A remark on the local Lipschitz continuity of vector hysteresis operators

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Abstract. It is known that the vector stop operator with a convex closed characteristic Z of class C^1 is locally Lipschitz in the space of absolutely continuous functions if the unit outward normal mapping n is Lipschitz on the boundary ∂Z of Z. We prove that in the regular case, this condition is also necessary.

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1 Introduction

Mathematical models of multidimensional hysteresis phenomena in elastoplasticity or ferromagnetism are often based on the variational inequality (see e. g. [Al, Be, Bro, BK, DL, K1, NH, V])

$$\left\{ \begin{array}{l} \langle \dot{u}(t) - \dot{x}(t), x(t) - \varphi \rangle \geq 0 \qquad \forall \varphi \in Z\,, \\ x(t) \in Z \qquad \forall t \in [0,T]\,, \\ x(0) = x^0 \in Z\,, \end{array} \right. \tag{1.1}$$

where $u \in W^{1,1}(0,T;X)$ is a given function, X a Hilbert space endowed with a scalar product $\langle \cdot, \cdot \rangle$, $Z \subset X$ is a convex closed set, $t \in [0,T]$ is the time variable and the dot denotes the derivative with respect to t.

The existence of a unique solution $x \in W^{1,1}(0,T;X)$ to problem (1.1) is a special case of classical results for evolution variational inequalities, cf. e. g. [Bre, DL].

In stochastics, inequality (1.1) is known as a special case of the *Skorokhod problem* ([DI, DN]). In the theory of hysteresis operators, the solution mapping

$$\mathcal{S}: \, Z \times W^{1,1}(0,T;X) \to W^{1,1}(0,T;X) : \, (x^0, u) \mapsto x \tag{1.2}$$

is called the *stop operator with characteristic* Z and its properties have been systematically studied (see [KP, V, K1, K2]) together with its extension to the space $C([0,T];X)$ of continuous functions. The dynamics described by the operator $\mathcal{S}$ is a special case of a *sweeping process*, see [M].

Analytical properties of the stop in the space $W^{1,1}(0,T;X)$ endowed with the norm

$$|u|_{1,1} \, := \, |u(0)| + \int_0^T |\dot{u}(t)|\, dt \tag{1.3}$$

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depend substantially on the geometry of the characteristic Z. The operator $\mathcal{S} : Z \times W^{1,1}(0,T;X) \to W^{1,1}(0,T;X)$ is always continuous, see Theorem I.3.12 of [K1]. It was conjectured without proof in [KP] that this mapping is Lipschitz if Z is a polyhedron and locally Lipschitz if the boundary ∂Z of Z is smooth. These statements have been rigorously proved only recently in [DT] and [D], respectively. In [D], it was shown that the Lipschitz continuity of the mapping

$$n : \partial Z \to \partial B_1(0) \tag{1.4}$$

(by $B_r(z)$ we denote the ball centered at $z \in X$ with radius $r > 0$), which with each $x \in \partial Z$ associates the unit outward normal $n(x)$ to Z at the point x, is sufficient for the local Lipschitz continuity of the stop. Another proof which also yields an explicit upper bound for the Lipschitz coefficient (optimal if Z is a ball) can be found in [K2] as a generalization of the technique used in [BK] for the ball.

Example 3.2 of [D] shows that the stop is not necessarily locally Lipschitz if the mapping n is only $1/2$-Hölder continuous. The aim of this paper is to fill the gap and to prove that the local Lipschitz continuity cannot be expected if ∂Z is of class C^1 and the ratio $|n(x) - n(y)|/|x - y|$, $x, y \in \partial Z$, is unbounded.

Let us note that this is not just an academic question. A precise upper bound for the Lipschitz coefficient of the stop has been substantially exploited in [BK] for proving the well-posedness of constitutive laws of elastoplasticity with nonlinear kinematic hardening.

2 Main result

We consider the simplest case $X = \mathbb{R}^2$ and fix a convex closed set $Z \subset X$ of class C^1 in such a way that there exists a point $x^* \in \partial Z$ for which we have

$$\lim_{\substack{x \to x^* \\ x \in \partial Z}} |n(x) - n(x^*)|/|x - x^*| = +\infty \,. \tag{2.1}$$

By shifting and rotating the coordinate system we may assume that $x^* = 0$ and that there exists $\varepsilon > 0$ such that

$$Z \cap \left([-\varepsilon, \varepsilon]^2\right) = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in [-\varepsilon, \varepsilon]^2 \,;\; b \ge G(a) \right\}, \tag{2.2}$$

where $G : [-\varepsilon, \varepsilon] \to \mathbb{R}^+$ is a convex function, $G(0) = 0$, and its derivative $g = G'$ is continuous, increasing, $g(0) = 0$ and $\lim_{a \to 0+} g(a)/a = +\infty$ (see Fig. 1).

We make the following simplifying assumptions.

Hypothesis 2.1

(i) $G : [-\varepsilon, \varepsilon] \to \mathbb{R}^+$ *is convex and even,* $G(0) = 0$,

(ii) $g = G'$ *is increasing and concave in* $[0, \varepsilon[$, $g(0) = 0$, $g'(0+) = +\infty$.

The rest of this paper is devoted to the proof of the following result.

Theorem 2.2 *Let $Z \subset \mathbb{R}^2$ be a convex closed set satisfying condition (2.2) and Hypothesis 2.1. Then for every $R > 0$ there exists a function $u \in W^{1,1}(0,1;\mathbb{R}^2)$ such that $|u|_{1,1} \le 1$, and initial conditions $x^0, y^0 \in Z$ such that the functions $x = \mathcal{S}(x^0, u)$, $y = \mathcal{S}(y^0, u)$, where $\mathcal{S}$ is the stop operator (1.2), satisfy the inequality*

$$\int_0^1 |\dot{x}(t) - \dot{y}(t)|\, dt \ge R\left|x^0 - y^0\right|. \tag{2.3}$$

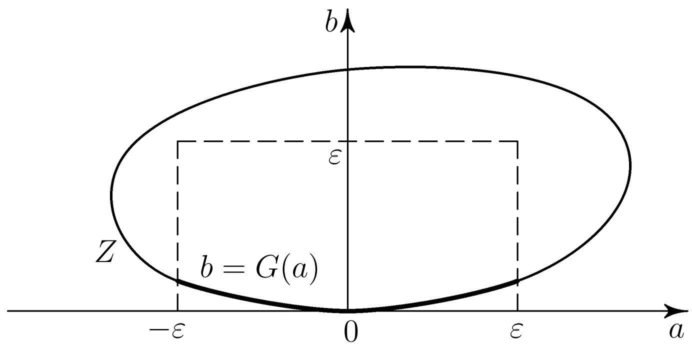

Figure 1: The convex characteristic Z

3 Proof of Theorem 2.2

We follow the construction from Example 3.2 of [D]. Taking a smaller $\varepsilon > 0$ if necessary, we may assume that

$$\varepsilon < \frac{1}{2\sqrt{2}}, \quad g(\varepsilon) < \frac{1}{\sqrt{2}}. \tag{3.1}$$

We fix some $a_0 \in]0, \varepsilon[$ (arbitrary, for the moment) and construct a sequence $\{a_k;\, k \in \mathbb{N} \cup \{0\}\}$ by induction in the following way. Let $a_0 > a_1 > \ldots > a_k > 0$ be already given and let us consider the differential equation

$$\dot{r}_k = \frac{1 - g(a_k - t)\, g(r_k)}{1 + g^2(r_k)}, \quad r_k(0) = 0, \tag{3.2}$$

in the domain $(t, r_k) \in \mathcal{D}_k := [0, a_k] \times [0, a_k]$. The function

$$F : (t, r_k) \mapsto \frac{1 - g(a_k - t)\, g(r_k)}{1 + g^2(r_k)}$$

is continuous in $\mathcal{D}_k$ and $0 < F(t, r_k) < 1$ whenever $(t, r_k) \in \mathcal{D}_k$, $r_k > 0$. Moreover, the function $r_k \mapsto F(t, r_k)$ is decreasing in $[0, a_k]$ for every $t \in [0, a_k]$; problem (3.2) therefore admits in $\mathcal{D}_k$ a unique maximal solution $r_k : [0, a_k] \to [0, a_k]$, $0 < \dot{r}_k(t) < 1$ for all $t \in]0, a_k[$. Putting

$$a_{k+1} := r_k(a_k) \tag{3.3}$$

we thus have $0 < a_{k+1} < a_k$ and the induction step is complete. By construction, we moreover have for every $k \in \mathbb{N} \cup \{0\}$

$$a_{k+1} \ \geq \ a_k \, \frac{1 - g^2(a_k)}{1 + g^2(a_k)} \ \geq \ a_k \left(1 - 2\, g^2(a_k)\right). \tag{3.4}$$

For $k \in \mathbb{N} \cup \{0\}$ put

$$t_0 := 0\,, \quad t_{k+1} := t_k + a_k\,, \quad T := \sum_{k=0}^{\infty} a_k \leq \infty\,. \tag{3.5}$$

We choose two points $x^0, y^0 \in Z$ in the form

$$x^0 := \begin{pmatrix} -a_0 \\ G(a_0) \end{pmatrix}, \quad y^0 := \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.6}$$

and define functions $\bar{u},\, \bar{x},\, \bar{y} \,:\, [0,T[\to \mathbb{R}^2$ by the formulas

$$\bar{u}(0) := 0\,, \quad \bar{x}(0) := x^0\,, \quad \bar{y}(0) := y^0\,, \tag{3.7}$$

$$\bar{u}(t) := \begin{cases} \bar{u}(t_j) + \begin{pmatrix} t - t_j \\ G(t_{j+1} - t) - G(a_j) \end{pmatrix} & \text{for } \ t \in \,]t_j, t_{j+1}]\,, \ \ j \text{ even}\,, \\[2ex] \bar{u}(t_j) + \begin{pmatrix} t_j - t \\ G(t_{j+1} - t) - G(a_j) \end{pmatrix} & \text{for } \ t \in \,]t_j, t_{j+1}]\,, \ \ j \text{ odd}\,, \end{cases} \tag{3.8}$$

$$\bar{x}(t) := \begin{cases} \bar{x}(t_j) + \bar{u}(t) - \bar{u}(t_j) & \text{for } \ t \in \,]t_j, t_{j+1}]\,, \ \ j \text{ even}\,, \\[2ex] \begin{pmatrix} -r_j(t - t_j) \\ G(r_j(t - t_j)) \end{pmatrix} & \text{for } \ t \in \,]t_j, t_{j+1}]\,, \ \ j \text{ odd}\,, \end{cases} \tag{3.9}$$

$$\bar{y}(t) := \begin{cases} \begin{pmatrix} r_j(t - t_j) \\ G(r_j(t - t_j)) \end{pmatrix} & \text{for } \ t \in \,]t_j, t_{j+1}]\,, \ \ j \text{ even}\,, \\[2ex] \bar{y}(t_j) + \bar{u}(t) - \bar{u}(t_j) & \text{for } \ t \in \,]t_j, t_{j+1}]\,, \ \ j \text{ odd}\,, \end{cases} \tag{3.10}$$

where $r_j \,:\, [0, a_j] \to [0, a_{j+1}]$ is the solution of equation (3.2) for $j \in \mathbb{N} \cup \{0\}$.

Let us check by induction that we have

$$\bar{x} \ = \ \mathcal{S}(x^0, \bar{u}), \quad \bar{y} \ = \ \mathcal{S}(y^0, \bar{u}) \qquad \text{in } \ [0, T[\,. \tag{3.11}$$

Assume that identities (3.11) hold for $t \in [0, t_k]$, and let for instance k be even, $k \geq 0$ (the case 'k odd' is analogous). For $k \geq 2$ we have

$$\bar{x}(t_k) \ = \ \begin{pmatrix} -r_{k-1}(t_k - t_{k-1}) \\ G(r_{k-1}(t_k - t_{k-1})) \end{pmatrix} = \begin{pmatrix} -a_k \\ G(a_k) \end{pmatrix}, \tag{3.12}$$

$$\begin{aligned} \bar{y}(t_k) \ &= \ \bar{y}(t_{k-1}) + \bar{u}(t_k) - \bar{u}(t_{k-1}) \\ &= \ \begin{pmatrix} r_{k-2}(t_{k-1} - t_{k-2}) \\ G(r_{k-2}(t_{k-1} - t_{k-2})) \end{pmatrix} - \begin{pmatrix} a_{k-1} \\ G(a_{k-1}) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{aligned} \tag{3.13}$$

for $k=0$ the above identities hold by the choice (3.6), (3.7) of initial conditions.

For $t \in]t_k, t_{k+1}]$ we have by definition

$$\bar{x}(t) := \bar{x}(t_k) + \bar{u}(t) - \bar{u}(t_k) = \begin{pmatrix} t - t_{k+1} \\ G(t_{k+1} - t) \end{pmatrix}, \qquad \bar{y}(t) := \begin{pmatrix} r_k(t - t_k) \\ G(r_k(t - t_k)) \end{pmatrix}.$$

In particular, both $\bar{x}$, $\bar{y}$ are absolutely continuous in $[0, t_{k+1}]$ and $\bar{x}(t)$, $\bar{y}(t)$ belong to Z for all $t \in [t_k, t_{k+1}]$. Since $\dot{\bar{x}}(t) = \dot{\bar{u}}(t)$ for all $t \in]t_k, t_{k+1}[$, the function $\bar{x}$ is automatically a solution of problem (1.1) in $[0, t_{k+1}]$. The same argument applies to $\bar{y}$ provided we check that the inequality

$$\langle \dot{\bar{u}}(t) - \dot{\bar{y}}(t), \bar{y}(t) - \varphi \rangle \ \geq\ 0 \qquad \forall \varphi \in Z \tag{3.14}$$

holds in $]t_k, t_{k+1}[$.

Equation (3.2) yields

$$\dot{r}_k(t - t_k) \ =\ \frac{1 - g(t_{k+1} - t)\, g(r_k(t - t_k))}{1 + g^2(r_k(t - t_k))} \qquad \text{for } t \in]t_k, t_{k+1}[\ , \tag{3.15}$$

hence

$$\dot{\bar{u}}(t) - \dot{\bar{y}}(t) \ =\ \frac{g(t_{k+1} - t) + g(r_k(t - t_k))}{\sqrt{1 + g^2(r_k(t - t_k))}}\ n(\bar{y}(t))\,, \tag{3.16}$$

where

$$n(\bar{y}(t)) \ :=\ \frac{1}{\sqrt{1 + g^2(r_k(t - t_k))}} \begin{pmatrix} g(r_k(t - t_k)) \\ -1 \end{pmatrix} \tag{3.17}$$

is the unit outward normal to Z at the point $\bar{y}(t)$ and inequality (3.14) follows from the convexity of Z. We have thus proved that identities (3.11) are fulfilled.

An elementary computation yields for all $j \in \mathbb{N} \cup \{0\}$

$$\begin{aligned} \int_{t_j}^{t_{j+1}} |\dot{\bar{u}}(t)|\ dt \ &=\ \int_{t_j}^{t_{j+1}} \sqrt{1 + g^2(t_{j+1} - t)}\ dt \\ &=\ \int_0^{a_j} \sqrt{1 + g^2(s)}\ ds \ \leq\ \sqrt{2}\ a_j\,, \end{aligned} \tag{3.18}$$

$$\begin{aligned} \int_{t_j}^{t_{j+1}} |\dot{\bar{x}}(t) - \dot{\bar{y}}(t)|\ dt \ &=\ \int_{t_j}^{t_{j+1}} \frac{g(t_{j+1} - t) + g(r_j(t - t_j))}{\sqrt{1 + g^2(r_j(t - t_j))}}\ dt \\ &\geq\ \frac{1}{\sqrt{2}} \int_{t_j}^{t_{j+1}} g(t_{j+1} - t)\, dt \ =\ \frac{1}{\sqrt{2}}\ G(a_j)\,. \end{aligned} \tag{3.19}$$

The proof of Theorem 2.2 consists in choosing an appropriate value of a_0 in the above construction and putting

$$u(t) \ :=\ \begin{cases} \bar{u}(t) & \text{for } t \in [0, t_n]\,, \\ \bar{u}(t_n) & \text{for } t \in]t_n, 1]\,, \end{cases} \tag{3.20}$$

with some n depending on a_0 such that $t_n < 1$. More precisely, we choose n to be the integer part of $1/(\sqrt{2}\, a_0)$,

$$n := \left[\frac{1}{\sqrt{2}\, a_0}\right], \tag{3.21}$$

and, according to assumption (3.1), we have

$$\frac{1}{2\sqrt{2}} \le n\, a_0 \le \frac{1}{\sqrt{2}}. \tag{3.22}$$

Definition (3.5) yields

$$t_n = \sum_{k=0}^{n-1} a_k \le n\, a_0 \le \frac{1}{\sqrt{2}} < 1,$$

hence formula (3.20) is meaningful. Inequality (3.18) yields

$$|u|_{1,1} = \int_0^1 |\dot u(t)|\, dt = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} |\dot{\bar u}(t)|\, dt \le \sqrt{2} \sum_{k=0}^{n-1} a_k \le 1. \tag{3.23}$$

Let now $R > 0$ be given. The proof will be complete if we check that inequality (2.3) holds for a suitable choice of a_0.

Let us first estimate the integral $\int_0^1 |\dot x(t) - \dot y(t)|\, dt$ from below. We obviously have $x = \bar x$, $y = \bar y$ in $[0, t_n]$, $\dot x = \dot y = 0$ in $]t_n, 1[$, consequently

$$\int_0^1 |\dot x(t) - \dot y(t)|\, dt = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} |\dot{\bar x}(t) - \dot{\bar y}(t)|\, dt \ge \frac{1}{\sqrt{2}} \sum_{k=0}^{n-1} G(a_k) \tag{3.24}$$

according to inequality (3.19).

We define auxiliary functions

$$\phi(s) := 2\, s\, g^2(s), \quad \Phi(s) := \int_s^\varepsilon \frac{dr}{\phi(r)} \qquad \text{for } s \in]0, \varepsilon]. \tag{3.25}$$

Then $\Phi' = -1/\phi$, $\Phi(\varepsilon) = 0$, $\Phi(0+) = +\infty$, $\phi(0) = 0$ and Hypothesis 2.1 (i) entails $\lim_{s\to 0+} \phi'(s) = 0$. Inequality (3.4) can be written in the form

$$a_{k+1} \ge a_k - \phi(a_k), \tag{3.26}$$

which implies that

$$\Phi(a_{k+1}) - \Phi(a_k) = \int_{a_{k+1}}^{a_k} \frac{dr}{\phi(r)} \le \frac{a_k - a_{k+1}}{\phi(a_{k+1})} \le \frac{\phi(a_k)}{\phi(a_k - \phi(a_k))} \tag{3.27}$$

for $k \in \mathbb{N} \cup \{0\}$. Note that

$$\lim_{s\to 0+} \frac{\phi(s) - \phi(s - \phi(s))}{\phi(s)} = \lim_{s\to 0+} \frac{1}{\phi(s)} \int_{s-\phi(s)}^{s} \phi'(r)\, dr = 0, \tag{3.28}$$

hence

$$\lim_{s\to 0+} \frac{\phi(s)}{\phi(s-\phi(s))} \;=\; 1\,. \tag{3.29}$$

Consequently, we can put

$$\alpha \;:=\; \sup_{s\in\,]0,\varepsilon]} \frac{\phi(s)}{\phi(s-\phi(s))} \;<\; \infty \tag{3.30}$$

and from inequality (3.27) it follows that

$$\Phi(a_{k+1}) - \Phi(a_k) \;\le\; \alpha \qquad \forall k \in \mathbb{N}\cup\{0\}\,. \tag{3.31}$$

Let $\Phi^{-1} : \mathbb{R}^+ \to]0,\varepsilon]$ be the inverse function to Φ. Summing up the above inequalities over k, we obtain

$$a_k \;\ge\; \Phi^{-1}(\Phi(a_0)+\alpha k) \qquad \forall k \in \mathbb{N}\cup\{0\}\,. \tag{3.32}$$

Combining relations (3.32) and (3.22), we have

$$\begin{aligned} \sum_{k=0}^{n-1} G(a_k) \;&\ge\; \sum_{k=0}^{n-1} G\left(\Phi^{-1}(\Phi(a_0)+\alpha k)\right) \;\ge\; \int_0^n G\left(\Phi^{-1}(\Phi(a_0)+\alpha x)\right)\,dx \\ &\ge\; \int_0^{\frac{1}{2\sqrt{2}a_0}} G\left(\Phi^{-1}(\Phi(a_0)+\alpha x)\right)\,dx\,. \end{aligned} \tag{3.33}$$

The estimates (3.33) and (3.24) together with the elementary inequality $|x^0 - y^0| = \sqrt{a_0^2 + G^2(a_0)} \le \sqrt{2}\,a_0$ show that Theorem 2.2 will be proved if

$$\limsup_{s\to 0+} \frac{1}{s}\int_0^{\frac{1}{2\sqrt{2}s}} G\left(\Phi^{-1}(\Phi(s)+\alpha x)\right)\,dx \;=\; \infty\,, \tag{3.34}$$

that is,

$$\limsup_{s\to 0+} \frac{1}{s}\int_{\Phi(s)}^{\Phi(s)+\frac{\beta}{s}} G\left(\Phi^{-1}(y)\right)\,dy \;=\; \infty \qquad \text{with} \qquad \beta = \frac{\alpha}{2\sqrt{2}}\,. \tag{3.35}$$

By Hypothesis 2.1 (ii), we have $2\,G(z) \ge z g(z)$ and $g(z) \le g(s)$ for $0 < z < s < \varepsilon$, hence

$$\begin{aligned} \frac{1}{s}\int_{\Phi(s)}^{\Phi(s)+\frac{\beta}{s}} G\left(\Phi^{-1}(y)\right)\,dy \;&=\; \frac{1}{2s}\int_{\Phi^{-1}\left(\Phi(s)+\frac{\beta}{s}\right)}^{s} \frac{G(z)}{zg^2(z)}\,dz \\ &\ge\; \frac{1}{4g(s)}\left(1 - \frac{1}{s}\,\Phi^{-1}\left(\Phi(s)+\frac{\beta}{s}\right)\right)\,. \end{aligned} \tag{3.36}$$

Let us define an auxiliary function $\psi(v) := 1/\Phi^{-1}(v)$ for $v > 0$. Then $\psi(0) = 1/\varepsilon$, $\lim_{v\to+\infty}\psi(v) = +\infty$, ψ is increasing in $\mathbb{R}^+$ and satisfies the differential equation

$$\psi'(v) \;=\; 2\,\psi(v)\;g^2\left(\frac{1}{\psi(v)}\right)\,. \tag{3.37}$$

By the change of variables $s = 1/\psi(v)$ we obtain

$$\frac{1}{s}\,\Phi^{-1}\left(\Phi(s) + \frac{\beta}{s}\right) \;=\; \frac{\psi(v)}{\psi(v+\beta\psi(v))}\,. \tag{3.38}$$

According to the Mean Value Theorem, for all $v > 0$ we have

$$\frac{\psi(v+\beta\psi(v))}{\psi(v)} \;=\; 1 + \beta\,\psi'(m(v)) \tag{3.39}$$

for some $m(v) \in [v, v+\beta\psi(v)]$. Using Eq. (3.37) and the fact that the function $s \mapsto g(s)/s$ is decreasing, we obtain

$$\begin{aligned} \frac{\psi(v+\beta\psi(v))}{\psi(v)} \;&=\; 1 + 2\,\beta\,\psi(m(v))\,g^2\left(\frac{1}{\psi(m(v))}\right) \\ &\geq\; 1 + 2\,\beta\,\frac{\psi^2(v)\,g^2\left(\frac{1}{\psi(v)}\right)}{\psi(m(v))} \\ &\geq\; 1 + 2\,\beta\,\frac{\psi^2(v)\,g^2\left(\frac{1}{\psi(v)}\right)}{\psi(v+\beta\psi(v))}\,, \end{aligned} \tag{3.40}$$

hence

$$\frac{\psi(v+\beta\psi(v))}{\psi(v)} \;\geq\; \frac{1}{2} + \left(\frac{1}{4} + 2\,\beta\,\psi(v)\,g^2\left(\frac{1}{\psi(v)}\right)\right)^{1/2} \qquad \forall v > 0\,. \tag{3.41}$$

In terms of $s = 1/\psi(v)$, the above inequality reads

$$\frac{1}{s}\Phi^{-1}\left(\Phi(s) + \frac{\beta}{s}\right) \;\leq\; \left(\frac{1}{2} + \left(\frac{1}{4} + 2\,\beta\,\frac{g^2(s)}{s}\right)^{1/2}\right)^{-1} \qquad \forall s \in\,]0,\varepsilon]\,, \tag{3.42}$$

and we conclude that for all $s \in\,]0,\varepsilon]$ we have

$$\frac{1}{g(s)}\left(1 - \frac{1}{s}\Phi^{-1}\left(\Phi(s) + \frac{\beta}{s}\right)\right) \;\geq\; 2\,\beta\,\frac{g(s)}{s}\,\left(\frac{1}{2} + \left(\frac{1}{4} + 2\,\beta\,\frac{g^2(s)}{s}\right)^{1/2}\right)^{-2}\,. \tag{3.43}$$

Taking into account estimates (3.36) and (3.43), we see that relation (3.35) is fulfilled provided

$$\limsup_{s\to 0+}\,\frac{g(s)}{s}\,\left(\frac{1}{2} + \left(\frac{1}{4} + 2\,\beta\,\frac{g^2(s)}{s}\right)^{1/2}\right)^{-2} \;=\; +\infty\,. \tag{3.44}$$

We distinguish two cases.

A. $\exists \gamma > 0 :\ \limsup_{s\to 0+}\, g^2(s)/s \;\geq\; \gamma$.

The function $x \mapsto x\,\left(1/2 + (1/4 + x)^{1/2}\right)^{-2}$ is increasing for $x > 0$, hence

$$\limsup_{s\to 0+}\,\frac{g^2(s)}{s}\,\left(\frac{1}{2} + \left(\frac{1}{4} + 2\,\beta\,\frac{g^2(s)}{s}\right)^{1/2}\right)^{-2} \;\geq\; \gamma\,\left(\frac{1}{2} + \left(\frac{1}{4} + 2\,\beta\,\gamma\right)^{1/2}\right)^{-2} \;>\; 0$$

and $\lim_{s\to 0+} 1/g(s) \;=\; +\infty$, which yields the assertion.

B. $\lim_{s \to 0+} g^2(s)/s \;=\; 0\,.$

Then

$$\lim_{s \to 0+} \left(\frac{1}{2} + \left(\frac{1}{4} + 2\,\beta\,\frac{g^2(s)}{s} \right)^{1/2} \right)^{-2} \;=\; 1$$

and $\lim_{s \to 0+} g(s)/s \;=\; +\infty$, with the same conclusion as above. Theorem 2.2 is proved.

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