Wellposedness of Kinematic Hardening Models in Elastoplasticity

Martin Brokate*
Mathematisches Seminar
Universität Kiel
24098 Kiel
Germany

Pavel Krejčí†‡
Institute of Mathematics
Academy of Sciences
Žitná 25
11567 Praha
Czech Republic

Abstract

We consider a certain type of rate independent elastoplastic constitutive laws for nonlinear kinematic hardening which include the models of Frederick-Armstrong, Bower and Mróz. We prove results concerning existence, uniqueness and continuous dependence for the stress-strain evolution considered as a function of time (but not of space). As an auxiliary result, we also prove a theorem concerning the Lipschitz continuity of the vector play operator.

1 Introduction

Depending on the material a solid body is made of, the relation between load and deformation may vary greatly in character. Any deeper understanding requires an analysis of the governing physical and molecular processes which take place on a microscopic scale. On the other hand, a study of the macroscopic behaviour, in particular numerical simulation, eventually has to rely upon some continuum model. One may analyze microscopic and macroscopic models separately, or concentrate on their interaction. Within this paper, we restrict ourselves to macroscopic models which are *rate independent* and assume *small strains*. Such a type of behaviour is typical e.g. for the elastoplastic deformation of commonly used ductile steels at room temperature. To model the elastoplastic stress-strain law, we use an operator formulation, namely

$$\varepsilon = \mathcal{F}(\sigma)\,, \quad \sigma = \mathcal{G}(\varepsilon)\,, \tag{1.1}$$

which automatically distinguishes between the *stress controlled* and the *strain controlled* situation. Here, the operators $\mathcal{F}$ and $\mathcal{G}$ map certain spaces of functions, defined on

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some time interval $[t_0, t_1]$ with values in some tensor space, into each other. In the rate independent case considered here, such operators are often called *hysteresis operators*. We consider the stress-strain law in isolation, that is, we concentrate on the evolution in time according to (1.1) at a single point; thus, the balance laws which specify the space interaction do not play any role here. We study the question whether the stress-strain law is *well posed* in the space $W^{1,1}(t_0, t_1; \mathbf{T_d})$, that is, whether the operators $\mathcal{F}$ respectively $\mathcal{G}$ are *well defined* and *continuous* with respect to the norm

$$\|\theta\|_{1,1} = |\theta(t_0)| + \int_{t_0}^{t_1} |\dot{\theta}(t)| \, dt \,. \tag{1.2}$$

Here, we discuss models which are of pure *kinematic hardening* type. The basic model, usually termed *linear kinematic hardening*, is due to Melan [27] and Prager [33]; during the last 40 years, many modifications and refinements have been developed in order to cope, for one thing, with the experimentally observed phenomenon of *ratchetting*. We refer to [7], [8], [9], [16], [17], [36] and [20] for discussions and comparisons. We show in this paper that some of these, in particular the models of Armstrong and Frederick [1], Bower [2] and Mróz [29] can be reduced to a differential equation of the type

$$\dot{u} = \dot{\theta} + \mathcal{M}(\theta, u)|\dot{\xi}| \,. \tag{1.3}$$

Here, θ stands for σ or ε, depending on whether we consider the stress controlled or the strain controlled case; u represents an artificial function and $\mathcal{M}$ denotes a certain operator, for each of the models considered. We will *not* require $\mathcal{M}$ to possess any monotonicity or convexity properties. The function ξ is related to u through the variational inequality which expresses the principle of maximum dissipation or, equivalently, the normality rule. The reduction to (1.3) as well as the wellposedness of the initial value problem for (1.3) constitutes the main content of this paper and is discussed in Sections 2 and 3. Some additional material related to the Mróz model is presented in Section 4. The appendix includes a result concerning the Lipschitz dependence of ξ upon u.

From the standpoint of mechanics, a proposal of a stress-strain law will be meaningful only if it is compatible with the second law of thermodynamics. For the isothermal case considered here, this means that the energy dissipation rate has to be nonnegative, that is

$$\dot{\varepsilon}\sigma - \dot{U} \geq 0 \tag{1.4}$$

has to hold along any possible trajectory of the system; here, U denotes the internal energy. For systems with memory, however, the construction of a suitable nonnegative U can be a tricky and nontrivial business. To tackle this problem in a somewhat general manner, the notion of a *dissipation potential* has been introduced. Within that framework, it is shown in [14] that for a certain class of *standard generalized materials* the second law is satisfied. We refer to [26] and [24] for an exposition and for remarks concerning the relation to the models treated here; we will be satisfied with a different explicit construction of U for those models, in form of a hysteresis operator.

We do not study multisurface models, except for some remarks. The Chaboche model will be treated in a subsequent paper.

2 Kinematic Hardening Models

In order to fix our notation, we start with a brief review of the ingredients of kinematic hardening models. We denote by $\mathbf{T}$ the space of symmetric $N \times N$ tensors endowed with the usual scalar product and the associated norm

$$\langle \tau, \eta \rangle = \sum_{i,j=1}^{N} \tau_{ij} \eta_{ij} \,, \quad |\tau| = \sqrt{\langle \tau, \tau \rangle} \,, \tag{2.1}$$

For $\tau \in \mathbf{T}$, we define its trace $\mathrm{Tr}\,\tau$ and its deviator τ_d by

$$\mathrm{Tr}\,\tau = \sum_{i=1}^{N} \tau_{ii} = \langle \tau, \delta \rangle \,, \quad \tau_d = \tau - \frac{\mathrm{Tr}\,\tau}{N} \delta \,, \tag{2.2}$$

where $\delta = (\delta_{ij})$ stands for the Kronecker symbol. We denote by

$$\mathbf{T_d} = \{\tau : \tau \in \mathbf{T} \,, \, \mathrm{Tr}\,\tau = 0\} \,, \quad \mathbf{T_d^\perp} = \{\tau : \tau = \lambda \delta \,, \, \lambda \in \mathbb{R}\} \,, \tag{2.3}$$

the space of all deviators respectively its orthogonal complement. Since we study the stress-strain law in isolation and do not discuss the spatial coupling described by the balance equations, we consider stress and strain as functions defined on some fixed time interval $[t_0, t_1]$. Most of our results concerning wellposedness will refer to the space of absolutely continuous functions, so we will usually consider

$$\sigma, \varepsilon \in W^{1,1}(t_0, t_1; \mathbf{T}) := \{\tau | \tau : [t_0, t_1] \to \mathbf{T} \,, \, \|\tau\|_{1,1} = |\tau(t_0)| + \int_{t_0}^{t_1} |\dot{\tau}(t)| \, dt < \infty\} \,. \tag{2.4}$$

In operator form, the stress-strain law becomes

$$\varepsilon = \mathcal{F}(\sigma) \,, \quad \sigma = \mathcal{G}(\varepsilon) \,, \tag{2.5}$$

depending on whether we study the stress controlled or the strain controlled case. The operators $\mathcal{F}$ and $\mathcal{G}$ will usually be defined on some subset D_F respectively D_G of $W^{1,1}(t_0, t_1; \mathbf{T})$, generically denoted by D (note that we already used $\mathbf{T}$ indiscriminately for stress and strain tensors). To ensure compatibility with the second law of thermodynamics, we require the existence of operators $\mathcal{U}_F$ defined on D_F respectively $\mathcal{U}_G$ defined on D_G, called *internal energy operators*, such that $\mathcal{U}_F(\sigma) \geq 0$ respectively $\mathcal{U}_G(\varepsilon) \geq 0$ in $W^{1,1}(t_0, t_1; \mathbb{R})$ and

$$\langle \frac{\mathrm{d}}{\mathrm{dt}} \mathcal{F}(\sigma), \sigma \rangle - \frac{\mathrm{d}}{\mathrm{dt}} \mathcal{U}_F(\sigma) \geq 0 \,, \quad \text{a.e. in } (t_0, t_1) \,, \tag{2.6}$$

respectively

$$\langle \frac{\mathrm{d}}{\mathrm{dt}} \varepsilon, \mathcal{G}(\varepsilon) \rangle - \frac{\mathrm{d}}{\mathrm{dt}} \mathcal{U}_G(\varepsilon) \geq 0 \,, \quad \text{a.e. in } (t_0, t_1) \,, \tag{2.7}$$

hold for all admissible arguments. Note that the left hand side of (2.6) respectively (2.7) represents the rate of dissipation of the energy.

In terms of *rheological models*, all the models studied below have the structure

$$\mathcal{E} - (\, \mathcal{R} \,|\, \mathcal{K} \,) \,, \tag{2.8}$$

that is, a linear elastic element $\mathcal{E}$ is connected in series with the parallel combination of a rigid plastic element $\mathcal{R}$ and a "kinematic" element $\mathcal{K}$; essentially, $\mathcal{R}$ defines the form of the yield surface, while $\mathcal{K}$ describes its movement. The rheological structure (2.8) is reflected in the decomposition

$$\varepsilon = \varepsilon^e + \varepsilon^p \,, \quad \sigma = \sigma^e + \sigma^p \,, \tag{2.9}$$

of the stress and strain tensor into an "elastic" and a "plastic" part, see Figure 1. (In order to conform with general usage, we write σ^e for the stress along $\mathcal{K}$, instead of the more proper notation σ^k, although $\mathcal{K}$ is not really an elastic element in the case of the Bower and the Mróz model below.)

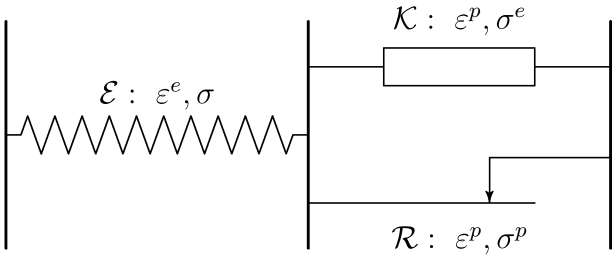

Figure 1: *The rheological model for kinematic hardening.*

The linear elastic element $\mathcal{E}$ relates the total stress σ and the elastic strain ε^e by

$$\sigma = A\varepsilon^e \,, \tag{2.10}$$

where $A = (A_{ijkl})$ is assumed to be constant in time and symmetric as well as positive definite with respect to the scalar product $\langle \cdot, \cdot \rangle$. The rigid plastic element $\mathcal{R}$ is characterized by a closed convex set $Z \subset \mathbf{T}$ which specifies the admissible values of the plastic stress, i.e. it is required that

$$\sigma^p(t) \in Z \,, \quad \text{for all } t \in [t_0, t_1] \,. \tag{2.11}$$

Its boundary ∂Z is called the *yield surface*. Plastic flow occurs according to the principle of maximum plastic work rate, that is, the plastic strain rate $\dot{\varepsilon}^p$ has to satisfy the evolution variational inequality

$$\langle \dot{\varepsilon}^p(t), \sigma^p(t) - \tilde{\sigma} \rangle \geq 0 \,, \quad \forall\, \tilde{\sigma} \in Z \,, \quad \text{a.e. in } (t_0, t_1) \,, \tag{2.12}$$

which implies that $\dot{\varepsilon}^p = 0$ as long as $\sigma^p \in \operatorname{Int} Z$, while $\dot{\varepsilon}^p$ points in the direction of the (or, in case of nonuniqueness, an) outward normal if $\sigma^p \in \partial Z$, see Figure 2.
For all models considered below, the plastic strain is volume invariant, that is,

$$\operatorname{Tr} \varepsilon^p(t) = \langle \varepsilon^p(t), \delta \rangle = 0 \,, \quad \varepsilon^p_d(t) = \varepsilon^p(t) \,, \quad \text{for all } t \in [t_0, t_1] \,. \tag{2.13}$$

In view of the normality rule (2.12), condition (2.13) requires Z to have the form

$$Z = Z_d \oplus \mathbf{T}_\mathbf{d}^\perp \,, \quad Z_d \subset \mathbf{T}_\mathbf{d} \text{ closed, convex} \,. \tag{2.14}$$

We will restrict ourselves to the *von Mises* yield condition

$$Z_d = B_r(0) \cap \mathbf{T_d} = \{\tau : \tau \in \mathbf{T_d}\,,\ |\tau| \le r\}\,. \tag{2.15}$$

Since $0 \in \operatorname{Int} Z_d$, the plastic work rate is always nonnegative, and there can be no plastic deformation if the plastic stress vanishes.

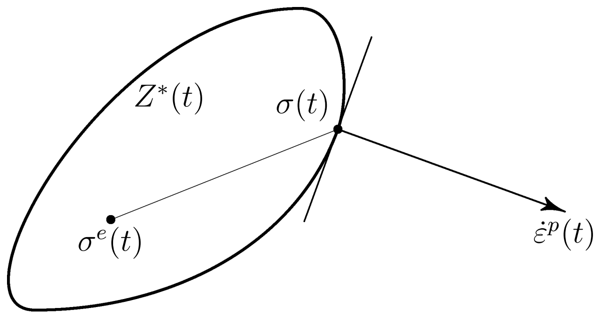

Figure 2: *The normality rule.*

The movement of the yield surface is related to the elastic stress $\sigma^e(t)$, commonly also called *backstress*, as follows. Since $\sigma^p(t) \in Z$ if and only if $\sigma(t) \in \sigma^e(t) + Z =: Z^*(t)$, the set

$$\partial Z^*(t) = \sigma^e(t) + \partial Z \tag{2.16}$$

represents the position of the yield surface within stress space at any given time t. Since $\partial Z = \partial Z_d \oplus \mathbf{T_d}^\perp$, only the movement in the deviatoric part plays any role, so one requires that

$$\sigma^e(t) = \sigma^e_d(t) \in \mathbf{T_d}\,, \quad \text{for all } t \in [t_0, t_1]\,. \tag{2.17}$$

In fact, for all models treated below, the requirement $\sigma^e(t_0) \in \mathbf{T_d}$ implies that (2.17) holds. We may write the time evolution of σ^e in operator form as

$$\sigma^e = \mathcal{H}_F(\sigma)\,, \qquad \text{resp.} \quad \sigma^e = \mathcal{H}_G(\varepsilon)\,; \tag{2.18}$$

the operators $\mathcal{H}_F$ respectively $\mathcal{H}_G$ are called the *hardening rule*.
The decomposition (2.9) introduces memory into the constitutive law; thus, the initial state of the memory has to be specified if one wants any stress or strain controlled evolution to be uniquely determined. Throughout this paper, we choose to prescribe the initial values

$$\varepsilon^p(t_0) = \varepsilon^p_0 \in \mathbf{T_d}\,, \quad \sigma^p_d(t_0) = \sigma^p_{0d} \in \mathbf{T_d}\,, \quad |\sigma^p_{0d}| \le r\,. \tag{2.19}$$

The second condition fixes the initial position of the yield surface with respect to the initial stress $\sigma_d(t_0)$. Once either $\sigma(t_0)$ or $\varepsilon(t_0)$ are given, the initial values for all variables in (2.9) are determined by (2.9), (2.10), (2.17) and (2.19). (In the case of linear kinematic hardening, equation (2.20) below replaces one of the two initial conditions.)
We now discuss specific choices for the kinematic element $\mathcal{K}$.

2.1 The model of Melan and Prager.

In this model, also referred to as *linear kinematic hardening*, one simply sets

$$\sigma^e = C\varepsilon^p \,, \tag{2.20}$$

where $C > 0$ is a constant. By (2.13), there holds $\sigma^e = \sigma^e_d$ a.e., and the evolution variational inequality (2.12) becomes

$$\langle \dot\sigma_d - \dot\sigma^p_d, \sigma^p_d - \tilde\sigma_d \rangle \geq 0 \,, \quad \forall\, \tilde\sigma_d \in Z_d \,, \quad \text{a.e. in } (t_0, t_1) \,, \tag{2.21}$$

$$\sigma^p_d(t) \in Z_d \,, \quad \text{for all } t \in [t_0, t_1] \,. \tag{2.22}$$

It is well known that (2.21), (2.22) has a unique solution σ^p_d for a given function σ_d and initial condition $\sigma^p_d(t_0) = \sigma^p_{0d} \in Z_d$; in our terminology, there holds

$$\sigma^p_d = \mathcal{S}(\sigma_d \,; \sigma^p_{0d}) \,, \tag{2.23}$$

where

$$\mathcal{S} : W^{1,1}(t_0, t_1; \mathbf{T_d}) \times Z_d \to W^{1,1}(t_0, t_1; \mathbf{T_d}) \tag{2.24}$$

denotes the stop operator with the characteristic Z_d as described in Definition A.2 of the appendix, with the choice $X = \mathbf{T_d}$. Since $\sigma^e = \sigma^e_d$, the hardening rule can be written as

$$\sigma^e(t) = \sigma_d(t) - \sigma^p_d(t) = \mathcal{P}(\sigma_d \,; \sigma^p_{0d})(t) \,, \tag{2.25}$$

where $\mathcal{P}$ denotes the play operator with the characteristic Z_d (again, we refer to the appendix). The stress-strain law in stress controlled form becomes

$$\varepsilon(t) = (\mathcal{F}(\sigma))(t) = A^{-1}\sigma(t) + \frac{1}{C}\mathcal{P}(\sigma_d \,; \sigma^p_{0d})(t) \,. \tag{2.26}$$

Thus, the wellposedness of $\mathcal{F}$ – with respect to a given pair of norms in stress and strain space – is equivalent to the wellposedness of the evolution variational inequality (2.21), (2.22). In particular, the estimate

$$|\sigma^p_d(t) - \bar\sigma^p_d(t)| \leq |\sigma^p_{0d} - \bar\sigma^p_{0d}| + \int_{t_0}^t |\dot\sigma_d(\tau) - \dot{\bar\sigma}_d(\tau)| \, d\tau \,, \tag{2.27}$$

which lies at the root of the theory initiated by Lions and Brézis (see [3] and [25] and the literature cited there), yields the Lipschitz continuity of

$$\mathcal{F} : W^{1,1}(t_0, t_1; \mathbf{T}) \to C([t_0, t_1]; \mathbf{T}) \,. \tag{2.28}$$

If one couples linear kinematic hardening with the balance equations of linearized elasticity, the resulting boundary value problem fits well into the framework of convex analysis, and the estimate (2.27) usually leads to uniqueness in a natural manner. We refer to [12], [14] and [32] for the general approach and to [13], [15], [19] and [31] for results concerning linear kinematic hardening.
In contrast to that, our proof of wellposedness of the models below requires stronger continuity properties of the operator $\mathcal{P}$, to be discussed in the appendix. By (2.26),

those results also furnish stronger results on continuous dependence for the Melan-Prager constitutive law.
Its compatibility with the second law follows from the inequality

$$\langle \dot\varepsilon, \sigma \rangle = \langle A^{-1}\dot\sigma, \sigma \rangle + \frac{1}{C}\langle \dot\sigma^e, \sigma^e \rangle + \langle \dot\varepsilon^p, \sigma^p \rangle \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(\langle A^{-1}\sigma, \sigma \rangle + \frac{1}{C}|\sigma^e|^2 \right) . \tag{2.29}$$

Thus, if we define an internal energy operator by

$$\mathcal{U}_F(\sigma) = \frac{1}{2}\langle A^{-1}\sigma, \sigma \rangle + \frac{1}{2C}|\sigma^e|^2 , \tag{2.30}$$

we see that (2.6) holds along arbitrary stress paths $\sigma \in W^{1,1}(t_0, t_1; \mathbf{T})$.

2.2 The Armstrong-Frederick model.

Armstrong and Frederick [1] proposed a modification, usually termed *nonlinear kinematic hardening*, of the model of Melan and Prager, namely

$$\dot\sigma^e = \gamma(R\dot\varepsilon^p - \sigma^e|\dot\varepsilon^p|) , \tag{2.31}$$

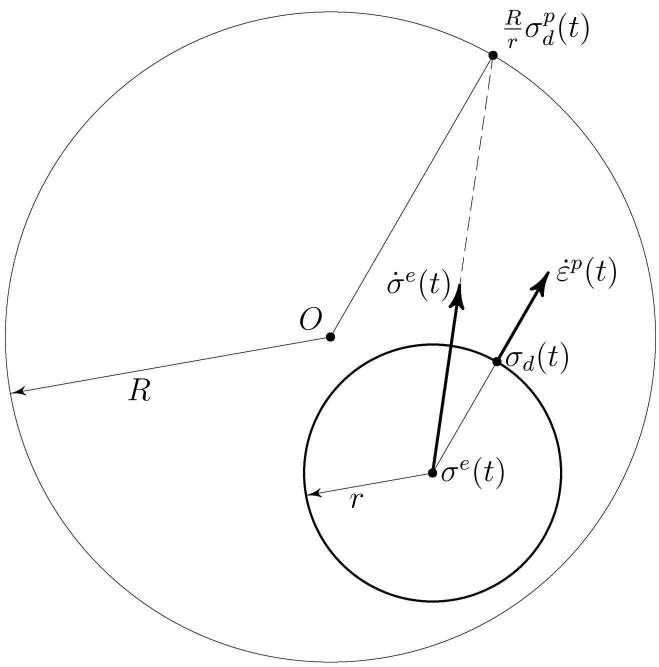

Figure 3: *The model of Armstrong and Frederick.*

where $\gamma, R > 0$ are constants. Obviously, (2.13) implies that (2.17) holds if $\sigma^e(t_0) \in \mathbf{T_d}$. Since

$$|\sigma^e|\frac{\mathrm{d}}{\mathrm{dt}}|\sigma^e| = \frac{\mathrm{d}}{\mathrm{dt}}\frac{1}{2}|\sigma^e|^2 = \gamma(R\langle \dot\varepsilon^p, \sigma^e \rangle - |\sigma^e|^2|\dot\varepsilon^p|) \leq \gamma|\dot\varepsilon^p|(R - |\sigma^e|)|\sigma^e| , \tag{2.32}$$

there holds $|\sigma^e(t)| \leq R$ for all t if it holds for $t = t_0$, and the yield surface will always lie inside the sphere of radius $R + r$. The restriction

$$|\sigma^e(t_0)| \leq R \tag{2.33}$$

thus appears to be natural, because otherwise the initial condition would not be reachable from the zero state. Since the normality rule in the von Mises case implies that

$$\dot{\varepsilon}^p = \frac{1}{r}\sigma_d^p \, |\dot{\varepsilon}^p| \,, \tag{2.34}$$

we can rewrite (2.31) in the form

$$\dot{\sigma}^e = \gamma|\dot{\varepsilon}^p| \left(\frac{R}{r}\sigma_d^p - \sigma^e\right) \,, \quad \text{a.e. in } (t_0, t_1) \,. \tag{2.35}$$

In particular, the vector $\dot{\sigma}^e(t)$ points in the direction of the vector $(R/r)\sigma_d^p(t) - \sigma^e(t)$ during plastic flow, see Figure 3.
To derive the wellposedness of the Armstrong-Frederick model, we employ a suitably chosen auxiliary variable. For the stress controlled case, we consider

$$u = \gamma R \varepsilon^p + \sigma_d^p \,. \tag{2.36}$$

Multiplying (2.12) by γR, we see that

$$\langle \dot{u} - \dot{\sigma}_d^p, \sigma_d^p - \tilde{\sigma}_d \rangle \geq 0 \,, \quad \forall\, \tilde{\sigma}_d \in Z_d \,, \quad \text{a.e. in } (t_0, t_1) \,, \tag{2.37}$$

has to be satisfied. In operator notation,

$$\sigma_d^p = \mathcal{S}(u\,; \sigma_{0d}^p) \,, \quad \varepsilon^p = \frac{1}{\gamma R}\mathcal{P}(u\,; \sigma_{0d}^p) \,. \tag{2.38}$$

The hardening rule becomes

$$\sigma^e = \sigma_d - \sigma_d^p = \sigma_d - \mathcal{S}(u\,; \sigma_{0d}^p) \,, \tag{2.39}$$

and the stress-strain law takes on the form

$$\varepsilon = A^{-1}\sigma + \frac{1}{\gamma R}\mathcal{P}(u\,; \sigma_{0d}^p) \,. \tag{2.40}$$

We replace σ_d^p in (2.36) by $\sigma_d - \sigma^e$, form the time derivative and evaluate $\dot{\sigma}^e$ according to (2.31). Using (2.38) and (2.39) we obtain the stress controlled differential equation for the unknown function u,

$$\dot{u} = \dot{\sigma}_d + \frac{1}{R}(\sigma_d - \mathcal{S}(u\,; \sigma_{0d}^p)) \left| \frac{\mathrm{d}}{\mathrm{dt}}\mathcal{P}(u\,; \sigma_{0d}^p) \right| \,, \tag{2.41}$$

which we have to solve subject to the initial condition

$$u(t_0) = \gamma R \varepsilon_0^p + \sigma_{0d}^p \,. \tag{2.42}$$

We will prove the wellposedness of this problem in Section 3.
A similar procedure works in the strain controlled case. We assume Hooke's law for the linear elastic part (2.10), that is,

$$\sigma = A\varepsilon^e = 2\mu\varepsilon^e + \lambda(\mathrm{Tr}\,\varepsilon^e)\delta \,, \tag{2.43}$$

holds with the Lamé constants $\lambda, \mu > 0$. Consequently, we have

$$\sigma^e = \sigma_d^e = 2\mu\varepsilon_d^e - \sigma_d^p = 2\mu\varepsilon_d - (2\mu\varepsilon^p + \sigma_d^p)\,. \tag{2.44}$$

We now choose the auxiliary function

$$v = (2\mu + \gamma R)\varepsilon^p + \sigma_d^p\,. \tag{2.45}$$

For the same reason as above, (2.37) continues to hold if we replace u by v, so

$$\sigma_d^p = \mathcal{S}(v\,;\sigma_{0d}^p)\,, \quad \varepsilon^p = \frac{1}{2\mu + \gamma R}\mathcal{P}(v\,;\sigma_{0d}^p)\,. \tag{2.46}$$

We form the time derivative in (2.45) and obtain

$$\dot v = \gamma R\dot\varepsilon^p + (2\mu\dot\varepsilon_d - \dot\sigma^e) = 2\mu\dot\varepsilon_d + \gamma\sigma^e|\dot\varepsilon^p|\,. \tag{2.47}$$

On the other hand, combining (2.44), (2.45) and (2.46) we get

$$\sigma^e = 2\mu\varepsilon_d - v + \rho\mathcal{P}(v\,;\sigma_{0d}^p)\,, \quad \rho := \frac{\gamma R}{2\mu + \gamma R}\,. \tag{2.48}$$

Putting together (2.47) and (2.48) we finally arrive at the strain controlled differential equation

$$\dot v = 2\mu\dot\varepsilon_d + \frac{\gamma}{2\mu + \gamma R}(2\mu\varepsilon_d - v + \rho\mathcal{P}(v\,;\sigma_{0d}^p))\left|\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{P}(v\,;\sigma_{0d}^p)\right|\,. \tag{2.49}$$

with the initial condition

$$v(t_0) = (2\mu + \gamma R)\varepsilon_0^p + \sigma_{0d}^p\,. \tag{2.50}$$

The thermodynamical consistency of the Armstrong-Frederick model follows from the inequality

$$\langle\dot\varepsilon, \sigma\rangle = \langle A^{-1}\dot\sigma, \sigma\rangle + \frac{1}{\gamma R}\langle\dot\sigma^e, \sigma^e\rangle + \frac{1}{R}|\sigma^e|^2|\dot\varepsilon^p| + \langle\dot\varepsilon^p, \sigma^p\rangle \geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(\langle A^{-1}\sigma, \sigma\rangle + \frac{1}{\gamma R}|\sigma^e|^2\right)\,. \tag{2.51}$$

2.3 Bower's model.

In order to improve the description of ratchetting effects which occur during the elasto-plastic deformation of railway rails, Bower [2] refined the Armstrong-Frederick model as

$$\dot\sigma^e = \gamma(R\dot\varepsilon^p - (\sigma^e - \sigma^\beta)|\dot\varepsilon^p|)\,, \tag{2.52}$$

$$\dot\sigma^\beta = c(\sigma^e - \sigma^\beta)|\dot\varepsilon^p|\,, \tag{2.53}$$

where $c > 0$ is an additional constant and

$$\sigma^\beta(t_0) = \sigma_0^\beta \in \mathbf{T_d} \tag{2.54}$$

is given.

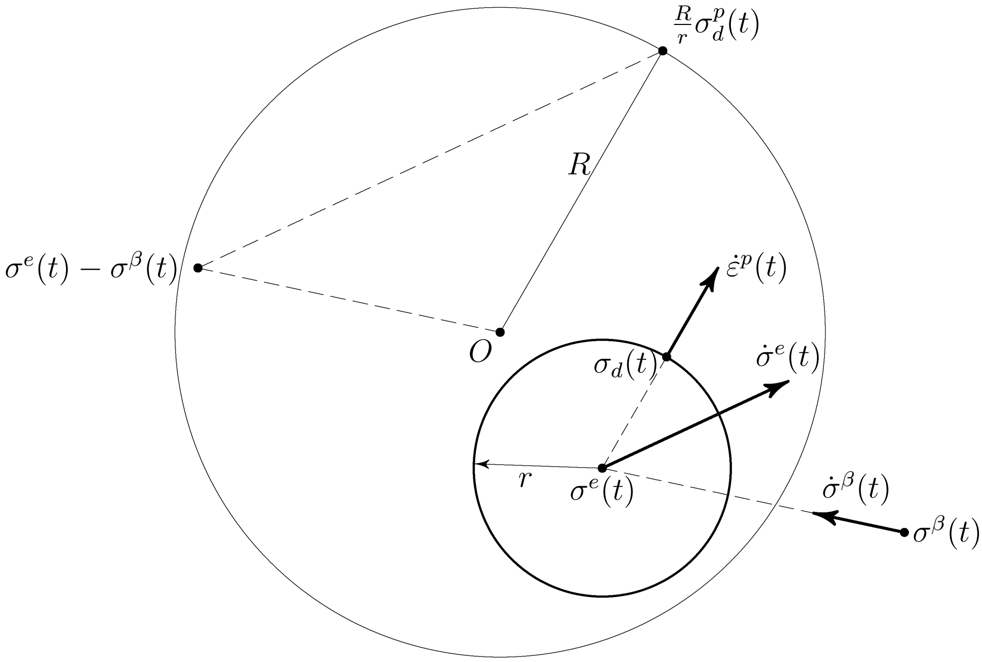

Figure 4: *The model of Bower.*

We have

$$\dot\sigma^e - \dot\sigma^\beta = \gamma R\dot\varepsilon^p - (\gamma+c)(\sigma^e-\sigma^\beta)|\dot\varepsilon^p|\,, \tag{2.55}$$

and the same argument as in (2.32) yields the natural restriction

$$|\sigma^e(t_0)-\sigma_0^\beta| \le \frac{\gamma R}{\gamma+c}\,. \tag{2.56}$$

Combining (2.52) and (2.53) we obtain

$$\sigma^\beta(t) = \sigma_0^\beta + cR(\varepsilon^p(t)-\varepsilon_0^p) - \frac{c}{\gamma}(\sigma^e(t)-\sigma^e(t_0))\,. \tag{2.57}$$

This enables us to eliminate σ^β in (2.52), and we obtain

$$\dot\sigma^e = \gamma R\dot\varepsilon^p - \Big((\gamma+c)\sigma^e - \gamma cR(\varepsilon^p-\varepsilon_0^p) - c\sigma^e(t_0) - \gamma\sigma_0^\beta\Big)\,|\dot\varepsilon^p|\,. \tag{2.58}$$

We now proceed similarly as we did for the Armstrong-Frederick model. In the stress controlled case, we put

$$u(t) = \gamma R(\varepsilon^p(t)-\varepsilon_0^p) + \sigma_d(t_0) - \sigma_{0d}^p + \frac{\gamma}{c}\sigma_0^\beta + \sigma_d^p(t)\,, \tag{2.59}$$

so

$$\mathcal{P}(u\,;\sigma_{0d}^p) = \gamma R(\varepsilon^p-\varepsilon_0^p) + \sigma_d(t_0) - \sigma_{0d}^p + \frac{\gamma}{c}\sigma_0^\beta\,,\quad \mathcal{S}(u\,;\sigma_{0d}^p) = \sigma_d^p\,. \tag{2.60}$$

Differentiating in time and inserting (2.58) we get the identity

$$\dot u - \dot\sigma_d = ((\gamma+c)(\sigma_d-\sigma_d^p) - c(u-\sigma_d^p))|\dot\varepsilon^p|\,, \tag{2.61}$$

and thus obtain

$$\dot u = \dot\sigma_d + \frac{1}{R}\left(\Big(1+\frac{c}{\gamma}\Big)(\sigma_d - u) + \mathcal{P}(u\,;\sigma^p_{0d})\right)\Big|\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{P}(u\,;\sigma^p_{0d})\Big|\,, \tag{2.62}$$

with the initial condition

$$u(t_0) = \sigma_d(t_0) + \frac{\gamma}{c}\sigma^\beta_0\,. \tag{2.63}$$

In the strain controlled case, we consider the auxiliary function

$$v(t) = (2\mu + \gamma R)\varepsilon^p(t) + \sigma^p_d(t) + c_0\,, \tag{2.64}$$

where

$$c_0 = \frac{c(2\mu+\gamma R)}{2\mu(\gamma+c)+\gamma c R}\Big(2\mu\varepsilon_d(t_0) - \sigma^p_{0d} - (2\mu+\gamma R)\varepsilon^p_0 + \frac{\gamma}{c}\sigma^\beta_0\Big)\,. \tag{2.65}$$

We differentiate (2.64) and obtain, assuming again that (2.43) holds,

$$\dot v = 2\mu\dot\varepsilon_d + \frac{\gamma+c}{2\mu+\gamma R}\left((2\mu\varepsilon_d - v) + \frac{\gamma}{\gamma+c}\cdot\frac{\gamma R}{2\mu+\gamma R}\mathcal{P}(v\,;\sigma^p_{0d})\right)\Big|\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{P}(v\,;\sigma^p_{0d})\Big|\,, \tag{2.66}$$

with the initial condition

$$v(t_0) = \frac{1}{2\mu(\gamma+c)+\gamma c R}\Big((2\mu+\gamma R)(2\mu(c\varepsilon_d(t_0)+\gamma\varepsilon^p_0) + \gamma\sigma^\beta_0) + 2\mu\gamma\sigma^p_{0d}\Big)\,. \tag{2.67}$$

The thermodynamical consistency is implied by the inequality

$$\begin{aligned}\langle\dot\varepsilon,\sigma\rangle &= \langle A^{-1}\dot\sigma,\sigma\rangle + \langle\dot\varepsilon^p,\sigma^p\rangle + \frac{1}{\gamma R}\langle\dot\sigma^e,\sigma^e\rangle + \frac{1}{cR}\langle\dot\sigma^\beta,\sigma^\beta\rangle + \frac{1}{R}|\sigma^e-\sigma^\beta|^2|\dot\varepsilon^p| \\ &\geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(\langle A^{-1}\sigma,\sigma\rangle + \frac{1}{\gamma R}|\sigma^e|^2 + \frac{1}{cR}|\sigma^\beta|^2\right)\,.\end{aligned} \tag{2.68}$$

Without further information, and in particular for the zero initial state, the natural initial value for σ^β will be $\sigma^\beta_0 = 0$. It turns out, however, that with the specific choice

$$\sigma^\beta_0 = cR\varepsilon^p_0 - \frac{c}{\gamma}\sigma^e(t_0)\,, \tag{2.69}$$

the kinematic element $\mathcal{K}$ becomes identical with a parallel combination of an Armstrong-Frederick element and a Melan-Prager element; in this manner, a special case of the two surface Chaboche model is obtained. To this end, we decompose the backstress σ^e as

$$\sigma^e = \sigma^a + \sigma^m\,, \tag{2.70}$$

where

$$\sigma^a = \frac{\gamma}{c+\gamma}(\sigma^e - \sigma^\beta)\,, \tag{2.71}$$

$$\sigma^m = \frac{c}{c+\gamma}\left(\sigma^e + \frac{\gamma}{c}\sigma^\beta\right)\,. \tag{2.72}$$

From (2.52) - (2.56) and (2.69) one easily computes that

$$\dot\sigma^a = (c+\gamma)\left(\frac{\gamma^2 R}{(c+\gamma)^2}\dot\varepsilon^p - \sigma^a|\dot\varepsilon^p|\right)\,, \tag{2.73}$$

$$\sigma^m = \frac{\gamma c R}{c+\gamma}\varepsilon^p\,. \tag{2.74}$$

2.4 The model of Mróz.

In contrast to the models above, the hardening rule $\sigma^e = \mathcal{H}_F(\sigma_d)$ of the Mróz model [29] is not based upon a formula involving the plastic strain rate $\dot{\varepsilon}^p$; instead, it employs a certain geometric construction involving an auxiliary surface, namely the sphere $\partial B_R(0)$ with the radius $R > r$ around 0. (We will not treat the case of several auxiliary surfaces as in the original paper [29], nor the version with a one parameter family of surfaces discussed in [10], [11] and [4].) Assume that there holds

$$|\sigma_d(t)| < R\,, \quad t \in (t_0, t_1)\,. \tag{2.75}$$

The Mróz hardening rule is defined by

$$\dot{\sigma}^e(t) = \mu(t)\Big(\frac{R}{r}\sigma_d^p(t) - \sigma_d(t)\Big)\,, \tag{2.76}$$

where $\mu(t) \geq 0$, if $|\sigma_d^p(t)| = r$ and $\langle \dot{\sigma}_d(t), \sigma_d^p(t)\rangle > 0$, and $\mu(t) = 0$ and hence $\dot{\sigma}^e(t) = 0$ otherwise. The actual value of $\mu(t)$ during plastic flow can be determined from the condition $|\sigma_d^p(t)| = r$. Moreover, in the case of the sphere, σ_d^p points into the direction of the outward normal if $|\sigma_d^p| = r$; consequently, the vector $\dot{\sigma}^e(t)$ defined by (2.76) points into the direction of the line which connects $\sigma_d(t)$ to the point having the same normal as $\sigma_d(t)$ on the auxiliary surface $\partial B_R(0)$, see Figure 5.

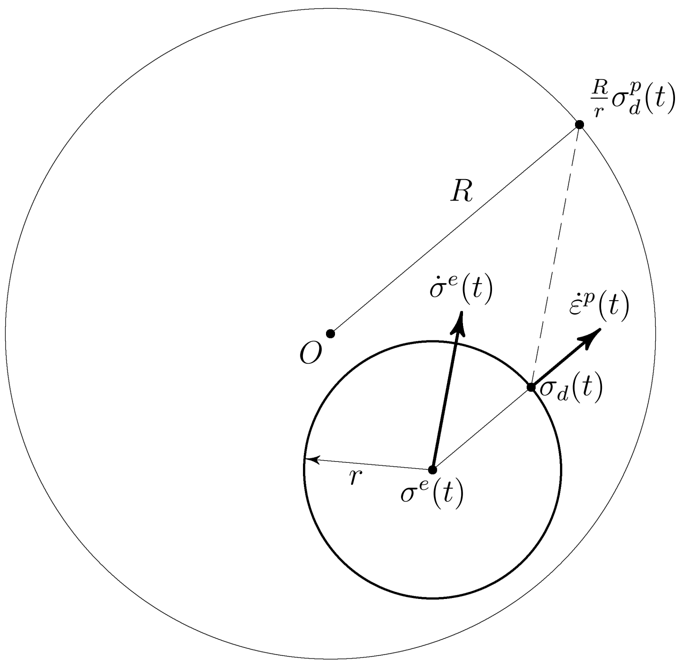

Figure 5: *The Mróz hardening rule.*

We now show how this construction is related to the stress controlled differential equation

$$\dot{u} = \dot{\sigma}_d + \frac{\sigma_d}{R}|\dot{\xi}|\,, \quad \xi = \mathcal{P}(u\,; \sigma_{0d}^p)\,. \tag{2.77}$$

To this end, let us first assume that the function μ is determined as described above. Let the auxiliary function u solve the equation

$$\dot{u}(t) = \dot{\sigma}_d(t) + \mu(t)\sigma_d(t) \,. \tag{2.78}$$

From (2.76) we obtain

$$\dot{u} - \dot{\sigma}_d^p = \dot{\sigma}^e + \mu\sigma_d = \mu\frac{R}{r}\sigma_d^p \,, \tag{2.79}$$

so that

$$\langle \dot{u} - \dot{\sigma}_d^p, \sigma_d^p - \tilde{\sigma}_d \rangle \geq 0 \,, \quad \forall\, \tilde{\sigma}_d \in B_r(0) \,, \quad \text{a.e. in } (t_0, t_1) \,. \tag{2.80}$$

By definition of the stop operator we get, setting $\xi = \mathcal{P}(u\,; \sigma_{0d}^p)$,

$$\sigma_d^p = \mathcal{S}(u\,; \sigma_{0d}^p) \,, \quad \dot{\xi}(t) = \dot{u}(t) - \dot{\sigma}_d^p(t) = \mu\frac{R}{r}\sigma_d^p(t) \,, \tag{2.81}$$

so that $R\mu(t) = |\dot{\xi}(t)|$, and (2.77) holds. Conversely, if u solves (2.77) and if we define

$$\mu(t) = R^{-1}|\dot{\xi}(t)| \,, \quad \sigma_d^p = \mathcal{S}(u\,; \sigma_{0d}^p) \,, \quad \sigma^e = \sigma_d - \sigma_d^p \,, \tag{2.82}$$

we see that (2.76) is satisfied. Thus, (2.77) characterizes the Mróz hardening rule. The initial condition for u can be chosen arbitrarily, for example as $u(t_0) = \sigma_{0d}^p$. The normality rule (2.12) requires that the plastic strain rate satisfies

$$\dot{\varepsilon}^p(t) = \lambda(t)\sigma_d^p(t) \,, \tag{2.83}$$

where $\lambda(t) \geq 0$ and $\lambda(t) = 0$ if $|\sigma_d^p(t)| < r$. The choice of λ is discussed in [24]; we only add the following remark concerning the thermodynamical consistency. If we solve (2.77) for $\dot{\sigma}_d$, we obtain

$$\dot{\sigma}^e = \dot{\sigma}_d - \dot{\sigma}_d^p = \dot{\xi} - \frac{\sigma_d}{R}|\dot{\xi}| \,. \tag{2.84}$$

Let t be such that $|\sigma_d^p(t)| = r$ and that the derivatives exist at t. Then there holds

$$\langle \dot{\sigma}_d(t), \sigma_d^p(t) \rangle = \langle \dot{\sigma}^e(t), \sigma_d^p(t) \rangle + \langle \dot{\sigma}_d^p(t), \sigma_d^p(t) \rangle \leq \langle \dot{\sigma}^e(t), \sigma_d^p(t) \rangle \,, \tag{2.85}$$

hence $\langle \dot{\sigma}_d(t), \sigma_d^p(t) \rangle > 0$ and $|\sigma_d^p(t)| = r$ imply that $\dot{\xi}(t) \neq 0$. Consequently, if we assume that there is no plastic deformation for unloading or neutral loading, that is, if

$$\dot{\varepsilon}^p(t) \neq 0 \quad \Rightarrow \quad \langle \dot{\sigma}_d(t), \sigma_d^p(t) \rangle > 0 \,, \tag{2.86}$$

we must have $\dot{\xi}(t) \neq 0$ if $\lambda(t) \neq 0$, so we can find a nonnegative function α such that

$$\dot{\varepsilon}^p(t) = \alpha(t)\dot{\xi}(t) \,. \tag{2.87}$$

We may combine (2.84) and (2.87) to obtain

$$\alpha\dot{\sigma}^e = \dot{\varepsilon}^p - \frac{\sigma_d}{R}|\dot{\varepsilon}^p| \,, \quad \text{a.e. in } (t_0, t_1) \,. \tag{2.88}$$

This enables us to estimate the rate of mechanical work from below as

$$\begin{aligned} \langle \dot{\varepsilon}, \sigma \rangle &= \langle A^{-1}\dot{\sigma}, \sigma \rangle + \frac{|\sigma_d|^2}{R}|\dot{\varepsilon}^p| + \langle \alpha\dot{\sigma}^e, \sigma_d \rangle \\ &\geq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\langle A^{-1}\sigma, \sigma \rangle + \alpha\langle \dot{\sigma}^e, \sigma^e \rangle \,. \end{aligned} \tag{2.89}$$

Thus, the choice

$$\alpha(t) = G(|\sigma^e(t)|^2)\,, \tag{2.90}$$

where G is a positive integrable function (in particular, it may be chosen as a constant) ensures the thermodynamical consistency of the model.
It is known that the Armstrong-Frederick model can be considered as a special case of the Mróz model. In the framework above, we see this if we select the function α in (2.87) to be a constant; in that case, (2.88) becomes

$$\dot\sigma^e = \frac{1}{\alpha R}\Big((R-r)\dot\varepsilon^p - \sigma^e|\dot\varepsilon^p|\Big)\,. \tag{2.91}$$

3 Existence, Uniqueness and Regularity Results

In this section we study the wellposedness of the Cauchy problem

$$\dot u(t) = \dot\theta(t) + \mathcal{M}(\theta,u)(t)|\dot\xi(t)|\,, \quad \text{a.e. in } (t_0,t_1)\,, \tag{3.1}$$

$$\xi(t) = \mathcal{P}(u\,;x^0)(t)\,, \quad t\in[t_0,t_1]\,, \tag{3.2}$$

$$u(t_0) = u^0\,. \tag{3.3}$$

The unknown functions are u and ξ, whereas the initial conditions u^0, x^0 as well as a source (or input) function θ are given. By $\mathcal{P}$ we denote the play operator with the characteristic $Z = B_r(0)$, $r>0$, as defined in the appendix. The operator $\mathcal{M}$ may have a rather general form, but it is required that all values $\mathcal{M}(\theta,u)(t)$ are uniformly smaller than 1 in absolute value. To be more precise, we consider

$$\mathcal{M} : \Theta\times C([t_0,t_1];X) \to C([t_0,t_1];X)\,, \tag{3.4}$$

where X is a finite dimensional Hilbert space, and Θ denotes a set of admissible input functions. In fact, $\mathcal{M}$ may also depend upon the initial value x^0; however, for simplicity we will suppress this dependence in the notation except in the statement and proof of Theorem 3.3. The operator $\mathcal{M}$ has to be *causal*, that is, it holds $\mathcal{M}(\theta_1,u_1) = \mathcal{M}(\theta_2,u_2)$ on $[t_0,t]$ whenever $(\theta_1,u_1) = (\theta_2,u_2)$ on $[t_0,t]$, if $t\in[t_0,t_1]$. Thus, $\mathcal{M}$ generates a family of operators

$$\mathcal{M}_t : \Theta_t\times C([t_0,t];X)\to C([t_0,t];X)\,, \quad \Theta_t = \{\theta\Big|[t_0,t] : \theta\in\Theta\}\,, \quad t\in[t_0,t_1]\,, \tag{3.5}$$

but we will usually drop the index t in the sequel. Since we will use the method of the retarded argument for the proof of the basic existence theorem, we also require Θ to be *shift invariant*, that is, $\tau^\delta\theta\in\Theta$ for every $\theta\in\Theta$ and $\delta>0$, where the shift $\tau^\delta f$ of a function f defined on $[t_0,t_1]$ is given by

$$(\tau^\delta f)(t) = \begin{cases} f(t-\delta)\,, & t_0+\delta\le t\le t_1\,,\\ f(t_0)\,, & t_0\le t\le t_0+\delta\,, \end{cases} \tag{3.6}$$

Assumption 3.1 *Let $\Theta\subset W^{1,1}(t_0,t_1;X)$ be shift invariant, let $\mathcal{M} : \Theta\times C([t_0,t_1];X)\to C([t_0,t_1];X)$ be causal and continuous with respect to the maximum norm. Moreover, assume that $u^0\in X$, $x^0\in B_r(0)$ and $\kappa>0$ are given such that*

$$\sup_{s\in[t_0,t]} |\mathcal{M}(\theta,u)(s)| \le 1-\kappa \tag{3.7}$$

holds for every $t \in [t_0, t_1]$, $\theta \in \Theta_t$ *and every* $u \in W^{1,1}(t_0, t; X)$ *with* $u(t_0) = u^0$ *and*

$$|\dot{u}(\tau)| \leq \frac{1}{\kappa}|\dot{\theta}(\tau)|\,, \quad \textit{a.e. in } (t_0, t)\,. \tag{3.8}$$

We present the basic existence theorem.

Theorem 3.2 *Let Assumption 3.1 hold, let* $\theta \in \Theta$ *be given. Then there exists a solution* (u, ξ) *of the Cauchy problem (3.1) - (3.3) such that* $u, \xi \in W^{1,1}(t_0, t_1; X)$ *and*

$$\| \mathcal{M}(\theta, u) \|_\infty \leq 1 - \kappa\,, \tag{3.9}$$

as well as

$$|\dot{u}(t)| \leq \frac{1}{\kappa}|\dot{\theta}(t)|\,, \quad \textit{a.e. in } (t_0, t_1)\,. \tag{3.10}$$

Moreover, every solution which satisfies (3.9) also satisfies (3.10).

Proof. We first consider the Cauchy problem

$$\dot{u}(t) = \dot{\theta}(t) + f(t)|\dot{\xi}(t)|\,, \quad \text{a.e. in } (a, a+\eta)\,, \tag{3.11}$$

$$\xi(t) = \mathcal{P}(u\,; x^a)(t)\,, \quad u(a) = u^a\,, \quad \xi(a) = u^a - x^a\,, \tag{3.12}$$

on some interval $[a, a+\eta] \subset [t_0, t_1]$, where $u^a \in X$, $x^a \in B_r(0)$, $f \in L^\infty(a, a+\eta)$ are given. We claim that (3.11), (3.12) has a unique solution $u, \xi \in W^{1,1}(a, a+\eta; X)$ satisfying (3.10), if

$$\| f \|_\infty \leq 1 - \kappa\,, \quad \int_a^{a+\eta} |\dot{\theta}(t)|\,dt \leq \frac{\kappa^2 r}{\sqrt{2}}\,. \tag{3.13}$$

This follows from the fact that the operator T defined by

$$(Tu)(t) = u^a + \theta(t) - \theta(a) + \int_a^t f(s)|\dot{\xi}(s)|\,ds \tag{3.14}$$

is a contraction on the subset

$$B = \{u : u \in W^{1,1}(a, a+\eta; X)\,, \, |\dot{u}(t)| \leq \frac{1}{\kappa}|\dot{\theta}(t)| \text{ a.e.}\,, \, u(a) = u^a\} \tag{3.15}$$

of $W^{1,1}(a, a+\eta; X)$. Indeed, T maps B into itself since $(Tu)(a) = u^a$ and, since $|\dot{\xi}| \leq |\dot{u}|$ holds pointwise a.e. by (A.12),

$$\Big|\frac{\mathrm{d}}{\mathrm{dt}}(Tu)(t)\Big| \leq |\dot{\theta}(t)| + (1-\kappa)|\dot{u}(t)| \leq \frac{1}{\kappa}|\dot{\theta}(t)|\,. \tag{3.16}$$

Moreover, if we apply Theorem A.5 and the estimate (A.4) on $[a, a+\eta]$, we obtain for any $u, v \in B$

$$\begin{aligned}
&\int_a^t \Big|\frac{\mathrm{d}}{\mathrm{dt}}(Tu) - \frac{\mathrm{d}}{\mathrm{dt}}(Tu)\Big|(s)\,ds \leq \\
&\quad \leq (1-\kappa)\left[\int_a^t |\dot{u} - \dot{v}|(s)\,ds + \frac{\sqrt{2}}{r}\int_a^t |\dot{u}(s)||\mathcal{S}[u\,; x^a] - \mathcal{S}[v\,; x^a]|(s)\,ds\right] \\
&\quad \leq (1-\kappa)\left[\int_a^t |\dot{u} - \dot{v}|(s)\,ds + \frac{\sqrt{2}}{r\kappa}\int_a^t |\dot{\theta}(s)|\int_a^s |\dot{u} - \dot{v}|(\sigma)\,d\sigma\,ds\right] \\
&\quad \leq (1-\kappa^2)\int_a^t |\dot{u} - \dot{v}|(s)\,ds\,.
\end{aligned} \tag{3.17}$$

In the second step, we consider the Cauchy problem

$$\dot{u}(t) = \dot{\theta}(t) + \mathcal{M}(\tau^\delta\theta, \tau^\delta u)(t)|\dot{\xi}(t)|\,, \quad \text{a.e. in } (t_0, t_1)\,, \tag{3.18}$$

$$\xi(t) = \mathcal{P}(u\,; x^0)(t)\,, \quad u(t_0) = u^0\,. \tag{3.19}$$

Since the constant η in (3.13) can be chosen independently from a, for every $\delta \in (0, \eta)$ we can use the result of the first step as well as Assumption 3.1 to construct an absolutely continuous solution (u_δ, ξ_δ) of (3.18), (3.19) successively on the intervals $[t_0, t_0 + \delta]$, $[t_0 + \delta, t_0 + 2\delta], \ldots$, such that

$$|\dot{u}_\delta(t)| \le \frac{1}{\kappa}|\dot{\theta}(t)| \tag{3.20}$$

holds almost everywhere. By (3.20), the family $\{\dot{u}_\delta : 0 < \delta < \eta\}$ is equiintegrable in $L^1(t_0, t_1; X)$ and $\{u_\delta\}$ is equicontinuous and uniformly bounded in $C([t_0, t_1]; X)$. By the Dunford-Pettis and the Arzelà-Ascoli theorems, there exists a $u \in W^{1,1}(t_0, t_1; X)$ and a sequence $\{u_{\delta_k}\}$ with $\delta_k \to 0$, denoted by $\{u_k\}$, such that $u_k \to u$ uniformly in $C([t_0, t_1]; X)$ as well as

$$\lim_{n\to\infty} \int_{t_0}^{t_1} \langle \dot{u}_k - \dot{u}, w\rangle\, dt = 0\,, \quad \text{for all } w \in L^\infty(t_0, t_1; X)\,. \tag{3.21}$$

Setting

$$V_k(t) = \operatorname{Var}_{[t_0,t]} \xi_k = \int_{t_0}^{t} |\dot{\xi}_k(s)|\, ds\,, \quad \xi_k = \mathcal{P}(u_k\,; x^0)\,, \tag{3.22}$$

we can rewrite (3.18), (3.19) in terms of a Stieltjes integral as

$$u_k(t) = u^0 + \theta(t) - \theta(t_0) + \int_{t_0}^{t} \mathcal{M}(\tau^k\theta, \tau^k u_k)(s)\, dV_k(s)\,, \quad \tau^k := \tau^{\delta_k}\,. \tag{3.23}$$

Since the sequence $\{V_k\}$ by Proposition A.9 converges pointwise (and, hence, uniformly) to

$$V(t) = \int_{t_0}^{t} |\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{P}(u\,; x^0)(s)|\, ds\,, \tag{3.24}$$

and since obviously $\tau^k u_k \to u$ uniformly, the continuity of $\mathcal{M}$ enables us to pass to the limit in (3.23), so (u, ξ) with $\xi = \mathcal{P}(u\,; x^0)$ yields a solution of (3.1) - (3.3). □

If the operator $\mathcal{M}$ is Lipschitz continuous, the solutions of the Cauchy problem (3.1) - (3.3) depend Lipschitz continuously upon the data (and, in particular, are unique), as the following theorem shows.

Theorem 3.3 *Let two sets of data* (θ_1, x_1^0, u_1^0), (θ_2, x_2^0, u_2^0) *with* $\theta_i \in \Theta$, $u_i^0 \in X$ *and* $x_i^0 \in B_r(0)$ *be given, let* (u_1, ξ_1) *and* (u_2, ξ_2) *be corresponding solutions in* $W^{1,1}(t_0, t_1; X)$ *of the Cauchy problem (3.1) - (3.3) which satisfy (3.9) and (3.10) for some* $\kappa > 0$. *Assume that*

$$\max_{s\in[t_0,t]} |\mathcal{M}(\theta_1, u_1; x_1^0)(s) - \mathcal{M}(\theta_2, u_2; x_2^0)(s)| \;\le\; A\Big(|x_1^0 - x_2^0| + \\ + |u_1(t_0) - u_2(t_0)| + \int_{t_0}^{t} |\dot{u}_1 - \dot{u}_2|\, ds + |\theta_1(t_0) - \theta_2(t_0)| + \int_{t_0}^{t} |\dot{\theta}_1 - \dot{\theta}_2|\, ds\Big)\,. \tag{3.25}$$

holds for all $t \in [t_0, t_1]$. *Then there holds*

$$\|\, u_1 - u_2 \,\|_{1,1} \le L(|x_1^0 - x_2^0| + |u_1^0 - u_2^0| + \|\, \theta_1 - \theta_2 \,\|_{1,1})\,. \tag{3.26}$$

where L *depends only upon* A, κ, r *and*

$$c := \max\{\| \theta_1 \|_{1,1}, \| \theta_2 \|_{1,1}\}. \tag{3.27}$$

Proof. From the differential equation (3.1) and from (3.9) we obtain, a.e. in (t_0, t_1),

$$\begin{aligned} |\dot u_1(t) - \dot u_2(t)| \quad &\leq \quad |\dot\theta_1(t) - \dot\theta_2(t)| + (1-\kappa)|\dot\xi_1(t) - \dot\xi_2(t)| + \\ &+ |\dot\xi_1(t)||\mathcal{M}(\theta_1, u_1; x_1^0)(t) - \mathcal{M}(\theta_2, u_2; x_2^0)(t)|. \end{aligned} \tag{3.28}$$

Theorem A.5 states that there holds, for every $t \in [t_0, t_1]$,

$$\int_{t_0}^t |\dot\xi_1 - \dot\xi_2|\, ds \leq |x_1^0 - x_2^0| + \int_{t_0}^t |\dot u_1 - \dot u_2|\, ds + \frac{\sqrt{2}}{r}\int_{t_0}^t |\dot u_1||x_1 - x_2|\, ds, \tag{3.29}$$

where $x_i = \mathcal{S}(u_i\,; x_i^0)$, $i = 1, 2$. By (A.12) and (3.10) we have

$$|\dot\xi_1(t)| \leq |\dot u_1(t)| \leq \frac{1}{\kappa}|\dot\theta_1(t)|. \tag{3.30}$$

Since (A.4) implies that

$$|x_1(s) - x_2(s)| \leq |x_1^0 - x_2^0| + \int_{t_0}^s |\dot u_1 - \dot u_2|\, d\tau, \quad s \in [t_0, t], \tag{3.31}$$

we obtain that

$$\begin{aligned} \int_{t_0}^t |\dot\xi_1 - \dot\xi_2|\, ds \quad &\leq \quad \Big(1 + \frac{\sqrt{2}}{r\kappa}\int_{t_0}^t |\dot\theta_1|\, ds\Big)|x_1^0 - x_2^0| + \int_{t_0}^t |\dot u_1 - \dot u_2|\, ds \\ &+ \frac{\sqrt{2}}{r\kappa}\int_{t_0}^t |\dot\theta_1(s)|\int_{t_0}^s |\dot u_1 - \dot u_2|\, d\tau\, ds. \end{aligned} \tag{3.32}$$

For a given $t \in [t_0, t_1]$, we integrate (3.28) over $[t_0, t]$, estimate the derivatives of ξ with the aid of (3.30), (3.32) and (3.27), rearrange and divide by κ to obtain

$$\int_{t_0}^t |\dot u_1 - \dot u_2|\, ds \leq B + \frac{1}{\kappa^2}\Big(A + \frac{(1-\kappa)\sqrt{2}}{r}\Big)\int_{t_0}^t |\dot\theta_1(s)|\int_{t_0}^s |\dot u_1 - \dot u_2|\, d\tau\, ds, \tag{3.33}$$

where B is the number given by

$$B = \frac{1}{\kappa}\Big(1 + \frac{c}{\kappa}\Big)\| \theta_1 - \theta_2 \|_{1,1} + \frac{cA}{\kappa^2}|u_1^0 - u_2^0| + \frac{1}{\kappa}\Big[(1-\kappa)\Big(1 + \frac{c\sqrt{2}}{r\kappa}\Big) + \frac{c}{\kappa}\Big]|x_1^0 - x_2^0|. \tag{3.34}$$

We define the functions β and w by

$$\beta(t) = \frac{1}{\kappa^2}\Big(A + \frac{(1-\kappa)\sqrt{2}}{r}\Big)|\dot\theta_1(t)|, \tag{3.35}$$

$$w(t) = \int_{t_0}^t \beta(s)\int_{t_0}^s |\dot u_1 - \dot u_2|\, d\tau\, ds. \tag{3.36}$$

In terms of those functions, (3.33) becomes

$$\dot w(t) \leq \beta(t)(B + w(t)), \quad \text{for all } t \in [t_0, t_1]. \tag{3.37}$$

Since $w(t_0) = 0$, Gronwall's inequality implies that

$$w(t) \leq B\Big[\exp\Big(\int_{t_0}^t \beta(s)\, dx\Big) - 1\Big]. \tag{3.38}$$

Inserting (3.38) into (3.37) we finally conclude that

$$\int_{t_0}^{t_1} |\dot u_1 - \dot u_2|\, dt \leq B\exp\Big(\frac{c}{\kappa^2}(A + (1-\kappa)\frac{\sqrt{2}}{r})\Big). \tag{3.39}$$

The proof is complete. □

We apply the results of Theorem 3.2 and of Theorem 3.3 to the models of Armstrong-Frederick, Bower and Mróz. We begin with the Mróz model which is particularly easy to treat, because in this case the operator $\mathcal{M}$ does not depend upon u. Let $\sigma \in W^{1,1}(t_0, t_1; \mathbf{T})$ be given. According to Subsection 2.4, we have to solve the initial value problem

$$\dot{u} = \dot{\sigma}_d + \frac{\sigma_d}{R}|\dot{\xi}|, \quad \xi = \mathcal{P}(u\,; \sigma_{0d}^p), \quad u(t_0) = \sigma_{0d}^p \in B_r(0). \tag{3.40}$$

Its solution (u, ξ) determines σ^e and σ^p by (2.78). The stress controlled constitutive law

$$\varepsilon = \mathcal{F}_M(\sigma) = A^{-1}\sigma + \varepsilon^p, \tag{3.41}$$

turns out to be well posed for rather general flow rules, for example (see the discussion in subsection 2.4)

$$\dot{\varepsilon}^p(t) = \alpha(t)\dot{\xi}(t), \quad \alpha(t) = G(\sigma^e(t), \sigma_d(t)), \quad \varepsilon^p(t_0) = \varepsilon_0^p. \tag{3.42}$$

Proposition 3.4 (Mróz model)
Let $G : \mathbf{T_d} \times \mathbf{T_d} \to \mathrm{I\!R}$ be locally Lipschitz continuous. Then the Mróz constitutive operator $\mathcal{F}_M$ given by (3.41) and (3.42) is well defined on the domain

$$D_M = \{\sigma : \sigma \in W^{1,1}(t_0, t_1; \mathbf{T}), \, \|\, \sigma_d \,\|_\infty < R\} \tag{3.43}$$

and Lipschitz continuous with respect to the norm $\|\cdot\|_{1,1}$ on every subset

$$D_M^{\kappa,C} = \{\sigma : \sigma \in D_M, \, \|\, \sigma_d \,\|_\infty \le R(1-\kappa), \, \|\, \sigma_d \,\|_{1,1} \le C\}, \quad 0 < \kappa < 1, C > 0. \tag{3.44}$$

Proof. We choose $X = \mathbf{T_d}$ and set

$$\theta = \sigma_d, \quad \mathcal{M}(\theta, u) = \frac{1}{R}\theta, \quad u^0 = x^0 = \sigma_{0d}^p. \tag{3.45}$$

We fix $\kappa \in (0, 1)$ and define Θ by

$$\Theta = \{\theta : \theta \in W^{1,1}(t_0, t_1; X), \, \|\, \theta \,\|_\infty \le R(1-\kappa)\}. \tag{3.46}$$

Then Assumption 3.1 as well as (3.25) hold, the latter with $A = R^{-1}$. The assertion follows from Theorems 3.2 and 3.3, since the assumption on G implies that the mapping $(u, \sigma_d) \mapsto \varepsilon^p$ is Lipschitz continuous w.r.t. the norm of $W^{1,1}$ on the set of pairs (u, σ_d) with solutions u for $\sigma_d \in D_M^{\kappa,C}$. □

Proposition 3.4 does not cover the case when $|\sigma_d| = R$, i.e. when the value of the stress reaches the boundary of the auxiliary surface. We will discuss that situation in Section 4.

For the Armstrong-Frederick and the Bower models, the operator $\mathcal{M}$ depends on u. To find out whether and how the input function θ must be restricted in order to ensure that $\|\, \mathcal{M}(\theta, u) \,\|_\infty < 1$, one needs a priori estimates. We first consider the stress controlled Armstrong-Frederick model. Here, problem (3.1) - (3.3) with $\theta = \sigma_d$ and $x^0 = \sigma_{0d}^p$ takes on the form

$$\dot{u} = \dot{\theta} + \frac{1}{R}(\theta - x)|\dot{\xi}|, \quad u(t_0) = u^0 = \gamma R \varepsilon_0^p + x^0, \tag{3.47}$$

where

$$\xi = \mathcal{P}(u\,; x^0), \quad x = \mathcal{S}(u\,; x^0), \tag{3.48}$$

so in particular

$$\mathcal{M}(\theta, u) = \frac{1}{R}(\theta - \mathcal{S}(u\,; x^0))\,. \tag{3.49}$$

If we assume that $R > r$ and restrict ourselves to stress inputs $\theta = \sigma_d$ satisfying $\| \sigma_d \|_\infty < R - r$, then $\| \mathcal{M}(\sigma_d, u) \|_\infty < 1$ holds since we have $\| \mathcal{S}(u\,; x^0) \|_\infty \le r$ regardless of the values of u, and we obtain the wellposedness of (3.47), (3.48) in the same straightforward manner as for the Mróz model above in 3.4. (The continuity of $\mathcal{M}$ with respect to u follows from (A.14).) However, from the model equations one would hope the less stringent restriction

$$\| \sigma_d \|_\infty < R + r \tag{3.50}$$

to suffice, since the Armstrong-Frederick equation (2.31) implies that $|\sigma^e| \le R$ if we have $|\sigma^e(t_0)| \le R$; on the other hand, the bound $|\sigma_d^p| \le r$ is already part of the definition of the plastic element. In fact, the following example (see [24], p. 222) shows that, in proportional loading, the plastic strain tends to infinity as we enforce $|\sigma_d|$ to approach the value $R + r$.

Example 3.5 Let $e \in \mathbf{T_d}$ be any tensor of unit norm, set

$$\theta(t) = \sigma_d(t) = (r + t)e\,, \quad x^0 = re\,, \quad t_0 = 0\,. \tag{3.51}$$

Then one easily checks that the ansatz

$$x(t) = re\,, \quad \dot{u}(t) = \dot{\xi}(t)\,, \tag{3.52}$$

reduces (3.47) to

$$\dot{u} = \left(1 + \frac{t}{R}|\dot{u}|\right) e\,, \quad u(0) = re\,. \tag{3.53}$$

From (3.53) we can compute the solution of (3.47) – uniqueness follows from Proposition 3.8 below – as

$$u(t) = \left(r + R \log \frac{R}{R - t}\right) e\,. \tag{3.54}$$

As σ_d^p is bounded, (2.36) shows that ε^p tends to infinity as t approaches R.

The following development up to Proposition 3.8 shows that the restriction (3.50) gives the correct bound also for arbitrary multiaxial loading.

Lemma 3.6 *Assume that $\theta, u, x, \xi \in W^{1,1}(t_0, b; \mathbf{T_d})$ solve (3.47), (3.48) in $[t_0, b]$ and that $|\theta(a) - x(a)| \le R(1 - \kappa)$ for some $a \in [t_0, b]$ and some $\kappa > 0$. Assume moreover that*

$$\| \theta \|_\infty \le r + R(1 - \kappa)^2\,. \tag{3.55}$$

Then

$$|\theta(t) - x(t)| \le R(1 - \kappa) \tag{3.56}$$

holds for all $t \in [a, b]$.

Proof. It suffices to prove that (3.56) holds for all t for which

$$\frac{\mathrm{d}}{\mathrm{dt}}|\theta(t) - x(t)|^2 > 0\,. \tag{3.57}$$

Assuming the latter, we get

$$0 < \langle \dot\theta(t) - \dot x(t), \theta(t) - x(t)\rangle = \langle \dot\xi(t), \theta(t) - x(t)\rangle - \frac{1}{R}|\dot\xi(t)||\theta(t) - x(t)|^2 , \tag{3.58}$$

hence $\dot\xi(t) \neq 0$ and $\langle \dot\xi(t), x(t)\rangle = r|\dot\xi(t)|$ by (A.16). We therefore conclude that

$$\frac{1}{R}|\theta(t) - x(t)|^2 + r \leq \Big\langle \frac{\dot\xi(t)}{|\dot\xi(t)|} , \theta(t)\Big\rangle \leq r + R(1-\kappa)^2 , \tag{3.59}$$

whence (3.56) follows. □

Lemma 3.7 *Let* $\theta \in W^{1,1}(t_0, t_1; \mathbf{T_d})$ *,* $u^0 \in X$ *and* $x^0 \in B_r(0)$ *be given. Assume that (3.55) as well as* $|\theta(t_0) - x^0| \leq R(1-\kappa)$ *hold for some* $\kappa \in (0,1)$ *. Then there exists a solution* (u,ξ) *of the Cauchy problem (3.47), (3.48) such that* $u, \xi \in W^{1,1}(t_0, t_1; \mathbf{T_d})$ *, and every solution satisfies*

$$\| \mathcal{M}(\theta, u) \|_\infty \leq 1 - \kappa , \quad |\dot u(t)| \leq \frac{1}{\kappa}|\dot\theta(t)| , \quad \textit{a.e. in } (t_0, t_1) . \tag{3.60}$$

Proof. We choose $\eta > 0$ such that there holds, for all $a \in [t_0, t_1 - \eta]$,

$$\int_a^{a+\eta} |\dot\theta(t)|\, dt < \frac{R\kappa^2}{4 + 2\kappa} . \tag{3.61}$$

It suffices to prove that, given any $a \in [t_0, t_1 - \eta]$ and any solution (u, ξ) on $[t_0, a]$ which satisfies

$$|\theta(t) - x(t)| < R(1-\kappa) \tag{3.62}$$

for $t = a$, that solution can be extended to a solution on $[t_0, a+\eta]$, and any such continuation satisfies (3.62) for all $t \in [a, a+\eta]$. To this end, we apply Theorem 3.2 on the interval $[a, a+\eta]$. We first show that (3.7), (3.8) hold with κ replaced by $\kappa/2$. Assume that $\tilde u \in W^{1,1}(a, a+\eta; \mathbf{T_d})$ satisfies $\tilde u(a) = u(a)$ and

$$|\dot{\tilde u}(t)| \leq \frac{2}{\kappa}|\dot\theta(t)| , \quad \text{a.e. in } (a, a+\eta) . \tag{3.63}$$

Setting $\tilde x = \mathcal{S}(\tilde u\,; x(a))$ on $[a, a+\eta]$, we have $|\dot{\tilde x}| \leq |\dot{\tilde u}|$ a.e. and

$$\begin{aligned} |\mathcal{M}(\theta, \tilde u)(t)| &= \frac{1}{R}|\theta(t) - \tilde x(t)| \\ &\leq \frac{1}{R}\left(|\theta(a) - x(a)| + \int_a^{a+\eta} |\dot\theta| + |\dot{\tilde x}|\, ds \right) \\ &\leq (1-\kappa) + \frac{1}{R}\Big(1 + \frac{2}{\kappa}\Big)\frac{R\kappa^2}{4+2\kappa} = 1 - \frac{\kappa}{2} , \end{aligned} \tag{3.64}$$

for all $t \in [a, a+\eta]$. Hence, Theorem 3.2 implies that there exists a solution on $[a, a+\eta]$. From Lemma 3.6 we conclude that (3.62) must hold on $[a, a+\eta]$ for any such continuation $\tilde u$. □

Proposition 3.8 (Stress controlled Armstrong-Frederick model)
The operator $\mathcal{F}_{AF}$ of the stress controlled Armstrong-Frederick model is well defined on the domain

$$D_{FAF} = \{\sigma : \sigma \in W^{1,1}(t_0, t_1; \mathbf{T}) \,,\, \| \sigma_d \|_\infty < R + r \,,\, |\sigma_d(t_0) - \sigma^p_{0d}| < R\} \,, \tag{3.65}$$

and Lipschitz continuous with respect to the norm $\| \cdot \|_{1,1}$ on every subset

$$D^{\alpha,C}_{FAF} = \{\sigma : \sigma \in D_{FAF} \,,\, \| \sigma_d \|_\infty \le R + r - \alpha \,,\, |\sigma_d(t_0) - \sigma^p_{0d}| \le R - \alpha \,,\, \| \sigma_d \|_{1,1} \le C\} \,. \tag{3.66}$$

Proof. This is a consequence of Lemma 3.7 and of Theorem 3.3 with $\Theta = D^{\alpha,C}_{FAF}$ and $\mathcal{M}$ given by (3.49); from the inequality (A.4) we see that $\mathcal{M}$ satisfies (3.25) with a constant A which does not depend on θ and u. □

The three remaining cases - the strain controlled Armstrong-Frederick model as well as both versions of the Bower model - can be treated similarly. Moreover, the initial value problem for the auxiliary variable arising from (2.49) respectively (2.62) or (2.66) takes on a common form, namely

$$\dot u = \dot\theta + \frac{1}{K}(\theta - u + z\xi)|\dot\xi| \,, \quad \xi = \mathcal{P}(u\,; x^0) \,, \quad u(t_0) = u^0 \,, \tag{3.67}$$

for certain constants $z \in (0,1)$ and $K > 0$, where $x^0 = \sigma^p_{0d}$ as before. In fact, the value of the constants are

$$K = R + \frac{2\mu}{\gamma} \,, \quad z = \frac{\gamma R}{2\mu + \gamma R} \,, \quad u^0 = x^0 + (2\mu + \gamma R)\varepsilon^p_0 \,, \tag{3.68}$$

for the strain controlled Armstrong-Frederick model,

$$K = \frac{\gamma R}{\gamma + c} \,, \quad z = \frac{\gamma}{\gamma + c} \,, \quad u^0 = \theta(t_0) + \frac{\gamma}{c}\sigma^\beta_0 \,, \tag{3.69}$$

for the stress controlled Bower model, and

$$K = \frac{2\mu + \gamma R}{\gamma + c} \,, \quad z = \frac{\gamma}{\gamma + c} \cdot \frac{\gamma R}{2\mu + \gamma R} \,, \tag{3.70}$$

$$u^0 = \frac{1}{2\mu(\gamma + c) + \gamma c R} \Big((2\mu + \gamma R)(c\theta(t_0) + 2\mu\gamma\varepsilon^p_0 + \gamma\sigma^\beta_0) + 2\mu\gamma x^0 \Big) \,. \tag{3.71}$$

For (3.67), we have the following a priori estimate.

Lemma 3.9 *Let $0 < z < 1$ and $K > 0$ be given. Let $u, \theta, \xi \in W^{1,1}(t_0, b; \mathbf{T_d})$ be a solution of (3.67), let $a \in [t_0, b]$. If*

$$|\theta(t) - u(t) + z\xi(t)| \le Kz \tag{3.72}$$

holds for $t = a$, then (3.72) holds for all $t \in [a, b]$.

Proof. It suffices to prove that, given any $t \in (a,b)$,

$$\frac{\mathrm{d}}{\mathrm{dt}}|\theta(t) - u(t) + z\xi(t)|^2 > 0 \tag{3.73}$$

implies that

$$|\theta(t) - u(t) + z\xi(t)| \le Kz\,. \tag{3.74}$$

Assume that (3.73) holds for some t. We then have

$$\begin{aligned} 0 &< \langle \dot\theta(t) - \dot u(t) + z\dot\xi(t), \theta(t) - u(t) + z\xi(t)\rangle \\ &= -\frac{1}{K}|\dot\xi(t)||\theta(t) - u(t) + z\xi(t)|^2 + z\langle\dot\xi(t), \theta(t) - u(t) + z\xi(t)\rangle\,, \end{aligned} \tag{3.75}$$

hence $\dot\xi(t) \neq 0$ and

$$|\theta(t) - u(t) + z\xi(t)|^2 < Kz\left\langle \frac{\dot\xi(t)}{|\dot\xi(t)|}\,,\, \theta(t) - u(t) + z\xi(t)\right\rangle, \tag{3.76}$$

so (3.74) holds. □

Lemma 3.10 *Let $0 < z < 1$, $K > 0$, $\theta \in W^{1,1}(t_0,t_1;\mathbf{T_d})$, $u^0 \in X$ and $x^0 \in B_r(0)$ be given. Assume that $|\theta(t_0) - (1-z)u^0 - zx^0| \le Kz$. Then there exists a solution (u,ξ) of the Cauchy problem (3.67) such that $u,\xi \in W^{1,1}(t_0,t_1;\mathbf{T_d})$, and every solution satisfies*

$$\|\,\mathcal{M}(\theta,u)\,\|_\infty \le z\,, \quad |\dot u(t)| \le \frac{1}{1-z}|\dot\theta(t)|\,, \quad \textit{a.e. in } (t_0,t_1)\,. \tag{3.77}$$

Proof. As $\xi(t_0) = u^0 - x^0$, condition (3.72) holds at $t = t_0$. The proof is now completely analogous to the proof of Lemma 3.7; we only sketch the pointwise estimate for

$$\mathcal{M}(\theta,u)(t) = \frac{1}{K}(\theta(t) - u(t) + z\xi(t))\,. \tag{3.78}$$

Assume that $u \in W^{1,1}(t_0, a+\eta\,; \mathbf{T_d})$ solves (3.67) on $[t_0,a]$ for some a and satisfies

$$|\dot u| \le \frac{1}{\kappa}|\dot\theta|\,, \quad \text{where} \quad \kappa = \frac{1-z}{2}\,, \tag{3.79}$$

a.e. in $(a, a+\eta)$. On the latter interval, we obtain the estimate

$$|\mathcal{M}(\theta,u)(t)| \le |\mathcal{M}(\theta,u)(a)| + \frac{1}{K}\left(1 + \frac{2(1+z)}{1-z}\right)\int_a^{a+\eta} |\dot\theta(t)|\,dt\,. \tag{3.80}$$

If we choose η such that the second summand on the right hand side is bounded by κ uniformly in a, Theorem 3.2 allows us to continue the solution up to $t = a+\eta$, and we can use Lemma 3.9 to obtain $|\mathcal{M}(\theta,u)(t)| \le z$ on $[a, a+\eta]$. We now continue as in the proof of Lemma 3.7 to obtain the result. □

Proposition 3.11 (Strain controlled Armstrong-Frederick and Bower's model) *The constitutive operators $\mathcal{F}_B$, $\mathcal{G}_B$ and $\mathcal{G}_{AF}$ for the stress controlled Bower, the strain controlled Bower and the strain controlled Armstrong-Frederick model are well defined and locally Lipschitz continuous with respect to the norm $\| \cdot \|_{1,1}$ on the respective domains*

$$D_{GAF} = \{\varepsilon : \varepsilon \in W^{1,1}(t_0, t_1; \mathbf{T}) \, , \, |2\mu(\varepsilon_d(t_0) - \varepsilon_0^p) - \sigma_{0d}^p| \leq R\} \, , \tag{3.81}$$

$$D_{FB} = \{\sigma : \sigma \in W^{1,1}(t_0, t_1; \mathbf{T}) \, , \, |\sigma_d(t_0) - \sigma_{0d}^p - \sigma_0^\beta| \leq \frac{\gamma R}{\gamma + c}\} \, , \tag{3.82}$$

$$D_{GB} = \{\varepsilon : \varepsilon \in W^{1,1}(t_0, t_1; \mathbf{T}) \, , \, |2\mu(\varepsilon_d(t_0) - \varepsilon_0^p) - \sigma_{0d}^p - \sigma_0^\beta| \leq \frac{\gamma R}{\gamma + c}\} \, , \tag{3.83}$$

Proof. This follows from Lemma 3.10 and Theorem 3.3 in the same manner as above. □

Remark 3.12 We note in particular that both versions of Bower's model as well as the strain controlled Armstrong-Frederick model are wellposed without any restriction concerning the input, except for the natural conditions resulting from (2.33) and (2.56).

4 Boundary Behaviour of the Mróz Model

The Mróz hardening rule $\sigma^e = \mathcal{H}_F(\sigma_d)$ determines the movement of the yield surface $\partial Z^*(t) = \sigma^e(t) + B_r(0)$. We have shown in Subsection 2.4 that σ^e is related to the auxiliary function u which solves the problem

$$\dot{u} = \dot{\sigma}_d + \frac{\sigma_d}{R}|\dot{\xi}| \, , \quad \xi = \mathcal{P}(u \, ; \sigma_{0d}^p) \, , \quad u(t_0) = \sigma_{0d}^p \, , \tag{4.1}$$

by

$$\sigma_d^p = \mathcal{S}(u \, ; \sigma_{0d}^p) \, , \quad \sigma^e = \sigma_d - \sigma_d^p \, . \tag{4.2}$$

Moreover, we have proved the wellposedness of (4.1) in Theorem 3.4 under the assumption that $\| \sigma_d \|_\infty < R$, that is, the stress input lies always within the auxiliary sphere $\partial B_R(0)$. Mathematical difficulties arise when $|\sigma^d(t)| = R$ for some t; however, that situation naturally occurs in the multisurface version of the model of Mróz. Indeed, an understanding of the case

$$|\sigma_d(t_0)| = |\sigma_d(t_1)| = R \, , \quad |\sigma_d(t)| < R \quad \text{for all } t \in (t_0, t_1) \, . \tag{4.3}$$

appears to be crucial for the study of the multisurface model, compare Remark 4.5 below. The *inclusion property*, often tacitly assumed to hold, states that the yield surface $\partial Z^*(t)$ always lies within $B_R(0)$. We present a formal proof.

Lemma 4.1 *Let $\sigma \in W^{1,1}(t_0, t_1; \mathbf{T})$ and $\sigma_{0d}^p \in B_r(0)$ be given, assume that $|\sigma_d(t)| < R$ for all $t > t_0$ and that*

$$|\sigma^e(t_0)| = |\sigma_d(t_0) - \sigma_{0d}^p| \leq R - r \, . \tag{4.4}$$

Then for every solution $(u, \xi, \sigma_d^p, \sigma^e)$ of (4.1) and (4.2) there holds the inclusion condition

$$|\sigma^e(t)| \leq R - r \, , \quad \textit{for all } t \in [t_0, t_1] \, . \tag{4.5}$$

Proof. Assume that $d/dt(|\sigma^e(t)|^2) > 0$ holds for some $t > t_0$. From (2.83) we obtain

$$0 < \langle \dot\sigma^e(t), \sigma^e(t)\rangle = \langle \dot\xi(t) - \frac{\sigma_d(t)}{R}|\dot\xi(t)|, \sigma^e(t)\rangle \,, \tag{4.6}$$

so in particular $|\dot\xi(t)| > 0$. Since $\sigma_d^p = \mathcal{S}(u\,;\sigma_{0d}^p)$ and hence $\dot\xi = r^{-1}\sigma_d^p|\dot\xi|$, (4.6) implies that

$$0 < \langle \frac{R}{r}\sigma_d^p(t) - \sigma_d(t), \sigma^e(t)\rangle = \langle \frac{R-r}{r}\sigma_d^p(t) - \sigma^e(t), \sigma^e(t)\rangle \le (R-r-|\sigma^e(t)|)|\sigma^e(t)|\,, \tag{4.7}$$

so $|\sigma^e(t)| < R-r$. Thus, $|\sigma^e(t)| > R-r$ cannot occur if $|\sigma^e(t_0)| \le R-r$. □

Lemma 4.2 *Under the hypotheses of Lemma 4.1 we have*

$$|\sigma_d^p(t) - \frac{r}{R}\sigma_d(t)|^2 \le \frac{r}{R}\left(1-\frac{r}{R}\right)(R^2 - |\sigma_d(t)|^2) \tag{4.8}$$

for all $t \in [t_0, t_1]$.

Proof. The algebraic identity

$$|\sigma_d^p - \frac{r}{R}\sigma_d|^2 + \frac{r}{R}\left(1-\frac{r}{R}\right)|\sigma_d|^2 = \left(1-\frac{r}{R}\right)|\sigma_d^p|^2 + \frac{r}{R}|\sigma^e|^2 \tag{4.9}$$

and Lemma 4.1 yield

$$|\sigma_d^p - \frac{r}{R}\sigma_d|^2 + \frac{r}{R}\left(1-\frac{r}{R}\right)|\sigma_d|^2 \le r^2\left(1-\frac{r}{R}\right) + \frac{r}{R}(R-r)^2\,, \tag{4.10}$$

which is nothing but (4.8). □

From Lemma 4.2 we see that the boundary values of σ^e and σ_d^p have to satisfy the equations

$$\sigma_d^p(t_i) = \frac{r}{R}\sigma_d(t_i)\,, \quad \sigma^e(t_i) = \left(1-\frac{r}{R}\right)\sigma_d(t_i)\,, \quad i = 0,1\,. \tag{4.11}$$

and that we might obtain solutions

$$\sigma_d^p, \sigma^e \in W^{1,1}_{loc}(t_0, t_1; \mathbf{T_d}) \cap C([t_0,t_1]; \mathbf{T_d})\,. \tag{4.12}$$

In fact, we can prove this only under the additional assumption of *transversality*,

$$\langle \dot\sigma_d(t_0+), \sigma_d(t_0)\rangle < 0\,, \tag{4.13}$$

where $\dot\sigma_d(t_0+) = \lim_{t\downarrow t_0}\dot\sigma_d(t)$ is assumed to exist. We first prove that (4.13) implies pure unloading near $t_0 = 0$.

Lemma 4.3 *Let* $\sigma_d \in W^{1,1}(t_0, t_1; \mathbf{T_d})$ *be given such that (4.3) as well as (4.13) hold, set*

$$\sigma_{0d}^p = \frac{r}{R}\sigma_d(t_0)\,. \tag{4.14}$$

Then there exists $\delta > 0$ *such that*

$$u(t) = \sigma_d(t) - \left(1-\frac{r}{R}\right)\sigma_d(t_0)\,, \quad \dot\xi(t) = 0\,, \tag{4.15}$$

constitute the unique solution (u,ξ) *of problem (4.1) within the space*

$$W^{1,1}_{loc}(t_0, t_0+\delta; \mathbf{T_d}) \cap C([t_0, t_0+\delta]; \mathbf{T_d})\,. \tag{4.16}$$

Proof. By virtue of (4.13), we can choose $\delta > 0$ such that the function u defined by (4.15) satisfies $\langle \dot{u}(t), u(t) \rangle < 0$ for every $t \in I := (t_0, t_0 + \delta)$. One then easily checks that (4.14), (4.15) together with $\dot{\sigma}_d^p = \dot{u} = \dot{\sigma}_d$, $\dot{\xi} = 0$, defines a solution of (4.1). Conversely, let (u, ξ) be any solution of (4.1) with regularity (4.16). From Lemma 4.2 we see, making δ smaller if necessary, that $\langle \dot{\sigma}_d, \sigma_d^p \rangle < 0$ holds within I, hence (2.83) yields

$$\begin{aligned} \langle \dot{\sigma}_d^p, \sigma_d^p \rangle &= \langle \dot{\sigma}_d, \sigma_d^p \rangle - \langle \xi, \sigma_d^p \rangle + \langle \frac{\sigma_d}{R}, \sigma_d^p \rangle |\dot{\xi}| \\ &= \langle \dot{\sigma}_d, \sigma_d^p \rangle - |\dot{\xi}| \left(r - \langle \frac{\sigma_d}{R}, \sigma_d^p \rangle \right), \end{aligned} \tag{4.17}$$

so $\langle \dot{\sigma}_d^p, \sigma_d^p \rangle < 0$ and therefore

$$|\sigma_d^p| < r\,, \quad \dot{\xi} = 0\,, \quad \dot{\sigma}_d^p = \dot{u} = \dot{\sigma}_d\,, \quad \text{a.e. in } I\,. \tag{4.18}$$

The continuity of u and (4.14) then imply the assertion. □

Theorem 4.4 (Unique Solvability up to the Boundary)
Let $\sigma_d \in W^{1,1}(t_0, t_1; \mathbf{T_d})$ be given such that (4.3) as well as (4.13) hold, let σ_{0d}^p be given by (4.14). Then the initial value problem (4.1) has a unique solution $u \in W_{loc}^{1,1}(t_0, t_1; \mathbf{T_d})$ which satisfies (4.15) near t_0; moreover, the functions $\sigma_d^p = \mathcal{S}(u\,; \sigma_{0d}^p)$ and $\sigma^e = \sigma_d - \sigma_d^p$ satisfy (4.12) as well as the boundary conditions (4.11).

Proof. This is a direct consequence of Lemma 4.3 and of Proposition 3.4. The validity of the boundary conditions at $t = t_1$ again follows from Lemma 4.2. □

Remark 4.5 (Multisurface Mróz Model)
Mróz [29] originally proposed a multisurface model which employs spheres $S_i(t)$, $0 \leq i \leq m$, moving around in $\mathbf{T_d}$, with radii $r_0 < \ldots < r_m$. The smallest surface $S_0(t)$ represents the yield surface $\partial Z^*(t)$. The inclusion property $\sigma_d(t) \in B_0(t) \subset \ldots \subset B_m(t)$ is assumed to hold for the corresponding closed balls. At any given time t, the active surface is defined by the largest index k such that $\sigma_d(t) \in S_k(t)$ but $\sigma_d(t) \in \operatorname{int} B_{k+1}(t)$; the movement of the active surface S_k with respect to S_{k+1} is determined by the geometric construction outlined in Subsection 2.4 above. If loading occurs, the smaller surfaces S_j with $j < k$ follow the movement of S_k, see Figure 6 for $k = 2$. In the case of unloading, none of the surfaces move. Thus, a general evolution decomposes into a sequence of problems of the type (4.1), (4.3), where r and R are replaced by r_k and r_{k+1} respectively. The question of existence, uniqueness and regularity of the plastic stress as well as of the plastic strain appears to be completely open. For the case of a continuous family $S_r(t)$ parametrized by $r \geq 0$, some results are available in [4].

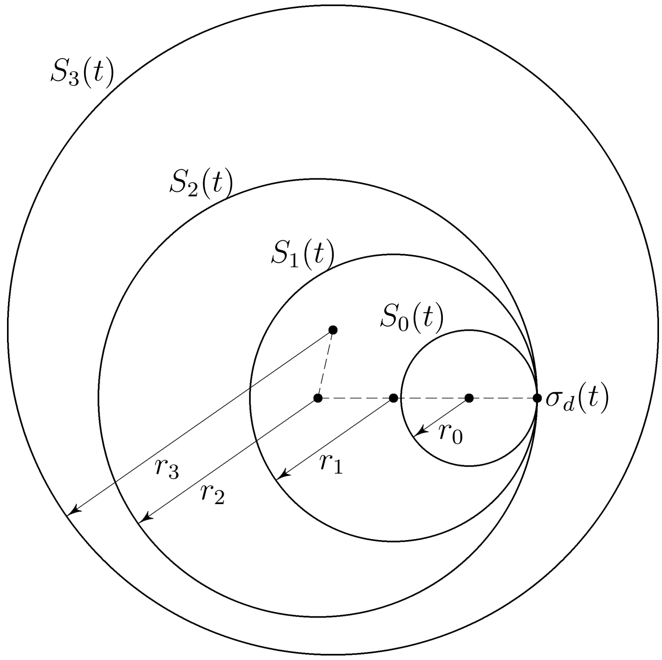

Figure 6: *The multisurface Mróz model for $m = 3$.*

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A Appendix: Properties of the Play Operator

The vector play operator with an arbitrary convex closed characteristic has already been the object of serious study, see e.g. the monographs [21], [38] and [23]. For the purposes of this paper, however, we need a result which is not adequately covered in the literature, namely the Lipschitz type estimate provided by Theorem A.5 below. Its statement and proof constitute the main purpose of this appendix. In addition, we cite some known results which we have used above in order to facilitate the reader's task.

The play operator as well as the stop operator are constructed by means of an evolution variational inequality with values in some space X; throughout this section we assume that X is a real separable Hilbert space endowed with a scalar product $\langle \cdot, \cdot \rangle$ and the corresponding norm $|x| = \langle x, x\rangle^{1/2}$. In the main body of this paper, X always represents some finite dimensional space of tensors; hence, the reader may very well be satisfied to assume that $X = \mathbb{R}^n$.

We begin with a variant of the classical result on the wellposedness of the evolution variational inequality.

Proposition A.1 *Let $Z \subset X$ be a convex closed set such that $0 \in \operatorname{Int} Z$, let $x^0 \in Z$ be given. Then for any function $u \in W^{1,1}(t_0, t_1; X)$ there exists a unique function $x \in W^{1,1}(t_0, t_1; X)$ such that*

$$\langle \dot{u}(t) - \dot{x}(t), x(t) - \tilde{x} \rangle \geq 0 \,, \quad \textit{for all } \tilde{x} \in Z \,, \quad \textit{a.e. in } (t_0, t_1) \,, \tag{A.1}$$

$$x(t) \in Z\,, \quad \textit{for all } t \in [t_0, t_1]\,, \tag{A.2}$$

$$x(t_0) = x^0\,. \tag{A.3}$$

Moreover, if $y \in W^{1,1}(t_0, t_1; X)$ *denotes the solution belonging to* $y^0 \in Z$ *and* $v \in W^{1,1}(t_0, t_1; X)$ *, then*

$$|x(t) - y(t)| \le |x^0 - y^0| + \int_{t_0}^{t} |\dot{u}(s) - \dot{v}(s)|\, ds \tag{A.4}$$

holds.

Proof. See e.g. [23], Theorem I.3.1 and Proposition I.3.9. The estimate (A.4) follows from (A.1) - (A.3), since (arguments t omitted) the inequalities

$$\langle \dot{u} - \dot{x}, x - y \rangle \ge 0\,, \quad \langle \dot{v} - \dot{y}, y - x \rangle \ge 0 \tag{A.5}$$

imply

$$|x - y| \frac{\mathrm{d}}{\mathrm{dt}} |x - y| = \frac{\mathrm{d}}{\mathrm{dt}} \frac{1}{2} |x - y|^2 \le |\dot{u} - \dot{v}|\, |x - y|\,. \tag{A.6}$$

$\square$

Thus, the evolution variational inequality (A.1), (A.2) together with the initial value (A.3) gives rise to an operator

$$x = \mathcal{S}(u\,; x^0)\,. \tag{A.7}$$

Definition A.2 *Let* $Z \subset X$ *be a convex closed set such that* $0 \in \operatorname{Int} Z$ *. The solution operator*

$$\mathcal{S} : W^{1,1}(t_0, t_1; X) \times Z \to W^{1,1}(t_0, t_1; X) \tag{A.8}$$

defined by (A.1) - (A.7) is called the stop*, the operator*

$$\mathcal{P} : W^{1,1}(t_0, t_1; X) \times Z \to W^{1,1}(t_0, t_1; X) \tag{A.9}$$

defined by

$$\mathcal{P}(u\,; x^0) = u - \mathcal{S}(u\,; x^0) \tag{A.10}$$

is called the play*. The set* Z *is called the characteristic of* $\mathcal{S}$ *respectively* $\mathcal{P}$ *.*

Proposition A.3 *Let* $Z \subset X$ *be a convex closed set such that* $0 \in \operatorname{Int} Z$ *. The play operator* $\mathcal{P}$ *with the characteristic* Z *has the following properties:*

(i) *The function* $\xi = \mathcal{P}(u\,; x^0)$ *satisfies*

$$\langle \dot{\xi}(t), \dot{u}(t) - \dot{\xi}(t) \rangle = 0\,, \quad \textit{a.e. in } (t_0, t_1)\,, \tag{A.11}$$

and, consequently,

$$|\dot{\xi}(t)| \le |\dot{u}(t)|\,, \quad \textit{a.e. in } (t_0, t_1)\,, \tag{A.12}$$

for all $u \in W^{1,1}(t_0, t_1; X)$ *and all* $x^0 \in Z$ *.*

(ii) $\mathcal{P}$ *maps* $W^{1,p}(t_0, t_1; X) \times Z$ *continuously into* $W^{1,p}(t_0, t_1; X)$ *for all* p *with* $1 \le p < \infty$ *.*

(iii) *$\mathcal{P}$ and $\mathcal{S}$ can be uniquely extended to operators*

$$\mathcal{P} : C([t_0, t_1]; X) \times Z \to C([t_0, t_1]; X) \cap BV(t_0, t_1; X)\,, \tag{A.13}$$

$$\mathcal{S} : C([t_0, t_1]; X) \times Z \to C([t_0, t_1]; X)\,, \tag{A.14}$$

which are continuous w.r.t the supremum norm

$$\| u \|_\infty = \sup_{t\in[t_0,t_1]} |u(t)|\,, \quad u \in C([t_0, t_1]; X)\,. \tag{A.15}$$

Proof. See [23], Section I.3. For a bounded set Z, part (iii) is due to [21]; the general case as well as (ii) have been proved in [22]. □

In the case where Z represents the ball $B_r(0)$ in X with radius $r > 0$ centered around 0, the play and the stop operator have additional regularity properties. Let us note first that, since the radius vector and the normal coincide for a ball, there holds

$$\dot{\xi}(t) = \alpha(t)x(t)\,, \quad \text{a.e. in } (t_0, t_1)\,, \tag{A.16}$$

where $\alpha \geq 0$ is a scalar function with $\alpha(t) = 0$ if $|x(t)| < r$. Moreover, the following estimate of Hölder type holds.

Proposition A.4 *Assume that $Z = B_r(0)$. Then for any $u, v \in C([t_0, t_1]; X)$ and any $x^0, y^0 \in Z$, the functions $\xi = \mathcal{P}(u\,; x^0)$ and $\eta = \mathcal{P}(v\,; y^0)$ satisfy the estimate*

$$\| \xi - \eta \|_\infty \leq \max\left\{ |\xi(0) - \eta(0)|\,,\ \sqrt{(\| u - v \|_\infty + r)^2 - r^2} \right\}\,. \tag{A.17}$$

Proof. See Sections 17.1 and 17.2 in [21], cf. also Theorem I.4.2 in [23]. □

We now present an estimate of Lipschitz type.

Theorem A.5 *Assume that $Z = B_r(0)$, let $u, v \in W^{1,1}(t_0, t_1; X)$ and $x^0, y^0 \in Z$ be given. Then the functions*

$$\xi = \mathcal{P}(u\,; x^0)\,, \quad \eta = \mathcal{P}(v\,; y^0)\,, \tag{A.18}$$

$$x = u - \xi = \mathcal{S}(u\,; x^0)\,, \quad y = v - \eta = \mathcal{S}(v\,; y^0)\,, \tag{A.19}$$

satisfy the estimate

$$\int_{t_0}^{t_1} |\dot{\xi} - \dot{\eta}|\, dt \leq |x^0 - y^0| + \int_{t_0}^{t_1} |\dot{u} - \dot{v}|\, dt + \frac{\sqrt{2}}{r} \int_{t_0}^{t_1} |\dot{u}||x - y|\, dt\,. \tag{A.20}$$

Proof. This will be given below. □

Corollary A.6 *The play operator $\mathcal{P}$ with the characteristic $Z = B_r(0)$ is Lipschitz continuous on bounded subsets of $W^{1,1}(t_0, t_1; X)$. Hence, the same is true for the stop operator $\mathcal{S}$.*

Proof. If we insert (A.4) into (A.20), we obtain

$$\int_{t_0}^{t_1} |\dot{\xi} - \dot{\eta}| \, dt \le \left(1 + \frac{\sqrt{2}}{r} \int_{t_0}^{t_1} |\dot{u}| \, dt \right) \left(|x^0 - y^0| + \int_{t_0}^{t_1} |\dot{u} - \dot{v}| \, dt \right) . \tag{A.21}$$

□

Theorem A.5 appears to be new. Corollary A.6 is a special case of Theorem 20.1 in [21]; however, in [21] there is no comment concerning its proof, nor is the value of the Lipschitz constant indicated. Note that in Section 3 we use the fact that the constant in front of the first integral on the right hand side of (A.20) equals 1. On the other hand, we do not know whether the Lipschitz constant given in (A.21) is optimal, cf. also Example A.8 below.
For the scalar case $\dim(X) = 1$, (A.20) can be improved to

$$\int_{t_0}^{t_1} |\dot{\xi} - \dot{\eta}| \, dt \le |x^0 - y^0| + \int_{t_0}^{t_1} |\dot{u} - \dot{v}| \, dt \, . \tag{A.22}$$

The proof of (A.22), given in [6], p. 46f., can be generalized to the vector case, if one takes into account the geometry of the sphere which is responsible for the rightmost integral in (A.20). This is done as follows. The normality rule

$$\dot{\xi} = |\dot{\xi}| \frac{x}{r} \, , \quad \text{a.e. in } (t_0, t_1) \, , \tag{A.23}$$

with $\dot{\xi} = 0$ a.e. on $\{t : |x(t)| < r\}$, follows from the variational inequality. Together with (A.11) it implies

$$|\dot{\xi}|^2 = \langle \dot{u}, \dot{\xi} \rangle = |\dot{\xi}| \frac{\langle \dot{u}, x \rangle}{r} \, , \quad \text{a.e. in } (t_0, t_1) \, , \tag{A.24}$$

so

$$\dot{\xi} = \frac{1}{r^2} \langle \dot{u}, x \rangle x \tag{A.25}$$

holds a.e. on $\{t : |x(t)| = r\}$.

Lemma A.7 *There holds*

$$|\dot{\xi} - \dot{\eta}| \le |\dot{u} - \dot{v}| + \frac{\sqrt{2}}{r} |\dot{u}| |x - y| \, , \tag{A.26}$$

a.e. on the set

$$A_r = \{t : t \in [t_0, t_1] \, , \, |x(t)| = |y(t)| = r\} \, . \tag{A.27}$$

Proof. From (A.25) and the corresponding formula

$$\dot{\eta} = \frac{1}{r^2} \langle \dot{v}, y \rangle y \, , \tag{A.28}$$

we infer that

$$|\dot{\xi} - \dot{\eta}| \le \frac{1}{r^2} \left(|\langle \dot{u} - \dot{v}, y \rangle y| + |\langle \dot{u}, x \rangle x - \langle \dot{u}, y \rangle y| \right) \tag{A.29}$$

holds a.e. on A_r. Using the identity

$$|\langle \dot{u}, x \rangle x - \langle \dot{u}, y \rangle y|^2 = r^2 \langle \dot{u}, x - y \rangle^2 + \langle \dot{u}, x \rangle \langle \dot{u}, y \rangle |x - y|^2 \, , \tag{A.30}$$

we easily derive the assertion. □

Proof of Theorem A.5. The identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(|x(t)|^2-|y(t)|^2\right)=-r(|\dot{\xi}(t)|-|\dot{\eta}(t)|)+(\langle\dot{u}(t),x(t)\rangle-\langle\dot{v}(t),y(t)\rangle)\,, \tag{A.31}$$

which holds a.e. on $[t_0,t_1]$, follows directly from (A.23). The crucial observation is that actually

$$\Big||\dot{\xi}(t)|-|\dot{\eta}(t)|\Big|+\frac{1}{2r}\frac{\mathrm{d}}{\mathrm{dt}}\Big||x(t)|^2-|y(t)|^2\Big|\le\frac{1}{r}\,|\langle\dot{u}(t),x(t)\rangle-\langle\dot{v}(t),y(t)\rangle| \tag{A.32}$$

holds a.e. on $[t_0,t_1]$. On the set $A=\{t:|x(t)|=|y(t)|\}$, (A.32) follows directly from (A.31) since in that case the left hand side of (A.31) is zero almost everywhere. To prove (A.32) on the complement of A, by virtue of

$$\left(|\dot{\xi}(t)|-|\dot{\eta}(t)|\right)\left(|x(t)|^2-|y(t)|^2\right)\ge 0\,,\quad \text{a.e. in } (t_0,t_1)\,, \tag{A.33}$$

which trivially follows from the fact that $|\dot{\xi}(t)|\ne 0$ only if $|x(t)|=r$, it suffices to multiply both sides of (A.31) with the sign of $|x(t)|^2-|y(t)|^2$. We now claim that

$$|\dot{\xi}-\dot{\eta}|+\frac{1}{2r}\frac{\mathrm{d}}{\mathrm{dt}}\Big||x|^2-|y|^2\Big|\le|\dot{u}-\dot{v}|+\frac{\sqrt{2}}{r}|\dot{u}||x-y|\,,\quad \text{a.e. in } (t_0,t_1)\,. \tag{A.34}$$

Indeed, (A.34) holds on A_r because of (A.26). On the complement of A_r, we have $|\dot{\xi}-\dot{\eta}|=||\dot{\xi}|-|\dot{\eta}||$, thus (A.34) follows from (A.32) and the inequality

$$\frac{1}{r}\,|\langle\dot{u},x\rangle-\langle\dot{v},y\rangle|\le|\dot{u}-\dot{v}|+\frac{1}{r}|\dot{u}||x-y|\,. \tag{A.35}$$

We now integrate (A.34) over $[t_0,t_1]$ to obtain the assertion (A.20) of Theorem A.5. □

One may ask whether the value of the Lipschitz constant given by (A.21) is best possible. It turns out that we can use Example I.4.3 of [23] to exhibit, for any $\varepsilon>0$, a pair of input functions u,v such that the inequalities

$$\int_0^1|\dot{v}(t)|\,dt>\frac{1}{\varepsilon}\,,\quad \int_0^1|\dot{u}(t)-\dot{v}(t)|\,dt<\varepsilon\,, \tag{A.36}$$

$$\int_0^T|\dot{\xi}(t)-\dot{\eta}(t)|\,dt\ge\left(\frac{1}{2r}\int_0^T|\dot{v}(t)|\,dt-\varepsilon\right)\int_0^T|\dot{u}(t)-\dot{v}(t)|\,dt\,, \tag{A.37}$$

holds. Thus, the gap between the constant in (A.21) and the optimal one is characterized by a factor of at most $2\sqrt{2}$. In particular, the play operator $\mathcal{P}$ is not globally Lipschitz continuous on $W^{1,1}(t_0,t_1;X)$, if $\dim(X)>1$.

Example A.8 Let $X=\mathbb{R}^2$, $Z=B_r(0)$. For $h<0$ and $\alpha>0$ we consider the inputs

$$u(t)=(r+h)\begin{pmatrix}\cos\alpha t\\ \sin\alpha t\end{pmatrix}\,,\quad v(t)=r\begin{pmatrix}\cos\alpha t\\ \sin\alpha t\end{pmatrix}\,, \tag{A.38}$$

with the intention to let h tend to 0 and α tend to infinity. If we choose $x(0)=y(0)=(1,0)$ as initial values, we obtain

$$x(t)=r\begin{pmatrix}\cos(\alpha t+\rho(t))\\ \sin(\alpha t+\rho(t))\end{pmatrix}\,,\quad \eta=0\,,\quad y=v\,, \tag{A.39}$$

where ρ is the solution of the initial value problem

$$\dot{\rho} = \alpha \left(\frac{r+h}{r} \cos(\rho) - 1 \right), \quad \rho(0) = 0, \tag{A.40}$$

that is,

$$\rho(t) = 2 \arctan \left(\sqrt{\frac{h}{2r+h}} \tanh \left(\frac{\sqrt{h(2r+h)}}{2r} \alpha t \right) \right). \tag{A.41}$$

We then obtain

$$\begin{aligned} |\dot{\xi}(t) - \dot{\eta}(t)|^2 &= |\dot{u}(t) - \dot{x}(t)|^2 \\ &= \alpha^2 (r+h)^2 [(\cos\rho \sin(\alpha t + \rho) - \sin \alpha t)^2 + (\cos \alpha t - \cos\rho \cos(\alpha t + \rho))^2] \\ &= \alpha^2 (r+h)^2 \sin^2 \rho(t), \end{aligned} \tag{A.42}$$

hence

$$|\dot{\xi}(t) - \dot{\eta}(t)| = 2\alpha(r+h) \frac{\tan \frac{\rho(t)}{2}}{1 + \tan^2 \frac{\rho(t)}{2}} = 2\alpha(r+h) \frac{\sqrt{h(2r+h)} f(t)}{2r + h + h f^2(t)}, \tag{A.43}$$

where

$$f(t) = \tanh \left(\frac{\sqrt{h(2r+h)}}{2r} \alpha t \right). \tag{A.44}$$

This yields the inequality

$$\alpha \sqrt{h(2r+h)} f(t) \le |\dot{\xi}(t) - \dot{\eta}(t)| \le \alpha \frac{2r+2h}{2r+h} \sqrt{h(2r+h)} f(t). \tag{A.45}$$

Note that we have

$$\alpha \sqrt{h(2r+h)} \int_0^1 f(t)\, dt = 2r \int_0^{\frac{\sqrt{h(2r+h)}}{2r}\alpha} \tanh s\, ds = 2r \log \left(\cosh \frac{\sqrt{h(2r+h)}}{2r} \alpha \right), \tag{A.46}$$

and

$$\int_0^1 |\dot{v}(t)|\, dt = \alpha r, \quad \int_0^1 |\dot{u}(t) - \dot{v}(t)|\, dt = h\alpha. \tag{A.47}$$

From (A.45) it therefore follows that for every fixed $\alpha > 0$ we have

$$\lim_{h \downarrow 0} \frac{\int_0^1 |\dot{\xi}(t) - \dot{\eta}(t)|\, dt}{\int_0^T |\dot{u}(t) - \dot{v}(t)|\, dt} = \frac{\alpha}{2} = \frac{1}{2r} \int_0^1 |\dot{v}(t)|\, dt. \tag{A.48}$$

With the pair (u, v) we thus achieve (A.36), (A.37) for a given $\varepsilon > 0$ if we choose $\alpha > 0$ sufficiently large and $h > 0$ sufficiently small.

Finally, we recall the following result, which goes back to Visintin ([38]) for the scalar (i.e., $\dim(X) = 1$) case.

Proposition A.9 *Let $\{u_n\}$ and $\{x_n^0\}$ be sequences in $C([t_0, t_1]; X)$ respectively $B_r(0)$ such that $\| u_n - u \|_\infty \to 0$ and $|x_n^0 - x| \to 0$ for some $u \in C([t_0, t_1]; X)$ and $x^0 \in B_r(0)$, set $\xi_n = \mathcal{P}(u_n ; x_n^0)$ and $\xi = \mathcal{P}(u ; x^0)$. Then there holds*

$$\lim_{n \to \infty} \mathrm{Var}_{[t_0, t_1]} \xi_n = \mathrm{Var}_{[t_0, t_1]} \xi. \tag{A.49}$$

Proof. See Proposition I.4.11 in [23]. □

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