\ &= t \,| f(0) | + | x |^{1/2} \left(\int_0^x | u_x(\xi,t) - u_x(\xi,0) |^2\, d\xi \right)^{1/2}. \end{aligned}$$

By Proposition 1.7, we have $u_x(x,t) = u_x(x,0) = V_+$ for $x > t\sqrt{b}$ and $u_x(x,t) = u_x(x,0) = V_-$ for $x < -t\sqrt{b}$, hence

$$\begin{aligned} \int_{-\infty}^{\infty} | u_x(x,t) &- u_x(x,0) |^2\, dx = \int_{-t\sqrt{b}}^{t\sqrt{b}} | u_x(x,t) - u_x(x,0) |^2\, dx \\ &= t \left[\int_0^{\sqrt{b}} | v(z) - V_+ |^2\, dz + \int_{-\sqrt{b}}^0 | v(z) - V_- |^2\, dz \right] \to 0 \end{aligned}$$

and similarly for u_t. □

Conclusion 1.9. The problem of finding weak self-similar solutions to equation (2) satisfying initial conditions (10) (the so-called *Riemann problem*) is equivalent to the problem of solving equation (5) with conditions

$$v(z) = \begin{cases} V_+ & \text{for } z \to +\infty \\ V_- & \text{for } z \to -\infty \end{cases}, \quad \int_{-\infty}^{\infty} (v(z) - P_0(z))\, dz = \bar{V}\ , \tag{11}$$

where $\bar{V} = D_+ - D_-$ and P_0 is given by (9).

2. Piecewise constant solutions.

We devote this section to an important class of piecewise constant solutions. We start with a well-known result.

Proposition 2.1. *Let $z_1 < \ldots < z_N$ be a given sequence and let $v_0, v_1, \ldots, v_N \in (U_-, U_+)$ be given. Then the piecewise constant function*

$$v(z) = \begin{cases} v_0 & \text{for } z < z_1, \\ v_k & \text{for } z \in (z_k, z_{k+1}),\ k = 1, \ldots, N-1, \\ v_N & \text{for } z > z_N \end{cases} \tag{12}$$

can be extended to a solution of equation (5) if and only if $z_k^2 = \frac{g(v_k) - g(v_{k-1})}{v_k - v_{k-1}}$ for all $k \in \{1, \ldots, N\}$ such that $v_k \neq v_{k-1}$.

Proof. The assertion is an immediate consequence of Lemma 1.5, since any constant function satisfies equation (5). □

The following statements show that piecewise constant solutions always exist and that they are not uniquely determined by the Cauchy data.

Theorem 2.2. *Let g be a nonlinear function satisfying Assumption 1.1 and let $V_+ = V_- \in (U_-, U_+)$ and $D_+ = D_- \in R^1$ be given. Then there exist infinitely many distinct piecewise constant solutions to (5), (11).*

For proving Theorem 2.2 we need some auxiliary results. Let us first mention the following elementary identity.

Lemma 2.3. *For all distinct real numbers p, q, r we have*

$$\frac{g(p) - g(q)}{p - q} - \frac{g(q) - g(r)}{q - r} = \frac{p - r}{q - r} \left(\frac{g(p) - g(q)}{p - q} - \frac{g(p) - g(r)}{p - r} \right). \tag{13}$$

We prove the following lemma.

Lemma 2.4. *Let $(\hat{U}_-, \hat{U}_+)$, $(\hat{G}_-, \hat{G}_+)$ be given open intervals such that $0 \in (\hat{U}_-, \hat{U}_+) \cap (\hat{G}_-, \hat{G}_+)$. Let $\hat{g} : (\hat{U}_-, \hat{U}_+) \to (\hat{G}_-, \hat{G}_+)$ be a nonlinear function such that $\hat{g}(r)r > 0$ for all $r \neq 0$. Then we have either*

$$\begin{array}{l} \text{(i) } \exists q < 0 < p;\ \forall r \in (q, 0]\ \ \frac{\hat{g}(p) - \hat{g}(q)}{p - q} > \frac{\hat{g}(p) - \hat{g}(r)}{p - r}, \\ \quad \text{or} \\ \text{(ii) } \exists q < 0 < p;\ \forall r \in [0, p)\ \ \frac{\hat{g}(p) - \hat{g}(q)}{p - q} > \frac{\hat{g}(r) - \hat{g}(q)}{r - q}. \end{array} \tag{14}$$

Proof of Lemma 2.4. Suppose that for every $q < 0 < p$ there exist $r_1 \in (q, 0]$ and $r_2 \in [0, p)$ such that

$$\frac{\hat{g}(p) - \hat{g}(q)}{p - q} \leq \frac{\hat{g}(p) - \hat{g}(r_1)}{p - r_1}, \quad \frac{\hat{g}(p) - \hat{g}(q)}{p - q} \leq \frac{\hat{g}(r_2) - \hat{g}(q)}{r_2 - q}.$$

For a fixed pair (p,q) put

$$\bar{r}_1 := \sup\left\{r_1 \in (q,0]; \quad \frac{\hat{g}(p)-\hat{g}(q)}{p-q} \le \frac{\hat{g}(p)-\hat{g}(r_1)}{p-r_1}\right\}$$

and

$$\bar{r}_2 := \inf\left\{r_2 \in [0,p); \quad \frac{\hat{g}(p)-\hat{g}(q)}{p-q} \le \frac{\hat{g}(r_2)-\hat{g}(q)}{r_2-q}\right\}.$$

We obviously have $\bar{r}_1 = \bar{r}_2 = 0$, hence

$$\frac{\hat{g}(p)-\hat{g}(q)}{p-q} - \frac{\hat{g}(p)}{p} \le 0, \quad \frac{\hat{g}(p)-\hat{g}(q)}{p-q} - \frac{\hat{g}(q)}{q} \le 0.$$

Identity (13) for $r=0$ then entails

$$\frac{\hat{g}(p)-\hat{g}(q)}{p-q} = \frac{\hat{g}(p)}{p} = \frac{\hat{g}(q)}{q} \quad \text{for all } q<0<p$$

which contradicts the assumption that $\hat{g}$ is nonlinear. □

Remark 2.5. If g is nonlinear, then there exist in fact infinitely many pairs (p,q) satisfying (i) or (ii) of Lemma 2.4. Let us assume for instance that (i) holds for some $q<0<p$. Then the function $\gamma(r) := \frac{g(p)-g(r)}{p-r}$ is continuous for $r \in [q,0]$ and satisfies $\gamma(r) < \gamma(q)$ $\forall r \in (q,0]$.
For each $c \in (\gamma(0), \gamma(q))$ we can put $q_c := \max\{r \in (q,0); \gamma(r) = c\}$. Then (i) holds for all pairs (p, q_c).

Proof of Theorem 2.2. First, by Proposition 2.1 it is clear that the function (12) is a solution of the problem (5), (11) if and only if the following conditions are fulfilled

$$\begin{aligned} &\text{(i)} \quad v_1 = V_-, v_N = V_+;\\ &\text{(ii)} \quad z_k^2 = \tfrac{g(v_k)-g(v_{k-1})}{v_k - v_{k-1}};\\ &\text{(iii)} \quad \textstyle\sum_{k=1}^{N-1} z_k(v_k - v_{k+1}) = D_+ - D_-. \end{aligned} \tag{15}$$

Put $V := V_+ = V_-, \hat{g}(r) := g(r+V) - g(V)$ for $r \in (\alpha,\beta) := (U_- - V, U_+ - V)$. We distinguish four cases (see Fig. 1).
A. $\hat{g}(r)r > 0$ for $r \in (\alpha,\beta)\setminus\{0\}$ and (14)(i) holds for some $\alpha<q<0<p<\beta$.
For some $r \in (q,0)$ which will be specified later we define

$$z_1 : = -\sqrt{\frac{\hat{g}(q)}{q}},\ z_2 : = -\sqrt{\frac{\hat{g}(p)-\hat{g}(q)}{p-q}},\ z_3 : = -\sqrt{\frac{\hat{g}(p)-\hat{g}(r)}{p-r}},\ z_4 : = \sqrt{\frac{\hat{g}(r)}{r}}, \tag{16}$$

and

$$v(z) : = \begin{cases} V & \text{for} \quad z < z_1,\\ V+q & \text{for} \quad z \in (z_1,z_2),\\ V+p & \text{for} \quad z \in (z_2,z_3),\\ V+r & \text{for} \quad z \in (z_3,z_4),\\ V & \text{for} \quad z > z_4. \end{cases} \tag{17}$$

Lemma 2.4 ensures that we have $z_1 < z_2 < z_3 < z_4$ and v defined by (17) is a solution to (5),(11) provided that the condition (15)(iii) holds. Here it reads

$$-\sqrt{\hat{g}(q)q} + \sqrt{(\hat{g}(p) - \hat{g}(q))(p-q)} - \sqrt{(\hat{g}(p) - \hat{g}(r))(p-r)} - \sqrt{\hat{g}(r)r} = 0. \tag{18}$$

Let us denote by $h(r)$ the left-hand side of equation (18). We have $h(0) > 0, h(q) < 0$, hence (18) is satisfied for a suitable $r \in (q, 0)$.

B. $\hat{g}(r)r > 0$ for $r \in (\alpha, \beta) \setminus \{0\}$ and (14)(ii) holds for some $\alpha < q < 0 < p < \beta$. Analogously as above we define for $s \in (0, p)$

$$z_1 := -\sqrt{\frac{\hat{g}(p)}{p}},\ z_2 := -\sqrt{\frac{\hat{g}(p) - \hat{g}(q)}{p-q}},\ z_3 := -\sqrt{\frac{\hat{g}(s) - \hat{g}(q)}{s-q}},\ z_4 := \sqrt{\frac{\hat{g}(s)}{s}} \tag{19}$$

and

$$v(z) := \begin{cases} V & \text{for} \quad z < z_1, \\ V + p & \text{for} \quad z \in (z_1, z_2), \\ V + q & \text{for} \quad z \in (z_2, z_3), \\ V + s & \text{for} \quad z \in (z_3, z_4), \\ V & \text{for} \quad z > z_4. \end{cases} \tag{20}$$

Similarly as in the case A we check that v solves (5),(11) provided $s \in (0, p)$ is a solution of the equation

$$\sqrt{p\hat{g}(p)} - \sqrt{(\hat{g}(p) - \hat{g}(q))(p-q)} + \sqrt{(\hat{g}(s) - \hat{g}(q))(s-q)} + \sqrt{s\hat{g}(s)} = 0. \tag{21}$$

Denoting by $\tilde{h}(s)$ the left-hand side of equation (21) we easily obtain $\tilde{h}(0) < 0$, $\tilde{h}(p) > 0$, hence (21) holds for some $s \in (0, p)$.

C. There exists $p > 0$ such that $\hat{g}(p) = 0$. We then put $r_0 := \min\{r \leq 0; \hat{g}(r) \geq 0\}$ and fix some $q_0 \in (\alpha, r_0)$.
For an arbitrary $\gamma \in \left(0, \frac{\hat{g}(q_0)}{q_0 - p}\right)$ put $q := \max\{r \in [q_0, r_0]; \frac{\hat{g}(r)}{r-p} = \gamma\}$. By Proposition 2.1, the function v defined by (16), (17) for some $r \in (q, r_0)$ is a solution to (5),(11) provided that the condition (18) holds. For the auxiliary function $h(r)$ as above we have $h(r_0) > 0$, $h(q) < 0$ with the same conclusion as above.
D. There exists $q < 0$ such that $\hat{g}(q) = 0$. We put $s_0 := \max\{r \geq 0; \hat{g}(r) \leq 0\}$ and fix some $p_0 \in (s_0, \beta)$. For an arbitrary $\gamma \in (0, \frac{\hat{g}(p_0)}{p_0 - q})$ put $p := \min\{s \in [s_0, p_0]; \frac{\hat{g}(s)}{s-q} = \gamma\}$. For $s \in (s_0, p)$ we define the function v by formulas (19),(20). Similarly as in the previous cases we choose s such that the equation (21) is satisfied.
It remains to check that there exist in fact infinitely many solutions of the form above. This is obvious in the cases C and D, where for each γ we obtain a different solution; cases A,B follow from Remark 2.5. □

Remark 2.6. The situation is simple if g is linear. Then $g(u) = bu$ and there is a unique solution of Problem (5),(11) with $\bar{V} = \sqrt{b}(2V_0 - V_+ - V_-)$.

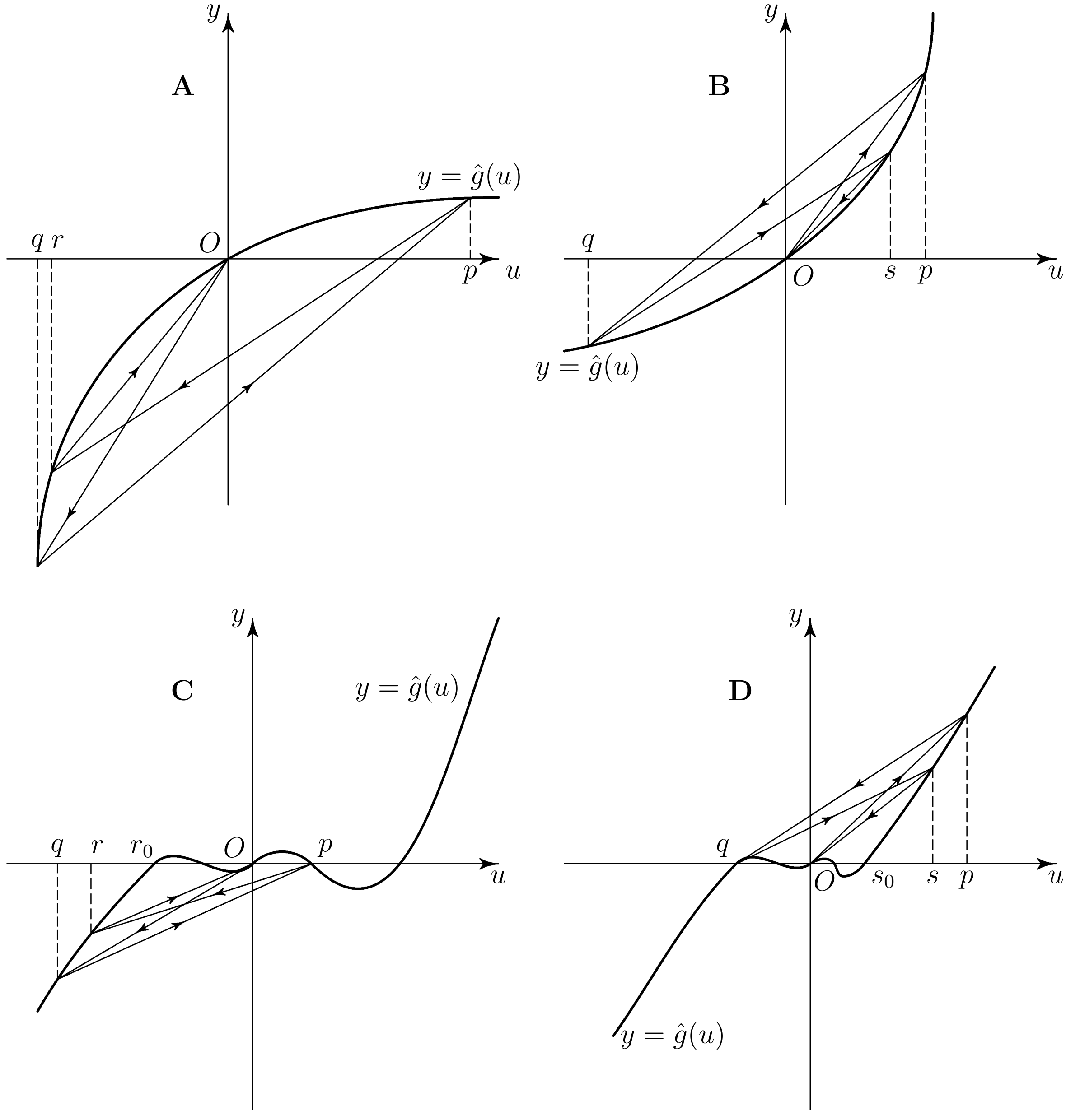


Fig. 1

3. Locally monotone solutions.

In this section we identify locally monotone solutions to equation (5) with their trajectories along the graph of the constitutive function g. We introduce the concept of convex (concave) path along g which corresponds to intervals where the solution increases (decreases, respectively).
Taking into account Proposition 1.7, it is convenient to investigate the equation (5) separately for $z > 0$ and $z < 0$. In fact, putting for $s > 0$

$$w_+(s) := v(\sqrt{s}),\ w_-(s) := v(-\sqrt{s}), \tag{22}$$

we see that both w_+ and w_- satisfy the equation

$$(sw(s) - g(w(s)))' = w(s) \quad \text{for a.e. } s > 0 \tag{23}$$

with boundary conditions

$$g(w(0)) = Q,\ w(s) = V \text{ for } s \text{ large}, \tag{24}$$

where $Q = g(v(0))$ and $V = V_+$ or V_- respectively.
Let us start with an auxiliary lemma.

Lemma 3.1. *Let* $w : [s_1, s_2]$ *be a monotone function,* $w(s_1) = v_1, w(s_2) = v_2$ *and let its inverse* w^{-1} *be defined by the formula*

$$w^{-1}(u) : = \begin{cases} \sup \Gamma_-(u) & \text{for } u \in [v_2, v_1] \quad \text{if } w \text{ is nonincreasing}, \\ \sup \Gamma_+(u) & \text{for } u \in [v_1, v_2] \quad \text{if } w \text{ is non-decreasing}, \end{cases} \tag{25}$$

where $\Gamma_\pm : = \{s \in [s_1, s_2]; \pm w(s) \le \pm u\}$. *Then we have*

$$\begin{array}{ll} (i) & \int_{s_1}^{s_2} w(s)ds + \int_{v_1}^{v_2} w^{-1}(u)du = s_2v_2 - s_1v_1, \\ (ii) & \int_{s_1}^{s_2} w^2(s)ds + 2\int_{v_1}^{v_2} uw^{-1}(u)du = s_2v_2^2 - s_1v_1^2. \end{array} \tag{26}$$

Proof. Both assertions follow from Fubini's theorem. We consider just the case of w non-decreasing (otherwise we pass from w to $-w$).
Let K be the rectangle $[s_1, s_2] \times [v_1, v_2]$. We define the maximal monotone graph $\Gamma_1 : = \{(s, u) \in K;\ w(s-) \le u \le w(s+)\}$, where we put $w(s_1-) : = w(s_1)$, $w(s_2+) : = w(s_2)$, and the sets $A_1 : = \{(s, u) \in K;\ v_1 \le u < w(s-)\}$, $B_1 : = \{(s, u) \in K; w(s+) < u \le v_2\}$. The function w^{-1} is non-decreasing in $[v_1, v_2]$ and we have $B_1 = \{(s, u) \in K;\ s_1 \le s < w^{-1}(u-)\}$, $K = \Gamma_1 \cup A_1 \cup B_1$, $A_1 \cap B_1 = \emptyset$, meas $\Gamma_1 = 0$, hence

$$(s_2 - s_1)(v_2 - v_1) = \int_{A_1} du\, ds + \int_{B_1} ds\, du = \int_{s_1}^{s_2} (w(s) - v_1)ds + \int_{v_1}^{v_2} (w^{-1}(u) - s_1)du$$

and (26) (i) follows easily.
To prove (26) (ii) we consider the cylinder in cylindrical coordinates

$$C : = \{(r, \varphi, s); r \in [0, v_2 - v_1], \varphi \in [0, 2\pi], s \in [s_1, s_2]\},$$

we define the sets

$$\Gamma_2 := \{(r,\varphi,s) \in C; \quad (s, r+v_1) \in \Gamma_1\}$$
$$A_2 := \{(r,\varphi,s) \in C; \quad (s, r+v_1) \in A_1\}$$
$$B_2 := \{(r,\varphi,s) \in C; \quad (s, r+v_1) \in B_1\}$$

and argue as above. □

Formulas (26) enable us to identify monotone solutions of (23) with their trajectories in the phase plane. This will be done in the next three lemmas.

Lemma 3.2. *Let* $V \in (U_-, U_+)$ *and* $Q \in (G_-, G_+)$ *be given and let* w *be a solution of (23), (24) which is monotone in* $(0,\infty)$, $w(0+) =: V_0 \in g^{-1}(Q)$, $w(+\infty) = V$. *Let* w^{-1} *be the inverse of* w *according to formula (25). For* $v \in [\min\{V_0, V\}, \max\{V_0, V\}]$ *put*

$$g^*(v) := Q + \int_{V_0}^{v} w^{-1}(u)du. \tag{27}$$

Then $g(w(s)) = g^*(w(s))$ *for all* $s > 0$.

Proof. By Lemma 3.1 and equation (23) we have for each $s > 0$

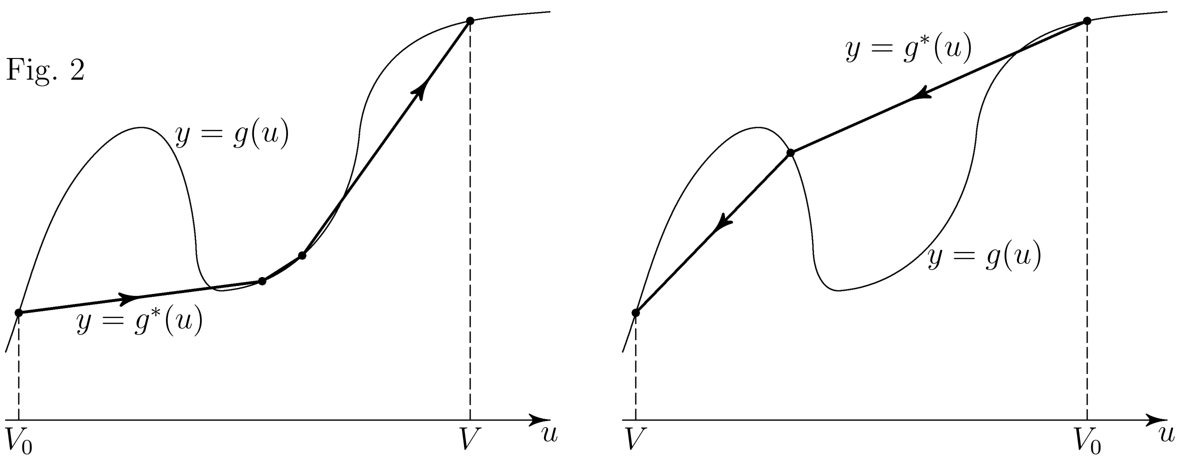
$$\int_{V_0}^{w(s)} w^{-1}(u)du = sw(s) - \int_0^s w(\sigma)d\sigma - g(w(s)) - Q,$$

hence $g^*(w(s)) = g(w(s))$ by definition of g^*. □

The function $y = g^*(v)$ describes the trajectory of the solution w along the strain-stress diagram $y = g(v)$ (see Fig. 2). From Lemma 3.2 we immediately derive two important properties, namely

$$\begin{aligned}
&\text{(i) } g^* \text{ is convex and increasing in } [V_0, V] \text{ if } w \text{ is non-decreasing} \\
&\quad \text{and concave and increasing in } [V_0, V] \text{ if } w \text{ is nonincreasing,} \\
&\text{(ii) if } g^*(v) \neq g(v) \text{ for some } v \in (\min\{V_0, V\}, \max\{V_0, V\}), \\
&\quad \text{then } g^* \text{ is affine in a neighborhood of } v.
\end{aligned} \tag{28}$$

Fig. 2



The proof of the converse of Lemma 3.2 is slightly more complicated.

Lemma 3.3. *(i) Let* $V_0, V \in (U_-, U_+)$ *be given such that* $V_0 < V, g(V_0) < g(V)$, *and let* $g^* : [V_0, V] \to (G_-, G_+)$ *be a convex increasing function such that* $g^*(V_0) = g(V_0)$, $g^*(V) = g(V)$ *and the implication (28)(ii) holds. Put* $\bar{s} : = g^{*'}(V-), w^*(s) : = \inf\{w \in [V_0, V]; g^{*'}(v) \geq s\}$ *for* $s \in (0, \bar{s}), w^*(s) : = V$ *for* $s \geq \bar{s}$. *Then* w^* *is a non-decreasing solution of (23) with* $Q = g(V_0)$, $w^*(0+) = V_0$ *and its trajectory* g^{**} *defined according to Lemma 3.2 by the formula*

$$g^{**}(v) : = g(V_0) + \int_{V_0}^{v} w^{*^{-1}}(u)du \quad \text{for } v \in [V_0, V] \tag{29}$$

coincides with g^*.
(ii) Let $V_0, V \in (U_-, U_+)$ *be given such that* $V_0 > V$, $g(V_0) > g(V)$, *and let* $g^* : [V, V_0] \to (G_-, G_+)$ *be a concave increasing function such that* $g^*(V_0) = g(V_0)$, $g^*(V) = g(V)$ *and the implication (28)(ii) holds. Put* $\bar{s} : = g^{*'}(V+)$, $w^*(s) : = \sup\{v \in [V, V_0]; g^{*'}(v) \geq s\}$ *for* $s \in (0, \bar{s})$, $w^*(s) : = V$ *for* $s \geq \bar{s}$. *Then* w^* *is a non-increasing solution of (23) with* $Q = g(V_0)$, $w^*(0+) = V_0$ *and its trajectory* g^{**} *defined by formula (29) coincides with* g^*.

Proof. It suffices to prove the first part. The statement concerning the concave case is then obtained by passing from $g(v)$ to $-g(-v)$.
The definition of w^* ensures that w^* is non-decreasing and

$$g^*(w^*(s)) - g^*(v) \leq s(w^*(s) - v) \quad \text{for all} \quad s > 0 \quad \text{and} \quad v \in [V_0, V], \tag{30}$$

hence

$$s_1(w^*(s_2) - w^*(s_1)) \leq g^*(w^*(s_2)) - g^*(w^*(s_1)) \leq s_2(w^*(s_2) - w^*(s_1))$$

for all $s_2 > s_1 > 0$. This yields

$$w^*(s_1)(s_2 - s_1) \leq s_2 w^*(s_2) - g^*(w^*(s_2)) - s_1 w^*(s_1) + g^*(w^*(s_1)) \leq w^*(s_2)(s_2 - s_1),$$

therefore the function $W^*(s) : = sw^*(s) - g^*(w^*(s))$ is Lipschitz in $(0, \infty)$, $W^{*'}(s) = w^*(s)$ a.e. To prove that w^* solves (23) it suffices to check that $g^*(w^*(s)) = g(w^*(s))$ for all $s > 0$. Assume on the contrary $g^*(w^*(s)) \neq g(w^*(s))$ for some $s > 0$. Then g^* is affine in a neighborhood of $w^*(s)$, say $g^{*'}(w^*(s) - \delta) = g^{*'}(w^*(s) + \delta) = s$, which contradicts the definition of w^*. It remains to verify that $g^{**} = g^*$. In fact, we prove more, namely $g^{*'}(u+) = w^{*^{-1}}(u)$ for all $u \in (V_0, V)$. Indeed, for an arbitrary $s > w^{*^{-1}}(u)$ we have by (25) $u < w^*(s)$ and the definition of $w^*(s)$ entails $g^{*'}(u+) < s$, hence $g^{*'}(u+) \leq w^{*^{-1}}(u)$ for all $u \in (V_0, V)$. Conversely, for $s > g^{*'}(u+)$ there exists $\delta > 0$ such that $w^*(s) \geq u + \delta$, hence $s \geq w^{*^{-1}}(u)$. Consequently, $w^{*^{-1}}(u) = g^{*'}(u+)$ and Lemma 3.3 is proved. □

For $(s, v) \in [0, \infty) \times R^1$ we next define the function

$$E(s, v) := \frac{1}{2}sv^2 + G(v) - vg(v), \tag{31}$$

where $G(v) := \int_{v_0}^{v} g(u)du$ for a fixed $v_0 \in (U_-, U_+)$.

Proposition 3.4. *Let w be a solution to (23) in $(s_1, s_2)(0 \le s_1 < s_2)$ with $w(s_1) = v_1, w(s_2) = v_2$. If w is monotone and g^* given by (27) then we have*

$$\frac{1}{2}\int_{s_1}^{s_2} w^2(s)ds = E(s_2, v_2) - E(s_1, v_1) + \int_{v_1}^{v_2} (g^*(u) - g(u))du. \tag{32}$$

Proof. We infer from Lemma 3.2

$$\int_{v_1}^{v_2} uw^{-1}(u)du = v_2 g(v_2) - v_1 g(v_1) - \int_{v_1}^{v_2} g^*(u)du.$$

Combining this last identity with Lemma 3.1(ii) we obtain the assertion. □

4. Energy.

It is well known ([Smo]) and we will see in the sequel that weak solutions of equation (2) do not necessarily satisfy the formal Energy Conservation Law

$$\left(\frac{1}{2}u_t^2(x,t) + G(u_x(x,t))\right)_t - (u_t g(u_x))_x = 0$$

in the sense of distributions.
The Second Principle of Thermodynamics requires the dissipation rate to be nonnegative. This leads to the entropy condition ([Lax-1971])

$$\left(\frac{1}{2}u_t^2(x,t) + G(u_x(x,t))\right)_t - (u_t g(u_x))_x \le 0 \tag{33}$$

in the sense of distributions. Rewriting condition (33) in terms of self-similar solutions we obtain the following statement.

Proposition 4.1. *Let u be a weak self-similar solution to equation (2) and let v be the corresponding solution of (5) defined in Proposition 1.4. Let us define functions w_+, w_- by the relations (22). Then u satisfies the entropy condition (33) if and only if the function (cf. (31))*

$$D(w)(s) := E(s, w(s)) - \frac{1}{2}\int_0^s w^2(\sigma)d\sigma \tag{34}$$

is non-decreasing in $[0, \infty)$ for both $w = w_+$ and $w = w_-$.

Before proving Proposition 4.1 we start with an easy lemma.

Lemma 4.2. *Let $[\alpha, \beta] \subset R^1$ be a closed interval and let $h \in L^1(\alpha, \beta)$ be a given function. Then h is non-decreasing in (α, β) if and only if*

$$\int_\alpha^\beta h(s)\rho'(s)ds \le 0 \qquad \forall \rho \in \mathcal{D}(\alpha, \beta), \quad \rho \ge 0. \tag{35}$$

Proof of Lemma 4.2. a) Let h be non-decreasing and let $\rho \in \mathcal{D}(\alpha,\beta)$ be given, $\mathrm{supp}\rho = [\bar\alpha,\bar\beta] \subset (\alpha,\beta)$. For each integer n we define $h_n : [\bar\alpha,\beta)$ as the solution of the equation $\frac{1}{n}h_n' + h_n = h$, $h_n(\bar\alpha) = h(\bar\alpha-)$. Then h_n is absolutely continuous and non-decreasing, $h_n \le h$ in $(\bar\alpha,\beta)$ and $h_n \to h$ in $L^1(\bar\alpha,\bar\beta)$ as $n\to\infty$. Inequality (35) holds for each h_n, $n = 1,2,\dots$ and it suffices to pass to the limit.
b) Let (35) hold, and let $s_1, s_2 \in (\alpha,\beta)$ be Lebesgue points of h, $s_1 < s_2$. By continuity, (35) holds for Lipschitz continuous functions $\rho = \rho_\varepsilon$, where for $\varepsilon > 0$ sufficiently small we put

$$\rho_\varepsilon'(s) = \begin{cases} \frac{1}{2\varepsilon} & \text{for} \quad s \in (s_1-\varepsilon, s_1+\varepsilon), \\ -\frac{1}{2\varepsilon} & \text{for} \quad s \in (s_2-\varepsilon, s_2+\varepsilon), \\ 0 & \text{otherwise.} \end{cases}$$

This yields $\frac{1}{2\varepsilon}\int_{s_1-\varepsilon}^{s_1+\varepsilon} h(s)\,ds \le \frac{1}{2\varepsilon}\int_{s_2-\varepsilon}^{s_2+\varepsilon} h(s)\,ds$, hence $h(s_1) \le h(s_2)$ and Lemma 4.2 is proved. $\square$

Proof of Proposition 4.1. Using the representation (4) with $v = f'$ we rewrite the condition (33) in the form

$$\int_{-\infty}^{\infty}\int_0^T \left[\left(\frac{1}{2}(f-zv)^2 + G(v)\right)\varphi_t(x,t) - ((f-zv)g(v))\,\varphi_x(x,t)\right] dtdx \ge 0$$

$$\forall \varphi \in \mathcal{D}(R^1\times(0,T)), \quad \varphi \ge 0, \quad \text{where} \quad z = \frac{x}{t}.$$

A computation analogous to the proof of Proposition 1.4 gives

$$\int_{-\infty}^{\infty}\left\{\left[\frac{z}{2}(f-zv)^2 + zG(v) + (f-zv)g(v)\right]\eta'(z) + \left(\frac{1}{2}(f-zv)^2 + G(v)\right)\eta(z)\right\} dz \le 0 \tag{36}$$

for every $\eta \in \mathcal{D}(R^1)$, $\eta \ge 0$.
The identities

$$\int_{-\infty}^{\infty} -f(z^2v - g(v))\eta' dz = \int_{-\infty}^{\infty}\left[v(z^2v - g(v)) + 2zfv\right]\eta dz,$$

$$\int_{-\infty}^{\infty}\left(\frac{1}{2}f^2\eta + \frac{z}{2}f^2\eta'\right) dz = -\int_{-\infty}^{\infty} zvf\eta dz$$

enable us to rewrite (36) in the form

$$\int_{-\infty}^{\infty}\left[\left(G(v) - vg(v) + \frac{z^2}{2}v^2\right)(z\eta(z))' + z^2v^2\eta(z)\right] dz \le 0 \tag{37}$$

$$\forall \eta \in \mathcal{D}(R^1), \quad \eta \ge 0.$$

Inequality (37) is equivalent to the system

$$\begin{cases} \int_0^\infty \left[\left(G(v) - vg(v) + \frac{z^2}{2}v^2\right)\xi_+'(z) + zv^2\xi_+(z)\right] dz \le 0, \\ \int_{-\infty}^0 \left[\left(G(v) - vg(v) + \frac{z^2}{2}v^2\right)\xi_-'(z) + zv^2\xi_-(z)\right] dz \ge 0 \end{cases} \tag{38}$$

$$\forall \xi_+ \in \mathcal{D}(0,\infty), \quad \forall \xi_- \in \mathcal{D}(-\infty,0) \quad \xi_+ \geq 0, \quad \xi_- \geq 0.$$

Substituting in (38) $z = \sqrt{s}$ in $(0,\infty)$ and $z = -\sqrt{s}$ in $(-\infty,0)$ for $s \in (0,\infty)$, we infer that system (38) has the form

$$\int_0^\infty \left[E(s,w(s))\, \rho'(s) + \frac{1}{2} w^2(s)\rho(s) \right] ds \leq 0 \tag{39}$$
$$\forall \rho \in \mathcal{D}(0,\infty), \quad \rho \geq 0$$

for each of the functions $w = w_+$, $w = w_-$, and Proposition 4.1 follows from Lemma 4.2. $\square$

Function $D(w)$ introduced in (34) can be interpreted as the total energy density of the system. Notice that the entropy condition (33) requires $D(w)$ to decrease at each space point $x \in R^1$ when t increases (i.e. positive dissipation). It enables us to exclude "non-physical" solutions of the Riemann problem. Nevertheless, in the general case it does not ensure the uniqueness of solutions unless g is globally convex or concave (see Section 6 below). Therefore, we propose a stronger "entropy" condition which covers the general case of Assumption 1.1 (the fact that it is stronger will also be proved in Section 6).
We first observe that solutions $w = w_+$, $w = w_-$ of (23), (24) are independent of each other for every choice of boundary conditions V_-, V_Q, V_+. We therefore formulate our condition separately for w_+ and w_-.

Definition 4.3. *A solution w to (23) with boundary conditions (24) is called* minimal, *if*

$$\int_0^\infty \left(w^2(s) - \hat{w}^2(s) \right) ds \leq 0 \tag{40}$$

for each solution $\hat{w}$ to (23), (24).

Note that integral (40) is meaningful, since by (24), $w = \hat{w}$ in a neighborhood of ∞, so we integrate bounded functions over a finite interval.
Condition (40) can be interpreted as a maximal dissipation principle. Indeed, the total dissipation difference of two solutions w and $\hat{w}$ with the same initial value $w(0) = \hat{w}(0) \in g^{-1}(Q)$ is

$$\Delta(w,\hat{w}) := D(w)(\infty) - D(\hat{w})(\infty). \tag{41}$$

So w dissipates maximally if $\Delta(w,\hat{w})$ is nonnegative for any solution $\hat{w}$. Theorem 5.1 below shows that the minimal solution is unique independently of the concrete choice of the initial condition $w(0) \in g^{-1}(Q)$.

5. Minimal solutions.

The aim of this section is to show that our minimality condition (40) ensures existence and uniqueness to the Riemann problem (5), (11) and to provide an explicit construction of the minimal solution. This is concentrated in the following two theorems.

Theorem 5.1. *For every $V \in (U_-, U_+)$ and $Q \in (G_-, G_+)$ there exists a unique minimal solution w^* of (23) with boundary conditions (24).*

Theorem 5.2. *Let $V_-, V_+ \in (U_-, U_+)$ be given. Then there exists an interval $(A, B) \subset R^1$ such that for every $D \in (A, B)$ there exists a unique $Q \in (G_-, G_+)$ and a unique solution to (5),(11) for $D_+ - D_- = D$ such that each of the functions w_+, w_- defined by formula (22) is a minimal solution of (23) with boundary conditions $g(w_+(0)) = g(w_-(0)) = Q$, $w_+(+\infty) = V_+$, $w_-(+\infty) = V_-$. Moreover, if $G_- = -\infty$, then $A = -\infty$ and if $G_+ = +\infty$, then $B = +\infty$.*

Lemma 3.1 enables us to express the value of the integral $\int_0^\infty (w_1^2(s) - w_2^2(s))ds$ for two monotone solutions w_1, w_2 of (23) in terms of their convex (concave) trajectories g_1^*, g_2^*. We first observe that integrating equation (23) we obtain

$$g(V) - Q = \int_0^\infty (V - w(s))ds \tag{42}$$

for each solution w of (23). If moreover we assume that w is monotone, then w is non-decreasing if $Q < g(V)$, non-increasing if $Q > g(V)$ and constant if $Q = g(V)$. Let now w_1, w_2 be two monotone solutions of (23) for given conditions $V \in (U_-, U_+)$ and $Q \in (G_-, G_+)$. We distinguish two cases.

A. $Q < g(V)$. Then both w_1 and w_2 are non-decreasing.
Assume for instance $w_1(0+) =: V_1 \le V_2 := w_2(0+) < V$, $g(V_1) = g(V_2) = Q$. The convex trajectories g_1^*, g_2^* corresponding to w_1, w_2 are given by a formula analogous to (27) and satisfy $g_i^{*'}(u) = w_i^{-1}(u)$ for a.e. $u \in (V_i, V)$, $i = 1, 2$. Identity (26)(ii) yields

$$\int_0^\infty (w_i^2(s) - V^2)ds + 2\int_{V_i}^V u g_i^{*'}(u)du = 0, \; i = 1, 2,$$

and integrating by parts we obtain

$$\frac{1}{2}\int_0^\infty (w_1^2(s) - w_2^2(s))ds = \int_{V_1}^{V_2} (g_1^*(u) - Q)du + \int_{V_2}^V (g_1^*(u) - g_2^*(u))du. \tag{43}$$

B. $Q > g(V)$. Then both w_1 and w_2 are non-increasing.
Assume $w_1(0+) =: V_1 \ge V_2 := w_2(0+) > V$, $g(V_1) = g(V_2) = Q$. For the corresponding concave trajectories g_1^*, g_2^* we have analogously as above

$$\frac{1}{2}\int_0^\infty (w_1^2(s) - w_2^2(s))ds = \int_{V_2}^{V_1} (Q - g_1^*(u))du + \int_V^{V_2} (g_2^*(u) - g_1^*(u))du. \tag{44}$$

We see that the minimization problem (40) in the class of monotone solutions consists in finding the minimal convex trajectory in the case A and the maximal concave trajectory in the case B. This suggests the following definition (cf. Fig. 3)

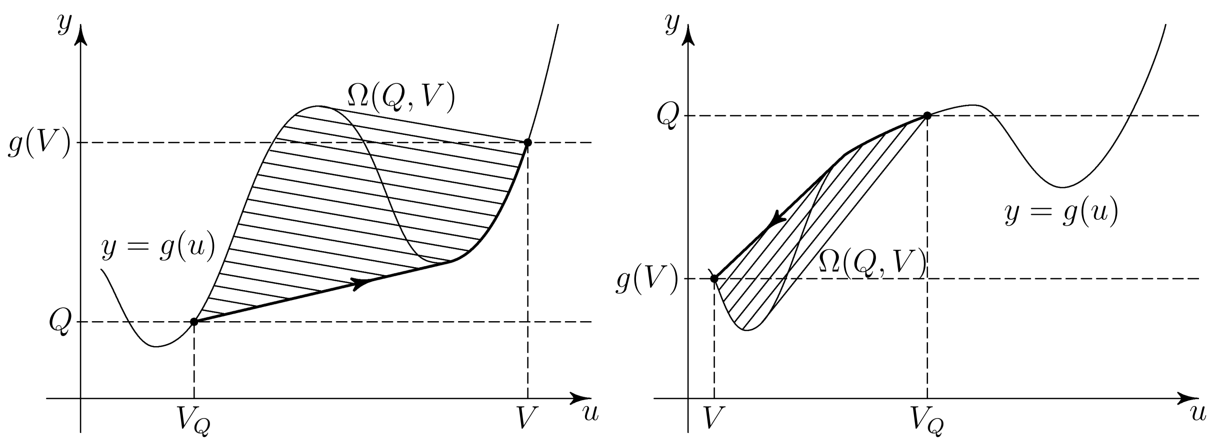
Definition 5.3. *Let* $V \in (U_-, U_+)$ *and* $Q \in (G_-, G_+)$ *be given. Put*

$$V_Q := \begin{cases} \max(g^{-1}(Q) \cap (U_-, V)) & \text{if } \; Q < g(V), \\ \min(g^{-1}(Q) \cap (V, U_+)) & \text{if } \; Q > g(V), \\ V & \text{if } \; Q = g(V), \end{cases}$$

$\Omega(Q,V) := \mathrm{Conv}\,\{(u,y) \in [V_Q, V] \times (G_-, G_+); y = g(u)\}$ *with the convention* $[V_Q, V] = [V, V_Q]$*, where* Conv *denotes the convex hull. Then the function* $g^* : [V_Q, V] \to (G_-, G_+)$ *defined for* $u \in [V_Q, V]$ *by the formula*

$$g^*(u) := \begin{cases} \min\{y \in (G_-, G_+);\, (u,y) \in \Omega(Q,V)\} & \text{if } Q < g(V), \\ \max\{y \in (G_-, G_+);\, (u,y) \in \Omega(Q,V)\} & \text{if } Q > g(V), \\ g(u) & \text{if } Q = g(V), \end{cases} \tag{45}$$

is called the minimal trajectory *from* Q *to* V*.*



We immediately see that the minimal trajectory satisfies the hypotheses of Lemma 3.3. From the identities (43), (44) we easily conclude that the solution w^* of (23),(24) associated to g^* by Lemma 3.3 is minimal with respect to all monotone solutions. We now prove that it is minimal in the sense of Definition 4.3.

Proof of Theorem 5.1
Theorem 5.1 will be proved in the following form.

Proposition 5.4. *Let $V \in (U_-, U_+)$ and $Q \in (G_-, G_+)$ be given and let g^* be the minimal trajectory from Q to V. Let w^* be the solution associated to g^* by Lemma 3.3 in the case $Q \neq g(V)$, $w^* \equiv V$ if $Q = g(V)$. Then for every solution $w \neq w^*$ of (23),(24) we have*

$$\int_0^\infty (w^{*^2}(s) - w^2(s))ds < 0. \tag{46}$$

This fact is less obvious. Its original proof in [KrStr-1993] is relatively complicated. We present here a simple and elegant proof which has been communicated us by V. Lovicar in private discussion. It consists of two steps (Lemmas 5.5 - 5.6).
We first observe that the case $Q = g(V)$ follows trivially from identity (42) which entails for every solution $w \not\equiv V$ of (23),(24)

$$\begin{aligned}\int_0^\infty (w^2(s) - V^2)ds &= \int_0^\infty (w(s) - V)^2 ds + 2V \int_0^\infty (w(s) - V)ds \\ &= \int_0^\infty (w(s) - V)^2 ds > 0.\end{aligned}$$

On the other hand, passing from w^* to $-w^*$ and from $g(v)$ to $-g(-v)$ we see that the cases $Q > g(V)$ and $Q < g(V)$ are equivalent. For the sake of definiteness we assume in the sequel $Q < g(V)$.
Let us suppose now that there exists a solution $w \neq w^*$ of (23),(24). We introduce the functions

$$\begin{cases} W^*(s) := sw^*(s) - g(w^*(s)) \\ W(s) := sw(s) - g(w(s)) \end{cases} \text{for } s > 0 \tag{47}$$

Both W and W^* are Lipschitz, $W' = w$, $W^{*'} = w^*$ a.e., W^* is convex and there exists $L > 0$ such that $W(s) = W^*(s) = sV - g(V)$ for $s \geq L$, $W(0+) = W^*(0+) = -Q$. We define the sets

$$\begin{cases} M_0 := \{s > 0; W^*(s) = W(s)\}, \\ M_+ := \{s > 0; W^*(s) > W(s)\}, \\ M_- := \{s > 0; W^*(s) < W(s)\}. \end{cases} \tag{48}$$

We have $[L, +\infty) \subset M_0$, hence both M_+ and M_- are open bounded sets. They have the form $M_+ = \bigcup_{k=1}^\infty (\alpha_k^+, \beta_k^+)$, $M_- = \bigcup_{k=1}^\infty (\alpha_k^-, \beta_k^-)$, with $\alpha_k^\pm, \beta_k^\pm \in M_0$, provided we include the case $\alpha_k^\pm = 0$.
For almost all $s \in M_0$ we have $w^*(s) = w(s)$, hence

$$\int_{M_0} (w^{*^2}(s) - w^2(s))ds = 0. \tag{49}$$

Lemma 5.5. *For all $k \in \mathbf{N}$ we have*

$$\int_{\alpha_k^+}^{\beta_k^+} (w^{*^2}(s) - w^2(s))ds < 0. \tag{50}$$

Proof. We have $W^*(s) > W(s)$ for all $s \in (\alpha_k^+, \beta_k^+)$, $W^*(\alpha_k^+) = W(\alpha_k^+)$, $W^*(\beta_k^+) = W(\beta_k^+)$, hence

$$\int_{\alpha_k^+}^{\beta_k^+} r(s)(w(s) - w^*(s))ds \geq 0$$

for each bounded non-decreasing function $r : (\alpha_k^+, \beta_k^+) \to R^1$. Indeed, this follows trivially from the integration by parts provided r is smooth. In the general case we approximate r by a pointwise convergent sequence $r_n \to r$ of smooth non-decreasing functions and pass to the limit.
This yields

$$\begin{aligned} 0 < \int_{\alpha_k^+}^{\beta_k^+} (w^*(s) - w(s))^2 ds &= \int_{\alpha_k^+}^{\beta_k^+} (w^2(s) - {w^*}^2(s))ds - 2\int_{\alpha_k^+}^{\beta_k^+} w^*(s)(w(s) - w^*(s))ds \\ &\leq \int_{\alpha_k^+}^{\beta_k^+} (w^2(s) - {w^*}^2(s))ds \end{aligned}$$

and Lemma 5.5 is proved. □

Lemma 5.6. *For all* $k \in \mathbf{N}$ *we have*

$$\int_{\alpha_k^-}^{\beta_k^-} ({w^*}^2(s) - w^2(s))ds < 0. \tag{51}$$

Proof. Lemmas 3.3, 3.2 and inequality (30) yield

$$sw^*(s) - g(w^*(s)) \geq sv - g^*(v) \geq sv - g(v) \tag{52}$$

for all $s > 0$ and $v \in [V_Q, V]$. On the other hand, for $s \in (\alpha_k^-, \beta_k^-)$ we have by hypothesis $sw^*(s) - g(w^*(s)) < sw(s) - g(w(s))$, hence $w(s) \notin [V_Q, V]$ for $s \in (\alpha_k^-, \beta_k^-)$. Put $A_+ := \{s \in (\alpha_k^-, \beta_k^-); w(s) > V\}$, $A_- := \{s \in (\alpha_k^-, \beta_k^-); w(s) < V_Q\}$. We have $(\alpha_k^-, \beta_k^-) = A_+ \cup A_-$ and

$$\begin{aligned} \int_{A_-} (w^2(s) - {w^*}^2(s))ds &> (V_Q + V)\int_{A_-} (w(s) - w^*(s))ds, \\ \int_{A_+} (w^2(s) - {w^*}^2(s))ds &> (V_Q + V)\int_{A_+} (w(s) - w^*(s))ds, \end{aligned}$$

therefore

$$\int_{\alpha_k^-}^{\beta_k^-} (w^2(s) - {w^*}^2(s))ds > (V_Q + V)(W(\beta_k^-) - W^*(\beta_k^-) - W(\alpha_k^-) + W^*(\alpha_k^-)) = 0,$$

and inequality (51) is proved. □

To finish the proof of Proposition 5.4 which in turn implies Theorem 5.1, it suffices to combine Lemmas 5.5, 5.6 and identity (49).

Let us state now two easy lemmas which enable us to prove Theorem 5.2.

Lemma 5.7. *Let $Q_1, Q_2 \in (G_-, G_+)$ and $V \in (U_-, U_+)$ be given such that $Q_1 < Q_2 < g(V)$. According to Definition 5.3 put $V_i := V_{Q_i}$ and $g_i^*(u) := \min\{y \in (G_-, G_+); (u, y) \in \Omega(Q_i, V)\}$ for $u \in [V_i, V], i = 1, 2$. Then $g_1^*(u) \le g_2^*(u)$ for all $u \in [V_2, V]$ and $g_1^{*'}(u) \ge g_2^{*'}(u)$ for a.e. $u \in (V_2, V)$.*

Proof. We obviously have $\Omega(Q_2, V) \subset \Omega(Q_1, V)$ and $V > V_2 > V_1$, hence $g_1^* \le g_2^*$ in $[V_2, V]$. Let us assume now $g_1^{*'}(u) < g_2^{*'}(u)$ for some Lebesgue point $u \in (V_2, V)$ of both $g_1^{*'}$ and $g_2^{*'}$. Then $g_1^*(u) < g_2^*(u) \le g(u)$, hence g_1^* is affine in a neighborhood of u. Put $\bar{u} := \min\{v \in (u, V); g(v) = g_1^*(v)\}$. The points $(\bar{u}, g(\bar{u}))$ and $(u, g_2^*(u))$ belong to $\Omega(Q_2, V)$, hence for all $\alpha \in (0, 1)$ we have

$$g_2^*(\alpha\bar{u} + (1-\alpha)u) \le \alpha g(\bar{u}) + (1-\alpha) g_2^*(u),$$

or equivalently

$$\frac{g_2^*(u + \alpha(\bar{u} - u)) - g_2^*(u)}{\alpha(\bar{u} - u)} \le \frac{g(\bar{u}) - g_2^*(u)}{\bar{u} - u}.$$

Passing to the limit as $\alpha \to 0+$ we obtain

$$g_2^{*'}(u) \le \frac{g(\bar{u}) - g_2^*(u)}{\bar{u} - u} < \frac{g(\bar{u}) - g_1^*(u)}{\bar{u} - u} = g_1^{*'}(u).$$

which is a contradiction. □

Lemma 5.8. *Let $V \in (U_-, U_+)$ and $G_- < Q_1 < Q_2 < G_+$ be given. Let w_1^*, w_2^* be the minimal solutions of (23),(24) for $Q = Q_1, Q = Q_2$, respectively. Then $w_1(s) \le w_2(s)$ for all $s > 0$.*

Proof. The cases $Q_1 \le g(V) < Q_2$ or $Q_1 < g(V) \le Q_2$ are obvious. We may therefore assume $Q_1 < Q_2 < g(V)$ (the opposite situation $g(V) < Q_1 < Q_2$ is again covered by the usual transformation $g(v) \mapsto -g(-v)$). By Lemma 3.3 we have for all $s > 0$

$$w_i^*(s) = \inf\{u \in [V_i, V]; g_i^{*'}(u) \ge s\},\ i = 1, 2,$$

where V_i, g_i^* are as in Lemma 5.7. For all $s > 0$ and $u \in [V_2, V]$ such that $u < w_1^*(s)$ we have by Lemma 5.7 $g_2^{*'}(u+) \le g_1^{*'}(u+) < s$. This entails $u < w_2^*(s)$, hence $w_1^*(s) \le w_2^*(s)$. □

Proof of Theorem 5.2. For an arbitrary $Q \in (G_-, G_+)$ we denote by w_+^Q, w_-^Q the minimal solutions of (23) with boundary conditions $g(w_+^Q(0)) = g(w_-^Q(0)) = Q, w_+^Q(+\infty) = V_+, w_-^Q(+\infty) = V_-$, and put

$$v_Q(z) := \begin{cases} w_+^Q(z^2) & \text{for} \quad z \ge 0, \\ w_-^Q(z^2) & \text{for} \quad z < 0. \end{cases}$$

By Lemma 1.5, $v_Q(z)$ solves (5) together with the condition $v_Q(\pm\infty) = V_\pm$ for all $Q \in (G_-, G_+)$ (cf.(11)). To handle the second condition in (11) we introduce the function

$$\varphi(Q) := \int_{-\infty}^{\infty} (v_Q(z) - P_0(z))dz \tag{53}$$

with the intention to put

$$A := \varphi(G_-), \quad B := \varphi(G_+). \tag{54}$$

The proof of Theorem 5.2 will be complete as soon as we prove that the function φ defined by (53) is continuous and increasing and the implications

$$G_- = -\infty \Rightarrow A = -\infty, \; G_+ = +\infty \Rightarrow B = +\infty \tag{55}$$

hold.
The fact that φ is increasing follows immediately from Lemma 5.8. To prove the continuity, we fix an arbitrary compact interval $[c,d] \subset (G_-, G_+)$ such that $g(V_+), g(V_-) \in (c,d)$. Put $a := \min\{v \in (U_-,U_+); g(v) \geq c\}$, $b := \max\{v \in (U_-,U_+); g(v) \leq d\}$, $L' := \sup\{|\frac{g(u)-g(v)}{u-v}|; a \leq v < u \leq b\} < +\infty$.
From Proposition 1.7 we infer that $v_Q(z) = P_0(z)$ for all $|z| > \sqrt{L'}$ and all $Q \in [c,d]$. Integrating equation (5) $\int_0^{\sqrt{L'}} dz$ and $\int_{-\sqrt{L'}}^0 dz$ we obtain for all $Q \in [c,d]$

$$\left.\begin{aligned} L'V_+ - g(V_+) + Q &= 2\textstyle\int_0^{\sqrt{L'}} z v_Q(z)dz, \\ -L'V_- + g(V_-) - Q &= 2\textstyle\int_{-\sqrt{L'}}^0 z v_Q(z)dz, \end{aligned}\right\} \tag{56}$$

hence

$$\int_{-\sqrt{L'}}^{\sqrt{L'}} |z|(v_{Q_1}(z) - v_{Q_2}(z))dz = \int_{-\infty}^{\infty} |z|(v_{Q_1}(z) - v_{Q_2}(z))dz = Q_1 - Q_2$$

for all $Q_1, Q_2 \in [c,d]$, $Q_1 > Q_2$. Note that by Lemma 5.8 we have $v_{Q_1}(z) \geq v_{Q_2}(z)$ for a.e. $z \in R^1$. Using the estimates

$$\begin{aligned} \varphi(Q_1) - \varphi(Q_2) &= \int_{-\infty}^{\infty} (v_{Q_1}(z) - v_{Q_2}(z))dz \\ &\leq \int_{-\sqrt{Q_1-Q_2}}^{\sqrt{Q_1-Q_2}} (v_{Q_1}(z) - v_{Q_2}(z))dz + \frac{1}{\sqrt{Q_1-Q_2}} \int_{-\infty}^{\infty} |z|(v_{Q_1}(z) - v_{Q_2}(z))dz \\ &\leq (2(b-a)+1)\sqrt{Q_1-Q_2} \end{aligned}$$

we conclude that φ is locally $\frac{1}{2}$-Hölder continuous in (G_-, G_+).
It is now clear that for every $D \in (A,B)$ the Riemann problem (5),(11) with $D_+ - D_- = D$ has a unique solution satisfying the requirements of Theorem 5.2. It remains to prove one of the implications (55), the other is analogous. Assume for instance $G_+ = +\infty$, $V_+ \geq V_-$, and put

$$L := \sup\{\frac{g(v) - g(V_+)}{v - V_+}; \, v \in (V_+, U_+)\} > 0.$$

We distinguish two cases.
A. $L < +\infty$. Then for every $Q > g(V_+)$ the slope of the minimal trajectory (45) from Q to V_+ does not exceed the value of L, and therefore $v_Q(z) = V_+$ for $z > \sqrt{L}$. Using formula (56) for $L' = L$ we obtain

$$\varphi(Q) \geq \int_0^{\sqrt{L}} (v_Q(z) - V_+)dz \geq \frac{1}{\sqrt{L}} \int_0^{\sqrt{L}} z(v_Q(z) - V_+)dz \geq \frac{1}{2\sqrt{L}}(Q - g(V_+)). \tag{57}$$

B. $L = +\infty$. Put $\hat{L} := \limsup_{v \to V_+} \frac{g(v) - g(V_+)}{v - V_+} < +\infty$. For $\lambda > \hat{L}$ we define

$$V_\lambda := \min\{v \in (V_+, U_+); \frac{g(v) - g(V_+)}{v - V_+} = \lambda\}, \; Q_\lambda := g(V_\lambda).$$

The minimal trajectory g^* from Q_λ to V_+ is then affine, namely $g^*(u) = g(V_+) + \lambda(u - V_+)$ for $u \in [V_+, V_\lambda]$. This yields

$$v_{Q_\lambda}(z) = \begin{cases} V_+ & \text{for} \quad z > \sqrt{\lambda}. \\ V_\lambda & \text{for} \quad z \in (0, \sqrt{\lambda}), \\ w_-^{Q_\lambda}(z^2) \geq V_- & \text{for} \quad z < 0, \end{cases}$$

therefore

$$\varphi(Q_\lambda) \geq \int_0^{\sqrt{\lambda}} (v_{Q_\lambda}(z) - V_+)dz = \sqrt{(Q_\lambda - g(V_+))(V_\lambda - V_+)}. \tag{58}$$

In both cases (57), (58) we obtain $\varphi(Q) \to +\infty$ as $Q \to +\infty$. Theorem 5.2 is proved. □

Remark 5.9. We observe an interesting phenomenon if the condition $\lim_{u\to\pm\infty} \varphi(u) = \pm\infty$ is not satisfied. For $g(u) = e^u - 1$, $V_+ = V_- = 0$ we have for instance $A = -4$, $B = +\infty$, so for $\bar{V} < -4$ no minimal solution exists. Since g is convex, we see from Theorem 6.2 below, Proposition 1.7 and Theorem 2.2 that this problem admits only solutions which violate the entropy condition (34)!

6. Entropy condition.

We first show that the entropy condition (34) for solutions of (23), (24) is a consequence of the minimality condition.

Proposition 6.1. *For every $V \in (U_-, U_+)$ and $Q \in (G_-, G_+)$ the minimal solution w^* of (23),(24) fulfils the dissipation condition (34).*

Proof. By Lemmas 3.1, 3.3 we have for all $s > 0$, (cf. Definition 5.3)

$$\begin{aligned} \frac{1}{2}\int_0^s {w^*}^2(\sigma)d\sigma &= \frac{1}{2}sw^*(s) - \int_{V_Q}^{w^*(s)} u{w^*}^{-1}(u)du \\ &= \frac{1}{2}sw^*(s) - w^*(s)g^*(w^*(s)) + V_Q g^*(V_Q) + \int_{V_Q}^{w^*(s)} g^*(u)du. \end{aligned}$$

The function $D(w^*)$ in (34) has therefore the form

$$D(w^*)(s) = \int_{V_Q}^{w^*(s)} (g(u) - g^*(u))du + G(V_Q) - QV_Q.$$

For $Q < g(V)$ we have $g(u) \ge g^*(u)$ for all $u \in [V_Q, V]$ and w^* is non-decreasing, for $Q > g(V)$ we have $g(u) \le g^*(u)$ for all $u \in [V, V_Q]$ and w^* is non-increasing, for $Q = g(V)$ the solution w^* is constant, hence in all cases condition (34) holds. □

Theorem 6.2. *Let g be convex and let $V \in (U_-, U_+), Q \in (G_-, G_+)$ be given. Let w be a solution of (23),(24) satisfying the dissipation condition (34) and let w^* be the minimal solution of (23),(24). Then $w = w^*$ a.e.*

In the proof we make use of an auxiliary lemma. Notice that a convex function satisfying Assumption 1.1 is increasing, hence every solution w of (23),(24) can be continuously extended to $s = 0$.

Lemma 6.3. *Let the hypotheses of Theorem 6.2 hold. Assume that there exist Lebesgue points s_1, s_2 of w such that $0 \le s_1 < s_2$ and $w(s_1) =: v_1 < v_2 := w(s_2)$. Then $s_2 \ge g'(v_2-)$, $s_1 \le g'(v_1+)$, $w(s) = \inf\{u \in [v_1, v_2]; g'(u) \ge s\}$ for a.e. $s \in [s_1, g'(v_2-))$, $w(s) = v_2$ for $s \in (g'(v_2-), s_2]$.*

Proof of Lemma 6.3. The function $w_0 : [0, \infty) \to (U_-, U_+)$ defined as $w_0(s) := w(s)$ for $s \in (s_1, s_2), w_0(s) := v_1$ for $s \in [0, s_1], w_0(s) := v_2$ for $s \in [s_2, +\infty)$ solves (23),(24) with $V = v_2, Q = g(v_1)$. The minimal convex trajectory g_0^* from $g(v_1)$ to v_2 coincides with g and the corresponding minimal solution w_0^* is given by the formula (cf. Lemma 3.3) $w_0^*(s) = \inf\{u \in [v_1, v_2], g'(u) \ge s\}$ for $s \in [0, g'(v_2-))$, $w_0^*(s) = v_2$ for $s > g'(v_2-)$, ${w_0^*}^{-1}(u) = g'(u)$ for a.e. $u \in (v_1, v_2)$. By (34) we have $D(w)(s_2) \ge D(w)(s_1)$, hence

$$\frac{1}{2}\int_{s_1}^{s_2} w^2(s)ds \le -\int_{v_1}^{v_2} ug'(u)du + \frac{1}{2}s_2v_2^2 - \frac{1}{2}s_1v_1^2.$$

Lemma 3.1 yields $-\int_{v_1}^{v_2} ug'(u)du = \frac{1}{2}\int_0^\infty ({w_0^*}^2(s) - v_2^2)ds$, therefore $\frac{1}{2}\int_0^\infty (w_0^2(s) - {w_0^*}^2(s))ds \le 0$.
By Proposition 5.4, the last inequality implies $w_0 = w_0^*$ a.e. and Lemma 6.3 follows. □

Proof of Theorem 6.2. The assertion is an immediate consequence of Lemma 6.3 if $Q < g(V)$ (we simply put $s_1 = 0$ and let s_2 tend to $+\infty$). The case $Q \ge g(V)$ is slightly more complicated. In fact, it suffices to prove that w is non-increasing in $[0, +\infty)$, since the only concave trajectory from Q to V in this case is the minimal one which is affine.
Let us suppose on the contrary that there exist Lebesgue points s_1, s_2 of w such that $0 \le s_1 < s_2$ and $w(s_1) := v_1 < v_2 := w(s_2)$. We distinguish 2 cases.
A. $g(v_1) < Q$. Put $\bar{v} := \sup\operatorname{ess}\{w(s); s \in [0, s_1]\}$. Then $\bar{v} \ge g^{-1}(Q) > v_1$ and there exists a sequence $\{\sigma_n\} \subset [0, s_1]$ of Lebesgue points of w such that $\sigma_n \to$

$\bar{s} < s_1, w(\sigma_n) \to \bar{v}$. Passing to the limit as $n \to \infty$ in the identity $\int_{\sigma_n}^{s_1} w(s)ds = s_1v_1 - g(v_1) - \sigma_n w(\sigma_n) + g(w(\sigma_n))$ we obtain $0 > \int_{\bar{s}}^{s_1}(w(s) - \bar{v})ds = g(\bar{v}) - g(v_1) - s_1(\bar{v} - v_1)$, hence $s_1 > \frac{g(\bar{v})-g(v_1)}{\bar{v}-v_1} \geq g'(v_1+)$, which is in contradiction with Lemma 6.3.
B. $g(v_1) \geq Q$. Then $v_2 > V$ and analogously as above we put $\underline{v} := \inf \operatorname{ess}\{w(s); s \in [s_2, +\infty)\}$. We have $\underline{v} \leq V$ and $w(s) = V$ for sufficiently large, therefore there exists a convergent sequence $\{\sigma_n\} \subset [s_2, +\infty)$ of Lebesgue points of w such that $\sigma_n \to \underline{s} > s_2, w(\sigma_n) \to \underline{v}$ as $n \to \infty$. Passing to the limit in the identity $\int_{s_2}^{\sigma_n} w(s)ds = \sigma_n w(\sigma_n) - s_2v_2 - g(w(\sigma_n)) + g(v_2)$ yields $0 < \int_{s_2}^{\underline{s}}(w(s) - \underline{v})ds = g(v_2) - g(\underline{v}) - s_2(v_2 - \underline{v})$ hence $s_2 < \frac{g(v_2)-g(\underline{v})}{v_2-\underline{v}} \leq g'(v_2-)$ which again contradicts Lemma 6.3. $\square$

We now present another negative result showing that the dissipation condition (34) does not guarantee the uniqueness of solutions of (5),(11) even in the "regular" case when g is increasing and smooth.

Proposition 6.4. *Let $g : (U_-, U_+) \to (G_-, G_+)$ be an increasing smooth function which has an inflection point $q_0 \in (U_-, U_+)$. Then there exist $V_+, V_- \in (U_-, U_+)$, $D_+, D_- \in R^1$ such that problem (5),(11) has infinitely many distinct solutions.*

Proof. We choose an interval $(q_0 - k_1, q_0 + k_2) \subset (U_-, U_+)$ such that one of the situations

(i) $g'' > 0$ in $(q_0 - k_1, q_0)$, $g'' < 0$ in $(q_0, q_0 + k_2)$,
(ii) $g'' < 0$ in $(q_0 - k_1, q_0)$, $g'' > 0$ in $(q_0, q_0 + k_2)$,

occurs. The construction will be different in each case (see Fig. 4)

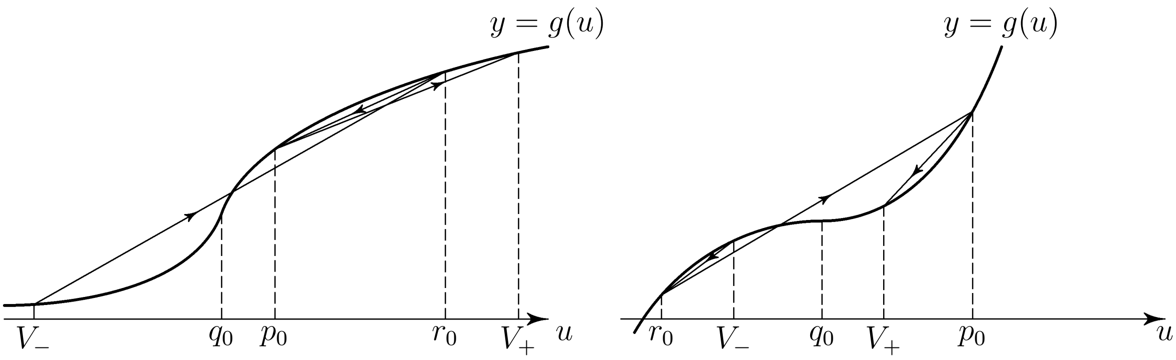


Fig. 4

(i) We fix some numbers $q_0 - k_1 < V_- < q_0 < p_0 < r_0 < V_+ < q_0 + k_2$ such that

$$\int_{V_-}^{r_0} g(v)dv < \frac{1}{2}(r_0 - V_-)(g(r_0) + g(V_-)), \tag{59}$$

$$\frac{g(r_0)-g(p_0)}{r_0-p_0} < \frac{g(r_0)-g(V_-)}{r_0-V_-} \tag{60}$$

and we define $v(z)$ by the formula

$$v(z) := \begin{cases} V_- & \text{for} \quad z < z_1, \\ r & \text{for} \quad z \in \ (z_1, z_2), \\ p & \text{for} \quad z \in \ (z_2, z_3), \\ V_+ & \text{for} \quad z > z_3, \end{cases} \tag{61}$$

$$z_1 := -\sqrt{\frac{g(r)-g(V_-)}{r-V_-}}, z_2 := -\sqrt{\frac{g(r)-g(p)}{r-p}}, z_3 := \sqrt{\frac{g(V_+)-g(p)}{V_+-p}} \tag{62}$$

for each (r,p) in a small neighborhood of (r_0,p_0) such that (59), (60) hold for (r,p). We put $D_- := 0$, $D_+ := h(p_0,r_0)$, and

$$h(p,r) := \sqrt{(g(r)-g(V_-))(r-V_-)} - \sqrt{(g(r)-g(p))(r-p)} - \sqrt{(g(V_+)-g(p))(V_+-p)}.$$

By Proposition 2.1, v is a solution of (5),(11) if and only if $h(p,r) = h(p_0,r_0)$. We obviously have $\frac{\partial}{\partial p}h(p,r) > 0$ and by the Implicit Function Theorem there exists a function $p(r)$ defined in a neighborhood of r_0 such that $h(p(r),r) = h(p_0,r_0)$ which determines a one-parametric family of solutions of (5),(11) satisfying condition (34).

(ii) Similarly as above, we fix some numbers $q_0-k_1 < r_0 < V_- < q_0 < V_+ < p_0 < q_0+k_2$ such that the inequalities (60) and

$$\int_{r_0}^{p_0} g(v)dv < \frac{1}{2}(p_0-r_0)(g(p_0)+g(r_0)) \tag{63}$$

hold. We put here $D_- := h(p_0,r_0), D_+ := 0$. We easily check that the argument of (i) remains valid for the function v defined by (61), (62). $\square$

7. Vanishing viscosity.

We show here that the maximal dissipation selection rule described above gives the same result as the *vanishing viscosity method* (see e.g. [Liu-1976]) which consists in considering the equation

$$u_{tt} - g(u_x)_x - \frac{\nu}{2}tu_{xxt} = 0 \quad \text{for } \nu > 0 \tag{64}$$

with the intention to pass to the limit as $\nu \to 0+$ as in [Daf - 1973 A]. It is also convenient to regularize the function g in (64) by means of the parameter ν. So we replace (64) by the equation

$$u^\nu_{tt} - g_{\nu,K}(u^\nu_x)_x - \frac{\nu}{2}tu^\nu_{xxt} = 0, \tag{65}$$

where $g_{\nu,K} \in C^1((U_-,U_+))$ is a regularization of the function g that we briefly describe here.
Let g satisfy Assumption 1.1. For a fixed compact set $K \subset (U_-,U_+)$ and a number $\nu > 0$ put

$$g_{\nu,K}(u) := e^{\frac{1}{\nu}(u-u_K)}g(u_K) + \int_{u_K}^{u} \frac{1}{\nu} e^{\frac{1}{\nu}(v-u)}g(v)dv \quad \text{for } u \in (U_-,U_+), \tag{66}$$

where $u_K := \min K$. The identity

$$\nu g'_{\nu,K}(u) = g(u) - g_{\nu,K}(u) \quad \forall u \in (U_-,U_+) \tag{67}$$

has the following immediate consequences (the proof is left to the reader).

Lemma 7.1. *Let $K \subset (U_-,U_+)$ be a compact set and let L_K be given by (1). Then for every $\nu > 0$ the function $g_{\nu,K}$ is continuously differentiable in (U_-,U_+) and for every $u \in K$ we have*

(i) $|g(u) - g_{\nu,K}(u)| \le \nu L_K$,
(ii) $|g'_{\nu,K}(u)| \le L_K$.

In terms of self-similar solutions, approximating equation (2) by (65) corresponds to the approximation of problem (23),(24) by the equation

$$\nu(sw'_\nu(s))' = w_\nu(s) - (sw_\nu(s) - g_{\nu,K}(w_\nu(s)))' \tag{68}$$

for a suitable choice of boundary conditions and of the compact set K. This can be done in the following way.

Let Q, V be given data in (24) and let us define V_Q as in Definition 5.3. We can assume for the sake of definiteness that $Q < g(V)$ leaving the other cases to the reader. We fix an open bounded interval $J \supset [V_Q, V]$, $\bar{J} \subset (U_-,U_+)$ and put $K := \bar{J}$. For an arbitrary $\beta > L_K$ we prescribe boundary conditions

$$w_\nu(\nu) = V_Q, \quad w_\nu(\beta) = V. \tag{69}$$

We first verify that problem (68), (69) cannot have multiple solutions.

Lemma 7.2. *Let $0 < \delta < s_1 < s_2$ be given and let $w, \tilde{w}$ be two solutions of (68) in the interval $(s_1 - \delta, s_2 + \delta)$. Assume $w(s_i) = \tilde{w}(s_i)$ for $i = 1, 2$. Then $w(s) = \tilde{w}(s)$ for all $s \in (s_1 - \delta, s_2 + \delta)$.*

Proof. If the set $B := \{s \in [s_1, s_2]; w(s) = \tilde{w}(s)\}$ is infinite, then it contains a convergent sequence and its limit point $\bar{s}$ satisfies $w(\bar{s}) = \tilde{w}(\bar{s}), w'(\bar{s}) = \tilde{w}'(\bar{s})$. The general theory of ordinary differential equations then yields $w \equiv \tilde{w}$.
Assume that B is finite. We choose two consecutive points $\sigma_1, \sigma_2 \in B$, so that for instance $w(\sigma_i) = \tilde{w}(\sigma_i)$ for $i = 1, 2$, $w(s) > \tilde{w}(s)$ for $s \in (\sigma_1, \sigma_2)$. Integrating $\int_{\sigma_1}^{\sigma_2} ds$ the identity

$$(\nu s(w' - \tilde{w}'))' = (w - \tilde{w}) - (s(w - \tilde{w}) - g(w) + g(\tilde{w}))'$$

we obtain

$$\nu[\sigma_2(w'(\sigma_2)-\tilde{w}'(\sigma_2))-\sigma_1(w'(\sigma_1)-\tilde{w}'(\sigma_1))]=\int_{\sigma_1}^{\sigma_2}(w-\tilde{w})ds>0,$$

hence either $w'(\sigma_2)>\tilde{w}'(\sigma_2)$ or $w'(\sigma_1)<\tilde{w}'(\sigma_1)$, which is a contradiction. □

For a fixed $\nu>0$ we have the following existence result.

Theorem 7.3. *Problem (68), (69) has a unique classical solution* w_ν*. Moreover, there exists* $\nu_0>0$ *such that for* $\nu<\nu_0$ *the solution* w_ν *can be extended to an interval* $(\alpha_\nu,+\infty)$ *for some* $\alpha_\nu\in(0,\nu)$*, it is twice continuously differentiable and increasing in its domain of definition.*

Proof. We define recursively for $s\in[\nu,\beta]$ a sequence $\{w^{(n)}(s);n\in N\cup\{0\}\}$ by the formula

$$\begin{aligned} w^{(0)}(s) &:= V_Q+(V-V_Q)\frac{s-\nu}{\beta-\nu},\\ w^{(n)}(s) &:= V_Q+c^{(n-1)}\int_\nu^s\frac{1}{\tau}e^{\frac{1}{\nu}\int_\nu^\tau\left(\frac{g'_{\nu,K}(w^{(n-1)}(\sigma))}{\sigma}-1\right)d\sigma}d\tau, \end{aligned}$$

where

$$c^{(n-1)}:=(V-V_Q)\Big[\int_\nu^\beta\frac{1}{\tau}e^{\frac{1}{\nu}\int_\nu^\tau\left(\frac{g'_{\nu,K}(w^{(n-1)}(\sigma))}{\sigma}-1\right)d\sigma}d\tau\Big]^{-1}.$$

We immediately see that $\{w^{(n)}\}\subset C^2([\nu,\beta])$ is a sequence of increasing functions satisfying boundary conditions (69) and that there exists a constant M_ν independent of n such that $0<c^{(n)}\le M_\nu, |w^{(n)'}(s)|\le M_\nu$ for all $s\in(\nu,\beta)$.
From the Arzelà-Ascoli theorem it follows that there exist convergent subsequences of $\{c^{(n)}\}$ and $\{w^{(n)}\}$ such that the limits $c_\nu:=\lim_{n\to\infty}c^{(n)}, w_\nu:=\lim_{n\to\infty}w^{(n)}$ satisfy

$$w_\nu(s)=V_Q+c_\nu\int_\nu^s\frac{1}{\tau}e^{\frac{1}{\nu}\int_\nu^\tau\left(\frac{g'_{\nu,K}(w_\nu(\sigma))}{\sigma}-1\right)d\sigma}d\tau, \tag{70}$$

hence w_ν is a solution of (68), (69).
The function w_ν can be extended to a maximal solution of (68) $w_\nu:(\alpha_\nu,\beta_\nu)\to(U_-,U_+)$ for some $\alpha_\nu<\nu$, $\beta_\nu>\beta$. Identity (70) remains valid for $s\in(\alpha_\nu,\beta_\nu)$, hence w_ν is twice continuously differentiable and increasing in its maximal domain of definition. Lemma 7.2 then entails that this solution is unique.
It remains to prove that $\beta_\nu=+\infty$ for ν sufficiently small. Put

$$\begin{aligned} \gamma_\nu &:= \sup\{s\in(\alpha_\nu,\beta_\nu);w_\nu(s)\in K\},\\ \delta &:= \frac{1}{4}(\beta-L_K). \end{aligned}$$

We have $\gamma_\nu > \beta$ and the identity

$$(sw_\nu'(s))' = \frac{1}{\nu}(\frac{g_{\nu,K}'(w_\nu(s))}{s} - 1)(sw_\nu'(s)) \tag{71}$$

combined with Lemma 7.1 (ii) entails for $s \in (L_K + \delta, \gamma_\nu)$

$$(sw_\nu'(s))' \le -\frac{\delta}{\nu(L_K + \delta)} sw_\nu'(s). \tag{72}$$

Integrating the last equation we obtain

$$e^{\frac{p}{\nu}s} sw_\nu'(s) \le e^{\frac{p}{\nu}t} tw_\nu'(t) \quad \text{for} \quad L_K + \delta < t < s < \gamma_\nu,$$

where we denote $p := \frac{\delta}{L_K+\delta} > 0$.
We now integrate $\int_{L_K+\delta}^{L_K+2\delta} dt$ the inequality $e^{\frac{p}{\nu}s} w_\nu'(s) \le e^{\frac{p}{\nu}t} w_\nu'(t)$ and for $s \in (L_K + 2\delta, \gamma_\nu)$ this yields

$$\delta e^{\frac{p}{\nu}s} w_\nu'(s) \le e^{\frac{p}{\nu}(L_k+2\delta)}(V - V_Q),$$

hence

$$w_\nu(s) \le w_\nu(L_K + 3\delta) + \frac{\nu(V - V_Q)}{\delta p} e^{-\frac{\delta p}{\nu}} \quad \text{for} \quad s \in (L_K + 3\delta, \gamma_\nu). \tag{73}$$

For $\nu > 0$ sufficiently small, say $\nu < \nu_0$, we thus have $w_\nu(s) \in K$ for all $s \in (\alpha_\nu, \beta_\nu)$, hence $\beta_\nu = +\infty$. This completes the proof of Theorem 7.3. $\square$

We now pass to the limit as $\nu \to 0+$. The following Theorem states that the solution obtained by the vanishing viscosity selection rule coincides with the minimal solution defined in Sect. 5.

Theorem 7.4. *Let $Q \in (G_-, G_+)$ and $V \in (U_-, U_+)$ be given and let w_ν be the solution of (68), (69) for $\nu \in (0, \nu_0)$. Let w^* be the minimal solution of (23),(24). Then $w_\nu(s) \to w^*(s)$ as $\nu \to 0+$ for all $s > 0$.*

Proof. For $\nu < \nu_0$ we define auxiliary functions

$$\hat{w}_\nu(s) := \begin{cases} w_\nu(s), & s \in [\nu, +\infty), \\ V_Q, & s \in [0, \nu). \end{cases} \tag{74}$$

It suffices to assume $Q < g(V)$ ($Q > g(V)$ is analogous and $Q = g(V)$ is trivial). By (73), the system $\{w_\nu; \nu < \nu_0\}$ converges uniformly to the constant V on $[\beta - \delta, +\infty)$ as $\nu \to 0+$. On $[0, \beta], \{\hat{w}_\nu; \nu > 0\}$ is an equibounded system of continuous non-decreasing functions, and from Helly's Selection Principle ([Kol]) we deduce the existence of a non-decreasing function $\bar{w} : [0, \beta] \to [V_Q, V]$ and of a sequence $\nu_k \to 0+$ as $k \to \infty$ such that

$$\hat{w}_{\nu_k}(s) \to \bar{w}(s) \quad \forall s \in [0, \beta] \quad \text{as} \quad k \to \infty. \tag{75}$$

Let $\varphi \in \mathcal{D}(0, \infty)$ be arbitrarily chosen. For k sufficiently large we have

$$\begin{aligned}\int_0^\infty [(s\hat{w}_{\nu_k}(s) &- g_{\nu_k,K}(\hat{w}_{\nu_k}(s)))\varphi'(s) + \hat{w}_{\nu_k}(s)\varphi(s)]ds = \\ &= \nu_k \int_0^\infty \hat{w}_{\nu_k}(s)(\varphi'(s) + s\varphi''(s))ds\end{aligned}$$

and passing to the limit as $k \to \infty$ we obtain

$$\int_0^\infty [(s\bar{w}(s) - g(\bar{w}(s)))\varphi'(s) + \bar{w}(s)\varphi(s)]ds = 0.$$

Consequently, $\bar{w}$ is a non-decreasing solution of (23),(24) with $\bar{w}(s) = V$ for $s \geq \beta$ and $\bar{w}(0+) = \bar{V} \in [V_Q, V]$.
For each $\nu > 0$ and $s > \nu$ we have

$$\nu^2 w'_\nu(\nu) = \nu s w'_\nu(s) + s w_\nu(s) - \nu V_Q - g_{\nu,K}(w_\nu(s)) + g_{\nu,K}(V_Q) - \int_\nu^s w_\nu(\sigma)d\sigma \qquad (76)$$

and integrating the last identity $\int_\nu^\beta ds$ we obtain

$$\begin{aligned}(\beta - \nu)\nu^2 w'_\nu(\nu) &= \nu[\beta V - \nu V_Q - \int_\nu^\beta w_\nu(s)ds] + \\ &+ \int_0^\beta (s\hat{w}_\nu(s) - g_{\nu,K}(\hat{w}_\nu(s)) + g_{\nu,K}(V_Q) - \int_0^s \hat{w}_\nu(\sigma)d\sigma)ds.\end{aligned}$$

For $\nu = \nu_k$ we pass to the limit as $k \to \infty$. This yields

$$\begin{aligned}\beta \lim_{k\to\infty} \nu_k^2 w'_{\nu_k}(\nu_k) &= \textstyle\int_0^\beta (s\bar{w}(s) - g(\bar{w}(s)) + g(V_Q) - \int_0^s \bar{w}(\sigma)d\sigma)ds \\ &= \textstyle\int_0^\beta g(V_Q) - g(\bar{V})ds = \beta(g(V_Q) - g(\bar{V})) \leq 0.\end{aligned}$$

We conclude

$$\bar{V} = V_Q, \quad \lim_{k\to\infty} \nu_k^2 w'_{\nu_k}(\nu_k) = 0. \qquad (77)$$

According to Lemma 3.2, we define the convex trajectory g^* of the solution $\bar{w}$ by the formula

$$g^*(u) := Q + \int_{V_Q}^u \bar{w}^{-1}(v)dv$$

analogous to (27). We are done if we prove

$$g(u) \geq g^*(u) \quad \forall u \in [V_Q, V]. \qquad (78)$$

Indeed, then g^* is the minimal trajectory from Q to V and by Proposition 5.4, $\bar{w}$ is the minimal solution of (23),(24). The limit function $\bar{w}$ is then independent of the choice of the sequence $\{\nu_k\}$, so the assertion of Theorem 7.4 holds.
To prove (78), we choose an arbitrary $u \in (V_Q, V)$ and find $s > 0$ such that $u \in [\bar{w}(s-), \bar{w}(s+)]$. Following Lemma 3.2 we have $g^*(\bar{w}(s\pm)) = g(\bar{w}(s\pm))$, hence it remains to consider the case.

$$\bar{w}(s-) < u < \bar{w}(s+). \qquad (79)$$

Let $\{s_k\}$ be the sequence such that $w_{\nu_k}(s_k) = u$ for all $k \in \mathbf{N}$ and let us assume that a subsequence (denoted again by s_k) converges to some $\bar{s} \neq s$. For $\bar{s} > s$ and $\sigma \in (s, \bar{s})$ we have $\bar{w}(s+) \leq \bar{w}(\sigma) = \lim_{k\to\infty} w_{\nu_k}(\sigma) \leq u$, which is a contradiction. The case $\bar{s} < s$ is analogous, so $s_k \to s$ as $k \to \infty$.
Put $\Delta = g(u) - g^*(u)$. Lemma 3.1 entails

$$\begin{aligned}
\Delta &= g(u) - g(V_0) - su + \int_0^{s_k} \bar{w}(\sigma)d\sigma \\
&= g_{\nu_k,K}(w_{\nu_k}(s_k)) - g_{\nu_k,K}(V_0) - s_k w_{\nu_k}(s_k) + \nu_k V_0 + \\
&+ \int_{\nu_k}^{s_k} w_{\nu_k}(\sigma)d\sigma + I_k, \qquad \text{where}
\end{aligned}$$

$$\begin{aligned}
I_k \quad := &\ (g(u) - g_{\nu_k,K}(u)) - (g(V_0) - g_{\nu_k,K}(V_0)) + (s_k - s)u + \\
&+ \textstyle\int_0^s (\bar{w}(\sigma) - \hat{w}_{\nu_k}(\sigma))d\sigma + \int_s^{s_k} w_{\nu_k}(\sigma)d\sigma.
\end{aligned}$$

We have $\lim_{k\to\infty} I_k = 0$ and identity (76) yields $\Delta = \nu_k s_k w'_{\nu_k}(s_k) - \nu_k^2 w'_{\nu_k}(\nu_k) + I_k$.
From (77) we conclude

$$\Delta = \lim_{k\to\infty} \nu_k s_k w'_{\nu_k}(s_k) \geq 0$$

which is nothing but inequality (78). Theorem 7.4 is proved. □

8. Other admissibility conditions.

In this section we compare our minimality criterion with other entropy conditions, namely with those of Lax [Lax - 1957], Liu [Liu - 1981] and Dafermos [Daf - 1973]. We consider only solutions to equation (5) which belong to the space $\mathcal{R}$ of regulated functions (cf. [Aum]), i.e. functions $v : R^1 \to R^1$ such that for every $z \in R^1$ there exist both limits $v(z-)$, $v(z+)$ and the limits $V_\pm := \lim_{z\to\pm\infty} v(z)$ exist and are finite.

A. Lax' condition. [Lax -1957] The following definition is the classical Lax' shock condition for systems of conservation laws adapted to our special situation.

Definition 8.1. *Let us assume $g' \in \mathcal{R}$. A solution $v \in \mathcal{R}$ to (5) is said to satisfy* Lax' entropy condition *at a point $z \in R^1$, if one of the following situations occurs:*

(i) $v(z-) = v(z+)$,
(ii) $v(z-) < v(z+)$, $\quad zg'(v(z-)+) \geq z^3 \geq zg'(v(z+)-)$,
(iii) $v(z-) > v(z+)$, $\quad zg'(v(z-)-) \geq z^3 \geq zg'(v(z+)+)$.

The fact that the minimal solution defined in Lemma 3.3 (i) follows the minimal convex (maximal concave) trajectory along g implies immediately the following result:

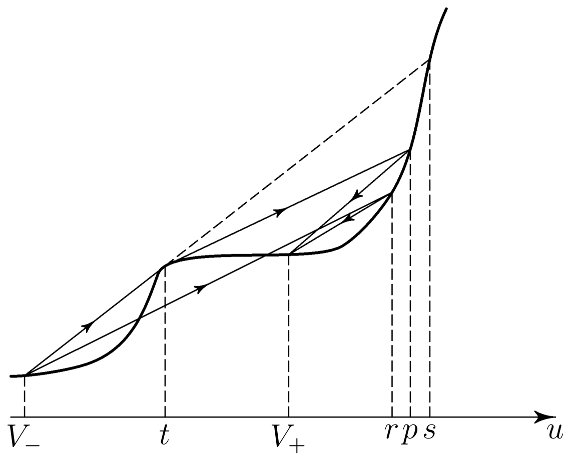
Proposition 8.2. *Let v be the minimal solution to the Riemann problem (5), (11) and assume $g' \in \mathcal{R}$. Then v satisfies Lax' entropy condition at each point $z \in R^1$.*

Example 8.3 below shows that in general, the converse is not true.

Example 8.3. Let $g : (U_-, U_+) \to (G_-, G_+)$ be an increasing smooth function and let there exist numbers $a < V_- < q < V_+ < s < b$ such that

(i) $g'' > 0$ in $(V_-, q) \cup (V_+, s), g'' < 0$ in (q, V_+),
(ii) there exists $t \in (q, V_+)$ such that $\frac{g(t)-g(V_-)}{t-V_-} = \frac{g(s)-g(V_-)}{s-V_-} = \max\{\frac{g(u)-g(V_-)}{u-V_-}; u \in (V_-, s]\}$ (see Fig. 5)

Fig. 5



We fix some $r \in (V_+, s)$ such that $\int_{V_-}^{r} g(u)du < \frac{1}{2}(r - V_-)(g(r) + g(V_-))$ and put

$$v(z) := \begin{cases} V_- & \text{for} \quad z < -\sqrt{\frac{g(r)-g(V_-)}{r-V_-}}, \\ r & \text{for} \quad z \in \Big(-\sqrt{\frac{g(r)-g(V_-)}{r-V_-}}, \sqrt{\frac{g(r)-g(V_+)}{r-V_+}} \Big), \\ V_+ & \text{for} \quad z > \sqrt{\frac{g(r)-g(V_+)}{r-V_+}}. \end{cases} \tag{80}$$

Then v is a solution of (5),(11) with

$$D_+ - D_- = \sqrt{(g(r) - g(V_-))(r - V_-)} + \sqrt{(g(r) - g(V_+))(r - V_+)}. \tag{81}$$

For $p \in [r, s)$ we further define

$$v_p(z) := \begin{cases} w_p^*(z^2) & \text{for} \quad z < 0, \\ p & \text{for} \quad z \in \Big[0, \sqrt{\frac{g(p)-g(V_+)}{p-V_+}}\Big), \\ V_+ & \text{for} \quad z > \sqrt{\frac{g(p)-g(V_+)}{p-V_+}}, \end{cases} \tag{82}$$

where w_p^* is the minimal solution of (23),(24) with $V = V_-, Q = g(p)$ and we check that the value of p can be chosen in such a way that v_p satisfies (5),(11) with $D_+ - D_-$ given by (81). Using Lemmas 3.1, 3.3 we obtain

$$\int_{-\infty}^{\infty} (v_p(z) - P_0(z))dz \; = \int_{-\infty}^{\infty} (w_p^*(z^2) - V_-)dz + \sqrt{(g(p) - g(V_+))(p - V_+)} =$$
$$= \int_{V_-}^{p} \sqrt{g_p^{*\prime}(u)}du + \sqrt{(g(p) - g(V_+))(p - V_+)},$$

where g_p^* is the minimal (concave) trajectory from $g(p)$ to V_-. Put

$$h(p) := \int_{V_-}^{p} \sqrt{g_p^{*\prime}(u)}du + \sqrt{(g(p) - g(V_+))(p - V_+)} -$$
$$-\sqrt{(g(r) - g(V_-))(r - V_-)} - \sqrt{(g(r) - g(V_+))(r - V_+)}.$$

We claim that $p \in [r, s)$ can be chosen in such a way that $h(p) = 0$. Indeed, we have

$$h(s) = \sqrt{(g(s) - g(V_-))(s - V_-)} + \sqrt{(g(s) - g(V_+))(s - V_+)} -$$
$$-\sqrt{(g(r) - g(V_-))(r - V_-)} - \sqrt{(g(r) - g(V_+))(r - V_+)} > 0,$$

and Hölder's inequality yields

$$\int_{V_-}^{r} \sqrt{g_r^{*\prime}(u)}du \leq \sqrt{(g(r) - g(V_-))(r - V_-)},$$

hence $h(r) \leq 0$. The function h is continuous in $[r, s)$, hence $h(p) = 0$ for some $p \in [r, s)$. We thus dispose of two solutions v, v_p of problem (5),(11) with $D_+ - D_-$ given by formula (81). Both v and v_p satisfy the Lax condition and the dissipation condition. To check that $v \neq v_p$ we notice that $g_p^*(t) = g(t)$, hence $\frac{g_p^*(p)-g_p^*(t)}{p-t} = \frac{g(p)-g(t)}{p-t} < \frac{g(s)-g(t)}{s-t} = \frac{g(t)-g(V_-)}{t-V_-} = \frac{g_p^*(t)-g_p^*(V_-)}{t-V_-}$. This implies that g_p^* is not affine, consequently the solutions v, v_p are distinct.

We can mention a positive result, namely

Proposition 8.4. *Let g' be monotone and let v be a piecewise constant solution to equation (5) satisfying Lax' entropy condition at each point $z \in R^1$. Then v is monotone in each interval $(-\infty, 0)$, $(0, +\infty)$.*

Proof. Let v be given by (12) and let us fix for instance $0 < z_k < z_{k+1} \leq \sqrt{b}$. Assume that $v_k > \max\{v_{k-1}, v_{k+1}\}$. Then Definition 8.1 yields

$$g'(v_{k-1}+) \geq z_k^2 \geq g'(v_k-) \geq z_{k+1}^2 \geq g'(v_{k+1}+),$$

which is a contradiction. The cases $z_k < z_{k+1} < 0$ and $v_k < \min\{v_{k-1}, v_{k+1}\}$ are quite analogous. □

B. Liu's condition [Liu - 1981, §3]

Definition 8.5. *A solution v to (15) is said to satisfy* Liu's shock admissibility criterion *at a point $z \in R^1$ if $v(z+) \neq v(z-)$ and the inequality*

$$z\left(\frac{g(v)-g(v(z-))}{v-v(z-)}-\frac{g(v(z+))-g(v(z-))}{v(z+)-v(z-)}\right)\geq 0 \tag{83}$$

holds for all v between $v(z-)$ and $v(z+)$.

It is obvious that the minimal solution of (5), (11) defined in Theorem 5.2 satisfies condition (83) at each point of discontinuity. The converse is true in the class of regulated functions.

Proposition 8.6. *Let the problem (5),(11) admit a solution $v \in \mathcal{R}$ such that condition (83) holds at each point $z \in R^1$ of discontinuity of v. Then v is minimal in the sense of Theorem 5.2.*

Proof. Let us first assume for instance that v is non-decreasing in $(0,\infty)$. Let $w_+(s) := v(\sqrt{s})$ be the corresponding solution of (23),(24) and let g^* be its trajectory according to Lemma 3.2. If for some $u \in (v(0+), V_+)$ we have $g(u) \neq g^*(u)$, then by Lemma 3.2 there exists $s > 0$ such that $u \in (w(s-), w(s+))$ and $g^*(u) = g(w(s-)) + (u - w(s-))\frac{g(w(s+))-g(w(s-))}{w(s+)-w(s-)}$, and condition (83) entails $g(u) \geq g^*(u)$. Consequently, g^* is the minimal trajectory. The same argument works for v non-increasing and for the interval $(-\infty, 0)$.
On the other hand, condition (83) excludes non-monotonicities of v in $(-\infty, 0)$ and $(0,\infty)$. This can be seen again by considering just the interval $(0,\infty)$ only. Let us assume for instance that there exist $z_3 > z_1 > 0$ and $z_2 \in [z_1, z_3]$ such that the values $v_1 := v(z_1-), v_3 := v(z_3+), v_2 := \inf\{v(z); z \in [z_1, z_3]\}$ satisfy $v_2 < v_1 < v_3, v_2 = v(z_2+)$ or $v_2 = v(z_2-), v(z) \in [v_2, v_1]$ for $z \in [z_1, z_2], v(z) \in [v_2, v_3]$ for $z \in [z_2, z_3]$ (the other cases, namely $v_2 < v_3 < v_1, v_2 > v_1 > v_3, v_2 > v_3 > v_1$ are analogous).
It is more convenient to work with the solution w of (23),(24) defined by the formula $w(s) := v(\sqrt{s})$ for $s > 0$. Put $s_i := z_i^2$ for $i = 1, 2, 3$, $A := \{s \in (s_2, s_3); w(s+) = v_1$ or $w(s-) = v_1\}$ and

$$s_A := \begin{cases} \inf A & \text{if } A \neq \emptyset, \\ s_2 & \text{if } A = \emptyset. \end{cases}$$

Integrating equation (23) we obtain

$$s_2(v_2 - v_1) - g(v_2) + g(v_1) = \int_{s_1}^{s_2} (w(s) - v_1)ds \leq 0, \tag{84}$$

$$s_2(v_1 - v_2) - g(v_1) + g(v_2) = \int_{s_2}^{s_1} (w(s) - v_1)ds. \tag{85}$$

Put $\bar{s} := \inf\{s \in [s_2, s_3]; w(s+) > v_1\}$. We have either $\bar{s} = s_2$ or $\bar{s} > s_2$. In the latter case it follows from (84), (85) that $[s_2, \bar{s}] \cap A = \emptyset$, hence in both cases we obtain

$w(\bar{s}-) < v_1 < w(\bar{s}+)$. Put $\bar{v} := w(\bar{s}-) \in [v_2, v_1)$. The hypotheses (83) and Lemma 1.5 then entail

$$\frac{g(v_1) - g(\bar{v})}{v_1 - \bar{v}} \geq \bar{s}. \tag{86}$$

This yields

$$\int_{s_1}^{\bar{s}} (w(s) - v_1) ds = \bar{s}(\bar{v} - v_1) - g(\bar{v}) + g(v_1) \geq 0. \tag{87}$$

By construction, we have $\int_{s_1}^{\bar{s}} (w(s) - v_1) ds < 0$, which is a contradiction. Proposition 8.6 is proved. □

C. Dafermos' condition

Definition 8.7. *A weak solution to (2), (10) is said to satisfy* Dafermos maximal entropy rate criterion, *if for every weak solution $\tilde{u}$ to (2), (10) we have*

$$\frac{d}{dt} \int_{-\infty}^{\infty} \left[\frac{1}{2} u_t^2 + G(u_x) - \frac{1}{2} \tilde{u}_t^2 - G(\tilde{u}_x) \right] (x,t) dx \leq 0 \tag{88}$$

in the sense of distributions.

Introducing the expression

$$\mathcal{E}(v) := G(v) - v g(v) + \frac{3}{2} z^2 v^2 \tag{89}$$

for $v \in L^\infty(R^1)$ we can rewrite condition (88) in the following form.

Proposition 8.8. *A self-similar solution u to (2), (10) satisfies condition (88) with respect to all self-similar solutions $\tilde{u}$ to (2), (10) if and only if*

$$\int_{-\infty}^{\infty} (\mathcal{E}(v) - \mathcal{E}(\tilde{v})) dz \leq 0 \tag{90}$$

where $v, \tilde{v}$ are solutions to (5), (11) associated to $u, \tilde{u}$, respectively, according to Proposition 1.4.

The proof of Proposition 8.8 is a simple exercise based on integration-by-parts formulae

$$\int_{-\infty}^{\infty} \frac{1}{2} \left(f^2 - \tilde{f}^2 \right) dz = - \int_{-\infty}^{\infty} z \left(v f - \tilde{v} \tilde{f} \right) dz,$$

$$\int_{-\infty}^{\infty} 2z \left(v f - \tilde{v} \tilde{f} \right) dz = - \int_{-\infty}^{\infty} \left[v \left(z^2 v - g(v) \right) - \tilde{v} \left(z^2 \tilde{v} - g(\tilde{v}) \right) \right] dz.$$

□

Comparison of the maximum principles (90) and (40).

Let us first consider the case of piecewise constant solutions given by (12). We have seen (as a consequence of formula (32)) that condition (40) consists in maximizing separately the expressions

$$\begin{aligned} &\text{(i)} \qquad \textstyle\sum_{z_i>0} \left[G(v_i) - G(v_{i-1}) - \frac{1}{2}\left(g(v_{i-1}) + g(v_i)\right)(v_i - v_{i-1})\right], \\ &\text{(ii)} \qquad -\textstyle\sum_{z_i<0} \left[G(v_i) - G(v_{i-1}) - \frac{1}{2}\left(g(v_{i-1}) + g(v_i)\right)(v_i - v_{i-1})\right] \end{aligned} \tag{91}$$

with an unknown intermediate condition $g(v(0)) = Q$. On the other hand, a straightforward computation shows that $\mathcal{E}(v)$ in (90) is minimal if we maximize an expression different from (91), namely the sum

$$\sum_{z_i} z_i \left[G(v_i) - G(v_{i-1}) - \frac{1}{2}\left(g(v_{i-1}) + g(v_i)\right)(v_i - v_{i-1})\right]$$

over all discontinuities $z_i \in (-\infty, \infty)$.

Open problem. Let g be monotone. Prove or disprove: A solution v to (5), (11) satisfies minimality criterion (40) if and only if it satisfies Dafermos' maximal entropy rate criterion (90)! On the other hand, one can construct examples of non-monotone functions g such that these criteria are not equivalent (see [Kre]).

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